Number Theory and Infinity Without Mathematics*

Uri Nodelman
and
Edward N. Zalta

Philosophy Department
Stanford University

Abstract

We address the following questions in this paper: (1) Which set or number existence axioms are needed to prove the theorems of 'ordinary' mathematics? (2) How should Frege's theory of numbers be adapted so that it works in a modal setting, so that the fact that equivalence classes of equinumerous properties vary from world to world won't give rise to different numbers at different worlds? (3) Can one reconstruct Frege's theory of numbers in a non-modal setting without mathematical primitives such as "the number of *F*s" (#*F*) or mathematical axioms such as Hume's Principle? Our answer to question (1) is 'None'. Our answer to question (2) begins by defining '*x* numbers *G*' as: *x* encodes all and only the properties *F* such that *being-actually-F* is equinumerous to *G* with respect to discernible objects. We answer (3) by showing that the mere existence of discernible objects allows one to reconstruct Frege's derivation of the Dedekind-Peano axioms in a non-modal setting.

1 Introduction

Our goal in this paper is to answer three questions in the philosophy of mathematics:

- Which set or number existence axioms are needed to prove the theorems of 'ordinary' mathematics?
- How should Frege's theory of numbers be adapted in a modal setting so that the fact, that equivalence classes of equinumerous properties vary from world to world, won't give rise to different numbers at different worlds?
- Can one reconstruct Frege's theory of numbers in a non-modal setting without mathematical primitives such as "the number of *F*s" (#*F*) or mathematical axioms such as Hume's Principle?

We lay out these questions in more detail in Sections 1.1 – 1.3.

1.1 Existence Questions

Much of the work devoted to the question "Which set or number existence axioms are needed to prove the theorems of 'ordinary' mathematics?" has been developed in the program of reverse mathematics (Friedman 1975, 1976). The question has been guided, in part, by the observation that all of 'ordinary' mathematics can be developed in *2nd-order Peano Arithmetic* (PA2 = Z_2).[1] A research program evolved to examine what can and can't be proved in the various subsystems weaker than PA2, beginning with Robinson's system Q (i.e., 1st-order PA without the principle of induction) and Primitive Recursive Arithmetic (PRA = 1st-order, quantifier-free PA).

It won't be important, for the purposes of this paper, to say exactly what is intended by 'ordinary' mathematics. In all of this work, it is assumed that some mathematical primitives and some mathematical axioms are needed to derive the theorems of Q, PRA, ..., and PA2. Indeed, it is just accepted wisdom that the formulation and derivation of

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[1]This observation is explicitly attributed to Hilbert & Bernays (1934/1939) in Raatikainen 2020 (87), and Simpson (2009, xiv) says that this this is where PA2 is first fully developed. But see the discussion in Dean & Walsh (2017), who point out (368) that that "the systems considered there contain neither the full nor a restrict[ed] version of the comprehension scheme".

the claim, *natural numbers exist*, requires distinctive mathematical primitives and mathematical axioms:

- The early versions of number theory postulated by Dedekind (1888 [1893]) and Peano (1889) both have mathematical primitives and axioms.[2]

- The now classic formulation of PA2 includes (a) the primitives 0, N, and S and the Dedekind/Peano axioms (including induction) stated in terms of them, (b) the primitive notions $+$ and $\times$ and the recursive axioms for addition and multiplication, and (c) a comprehension principle for properties or sets of numbers, depending on the formulation (Simpson 2009, 4).

- A consistent 2nd-order fragment of Frege 1893/1903, known as *Frege Arithmetic*, either (a) uses the primitive mathematical notion $\#F$ ('the number of Fs') and asserts Hume's Principle ($\#F = \#G \equiv F \approx G$), as in Wright 1983 and Heck 1993, or (b) uses the primitive *F is in x* ('$F\eta x$') and guarantees that there are objects exemplifying a defined notion of *number* by asserting $\forall G \exists! x \forall F(F\eta x \equiv F \approx G)$ as an axiom, from which the existence of numbers follows (Boolos 1986/1987, 1987) – more on this below.[3]

- Whitehead & Russell, in *Principia Mathematica* (1910–13), used higher-order functions to define classes, then define the number sequence in terms of the empty class, the class of all singletons ($*52\cdot 01$), the class of all pairs ($*54\cdot 02$), etc. They then complete the construction by asserting an axiom of infinity ($*120\cdot 03$).

- In set theory with an axiom of infinity, one can define the numbers in a variety of ways (the two most popular being inspired by Zermelo 1908 and von Neumann 1923), thus reducing the primitives and axioms of number theory to those of set theory.[4]

[2] Dedekind uses set-theoretic primitives and defines a simply infinite system as a set N with a distinguished element ('1'), and a function ('transformation') from N to N meeting specific conditions that force N to contain an unbounded sequence; see 1888 §71 [1893, 20; 1939, 16]. By contrast, Peano starts with mathematical primitives N, 1, and successor notation ('+1'), and then asserts axioms that govern them (1889, 1).

[3] Here, $\approx$ stands for *equinumerosity*, i.e., the condition on properties F and G that holds when there is a relation R that is a witness to the one-to-one correspondence of the Fs and the Gs. This condition can be defined in logical terms, without any mathematical primitives.

