

Frege, Boolos, and Logical Objects*

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In [1884], Frege formulated some ‘abstraction’ principles that imply the existence of abstract objects in classical logic. The most well-known of these is:

Hume’s Principle:

The number of F s is identical to the number of G s iff there is a one-to-one correspondence between the F s and the G s.

$$\#F = \#G \leftrightarrow F \approx G$$

When added to classical second-order logic (but not *free* second-order logic), this implies the existence of numbers, which Frege regarded as ‘logical objects’. He also developed analogous principles for such abstract objects as directions and shapes:

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Directions:

The direction of line a is identical to the direction of line b iff a is parallel to b .

$$\vec{a} = \vec{b} \leftrightarrow a \parallel b$$

Shapes:

The shape of figure a is identical to the shape of figure b iff a is (geometrically) similar to b .

$$\check{a} = \check{b} \leftrightarrow a \cong b$$

With a system that allows for propositions (intensionally conceived) and distinguishes them from truth-values, one could also propose a principle governing truth-values in a way analogous to the above. Of course, Frege wouldn’t have formulated such a principle. In his system, sentences denote truth values and the identity symbol can do the job of the biconditional. Thus, in his system, the truth value of p equals the truth value of q just in case $p = q$. But if we use a modern-day predicate logic instead of a term logic, distinguish propositions from truth values, and allow the propositional variables ‘ p ’ and ‘ q ’ to range over propositions, something like the following principle governing truth values would be assertible for a modern-day Fregean (Boolos [1986], 148):

Truth Values:

The truth value of p is identical with the truth value of q iff p is equivalent to q .

$$(p^\circ = q^\circ) \leftrightarrow (p \leftrightarrow q)$$

Here, and in what follows, the biconditional is *not* to be construed as an identity sign.

Frege might have called all of these objects ‘logical objects’, since in [1884], he thought he had a way of defining them all in terms of a paradigm logical object, namely, extensions. Let us for the moment use α, β as metavariables ranging over variables for objects, concepts, or propositions, φ as a metavariable ranging over formulas, and $\check{[\lambda\alpha\varphi]}$ for the extension of the (first- or second-level) concept $[\lambda\alpha\varphi]$. Frege then defined ([1884], §68) ‘the number of F s’ ($\#F$), ‘the direction of line a ’ (\vec{a}), and ‘the shape of figure c ’ (\check{c}) as follows:

$$\#F =_{df} \check{[\lambda G G \approx F]}$$

$$\vec{a} =_{df} \sim[\lambda x x || a]$$

$$\check{c} =_{df} \sim[\lambda x x \cong c]$$

If we ignore the infamous Section 10 of Frege's *Grundgesetze*, then the above definitions suggest the following definition of 'the truth value of proposition p ' (p°), given the principle Truth Values:

$$p^\circ =_{df} \sim[\lambda q q \leftrightarrow p]$$

All of these logical objects would thereby have been systematized by:

Basic Law V:

The extension of the concept $[\lambda \alpha \varphi]$ is identical to the extension of the concept $[\lambda \alpha \psi]$ iff all and only the objects falling under the concept φ fall under the concept ψ .

$$\sim[\lambda \alpha \varphi] = \sim[\lambda \alpha \psi] \leftrightarrow \forall \beta ([\lambda \alpha \varphi] \beta \leftrightarrow [\lambda \alpha \psi] \beta)$$

Since it follows from Basic Law V that for any formula φ , $\exists x (x = \sim[\lambda \alpha \varphi])$ ¹ the logical objects defined above ($\#_F$, \vec{a} , \check{c} , and p°) would all be well-defined. Moreover, a Fregean would then suggest that we derive Hume's Principle, Directions, Shapes, and Truth Values from equivalence relations like equinumerosity, parallelism, geometric similarity, and material equivalence.

Frege's program was undermined by the inconsistency of Basic Law V with second-order logic. Recently, there has been a renaissance of research on consistent Frege-style systems.² In an important series of papers, George Boolos also developed systems for reconstructing Frege's work. We'll focus on the work in Boolos [1986], [1987], [1989], and [1993]. Although in [1986] and [1993] Boolos offers reconstructions of Basic Law V that are consistent with second-order logic, we plan to show that these systems can be made to follow the pattern of his [1987] paper, and that once this pattern is recognized, a comparison becomes possible between the systems Boolos formulated and the 'object theory' formulated in Zalta [1983], [1988], [1993], and [1999]. As we shall see, each of Boolos' systems can be grounded in one non-logical axiom in the form of an explicit

¹Substitute φ for ψ in Basic Law V, derive the right condition by logic alone, from which the left condition then follows, and existentially generalize on $[\lambda \alpha \varphi]$.

²See Schroeder-Heister [1987], T. Parsons [1987], Bell [1994], Heck [1996], Burgess [1998], Wehmeier [1999], Goldfarb [2001], and Ferreira & Wehmeier [2002].

existence assertion. These explicit existence assertions can be directly compared with the comprehension principle for abstract objects in object theory. This comparison allows one to evaluate the theories of Fregean logical objects formulable in the Boolos and object-theoretic frameworks.³ As part of this evaluation, we shall develop a series of new results in object theory pertaining to logical objects.

1. Boolos' Systems: Numbers, New V, and New V*

In examining Frege's work in the *Die Grundlagen der Arithmetik*, Boolos arrived at the following insight:

Thus, although a division into two types of entity, concepts and objects, can be found in the *Foundations*, it is plain that Frege uses not one but two instantiation relations, 'falling under' (relating some objects to some concepts), and 'being in' (relating some concepts to some objects), and that both relations sometimes obtain reciprocally: the number 1 is an object that falls under 'identical with 1', a concept that is in the number 1. (Boolos [1987], p. 3)

Boolos proceeds to salvage the work in the *Foundations* by introducing the notion ' G is in the extension x ' ($G\eta x$) as the second instantiation relation ('being in') mentioned in the above passage. He then develops what he calls 'Frege Arithmetic' by adding the following assertion to second-order logic:

Numbers:

For every concept F , there is a unique (extension) x which has 'in' it all and only those concepts G which are in one-to-one correspondence with F .

$$\forall F \exists ! x \forall G (G\eta x \leftrightarrow G \approx F)$$

³Wright [1983] and Hale [1987] have developed somewhat different method for asserting the existence of logical objects, namely, by adding Fregean-style biconditionals to second-order logic whenever an appropriate equivalence condition on objects or concepts presents itself. However, in the present paper, we shall be focusing on those systems which try to reconstruct Frege's theory of logical objects by adding a *single* additional axiom to second-order logic—one which is sufficient for defining a wide variety of Fregean logical objects and deriving their governing biconditionals.

Boolos then goes on to show that the Dedekind/Peano axioms are derivable from this claim, by first deriving Hume's Principle and then following Frege's plan in the *Grundlagen* and *Grundgesetze*.⁴ Boolos notes that the explicit assertion of the existence of numbers embodied by Numbers is a way of making clear the commitment implicit in the use of the definite article in 'the number of F s'.⁵

In his papers of [1986] and [1993], Boolos returned to the idea of salvaging Frege's work by using biconditionals which are weakenings of Basic Law V. But as we now plan to show, these other developments can be recast in the terms Boolos used for the development of Frege Arithmetic, namely, as explicit existence assertions. To see this, let us rehearse the definitions and principles Boolos provides, first in [1986] and then in [1993].