[4] In Zermelo's reconstruction:

Clearly, all of these approaches require the primitives and axioms of either number theory or set theory.

By contrast, our answer to the question, 'What set or number existence axioms are needed?' is 'None'. Of course, we'll need to assert *some* existence axioms, but we'll show that none of these are specifically mathematical or utilize mathematical primitives. Our methodology is therefore very different from that of reverse mathematics and so not part of the reverse mathematics research program. We shall not be presupposing *any* mathematics and, in particular, we shall not be presupposing any number theory or set theory. Instead, we plan to (a) derive, *without* mathematical primitives or mathematical axioms, the Dedekind-Peano axioms for number theory and the additional axioms of PA2 (the axioms for recursive addition and multiplication, and comprehension for properties of numbers), and (b) show that one can prove the existence of an infinite number and an infinite set without assuming any mathematics.

1.2 Frege Arithmetic in a Modal Setting

Though Frege Arithmetic has given rise to a number of problems,[5] we focus here on one problem that hasn't received much attention. As we saw in Section 1.1, Frege Arithmetic has traditionally been formulated either (a) with the primitive mathematical notion $\#F$ ('the number of Fs') and the axiom:

$$\#F = \#G \equiv F \approx G,$$

or (b) with the primitives $Number(x)$ and $F\eta x$ and the axiom:[6]

$$\forall G \exists! x(Number(x) \,\&\, \forall F(F\eta x \equiv F \approx G))$$

- $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\{\emptyset\}\}$, $3 = \{\{\{\emptyset\}\}\}$, …

In von Neumann's reconstruction:

- $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$, $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, …

Either way, the mathematical background of set theory is presupposed.

[5] See, for example, the literature that has arisen on the Julius Caesar problem (Frege 1884, §56), the Bad Company objection (Dummett 1998), the Embarassment of Riches objection (Weir 2003), etc. These problems do not affect the present approach; see the discussions in Linsky & Zalta 2006 (81ff.) and Nodelman & Zalta 2014 (61, footnote 18).

[6] Boolos does, in places, *define* $F\eta x$ in third-order logic supplemented with the notion of the *extension* of F, written ${}'F$. In Boolos 1986/1987, he writes (140):

Define η by: $F\eta x$ iff for some second level concept D, $x = {}'D$ and DF,

where DF is to be read (first-level concept) F falls under the (second-level concept) D.

If one reconstructs Frege Arithmetic in terms of (a), then a cardinal number is any object x such that $\exists G(x = \#G)$. Alternatively and equivalently, if one reconstructs Frege Arithmetic in terms of (b), then one first defines $Numbers(x,G)$ as $\forall F(F\eta x \equiv F \approx G)$, and then asserts $\forall G\exists!xNumbers(x,G)$. (This guarantees the existence of numbers once one defines $Number(x)$ as $\exists G(Numbers(x,G)$.) In either case, cardinal numbers are abstracted from equivalence classes of equinumerous properties. For example, the number of planets is abstracted from the equivalence class of properties that satisfy the higher-order property *being equinumerous to the property of being a planet* (*in our solar system*). Frege then defined 0, *being a cardinal number*, *predecessor*, the strong and weak ancestrals of a relation, and *being a natural number*. He subsequently derived the Dedekind-Peano axioms from these axioms and definitions. We won't rehearse this derivation just now, since we'll reconstruct it later in the paper.

However, Frege's definitions, if transferred directly to a *modal* context, yield different numbers at different possible worlds. For at each possible world, the equivalence classes of equinumerous properties will vary. In terms of our example, properties equinumerous with the property *being a planet* will differ from world to world. Thus, the identity of the number of planets, in so far as it is abstracted from the extension of a higher-order property, will differ at possible worlds where there are different properties that are equinumerous to being a planet.[7] Thus, the natural numbers that emerge from the equivalence classes of properties at the actual world are different from the natural numbers that emerge at other possible worlds. The number of planets at the actual world, namely, eight, isn't the same number as the number of planets at some other possible world, even if there are eight planets there! Our second goal, then, is to develop a version of Frege Arithmetic that yields universal numbers – natural numbers that emerge at the actual world but which can be used to count the objects falling under properties at other possible worlds.

1.3 Reconstructing Non-Modal Frege Arithmetic

Our final goal is to show that one can reconstruct Frege's number theory in a *non-modal* setting without mathematical primitives or mathematical axioms. In Section 6.2, we'll show how to eliminate modality while preserving the key elements of our approach. Our non-modal reconstruction does not use $\#F$ as a primitive, Hume's Principle, Boolos's relation $F\eta x$, or its associated axiom that asserts the existence of numbers.[8] As we shall see, the modal axiom we use to reconstruct Frege's theory in a modal context can be replaced by a non-modal axiom asserting the existence of a *discernible* object. All of the other modal notions, including the appeal to actuality, can be eliminated.