In [1986], Boolos defines the notion of a *small* concept based on a 'limitation of size' conception of sets. We may paraphrase his definition

⁴Define $\#F$ as the object x which has in it all and only those concepts G which are equinumerous with F . Then Hume's Principle is derivable from Numbers. As is now well-known, the Dedekind/Peano axioms become derivable from Hume's Principle using Frege's techniques. See Frege [1893], Parsons [1965], Wright [1983], and Heck [1993].

⁵Boolos says,

In §68 Frege first asserts that F is equinumerous with G iff the extension of 'equinumerous with F ' is the same as that of 'equinumerous with G ' and then defines the number belonging to the concept F as the extension of the concept 'equinumerous with the concept F '. Since Frege, like Russell, holds that existence and uniqueness are implicit in the use of the definite article, he supposes that for any concept F , there is a unique extension of the concept 'equinumerous with F '. Thus, the sentence Numbers expresses this supposition of uniqueness in the language of Frege Arithmetic; it is the sole nonlogical assumption utilized by Frege in the course of the mathematical work done in §§68-83.

([1987], 5-6 = [1998], 186)

as follows:⁶

$$\text{Small}(F) =_{df} F \not\approx V,$$

where V is the universal concept $[\lambda x x = x]$. He then defines the equivalence condition F is (logically) similar to G ($F \sim G$) as follows:

$$F \sim G =_{df} (\text{Small}(F) \vee \text{Small}(G)) \rightarrow F \equiv G,$$

where $F \equiv G$ just means $\forall x(Fx \leftrightarrow Gx)$. Now Boolos introduces the logical object called the *subtension of F* ($*F$) by adding the following Fregean biconditional principle to second-order logic:

New V:

The subtension of F is identical to the subtension of G iff F and G are (logically) similar.

$$*F = *G \leftrightarrow F \sim G$$

He then derives number theory by first deriving finite set theory (extensionality, separation and adjunction).⁷

Note that one can reformulate New V in terms of an explicit existence assertion in the same way that Boolos reformulated Hume's Principle as Numbers, using the second 'instantiation' relation η , as follows:

⁶This definition is equivalent to the one Boolos uses, namely, that the universal concept can't be put in one-one correspondence with a subconcept of a small concept, as the following proof of equivalence shows:

(\rightarrow) Assume F is small under Boolos' definition, to show F is small in our sense. So we assume $\neg\exists G(G \subseteq F \ \& \ V \approx G)$; i.e., $\forall G(G \subseteq F \rightarrow V \not\approx G)$. Since F is a subconcept of itself, $V \not\approx F$.

(\leftarrow) Assume F is small under our definition, to show F is small in Boolos' sense. So we assume $V \not\approx F$. To show that $\forall G(G \subseteq F \rightarrow V \approx G)$, assume that $G \subseteq F$, to show $V \approx G$. For *reductio*, assume $V \not\approx G$. But, it is a corollary to the Cantor-Schröder-Bernstein theorem in second order logic (see Shapiro [1991], 102-103), that if $G \subseteq F \ \& \ V \approx G \rightarrow V \approx F$. So it follows that $V \approx F$, contrary to hypothesis.

⁷An object is a 'set' whenever it is the subtension of a small concept; ' $x \in y$ ' is defined to be true whenever $\exists F(y = *F \ \& \ Fx)$; '0' is defined to be subtension of the concept $[\lambda x x \neq x]$ (i.e., $0 = *[\lambda x x \neq x]$); the object ' $z+w$ ' is defined as the subtension of the concept $[\lambda x x \in z \vee x = w]$ and it is proved that if z is a set, so is $z+w$; the concept of 'hereditarily finite' is then defined as $[\lambda x \forall F(F0 \ \& \ \forall z \forall w(Fz \ \& \ Fw \rightarrow Fz+w) \rightarrow Fx)]$ (i.e., as the property of having every property F which is had by 0 and which is had by $z+w$ whenever z and w have it); it is then proved that all hereditarily finite objects and their members are (hereditarily finite) sets; and, finally, it is shown that the hereditarily finite sets validate the axioms of finite set theory.

Explicit New V:

$$\forall F \exists! x \forall G (G \eta x \leftrightarrow G \sim F)$$

If Explicit New V were now to replace Numbers in the language Boolos used for Frege Arithmetic, and we define:

$$*F =_{df} \iota x \forall G (G \eta x \leftrightarrow G \sim F),$$

then New V is derivable from Explicit New V.⁸

We may do something similar for the Basic Law V replacement proposed in Boolos [1993]. In this paper (231), Boolos builds upon an idea of Terence Parsons to develop an alternative to New V. We will call this alternative New V*. Boolos defines ([1993], 231 = [1998], 234):

$$Bad(F) =_{df} F \approx V \approx \bar{F},$$

where \bar{F} is $[\lambda x \neg Fx]$. In other words, a *good* (i.e., non-bad) concept F is such that either F or its complement fails to be in one-to-one correspondence with the universal concept V . Boolos then essentially defines the equivalence condition ' F is (logically) similar* to G ' ($F \sim^* G$):

$$F \sim^* G =_{df} (Good(F) \vee Good(G)) \rightarrow F \equiv G$$

Finally Boolos introduces a logical object that we shall call the *the subtension** of F by adding the following Fregean biconditional principle to second-order logic:

New V*:

The subtension* of F = the subtension* of G iff F is (logically) similar* to G .

$$*F = *G \leftrightarrow F \sim^* G$$

Boolos then notes that number theory can be derived by defining 0 as $*[\lambda y y \neq y]$ and successor in either the Zermelo or von Neumann way ([1993], 231 = [1998], 234).

⁸Note that the definition of $*F$ is well-formed because *Explicit New V* guarantees the existence and uniqueness of an individual satisfying the condition. From left to right, let $a = *A = *B$, then by the definition of $*A$, we know $\forall G (G \eta a \leftrightarrow A \sim G)$. Since $A \sim A$ we know $A \eta a$. By the definition of $*B$, $\forall G (G \eta a \leftrightarrow B \sim G)$ so since $A \eta a$ then $B \sim A$. From right to left, assume $A \sim B$ and let $a = *A$. By the definition of $*A$, $\forall G (G \eta a \leftrightarrow A \sim G)$. Since $A \sim B$ this is equivalent to $\forall G (G \eta a \leftrightarrow B \sim G)$ which is just the definition of $*B$ so $*A = *B$.

It is straightforward to reformulate New V* to explicitly assert the existence of subtensions*, in the same way that Explicit New V constitutes the explicit existence assertion underlying New V. The relevant axiom is:

Explicit New V*:

$$\forall F \exists! x \forall G (G \eta x \leftrightarrow G \overset{*}{\sim} F)$$

Again, one can show that New V* is derivable from Explicit New V*.

The trend here is generalizable. Whenever a replacement for Basic Law V (preserving its form) is proposed, one can reformulate that replacement in terms of an explicit existence assertion using Boolos' η relation. In other words, if one has an equivalence relation R on concepts which can be used to construct an operator $\%F$ for which the biconditional $\%F = \%G \leftrightarrow R(F, G)$ holds, one can show that the following explicit existence claim is equivalent to the biconditional:⁹

$$\forall F \exists! x \forall G (G \eta x \leftrightarrow R(F, G))$$

It strikes us that an obvious question to ask is, why not just take these explicit existence assertions like Explicit New V and Explicit New V* to the logical limit? That is, why not allow arbitrary conditions φ on the right-hand side of the explicit existence claim, as follows:

Explicit Logical Objects:

$$\exists! x \forall G (G \eta x \leftrightarrow \varphi)$$

Boolos himself must have considered this question, for he showed that Explicit Logical Objects is inconsistent with second-order logic. He notes

⁹To see this, let the principles Biconditional and Explicit be defined as follows:

$$\text{Biconditional: } \forall F \forall G (\%F = \%G \leftrightarrow R(F, G))$$

$$\text{Explicit: } \forall F \exists! x \forall G (G \eta x \leftrightarrow R(F, G))$$

Then, in second-order logic, (1) *Explicit* implies *Biconditional* under the definition

$$\%F =_{df} \iota x \forall G (G \eta x \leftrightarrow R(F, G))$$

and (2) *Biconditional* implies *Explicit* under the definition

$$F \eta x =_{df} x = \%F$$

The proof is straightforward. For (1), just generalize the proof of the derivation of New V from Explicit New V. For (2), let $x = \%F$, for any F . We need to show $\forall G (G \eta \%F \leftrightarrow R(F, G))$. Substituting in definition of η this becomes $\forall G (\%G = \%F \leftrightarrow R(F, G))$ which is just *Biconditional*. To show x is unique, let y be such that $\forall G (y = \%G \leftrightarrow R(F, G))$. Then in particular $y = \%F \leftrightarrow R(F, F)$. Since R is an equivalence relation, $R(F, F)$, and so $y = \%F = x$.

that one instance of Explicit Logical Objects is the following, which essentially restates Basic Law V:

Explicit Rule V:

$$\forall F \exists ! x \forall G (G \eta x \leftrightarrow G \equiv F)$$

Boolos points out ([1987], 17 = [1998], 198) that this instance leads to a contradiction when you substitute $[\lambda z \exists H (H \eta z \& \neg H z)]$ (i.e., the property *containing a concept which you don't fall under*) for F . Similarly, Boolos notes ([1987], 17 = [1998], 198-99) that the following

SuperRussell:

$$\exists ! x \forall G (G \eta x \leftrightarrow \exists y (G y \& \neg G \eta y))$$

asserts the existence of an object that has in it all and only those properties G which fail to be contained in something which falls under G . This leads to a contradiction too.¹⁰

One might think that this is the end of the story—the comprehension schema Explicit Logical Objects is just too strong. However, as we shall see in the next section, if you place a minor restriction on comprehension for properties, one can retain Explicit Logical Objects and show that it has a wide variety of applications.

2. Object Theory

Suppose we were to keep Explicit Logical Objects, as stated above, but weaken the comprehension principle over concepts. Instead of full comprehension over concepts, i.e.,

$$\exists F \forall x (F x \leftrightarrow \varphi), \text{ where } \varphi \text{ has no free } F\text{s,}$$

let us use a restricted comprehension principle in which φ is η -free:

$$\exists F \forall x (F x \leftrightarrow \varphi), \text{ where } \varphi \text{ has no free } F\text{s and is } \eta\text{-free}$$

This restriction is attractive because it preserves many of Frege's logical objects; the remainder of this paper will show how, exactly, it preserves these objects. In this section, we will show that a theory very much like this has been proposed in research conducted independently of Boolos' papers.

¹⁰Let a be such an object; by property comprehension, $[\lambda z z = a]$ exists; but $[\lambda z z = a] \eta a$ iff not.

In [1983], Zalta proposed an axiomatic metaphysics that formalized and systematized E. Mally's [1912] distinction between two modes of predication. Mally distinguished between an object's 'satisfying' a property and an object's being determined by a property. The formal reconstruction of Mally's distinction was used to identify 'intentional objects' (such as fictions), possible worlds, Leibnizian complete individual concepts, and other abstract objects of interest to the formal ontologist.

Of particular interest here is the elementary version of the system in [1983], extended so as to include the theory of propositions used in the modal version of the system. We will call this subsystem Object Theory, or OT for short. OT is formulated in a second-order language that has been modified only by the inclusion of two kinds of atomic formulas: Fx (read either ' x exemplifies the property F ' or ' x falls under the concept F ') and xF (read ' x encodes F ').¹¹ In addition, OT contains a primitive theoretical predicate $E!$ ('being concrete') whose complement is $A!$ ('being abstract').¹² It is axiomatic that concrete objects don't encode properties.

The proof theory of OT is just second-order logic with the restricted comprehension principle for concepts we considered above.¹³

OT Concept Comprehension

$$\exists F \forall x (F x \leftrightarrow \varphi),$$

where φ has no free F s and has no encoding subformulas.

A version of Explicit Logical Objects is the system's most fundamental group of non-logical axioms:

¹¹We've simplified here a bit. We may assume that in OT, the exemplification mode of predication generalizes to n objects exemplifying an n -place relation. So ' $F^n x_1, \dots, x_n$ ' is well formed.

¹²In [1983], Zalta reads ' $E!$ ' as 'exists', but notes (50-52) that the system could be interpreted somewhat differently. In the present paper, we will read ' $E!$ ' as 'concrete', and so there is a simple distinction between concrete and abstract objects. We reserve 'there exists' for the now standard, Quinean reading of the existential quantifier ' \exists '. Note that in Zalta's modal system, the 'abstract objects' are necessarily non-concrete, not simply non-concrete, and 'ordinary objects' are defined as possibly concrete.

¹³In Zalta's systems, the more general comprehension principle for n -place relations, $n \geq 0$, is used. Indeed, Concept Comprehension is derived as consequence of the abstraction principle for λ -expressions, namely, $[\lambda x_1 \dots x_n \varphi] y_1 \dots y_n \leftrightarrow \varphi_{x_1, \dots, x_n}^{y_1, \dots, y_n}$. Since Zalta uses a restriction on the formation of λ -expressions (namely, they may not contain encoding subformulas), both the explicit comprehension principle for relations and the derivable 1-place OT Concept Comprehension principle inherit that restriction when they are derived from λ -abstraction.

OT Object Comprehension

$\exists x(A!x \& \forall F(xF \leftrightarrow \varphi))$, where φ has no free xs

Finally, in OT, identity is a defined notion for concrete objects, abstract objects, and concepts.

$$x =_E y =_{df} E!x \& E!y \& \forall F(Fx \leftrightarrow Fy)$$

$$x =_A y =_{df} A!x \& A!y \& \forall F(xF \leftrightarrow yF)$$

$$x = y =_{df} x =_E y \vee x =_A y$$

$$F = G =_{df} \forall x(xF \leftrightarrow xG)$$

Note that the definition of identity for objects has a consequence for OT Object Comprehension—there can't be two distinct abstract objects which encode exactly the properties satisfying φ . So each instance of OT Object Comprehension yields a unique abstract object encoding just the properties satisfying φ . Consequently, the definite description:

$$ix(A!x \& \forall F(xF \leftrightarrow \varphi))$$

is always well-defined, for any formula φ with no free xs .

In short, OT is an extension of second-order logic that provides unrestricted object comprehension at the cost of restricting concept comprehension. Our interest in this system has two dimensions. First, OT shares deep similarities with Boolos' systems. Second, the system demonstrates that an axiom like Explicit Logical Objects *can* be consistently added to a suitably weakened second-order logic.