It is important to emphasize, though, why we shall focus in what follows on the *modal* reconstruction of the Fregean natural numbers, rather than on the simpler *non-modal* reconstruction. Our view is that alethic modalities (often in the form of possibility statements) play an important role, not just in our ordinary cognitive life but also in the pursuit of the sciences. Since modal notions are so ubiquitous, we think it is important to show that Frege's theory can work in a modal context. But since Frege's theory doesn't obviously adapt to such a setting, an important part of what follows is to improve the modal reconstruction of Frege Arithmetic that was put forward in Zalta 1999. We'll use a simpler modal axiom – one that (a) doesn't force the domain of ordinary objects to be infinite, but only guarantees the existence of a single ordinary object (which doesn't imply the existence of a concrete object!) and (b) allows us to use Frege's derivation that every number has a successor. After we've derived the Dedekind-Peano axioms and all of PA2 using our new, simplified modal axiom (Sections 4.4 – 5.2), we'll show, at the end of the paper (Section 6.2), how to reproduce all the derivations in a non-modal setting, for those who believe that Frege's theory doesn't need to be adapted for a modal context.

1.4 Our Approach

We conclude this introduction with a few remarks that further characterize what we are attempting to do. First, our background theory for achieving the above goals is 2nd-order object theory (OT). The summary of this theory, provided in Section 2, reveals that it has no distinctively-mathematical primitives and no mathematical axioms. Our derivation

[7]See Panza 2018 (99–100), who concludes from this observation that Frege numbers aren't logical objects.

[8]Note that, if primitive, η is a mathematical relation given that Boolos introduces it solely to formulate existence conditions for numbers, and if defined as in footnote 6, η becomes defined in terms of a primitive mathematical notion of an extension or set.

of PA2 in OT follows Fregean lines, for the most part. We will *define* the primitives of natural number theory and natural set theory in object-theoretic terms, *reconstruct* the arithmetical notions defined in terms of these primitives, and *derive* interesting mathematical principles about numbers and sets from the principles of object theory. We're not, however, suggesting that all of theoretical mathematics can (or should) be analyzed in this way; our approach to the analysis of *arbitrary* mathematical theories has been developed elsewhere.[9]

One final remark concerns the strength of the background theory we shall employ – it is developed within axiomatic second-order logic. We do not presuppose a second-order consequence relation or full (standard) second-order models of our axioms. General (Henkin) models suffice and, indeed, are the natural model for the theory we develop in the next section. Since our background theory has a model and is thus consistent,[10] we omit discussion of a background semantics. As noted by Väänänen (2001, 505):

> ... if second-order logic is used in formalizing or axiomatizing mathematics, the choice of semantics is irrelevant: it cannot meaningfully be asked whether one should use Henkin semantics or full semantics. This question arises only if we formalize second-order logic after we have formalized basic mathematical concepts needed for semantics.

In this paper, we formalize a second-order logic and the theory of objects *before* we formally define the basic mathematical concepts.

2 Our Background Theory

Since there have been a significant number of publications on object theory over the past 30 years, we'll assume some familiarity with it. However, the following presentation will contain some new elements, since the theory is still being refined and improved. Object theory is best

[9]OT offers an analysis of theoretical mathematics; see Linsky & Zalta 1995, Zalta 2000, Linsky & Zalta 2006, Nodelman & Zalta 2014, Zalta 2023a, and Leitgeb, Nodelman, & Zalta m.s.

[10]In the natural 'Aczel models' of our background theory, the domain of the 2nd-order quantifier (i.e., the domain of properties) is *not* the full power set of the domain of the 1st-order quantifier. See the discussion in Section 6.3 and the Appendix.

couched in a language that uses two kinds of variables: $x, y, z, \ldots$ range over *individuals* or *objects*, and $F^n, G^n, H^n, \ldots$ range over primitive n-ary *relations* ($n \geq 0$), where unary relations are called *properties* and 0-ary relations are called *propositions*. The primitive relations are to be understood hyperintensionally. That is, the theory does not assume that necessarily equivalent relations are identical – one may consistently assert $\exists F \exists G(\Box\forall x(Fx \equiv Gx) \,\&\, F \neq G)$. There is one distinguished unary relation (i.e., property) constant, namely, $E!$, which one may interpret, in the first instance, as representing the property *being concrete*.

A second-order, quantified modal language is then formulated, with two atomic formulas: exemplification formulas of the form $F^n x_1 \ldots x_n$ ('$x_1, \ldots, x_n$ *exemplify* F^n'), for $n \geq 0$, and encoding formulas $x_1 \ldots x_n F^n$ ('$x_1, \ldots, x_n$ *encode* F^n'), for $n \geq 1$.[11] Intuitively, the unary encoding predication xF represents the fact that F is one of the properties by which we conceive or theoretically define x and so one of the properties that constitutes x. When $n = 0$, F^0 is a formula that asserts F^0 is true. In what follows, we write the propositional variables $F^0, G^0, \ldots$ as $p, q, \ldots$. Identity is defined and so isn't a primitive of the language (see below). In addition to the standard connectives, quantifier, and modal operator ($\neg$, $\rightarrow$, $\forall$, and $\Box$) found in a quantified modal language, the theory also employs an actuality operator $\mathcal{A}$ and two kinds of complex terms: where φ is any formula (called the *matrix*), we include complex individual terms (rigid definite descriptions) of the form $\imath x\varphi$ and, for $n \geq 0$, complex n-ary relation terms (λ-expressions) of the form $[\lambda x_1 \ldots x_n\, \varphi]$ (and φ itself is a 0-ary relation term).[12]

Using this language, we define several notions that are needed to state the axioms. For readability, our definitions use object language variables; strictly speaking, though, such variables function as metavariables. The reason is that we'll be assuming a negative free logic in which definite descriptions and λ-expressions may fail to denote. In such a context, we want our definitions to be instanced even by terms that don't have denotations.