The most striking similarity between the explicit versions of Boolos' systems and OT is their use of two instantiation relations. Recall the quotation from Boolos where he notes that Frege was using two instantiation relations in the *Foundations*. Where Boolos uses Fx and $F\eta x$ to represent these kinds of predication, OT uses Fx and xF . In fact, the analogy between $F\eta x$ and xF is more than superficial. The paradoxes of η created by Explicit Logical Objects are exactly the paradoxes of encoding that would result from adding OT Object Comprehension without also restricting concept comprehension.¹⁴ For example, compare Boolos' contradictory instance of Explicit Logical Objects:

$$\exists!x\forall F(F\eta x \leftrightarrow F \equiv [\lambda z \exists H(H\eta z \& \neg H z)])$$

¹⁴The paradoxes of encoding are presented in more detail in Zalta [1983], 158-159.

with the following paradox of encoding (Zalta [1983], 158-159):

$$\exists!x(Ax \& \forall F(xF \leftrightarrow F = [\lambda x \exists G(xG \& \neg Gx)]))$$

Furthermore, compare Boolos' SuperRussell

$$\exists!x\forall F(F\eta x \leftrightarrow \exists y(Fy \& \neg F\eta y))$$

with 'McMichael's Paradox' of encoding (Zalta [1983], 159):

$$\exists x(A!x \& \forall F(xF \leftrightarrow \exists y(F = [\lambda z z = y] \& \neg yF)))$$

Of course, none of these contradictions are derivable in OT when encoding subformulas are banished from property comprehension conditions. Moreover, the theory is provably consistent, as shown in Zalta [1983] (164-166) and [1999] (626-7), which are based on models of D. Scott and P. Aczel, respectively.

The pattern we identified in Boolos' systems, of grounding abstraction principles for logical objects in explicit existence claims which are formulated with a second mode of predication, has its natural endpoint in OT. Working with the same, or at least similar, notions of predication, Boolos formulated theories of logical objects with unrestricted comprehension for relations but restricted comprehension conditions for logical objects, whereas Zalta independently developed a system with unrestricted comprehension for logical objects but restricted comprehension conditions for relations. In the remaining sections of this paper, we compare these two ways of developing systems for Fregean logical objects. We first develop some new results in OT concerning the range of definable logical objects, and then consider whether those results are obtainable in any of Boolos' systems.

3. Fregean Logical and Abstract Objects

In this section, we discuss the following kinds of logical object: natural cardinals, extensions, directions, shapes, and truth values. The material concerning the latter four kinds of logical objects are presented here as new results of OT. However, before introducing those results, we first briefly rehearse the development of number theory in Zalta [1999].

3.1 Natural Cardinals

In Zalta [1999], the relation of ‘equinumerous with respect to ordinary objects’ ($F \approx_E G$) is shown to be an equivalence relation on properties.¹⁵ Then, ‘the number of ordinary G s’ ($\#G$) is given a Frege-style definition as, follows

$$\#G =_{df} \iota x(A!x \ \& \ \forall F(xF \leftrightarrow F \approx_E G))$$

Then, natural cardinals are defined as:

$$NaturalCardinal(x) =_{df} \exists F(x = \#F)$$

A version of Hume’s Principle is derivable from the above definitions.

Hume’s Principle in OT:

$$\#F = \#G \leftrightarrow F \approx_E G$$

Moreover, with two additional assumptions, the Dedekind/Peano axioms for number theory are then derived, mostly following Frege’s plan.¹⁶ Thus, no mathematical primitives are needed for the development of this conception of the natural numbers.

3.2 Extensions

Our new results concern extensions, directions, shapes, and truth values. First we derive a theory of extensions in OT. We begin with the definition of the equivalence condition ‘ F is materially equivalent to G ’ in the usual way:

$$F \equiv G =_{df} \forall x(Fx \leftrightarrow Gx)$$

We then define ‘the extension of G ’ as follows:

¹⁵In [1999], ‘ordinary’ objects are defined as ‘possibly concrete’, i.e., as $\diamond E!x$. Now let u, v range over ordinary objects. Zalta defines:

$$F \approx_E G =_{df} \exists R[\forall u(Fu \rightarrow \exists!v(Gv \ \& \ Ruv)) \ \& \ \forall u(Gu \rightarrow \exists!v(Fv \ \& \ Rvu))]$$

¹⁶See Zalta [1999] for details. The two additional assumptions are: (1) that the condition which defines *Predecessor* constitutes a relation, and (2) (the modal claim) that if (natural number) n numbers the G , then there might have been something distinct from all of the actual G s. Note that the system of Zalta [1999] was developed so as to include a primitive modality, and the axioms of second-order S5 modal logic were included. This modal reformulation was essential to the proof that every number has a successor, since the proof required the interplay of the Barcan-formulas with the modal claim just described.

$$\mathcal{G} =_{df} \iota x(A!x \ \& \ \forall F(xF \leftrightarrow F \equiv G))$$

Our earlier discussion established that this description is well-defined. Using this definition, we can then define the notion of an extension as any object that is the extension of some concept:

$$Extension(x) =_{df} \exists F(x = \mathcal{F})$$

Now, a consistent version of Basic Law V is derivable:¹⁷

Basic Law V in OT:

$$\mathcal{F} = \mathcal{G} \leftrightarrow F \equiv G$$

The reason this is a consistent version of Law V is that only genuine properties (i.e., ones without encoding subformulas) are substitutable for F or G —paradoxical conditions of the kind Boolos discussed (e.g., SuperRussell) do not define genuine properties.

To see that a proper notion of ‘extension’ has been defined here, we now show that the following principles are derivable: Extensionality, Empty Extension, Unions, Complements, Intersections, and Comprehension. First we define membership in an extension in the usual way:

$$y \in x =_{df} \exists G(x = \mathcal{G} \ \& \ Gy)$$

The following abstraction principle for extensions immediately follows:

$$Lemma: x \in \mathcal{G} \leftrightarrow Gx$$

From this Lemma, it is easy to prove the following, where the e_i are variables ranging over extensions:¹⁸

¹⁷For the proof:

(\rightarrow) Suppose $\mathcal{A} = \mathcal{B}$. By the definition of \mathcal{A} , $\forall F(\mathcal{A}F \leftrightarrow F \equiv A)$. Since $\mathcal{A} = \mathcal{B}$ we can substitute to get $\forall F(\mathcal{B}F \leftrightarrow F \equiv A)$. In particular, this implies $\mathcal{B}\mathcal{B} \leftrightarrow B \equiv A$. Since clearly $\mathcal{B}\mathcal{B}$ it follows that $B \equiv A$.

(\leftarrow) Suppose $A \equiv B$. To show $\mathcal{A} = \mathcal{B}$ it suffices to show that $\forall F(\mathcal{A}F \leftrightarrow F \equiv B)$. By the definition of \mathcal{A} , $\forall F(\mathcal{A}F \leftrightarrow F \equiv A)$. Since $A \equiv B$ it follows that $\forall F(\mathcal{A}F \leftrightarrow F \equiv B)$. \boxtimes

¹⁸For Extensionality:

Suppose $\forall x(x \in e_1 \leftrightarrow x \in e_2)$ and e_1 and e_2 are \mathcal{P} and \mathcal{Q} , respectively. Then, by the *Lemma* on extensions, $\forall x(Px \leftrightarrow Qx)$, i.e. $P \equiv Q$. So by Basic Law V, $e_1 = e_2$. \boxtimes

For Empty Extension:

Extensionality: $\forall e_1 \forall e_2 [\forall x (x \in e_1 \leftrightarrow x \in e_2) \rightarrow e_1 = e_2]$

Empty Extension: $\exists! e \forall x (x \notin e)$

Unions: $\forall e_1 \forall e_2 \exists e_3 \forall x (x \in e_3 \leftrightarrow x \in e_1 \vee x \in e_2)$

Complements: $\forall e_1 \exists e_2 \forall z (z \in e_2 \leftrightarrow z \notin e_1)$

Intersections: $\forall e_1 \forall e_2 \exists e_3 \forall x (x \in e_3 \leftrightarrow x \in e_1 \& x \in e_2)$

Let $e = \ulcorner [\lambda x E!x \& \neg E!x]$ (or the extension of any other empty concept). Then, by the *Lemma* on extensions, $\forall x (x \notin e)$. For uniqueness, suppose there exists extension $e' = \ulcorner P$ for some P such that $\forall y (y \notin e')$. Then, by the *Lemma*, $\forall x (\neg Px)$. So $P \equiv [\lambda z E!x \& \neg E!x]$. But then by Basic Law V, $e = e'$. \bowtie

For Unions:

Consider arbitrarily chosen extensions e and e' . Then there are properties P and Q such that $e = \ulcorner P$ and $e' = \ulcorner Q$. Consider the property $[\lambda z Pz \vee Qz]$, which exists by property abstraction. Call this property H . We show that for any object x , $x \in H$ iff $x \in e$ or $x \in e'$. But this follows by the λ -conversion principle and the definition of membership. \bowtie

For Complements:

Consider an arbitrarily chose extension e . Then e is the extension of some concept, say P . But now consider $[\lambda y \neg Py]$, which exists by property comprehension. Then $\ulcorner [\lambda y \neg Py]$ also exists. It is straightforward to show that something is in this extension iff it fails to be in e .

For Intersection:

Follow the same proof as for Unions, but instead using the property $[\lambda z Pz \& Qz]$. Alternatively, use the complement of the union of the complements of e and e' .

For Comprehension:

Let $\varphi[z]$ be any formula without encoding subformulas or definite descriptions and in which the variable z may or may not be free. Then $[\lambda z \varphi]$ denotes a property. So $\ulcorner [\lambda z \varphi]$ is well-defined and so exists. So by the definition of membership

$$x \in \ulcorner [\lambda z \varphi] \leftrightarrow [\lambda z \varphi]x,$$

where x is any object. By λ -conversion, it then follows that:

$$x \in \ulcorner [\lambda z \varphi] \leftrightarrow \varphi_x^x$$

So by universally generalizing on x and existentially generalizing on our extension abstract, it follows that there is a extension that contains as members all and only those objects that are such that φ , i.e.,

$$\exists e \forall x (x \in e \leftrightarrow \varphi[x])$$

\bowtie

Comprehension: $\exists e \forall x (x \in e \leftrightarrow \varphi[x])$,

where $\varphi[x]$ is any formula without encoding subformulas.

Note that Pairs and Adjunction are not straightforwardly derivable—in each case, a proviso is needed. Consider the following:¹⁹

Pairs: $\forall x \forall y [E!x \& E!y \rightarrow \exists e \forall z (z \in e \leftrightarrow z =_E x \vee z =_E y)]$

Adjunction: $\forall e_1 \forall x [E!x \rightarrow \exists e_2 \forall y (y \in e_2 \leftrightarrow y \in e_1 \vee y =_E x)]$

The restrictions here are required because properties of the form $[\lambda y y = x]$ are not guaranteed to exist, given that ' $x = y$ ' is defined in terms of encoding subformulas. It should be easy to see that the property $[\lambda w w =_E x \vee w =_E y]$ figures into the proof of Pairs, and the property $[\lambda w Fw \vee w =_E y]$ (where $e_1 = \ulcorner F$) figures into the proof of Adjunction. These properties are guaranteed to exist, by property comprehension.

Note, finally, that the following schema,

Separation: $\forall e_1 \exists e_2 \forall x (x \in e_2 \leftrightarrow x \in e_1 \& \varphi[x])$,

where $\varphi[x]$ is any formula without encoding subformulas.

is easily derivable from Comprehension and Intersection—take the intersection of any given extension with $\ulcorner [\lambda z \varphi[z]]$, where $\varphi[z]$ satisfies the restrictions on comprehension.

3.3 Directions, Shapes, etc.

Next we turn to directions, shapes, and other abstract, logical objects based on equivalence relations among individuals. We may define directions and shapes in just the way Frege suggests in [1884] (§68 = p. 79):

$$\vec{v} =_{df} \ulcorner [\lambda z z \parallel v]$$

¹⁹For Pairs:

Consider arbitrarily chosen concrete objects a and b . Consider the property $[\lambda y y =_E a \vee y =_E b]$. This property exists by property abstraction. Call this property H . Then there is a extension $\ulcorner H$. It is straightforward to prove that something z is in $\ulcorner H$ iff $z =_E a$ or $z =_E b$. \bowtie

For Adjunction:

Let e be an extension and suppose $E!a$. Then there is some property, say P , such that $e = \ulcorner P$. Now consider the property $[\lambda z Pz \vee z =_E a]$. Call this property H . Then it is easy to show that something y is in $\ulcorner H$ iff $y \in e \vee y =_E a$. \bowtie

$$\check{v} =_{df} \ulcorner [\lambda z z \cong v] \urcorner$$

From these definitions, one may define the associated notions:

$$Direction(x) =_{df} \exists v(x = \vec{v})$$

$$Shape(x) =_{df} \exists v(x = \check{v})$$

Next, it is straightforward to prove the corresponding Fregean biconditionals:²⁰

Directions in OT:

$$\vec{u} = \vec{v} \leftrightarrow u \parallel v$$

Shapes in OT:

$$\check{u} = \check{v} \leftrightarrow u \cong v$$

One can easily see that this generalizes to any equivalence relation R on objects:

$$\tilde{x} =_{df} \ulcorner [\lambda z z R x] \urcorner$$

The corresponding biconditional for the logical object \tilde{x} is:²¹

$$\tilde{a} = \tilde{b} \leftrightarrow a R b$$

It is of some interest here that although we have followed Frege in defining directions and shapes as extensions, one might argue that, strictly speaking, directions and shapes should not be reduced to extensions but rather defined as *sui generis* logical objects. OT can capture this idea—we may alternatively define:

$$\vec{v} =_{df} \ulcorner \lambda x(A!x \ \& \ \forall F(xF \leftrightarrow \forall y([\lambda z z \parallel v]y \rightarrow Fy))) \urcorner$$

$$\check{v} =_{df} \ulcorner \lambda x(A!x \ \& \ \forall F(xF \leftrightarrow \forall y([\lambda z z \cong v]y \rightarrow Fy))) \urcorner$$

This would define \vec{v} , for example, as the abstract object that encodes all the properties that are implied by the property *being parallel to v*. It is easy to see that the corresponding Fregean biconditionals are still derivable for these alternative definitions.²²

²⁰For the proof, see the more general case, discussed next in the text and proved in the next footnote.

²¹For the proof:

We know $\tilde{a} = \tilde{b}$ iff (by definition of \tilde{a}, \tilde{b}) $\ulcorner [\lambda z z R a] \urcorner = \ulcorner [\lambda z z R b] \urcorner$ iff (by Basic Law V) $[\lambda z z R a] \equiv [\lambda z z R b]$ iff (by λ -conversion and the fact that R is an equivalence relation) $a R b$. \square

²²For example, we prove Directions as follows:

3.4 Truth Values

Truth values have been familiar to philosophers and logicians for so long that few have been interested in the question of what they are and why we are justified in assuming that they exist. Frege introduced the two truth values, The True and The False, as *primitive* objects. Logicians ever since have used them to give the semantics of propositional logic, while mathematicians typically just use the numbers 0 and 1. But truth values are not numbers, and logicians have never justified the existence of truth values within the wider context of a theory of logical objects.