In many free logics it is common to introduce the expression $\tau\!\downarrow$ (where

[11]Readers familiar with OT should note that it now allows n-ary encoding formulas instead of just unary encoding formulas of the form xF. However, in this paper, we make only minimal use of n-ary encoding formulas.

[12]Readers familiar with previous versions of OT will recognize that we've extended the theory to allow any formula φ to be the matrix of a λ-expressions. But it is to be emphasized that not all λ-expressions are guaranteed to have a denotation.

τ ranges over terms) and give this the metatheoretic reading that τ is *significant* or *defined*. However, we also give it the theoretic reading that τ *exists*. We introduce $\tau\!\downarrow$ by cases as follows (using object language variables, for the reasons noted above):[13]

$$x\!\downarrow \equiv_{df} \exists F Fx$$

$$F^n\!\downarrow \equiv_{df} \exists x_1 \ldots \exists x_n (x_1 \ldots x_n F^n) \qquad (n \geq 1)$$

$$p\!\downarrow \equiv_{df} [\lambda x\, p]\!\downarrow$$

As an example of the first of these: $\imath x Mx\!\downarrow$ ("the mayor of the town *exists*") just in case the mayor of the town exemplifies a property. As a unary example of the second: $[\lambda x\, \neg Rx]\!\downarrow$ ("being non-red *exists*") just in case some individual encodes being non-red. As an example of the third: $(Pb)\!\downarrow$ ("Biden is president *exists*") just in case the property being such that Biden is president *exists*. In all these cases, $\neg\tau\!\downarrow$ has the theoretic reading that τ doesn't exist, and the metatheoretic reading that τ is *empty* or *undefined*.

We next distinguish ordinary (i.e., possibly concrete) and abstract (i.e., not possibly concrete) objects by introducing two property constants, as follows:

$$O! =_{df} [\lambda x\, \Diamond E!x]$$

$$A! =_{df} [\lambda x\, \neg\Diamond E!x]$$

We then define identity by cases. The only two cases of the definition that we'll need in what follows are:

$$x = y \ \equiv_{df} \ (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)) \qquad (1)$$

$$F = G \ \equiv_{df} \ F\!\downarrow \,\&\, G\!\downarrow \,\&\, \Box\forall x(xF \equiv xG)$$

We take the closures of the following as axioms of the system:

- The axioms of propositional logic.

[13]Unlike free logic, we shall distinguish between $\tau\!\downarrow$ and $\exists\alpha(\alpha = \tau)$. The two are equivalent in OT but the former is defined without appeal to the notion of identity; rather existence is defined using predication and quantification, as follows in the text. Note that whereas $\exists F^n\varphi$ asserts that there exists an F^n such that φ, $F^n\!\downarrow$ asserts that F^n exists.

- The axioms of classical quantification theory, extended to a negative free logic of *complex* terms. Thus, where κ ranges over individual terms and Π ranges over relation terms, there are axioms asserting that: (a) one may instantiate a term τ into a universal generalization only when $\tau\!\downarrow$, (b) $\tau\!\downarrow$, if τ is a primitive constant, a variable, or a λ-expression that doesn't build an encoding condition directly into its exemplifications conditions,[14] and (c) true atomic formulas of the form $\Pi^n\kappa_1 \ldots \kappa_n$ and $\kappa_1 \ldots \kappa_n\Pi^n$ imply all of the following: $\Pi\!\downarrow$, $\kappa_1\!\downarrow, \ldots$, and $\kappa_n\!\downarrow$.

- The principle asserting the substitution of identicals in any context. This principle will govern the defined notion of identity for both individuals and n-ary relations. (Reflexivity of identity is derivable from these definitions.)

- The axioms for S5 modal logic (with derived 1st- and 2nd-order Barcan formulas) supplemented with the principle that there might have been a concrete object that is not actually concrete, formalized as:

$$\Diamond\exists x(E!x \,\&\, \neg\mathcal{A}E!x) \qquad (2)$$

This axiom guarantees that truth and necessity won't collapse.[15]

[14]This condition on λ-expressions can be spelled out formally. First, we say that a term τ *occurs in encoding position in* φ iff φ has a subterm of the form $\kappa_1 \ldots \kappa_n\Pi^n$ and τ is one of $\kappa_1, \ldots, \kappa_n$, or Π^n. Then clause (b) in the text would become: $\tau\!\downarrow$ if τ is a primitive constant, a variable, or a λ-expression $[\lambda x_1 \ldots x_n\, \varphi]$ in which no variable bound by the λ occurs in encoding position in φ. So, for example, let the matrix φ be $\exists G(xG \,\&\, \neg Gx)$ and consider $[\lambda x \exists G(xG \& \neg Gx)]$. Since the λ binds a variable in encoding position, the axiom of quantification theory we're discussing won't assert $[\lambda x\, \exists G(xG \,\&\, \neg Gx)]\!\downarrow$. Such an assertion would be inconsistent with the comprehension principle for abstract objects introduced below.