By contrast, we may prove that there are truth values, identify The True and The False as particular truth values and, finally, prove that there are exactly two truth values. In proving these claims, we shall assume that the reader has some familiarity with the subtheory of propositions in Zalta [1983], and elsewhere, which we have incorporated into OT. It is important to note that this framework is a predicate logic and not a term logic like Frege's. Sentences do *not* denote truth values, but receive recursively defined truth conditions. Some sentences, however, do qualify as 0-place terms (namely, the ones which have no encoding subformulas), and these do denote propositions (intensionally conceived). In particular, the theory of propositions begins with a comprehension principle for propositions. Where ' p ' is a variable ranging over propositions (i.e., 0-place relations), the following is actually a 'degenerate' 0-place sub-schema of Concept Comprehension in OT:

$$\exists p(p \leftrightarrow \varphi), \text{ where } \varphi \text{ has no free } ps \text{ and no encoding subformulas}$$

In addition, there is a definition of proposition identity:

$$p = q =_{df} [\lambda y p] = [\lambda y q]$$

In this definition, the identity of p and q is defined in terms of the identity

(\rightarrow) Let $\vec{a} = \vec{b}$. Since clearly $\forall y([\lambda z z \parallel a]y \rightarrow [\lambda z z \parallel a]y)$ we know $\vec{a}[\lambda z z \parallel a]$. Substituting \vec{b} for \vec{a} yields $\vec{b}[\lambda z z \parallel a]$. By the definition of \vec{b} we know $\forall y([\lambda z z \parallel b]y \rightarrow [\lambda z z \parallel a]y)$ and in particular $[\lambda z z \parallel b]b \rightarrow [\lambda z z \parallel a]b$ which is equivalent, by λ -abstraction, to $b \parallel b \rightarrow b \parallel a$. Since $b \parallel b, b \parallel a$.

(\leftarrow) Let $a \parallel b$. It suffices to show that for any $P, \vec{a}P \leftrightarrow \vec{b}P$. Suppose $\vec{a}P$. Then by the definition of $\vec{a}, \forall y([\lambda z z \parallel a]y \rightarrow Py)$ Since $a \parallel b$ this is equivalent to $\forall y([\lambda z z \parallel b]y \rightarrow Py)$. By the definition of \vec{b} this implies $\vec{b}P$.

of certain propositional properties.²³ (Recall that identity for properties, or concepts, has already been defined.) In the expression ‘ $[\lambda y p]$ ’, since p is a variable ranging over propositions, the variable ‘ y ’ bound by the λ is vacuously bound. For example, ‘ $[\lambda y Pb]$ ’ might denote the property *being such that Bush is President*. Complex properties of this form are logically governed by λ -abstraction; the following is the relevant general instance: $[\lambda y p]x \leftrightarrow p$.

Although there may be alternative definitions of truth values that will do the job we want, we prefer the following definition of ‘the truth value of proposition p ’ (p°):

$$p^\circ =_{df} \lambda x(A!x \ \& \ \forall F(xF \leftrightarrow \exists q(q \leftrightarrow p \ \& \ F = [\lambda y q])))$$

This says that the extension of p is the abstract object that encodes all and only the properties of the form $[\lambda y q]$ which are constructed out of propositions q materially equivalent to p . To get a more natural sense of what this definition does, let us extend our notion of encoding so that objects can encode propositions:

$$x \text{ encodes } p =_{df} x[\lambda y p]$$

Using this extended notion of encoding, we may read our definition of p° more simply as follows: the truth value of p is the abstract object that encodes exactly the propositions materially equivalent to p .²⁴

It is now straightforward to define the notion of a truth value as any object that is the truth value of some proposition:

$$T\text{-value}(x) =_{df} \exists p(x = p^\circ)$$

The Fregean biconditional governing truth values is a consequence of our work so far:²⁵

²³The resulting theory of propositions is consistent with the claim that there are distinct propositions which are nevertheless materially equivalent. That is, one may consistently assert the existence of propositions p, q such that both $p \neq q$ and $p \leftrightarrow q$. In modal developments of object theory, the definition of proposition identity becomes consistent with the claim that there are distinct propositions which are *necessarily* equivalent.

²⁴It is for this reason that we do not define p° as $\lambda x[\lambda y p]$. This latter definition would make the truth value of a proposition an extension containing objects, whereas on our present definition, the truth of a proposition more like an extension ‘containing’ (encoding) propositions.

²⁵Here is the proof:

Truth Values in OT:

$$p^\circ = q^\circ \leftrightarrow p \leftrightarrow q$$

Now the two truth values The True (\top) and The False (\perp) are identifiable:

$$\top =_{df} \lambda x(A!x \ \& \ \forall F(xF \leftrightarrow \exists p(p \ \& \ F = [\lambda y p])))$$

$$\perp =_{df} \lambda x(A!x \ \& \ \forall F(xF \leftrightarrow \exists p(\neg p \ \& \ F = [\lambda y p])))$$

We may prove that these two objects are indeed truth values:²⁶

Theorem: T-value(\top) & *T-value*(\perp)

Finally, we may establish the fundamental theorem governing truth values, namely, that there are exactly two of them:²⁷

It is a simple lemma to establish that q° encodes p iff $p \equiv q$ (exercise). Now pick arbitrary propositions p_1 and q_1 . (\rightarrow) Assume p_1° is identical to q_1° . Given the simple lemma and the self-equivalence of p_1 , it is easy to show that p_1° encodes p_1 . So q_1° encodes p_1 . So, again, by our simple lemma, it follows that p_1 is materially equivalent to q_1 . (\leftarrow) Assume p_1 is materially equivalent to q_1 . To show that p_1° is identical to q_1° , we show they encode the same properties. (\rightarrow) Assume p_1° encodes P . Then there is a proposition, say r_1 , such that r_1 is materially equivalent to p_1 and such that P is identical to *being such that* r_1 . So there is a proposition r (namely r_1) such that r is materially equivalent to q_1 and such that P is identical to $[\lambda y r]$. So, by the definition of q_1° , it follows that q_1° encodes P . (\leftarrow) By analogous reasoning.

²⁶Here is the proof that \top is a truth value:

We want to show that \top is identical with the truth value of some proposition. So pick any arbitrary proposition you please, say p_1 , and consider the proposition $p_1 \rightarrow p_1$. Call this proposition p_2 . Since p_2 is a logical truth, it is true. Now to show that \top is identical with p_2° , we need to show that \top encodes a property Q iff p_2° encodes Q . (\rightarrow) Assume \top encodes Q . Then there is a proposition, say r_1 , such that r_1 is true and Q is the property *being such that* r_1 . Since p_2 and r_1 are both true, they are materially equivalent. So there is a proposition r (namely, r_1) such that r is materially equivalent to p_2 and such that Q is the property *being such that* r . So, by the definition of p_2° , it follows that p_2° encodes Q . (\leftarrow) Assume that p_2° encodes Q . So, there is a proposition, say r_1 , such that r_1 is materially equivalent to p_2 and such that Q is *being identical with* r_1 . But since p_2 is true, r_1 is true. So there is a proposition r (namely, r_1) which is true and such that Q is *being such that* r . So \top encodes Q .

The proof that \perp is a truth value is left as an exercise.

²⁷Here is the proof:

Since by our previous theorems we know that \top and \perp are both truth values, it suffices to show that they are distinct and that any other truth value is identical to either \top or \perp . Since it is obvious that they are distinct, we simply prove

Theorem: $\exists x, y [T\text{-value}(x) \ \& \ T\text{-value}(y) \ \& \ x \neq y \ \& \ \forall z (T\text{-value}(z) \rightarrow z = x \vee z = y)]$

4. Logical Objects in Boolos' Systems

In his work in [1986], [1989], and [1993], Boolos has shown some of the strengths of New V and New V* as a ground for set theory. This work has been refined by Shapiro and Weir [1999], where they consider which standard axioms of set theory are derivable from New V and which are not. In this section, we ask to what extent the systems of (Explicit) New V and New V* can be used to identify other kinds of logical objects.²⁸ More precisely, we will examine whether Boolos' systems can yield biconditionals like Hume's Principle, Directions, and Shapes when the relevant objects are defined as Frege suggested. We will see that New V and New V* often identify as 'large' or 'bad' the concepts that Frege used to define logical objects.

4.1 Hume's Principle, New V, and New V*

Shapiro and Weir ([1999], 301) offer the following counterexample to Hume's Principle in New V. They note that the property, A , of being a singleton concept and the property, B , of being a doubleton concept are both large, and so they are logically similar and therefore have the same extension. Thus, where 1 is the extension of A , and 2 is the extension of B , it would follow that $1 = 2$. This quick counterexample does demonstrate a problem that faces New V for the reconstruction of Fregean number theory. But a more in-depth analysis will offer an insight into the obstacles that lie in the way of reconstructing numbers and other logical objects using either New V or New V*.

On our analysis, Hume's Principle fails in these systems because the

that every truth value is identical to either \top or \perp . So assume that z is a truth value. So there is some proposition, say p_1 , such that $z = p_1^\circ$. Now it is easy to show that, in general, $p \rightarrow (p^\circ = \top)$ and $\neg p \rightarrow (p^\circ = \perp)$. So, given that either p_1 or $\neg p_1$, it follows by disjunctive syllogism, that either $p_1^\circ = \top$ or $p_1^\circ = \perp$. So either $z = \top$ or $z = \perp$.

²⁸We do not consider whether Numbers could serve as a basis for a theory of logical objects because, unlike New V and New V*, it is intended only to make the existence assertion implicit in Hume's Principle explicit. Numbers was never intended to provide a foundation for logical objects generally.

concept used to define cardinal numbers is 'large' and 'bad'. Remember that Frege defined the finite cardinal 'the number of F s' as the extension of the higher-order concept *being equinumerous with F* . Since Boolos' system does not contain λ -expressions for higher-order concepts, this Fregean definition for the finite cardinal can't be reconstructed. However, Frege suggested a suitable substitute in [1893], §34. Let us define 'the number of F 's' as the subtension of the concept *being a subtension of a concept equinumerous to F* (Q_F):

$$Q_F =_{df} [\lambda x \exists G (x = {}^*G \ \& \ G \approx F)]$$

We may therefore define:

$$\#F =_{df} {}^*Q_F$$

Now it is a fundamental fact about the system of New V and New V* that the following holds:²⁹

$$[F \not\approx V \ \& \ \overline{F} \approx V \ \& \ \exists x Fx] \rightarrow Q_F \approx V \approx \overline{Q_F} \quad (\delta)$$

where \overline{F} is the complement of F . In words, if F is a small, nonempty concept with a large complement then the concept *being the subtension of a concept equinumerous with F* is large and bad.

Let's see how this fact undermines Hume's Principle. Any singleton (S) or doubleton (D) concept satisfies the antecedent of (δ) . Thus, the concepts *being the subtension of a concept equinumerous with singleton S* (Q_S) and *being the subtension of a concept equinumerous with doubleton D* (Q_D) are large and bad. But if these concepts are both large and bad then they are logically similar ($Q_S \sim Q_D$) and logically similar* ($Q_S \sim^* Q_D$). But under New V and New V* this means the subtensions of Q_S and Q_D are identical despite the fact that $S \not\approx D$. But this is a counterexample to Hume's Principle which, given the definition of $\#F$ as *Q_F , implies that ${}^*Q_S = {}^*Q_D \leftrightarrow S \approx D$. So in one sense, Hume's Principle fails in New V and New V* because finite cardinals are extensions of concepts that, in Boolos' systems, are large and bad.³⁰

Even though Boolos' approach marks as large and bad concepts (like Q_S) that are essential to Frege's definition of number, it is still quite astonishing what can be accomplished from New V and New V*. For

²⁹For the proof, see the Appendix.

³⁰This is presumably why Boolos does not use Frege's own development of arithmetic in the system of New V.

example, Boolos has shown [1989] that New V is sufficient for the development of Zermelo-Fraenkel set theory without the powerset and infinity axioms. Here, we have only tried to explain why Frege’s own development of arithmetic is not available in New V and New V*.

4.2 Directions and Shapes

In both New V and New V*, the provability of the biconditionals Directions and Shapes depends upon the assumptions one makes about directions and shapes. In Boolos’s systems, \vec{a} would be identified as $*[\lambda z z||a]$ (New V) or $*[\lambda z z||a]$ (New V*), and \check{a} would be identified as $*[\lambda z z \cong a]$ (New V) or $*[\lambda z z \cong a]$ (New V*). In New V (New V*), the corresponding biconditionals are provable only when the concepts $[\lambda z z||a]$ and $[\lambda z z \cong a]$ are small (good), for any a . So if it is possible to show—perhaps by employing some axioms of geometry—that $[\lambda z z||a]$ and $[\lambda z z \cong a]$ are well-behaved in the appropriate way, then New V and New V* imply the biconditionals. However, without additional geometric axioms, there is a model of the New V system where the biconditionals do not hold. Consider Boolos’ model for the New V system ([1986], 150-151) in which object variables range over natural numbers and concept variables range over sets of natural numbers. If F denotes a finite set then $*F$ denotes some natural number that encodes this set. If F denotes an infinite set, then $*F$ denotes 0. Now interpret $||$ such that $x||y$ iff x and y are either both even or both odd. Then $[\lambda z z||2]$ and $[\lambda z z||3]$ are both large so they have the same subtension, however, $\neg(2||3)$ —contra Directions.

4.3 Truth Values

In Boolos [1986] (148), Boolos considers adding propositional variables along with a special axiom $p^\circ = q^\circ \leftrightarrow (p \leftrightarrow q)$ to define truth values. However, if propositional variables are added to the New V system and degenerate 0-place instances of concept comprehension are allowed, this biconditional can be proven using the following definition of truth value:

$$p^\circ =_{df} *[\lambda z p]$$

This definition maps all true propositions to the universal concept and all false propositions to the empty concept. From this definition the biconditional

$$p^\circ = q^\circ \leftrightarrow (p \leftrightarrow q)$$

holds in the New V system (and a corresponding version involving subtensions* holds in New V*).³¹ The definition is materially adequate, not only because it yields the above biconditional, but also because one can prove that there are truth values, define The True and The False, show they are truth values, and show that there are exactly two truth values.³² The above definition, however, does not make truth values *sui generis* objects in the way that Frege thought (and most logicians think) of them.