[15]Without it, the axioms of S5 are consistent with there being a single possible world. Axiom (2) requires any model to have at least two possible worlds and also guarantees that there are at least 4 propositions: one necessary, one impossible, one contingently true and one contingently false (for the latter, note that the unmodalized claim $\exists x(E!x \,\&\, \neg\mathcal{A}E!x)$ is false but, given the axiom, possible true). Note that this additional axiom is far weaker than the logician's implicit assumption that the domain might be of any size. It also has one added benefit for our particular analysis: it guarantees the existence of at least one possibly concrete object and, hence, the existence of at least one *discernible* object, as this latter notion is defined in Section 4.1 below. As we'll see, discernible objects play an important role in the new analysis.

- The *necessary* axioms for the actuality operator $\mathcal{A}$, including its interaction with the modal operator $\Box$.[16] These assert (a) that $\mathcal{A}$ is idempotent, (b) that $\neg$ and $\forall\alpha$ commute with $\mathcal{A}$, (c) that $\mathcal{A}$ distributes over conditionals, (d) that $\mathcal{A}\varphi \to \Box\mathcal{A}\varphi$, and (e) that $\Box\varphi \equiv \mathcal{A}\Box\varphi$.

- An axiom such as the Russell (1905) axiom or the Hintikka (1959) axiom governing definite descriptions, suitably modified to ensure that descriptions are rigid.[17]

- The axioms for the *relational* λ-calculus, namely, α-, β-, and η-Conversion, all conditionalized to λ-expressions that are significant. For example, when we conditionalize β-Conversion to significant λ-expressions, the axiom asserts:

  $$[\lambda x_1 \ldots x_n\, \varphi]\!\downarrow\; \to ([\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi) \tag{3}$$

  In addition, we stipulate that λ-expressions, whose matrices are necessarily and universally equivalent to the matrices of significant λ-expressions, are themselves significant, i.e., for $n \geq 1$, that:

  $$([\lambda x_1 \ldots x_n\, \varphi]\!\downarrow \,\&\, \Box\forall x_1 \ldots \forall x_n(\varphi \equiv \psi)) \;\to\; [\lambda x_1 \ldots x_n\, \psi]\!\downarrow \tag{4}$$

The above axioms should be reasonably straightforward, if not familiar. However, the final group of four axioms that govern the logic of encoding may be less familiar. The first three assert:[18]

[16]For the purposes of this paper, we omit the single *contingent* axiom governing the actuality operator, namely $\mathcal{A}\varphi \to \varphi$. It won't play a role in the proofs that follow. This allows us to simplify the deductive system, since we no longer have to restrict the Rule of Necessitation from being applied to $\mathcal{A}\varphi \to \varphi$ or any theorem derived from it. See Zalta 1988 for a discussion of details such logical truths that aren't necessary.

[17]By making descriptions rigid, we can suppose that all the closed terms in the language are rigid designators. So we don't need to define a denotation function in the semantics that assigns each term a (possibly distinct) denotation at each world. This is what allows us to assert the substitution of identicals even in modal contexts.

We therefore formulate the axiom for definite descriptions with an actuality operator. The following axiom does the job:

$$y = \imath x\varphi \equiv \forall x(\mathcal{A}\varphi \equiv x = y)$$

This axiom strategically adds the $\mathcal{A}$ operator to the axiom in Hintikka 1959.

[18]The first of these axioms references ordinary objects, and so it should be mentioned that an identity relation on ordinary objects, $=_E$, can be defined as:

$$=_E \;=_{df}\; [\lambda xy\; O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]$$

The definiens is significant, since the λ doesn't bind any variables in encoding position.

- $O!x \to \neg\exists F\, xF$

- $xF \to \Box xF$ (5)

- $x_1 \ldots x_n F^n \equiv x_1[\lambda y\, F^n y x_2 \ldots x_n] \,\&\, x_2[\lambda y\, F^n x_1 y x_3 \ldots x_n] \,\&\, \ldots \,\&\, x_n[\lambda y\, F^n x_1 \ldots x_{n-1} y]$

The first states that ordinary objects don't encode any properties, and the second states that encoding predications are necessary if true. The third asserts that objects $x_1, \ldots, x_n$ encode the n-ary relation F just in case x_1 encodes the property that results when $x_2, \ldots, x_n$ respectively fill F's 2nd-through-nth places, and ..., and x_n encodes the property that results when $x_1, \ldots, x_{n-1}$ respectively fill F's first $n-1$ places.

The fourth axiom of encoding is the comprehension principle for abstract objects; it asserts that for any formula φ that places a condition on properties (i.e., φ has no free x variables, but may have free individual variables other than x and free relation variables), there is an abstract object that encodes just the properties satisfying the condition:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)) \tag{6}$$

By taking, as axioms, the closures of all the above axioms (and only the non-modal closures of the first axiom for actuality), the logic requires only one primitive rule of inference, Modus Ponens (MP). Derivations and theoremhood are defined in the usual way. The Rule of Generalization (GEN), the Rule of Necessitation (RN), and the Rule of Actualization (RA) are then derived as metarules. In this system, the Comprehension Principle for Relations becomes derivable as a theorem: for any formula φ, there is an n-ary relation F such that necessarily, objects $x_1, \ldots, x_n$ exemplify F if and only if φ, provided F doesn't occur free in φ and $x_1, \ldots, x_n$ don't occur free in an encoding formula subterm of φ, i.e., $\exists F \Box\forall x_1 \ldots \forall x_n(F^n x_1 \ldots x_n \equiv \varphi)$ (φ may contain free relation variables other than F, quantified relation variables, and free individual variables).