5. Observations

We saw earlier that the nonlogical axioms in Boolos’ systems can be grounded in the existence assertions Explicit New V and Explicit New V*. These axioms involve a second mode of predication and provide a basis of comparison with OT’s comprehension for abstract objects. In particular, whereas Boolos has unrestricted property comprehension and restricted object comprehension, OT restricts property comprehension and leaves object comprehension unrestricted. We asked to what extent each approach was capable of developing Fregean logical objects such as extensions, natural cardinals, directions, shapes, and truth values.

A lot of work has been done to reveal the strengths of Boolos’ systems as a basis for a ‘limitation of size’ set theory. It is from set theory

³¹For the proof in New V:

(\rightarrow) Assume $p^\circ = q^\circ$. By the definition, this amounts to $*[\lambda y p] = *[\lambda y q]$. So by New V, the concepts $[\lambda y p]$ and $[\lambda y q]$ are logically similar, i.e., $Small([\lambda y p]) \vee Small([\lambda y q]) \rightarrow [\lambda y p] \equiv [\lambda y q]$. Case 1. If both concepts are large, then they are both nonempty and so p and q are both true. Thus, they are equivalent. Case 2. Both concepts are materially equivalent and at least one is small. Subcase (a). If $[\lambda y p]$ is nonempty, then p is true, so since the concepts are equivalent, $[\lambda y q]$ is nonempty, and so q is true. Subcase (b). If $[\lambda y p]$ is empty, then p is false. So, by analogous reasoning, q is false.

(\leftarrow) If p and q are equivalent, then the concepts are equivalent. So they are logically similar, and by New V, their subtensions are identical. \boxtimes

For New V*:

(\rightarrow) By the same reasoning as above, one can reach the conclusion that if the concepts are good, then they are materially equivalent. Note that for any proposition p , the concept $[\lambda y p]$ is always good (because if p is true, the $[\lambda y \neg p] \not\approx V$; if p is not true, then $[\lambda y p] \not\approx V$). So the concepts are materially equivalent and so the propositions are equivalent.

(\leftarrow) By the same reasoning as in New V. \boxtimes

³²The True, for example, would be defined as $*[\lambda z \forall x(x = x)]$.

that Boolos can develop arithmetic and other mathematical constructs. However, because New V and New V* conflate the extensions of ‘bad’ concepts these systems prove unsuitable for defining Fregean natural cardinals. The story for other logical objects is similar. For directions and shapes, additional geometric axioms are necessary to forestall a counter-model to Directions and Shapes. A simulation of truth values is possible, although truth values look more like extensions of one-place concepts than the extensions of zero-place propositions.

Contrast these results with the theory of logical objects in OT. We saw that *natural* cardinals have a Fregean definition in a modal extension of OT and a version of Hume’s Principle is derivable under this definition. However, there are two caveats. The first is that equinumerosity on concepts must be restricted to ordinary objects, and so the resulting cardinals count only the ordinary objects falling under a concept, though Hume’s Principle is derivable and governs this notion of natural cardinal. Second, the derivation of number theory requires two additional assumptions (see Zalta [1999]).

A second group of paradigm logical objects, extensions, is definable and is governed by a provably consistent version of Basic Law V. In addition some basic set-theoretic principles are shown to hold for extensions. Because shapes and directions are definable in terms of extensions, their associated biconditionals are provable. Indeed, for any equivalence relation on objects, one can define logical objects that correspond to the cells of the partition and prove the corresponding Fregean biconditional. Finally, we identified truth values as distinctive logical objects in their own right. They satisfy the corresponding Fregean biconditional and provably have the other properties typically associated with truth values.

The inconsistency of Basic Law V with second-order logic shows that extensions cannot be well-behaved for *all* concepts (i.e., well-behaved in the sense that only extensions of materially equivalent concepts are equal). Most attempts at saving Frege’s work have made extensions well-behaved in a smaller class of concepts.³³ By comparing extensions in OT with subtensions in Boolos, it is clear that in OT, extensions are well-behaved for the concepts that are definable without encoding subformulas, while in Boolos’s systems, subtensions are well-behaved for concepts that are small or good.

A suitable reconstruction of Frege’s work, we argue, must provide a

³³Consider, for example, the systems in Heck [1996] and Wehmeier [1999].

basis for defining logical objects like the one offered here. Further research should be done comparing other reconstructions of Fregean logical objects, as well as comparing the modal versions of these neo-Fregean systems.

Appendix

Here we show the fact about New V and New V* mentioned in the text; namely that

$$F \not\approx V \ \& \ \overline{F} \approx V \ \& \ F \text{ nonempty} \rightarrow Q_F \approx V \approx \overline{Q_F} \quad (\delta)$$

holds in both New V and New V*.

Proof

Assume the antecedent to show (A) $Q_F \approx V$ and (B) $\overline{Q_F} \approx V$.

(A) Since F is nonempty, let a be such that Fa . Now define a map $g : \overline{F} \rightarrow Q_F$ by $g(x) = *F^{-a+x}$ where $F^{-a+x} =_{df} [\lambda z (Fz \ \& \ z \neq a) \vee z = x]$. To show Q_F is large it suffices to verify that g both (1) has range in Q_F and (2) is injective. To show (1), supposing $\overline{F}b$, we need to show that $F \approx F^{-a+b}$. This is verified by the bijection $h : F \rightarrow F^{-a+b}$ defined by

$$h(x) = \begin{cases} b & \text{if } x = a \\ x & \text{otherwise} \end{cases}$$

For (2) it suffices to show that if $\overline{F}x$, $\overline{F}x'$, and $x \neq x'$, then $g(x) \neq g(x')$ (i.e., $*F^{-a+x} \neq *F^{-a+x'}$). By part (1), $F \approx F^{-a+x}$, so since F is small, F^{-a+x} is also small. Clearly, $\neg F^{-a+x'}x$ because $x \neq x'$ and $\neg Fx$. Thus, F^{-a+x} and $F^{-a+x'}$ are not materially equivalent. By New V, since F^{-a+x} is small and not materially equivalent to $F^{-a+x'}$, $*F^{-a+x} \neq *F^{-a+x'}$.

(B) Two cases. In the first case, suppose F has more than one member. Define a map $g : V \rightarrow \overline{Q_F}$ by $g(x) = *[\lambda z z = x]$. To show $\overline{Q_F}$ is large it suffices to verify that g both (1) has range in $\overline{Q_F}$ and (2) is injective. For (1), since objects in the range of g are subtensions of singleton concepts, and F is not a singleton, it follows that the range of g is in $\overline{Q_F}$. For (2) it suffices to show that if $x \neq x'$ then $*[\lambda z z = x] \neq *[\lambda z z = x']$, but this follows from New V.

In the second case, suppose F has exactly one member, a . Then we can define a similar map $g : \overline{F} \rightarrow \overline{Q_F}$ by $g(x) = *[\lambda z z = a \vee z = x]$. To show $\overline{Q_F}$ is large it suffices to verify that g both (1) has range in $\overline{Q_F}$ and (2) is injective. For (1), since objects in the range of g are subtensions of doubleton concepts, and F is a singleton, it follows that the range of

g is in $\overline{Q_F}$. For (2) it suffices to show that if $\overline{F}x$, $\overline{F}x'$, and $x \neq x'$ then $*[\lambda z z = a \vee z = x] \neq *[\lambda z z = a \vee z = x']$, but this follows from New V. \boxtimes

Since small concepts are good concepts, the same proof goes through for New V*.

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