Models that establish the (relative) consistency of the foregoing system were developed by both Dana Scott and by Peter Aczel.[19] We discuss

[19]Scott sketched models on note paper in 1981 (available on request), and a sketch of the model appears in Zalta 1983 (Appendix A). Aczel described models in personal communications of January 10, 1991 and November 11, 1996 and a sketch of his model construction was described in Zalta 1997 and amended slightly in Zalta 1999 (626–627).

and extend the Aczel model in Section 6.3 and the Appendix below, to verify the consistency of the work in the remainder of this paper.

3 Previous Analysis

In this section, we summarize the definitions and the three axioms introduced in Zalta 1999 to reconstruct the natural numbers (Section 3.1). We then adumbrate a number of issues that this reconstruction faces (Section 3.2). This will prepare us for the next section in which we develop our new analysis and show how it more elegantly approaches the derivation of the natural numbers without mathematical primitives.

3.1 Summary of Zalta 1999

To understand how the axioms of Dedekind-Peano number theory were derived in Zalta 1999, it is important to understand a key theorem of OT, namely, that some distinct abstract objects are "indiscernible with respect to exemplification", i.e., that there are abstract objects that exemplify all and only the same properties. Formally:

$$\exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, \forall F(Fx \equiv Fy)) \tag{7}$$

For the proof, see footnote 16 in Zalta 1999. We'll say more about this theorem in Section 6.3, but for now, note that the theorem implies that when equinumerosity is defined in the classical way, it fails to be an equivalence condition on properties. Specifically, if one defines:

$$F \approx G \;=_{df} \exists R[\forall x(Fx \rightarrow \exists!y(Gy \,\&\, Rxy)) \,\&\, \forall y(Gy \rightarrow \exists!x(Fx \,\&\, Rxy))]$$

then it becomes derivable that:[20]

$$\neg\exists G(A! \approx G)$$

i.e., $\forall G \neg(G \approx A!)$ and, in particular, $\neg(A! \approx A!)$. Since $\approx$ is not an equivalence condition, it doesn't partition the properties.

[20] *Proof*. Suppose $A!a$, $A!b$, $a \neq b$, and $\forall F(Fa \equiv Fb)$. Suppose, for reductio, that $\exists G(A! \approx G)$. Let Q be such a property, i.e., $A! \approx Q$. Then there is a witness R (one-one and onto) from $A!$ to Q. So Rac for some object c such that Qc. So $[\lambda z\, Rzc]a$. But, since a and b are indiscernible, $[\lambda z\, Rzc]b$, i.e., Rbc. But this contradicts the one-one character of R, for we have both Rac and Rbc and yet $a \neq b$.

But one can define a related condition that does partition the properties, namely, *equinumerosity with respect to the ordinary objects*. To define this notion, Zalta used the relation of *identity with respect ordinary objects*, written $=_E$, which is an equivalence condition on ordinary objects. $=_E$ is defined as $[\lambda xy\, O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]$ (see footnote 18). This relation is well-behaved: $=_E$ is reflexive, symmetric, and transitive with respect to the ordinary objects. Moreover, ordinary objects that exemplify the same properties exemplify $=_E$ and the property *being identical*$_E$ *to* x is distinct from the property *being identical*$_E$ *to* y whenever x and y are distinct ordinary objects.[21] Given these facts, we can define equinumerosity with respect to the ordinary objects, written $F \approx_E G$, and show that it is an equivalence condition on properties. Where u, v range over ordinary objects, we define:

$$F \approx_E G =_{df} \exists R[\forall u(Fu \rightarrow \exists!v(Gv \,\&\, Ruv)) \,\&\, \forall v(Gv \rightarrow \exists!u(Fu \,\&\, Ruv))]$$

It then follows that $F \approx_E G$ is a reflexive, symmetric, and transitive condition on properties (Zalta 1999, 629).

With this framework, Zalta defined (1999, 630–632):

$$Numbers(x, G) \equiv_{df} A!x \,\&\, \forall F(xF \equiv F \approx_E G) \tag{8}$$

$$\#G =_{df} \iota x Numbers(x, G)$$

$$NaturalCardinal(x) \equiv_{df} \exists F(x = \#F) \tag{9}$$

$$Precedes(x, y) \equiv_{df} \exists F \exists u(Fu \,\&\, y = \#F \,\&\, x = \#[\lambda z\, Fz \,\&\, z \neq_E x])$$

These definitions anchored the subsequent analysis of numbers in Zalta 1999, but they required one to assert that *Predecessor* and its weak ancestral are relations, and assert a modal axiom that helped to establish that every number has a successor. The modal axiom asserted (1999, 635): if there is a natural number of Gs, there might have been a concrete object distinct from all the actual Gs, i.e.,

$$\exists x(NaturalNumber(x) \,\&\, x = \#G) \rightarrow \Diamond\exists y(E!y \,\&\, \forall u(\mathcal{A}Gu \rightarrow u \neq_E y)) \tag{10}$$

We now examine some of the issues that arise for this reconstruction.

[21] These last two facts are formalized as:

$$O!x \,\&\, O!y \,\&\, \forall F(Fx \equiv Fy) \rightarrow x =_E y$$

$$O!x \,\&\, O!y \,\&\, x \neq_E y \rightarrow [\lambda z\, z =_E x] \neq [\lambda z\, z =_E y]$$

3.2 The Problems for This Analysis

The first problem to note is the inappropriate definition of the weak ancestral of *predecessor*, given in Zalta 1999. To see the problem, suppose $\mathbb{P}$ is the *relation* of *predecessor*, as defined in the usual way. We noted above that in Zalta 1999, the existence of the strong ancestral of $\mathbb{P}$, namely, $\mathbb{P}^*$, is derivable, but the existence of the weak ancestral, namely $\mathbb{P}^+$, was asserted as an axiom, where $\mathbb{P}^+$ is defined as $[\lambda xy\, \mathbb{P}^*xy \vee x = y]$. From these claims, one can form the relation $[\lambda xy\, \mathbb{P}^+xy \,\&\, \neg\mathbb{P}^*xy]$, i.e., the relation of being an x and y such that x is a weak $\mathbb{P}$-ancestor of y but not a strong $\mathbb{P}$-ancestor of y. But this relation turns out to be equivalent to a *completely general* relation of identity given that the definitions of $\mathbb{P}$, $\mathbb{P}^*$, and $\mathbb{P}^+$ were defined for any objects whatsoever.[22] A completely general relation of identity would introduce the McMichael-Boolos Paradox. This result could have been avoided if Zalta had asserted that the weak ancestral is a relation only relative to the domain of *Precedes*. So we have to take greater care when defining the weak ancestral of a relation R – the notion of identity needed (in the definition of the weak ancestral of R) only has to be an equivalence relation on the objects in the domain of R. The weak ancestral of R doesn't require a completely general notion of identity for its definition.

A second problem (one mentioned previously) is that Frege's definition of natural cardinals has an unintuitive consequence when naively used in a non-trivial modal setting (i.e., one where there are at least two possible worlds), namely, that the equivalence classes of equinumerous (and equinumerous$_E$) properties vary from world to world, thereby giving rise to different numbers at different worlds. If there were, say, just a pair of ordinary G things at one possible world and just a pair of ordinary G things at a different possible world, we *don't* want to be forced to say that the number that numbers G is different at both worlds. But since G will be a member of distinct equivalence classes of equinumerous properties at distinct worlds, the object that numbers G at the former world will be distinct from the object that numbers G at the latter world.

[22] To see this, we can reason as follows:

$$\begin{array}{rcll} [\lambda xy\, \mathbb{P}^+xy \,\&\, \neg\mathbb{P}^*xy]zw & \equiv & \mathbb{P}^+zw \,\&\, \neg\mathbb{P}^*zw & \text{by } \beta\text{-Conversion} \\ & \equiv & (\mathbb{P}^*zw \vee z = w) \,\&\, \neg\mathbb{P}^*zw & \text{by definition of } \mathbb{P}^+zw \\ & \equiv & z = w & \text{by propositional logic} \end{array}$$

Hence the defined relation would hold between any two objects if and only if they are equal. But there can be no such relation in object theory.

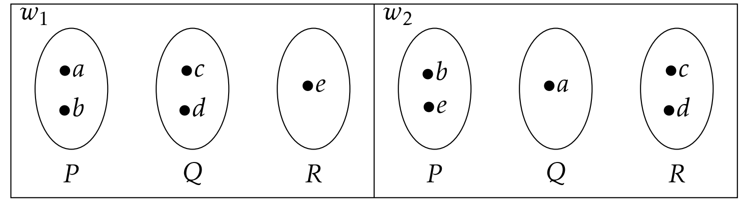

Figure 1: Emergence of world-bound natural cardinals.

This runs contrary to our general understanding of how natural cardinals work and it shows that Frege's picture doesn't generalize in a modal context without some adjustment.

The problem here isn't that some object, say x, numbers G at possible world w_1 and some different object, say y, numbers G at possible world w_2. That is only to be expected, since G might be exemplified by two ordinary objects in w_1 and three ordinary objects in w_2. Rather, the problem is that G might be exemplified by two ordinary objects in both w_1 and w_2, but the object that numbers G in w_1 is different from the object that numbers G in w_2. To see why, consider the object-theoretic representation of Frege's picture, as captured by (8) above. Then note that (8) gives rise to the following necessary equivalence:

$$\Box(Numbers(x, G) \equiv (A!x \,\&\, \forall F(xF \equiv F \approx_E G)))$$

Intuitively, then, the central material biconditional should hold at every possible world. But consider the scenario depicted in Figure 1, which shows only the ordinary objects that exemplify P, Q, and R in possible worlds w_1 and w_2. In this scenario, P and Q are equinumerous$_E$ in w_1, but not in w_2. Now, intuitively, the number of Ps at w_1 should be identical to the number of Ps at w_2, since there are exactly two objects exemplifying P at both w_1 and w_2. But suppose x numbers P at w_1. Then, by (8), x encodes all and only the properties equinumerous$_E$ to P at w_1. Hence, x encodes Q as well, though not R. Now suppose y numbers P at w_2. Intuitively, it should be the case that $y = x$. But then, y encodes R, not Q, since R is equinumerous$_E$ to P at w_2 and Q isn't. Hence, $y \neq x$, since x and y are abstract objects that encode different properties. In general, the objects that number properties at w_1 according to (8) are different from the objects that number properties at w_2 according to (8).

This is not just a consequence of identifying an object that numbers

G as an abstract object that encodes the properties equinumerous$_E$ to G. The problem arises in adopting Frege's conception of numbers in a modal setting. The second-level concept *being equinumerous to P* has different properties falling under it at w_1 and at w_2. So the object that is the extension of this second-level concept at w_1 is different from the object that is the extension of the second-level concept at w_2. Thus, the number that belongs to P at w_1 would be different from the number that belongs to P at w_2, i.e., the number 2 abstracted from w_1 would be distinct from the number 2 abstracted from w_2. Since equivalence classes of equinumerous properties may vary from world to world, the Fregean abstractions on the basis of the simple equivalence condition of equinumerosity (or equinumerous$_E$) will yield *world-bound* numbers.

The third and final problem for the analysis in Zalta 1999 is that the natural cardinals defined there count only the ordinary objects falling under a property and so can't count the numbers themselves. Thus, Zalta couldn't reconstruct Frege's proof that every number has a successor by establishing that n precedes the number of the concept *being less than or equal to n*. A modal axiom, labeled above as (10), was introduced to address this problem; with the modal axiom, the system as a whole implied that there is an infinite supply of ordinary objects and, hence, that every number has a successor. And (10) was justified independently on the grounds that it, in part at least, captured an intuition accepted almost universally by logicians, namely, that the domain might be of any size. But we've now discovered that (10) isn't really needed; the axiom we introduce below only implies the existence of a single ordinary (i.e., *possibly concrete*) object.

4 The New Analysis

Our work in this section may be summarized as follows:

- Section 4.1 consists of an explanation how the notion of a *discernible* object gives us a broader group of objects to count.
- Section 4.2 introduces a way to use the natural numbers that emerge at the actual world as universal natural numbers that can count the objects falling under a property at any world.
- Section 4.3 is a study of relations on discernibles and how the strong and weak ancestrals of those relations are well-behaved.
- Section 4.4 shows that the numbers are, themselves, discernible, thus allowing us to reconstruct Frege's derivation of the Dedekind-Peano postulates and, in the process, eliminates the need for the modal axiom (10) used in Zalta 1999.

4.1 A Broader View: Discernibility

The key to our new research results is to define the *discernible* objects. Whereas *indiscernibility* relates two objects, discernibility is a condition that a single object x satisfies just in case, necessarily, for any object y distinct from x, there is some property that distinguishes x and y. Formally, x is *discernible* iff $\Box\forall y(y \neq x \to \exists F\neg(Fy \equiv Fx))$. However, it turns out that we can establish that the condition just stated defines a property, i.e., that:

$$[\lambda x\, \Box\forall y(y \neq x \to \exists F\neg(Fy \equiv Fx))]\!\downarrow \qquad (11)$$

The existence of this property appeals to Kirchner's Theorem, which is derivable in object theory:[23]

$$[\lambda x\, \varphi]\!\downarrow\, \equiv \Box\forall x\forall y(\forall F(Fx \equiv Fy) \to (\varphi \equiv \varphi^{y}_{x})), \qquad (12)$$
provided y doesn't occur free in φ.

We shall leave it to the reader to prove that when φ is set to $\Box\forall y(y \neq x \to \exists F\neg(Fy \equiv Fx))$, then the resulting instance of (12) implies (11). This fact allows us to define the property *being discernible*:

$$D! =_{df} [\lambda x\, \Box\forall y(y \neq x \to \exists F\neg(Fy \equiv Fx))] \qquad (13)$$

An identity relation on discernibles is also definable:

$$=_D\ =_{df}\ [\lambda xy\, D!x \,\&\, D!y \,\&\, x = y]$$

These definitions yield a number of facts, including:

[23]Intuitively, Kirchner's Theorem says: the relation $[\lambda x\, \varphi]$ exists if and only if necessarily, φ can't distinguish between objects that are indiscernible. The left-to-right direction is trivial: if $[\lambda x\, \varphi]$ exists, then we can instantiate it into $\forall F(Fx \equiv Fy)$ and infer that x satisfies φ if and only if y does. The right-to-left direction requires more reasoning. Assume φ can't distinguish indiscernibles. Then the claim $\exists x(\forall F(Fx \equiv Fy) \,\&\, \varphi)$ is necessarily and universally equivalent to φ^{y}_{x}. Since the property $[\lambda y\, \exists x(\forall F(Fx \equiv Fy) \,\&\, \varphi)]$ exists (y doesn't occur free in φ, and so the variable bound by the λ only occurs in exemplification position), it follows by axiom (4) that $[\lambda y\, \varphi^{y}_{x}]$ exists. So $[\lambda x\, \varphi]$ exists, since it is an alphabetic variant.

- Ordinary objects are discernible.
  $O!x \rightarrow D!x$ (14)

- Discernibility is a rigid property.
  $D!x \rightarrow \Box D!x$

- Indiscernible discernibles are identical.
  $(D!x \vee D!y) \rightarrow (\forall F(Fx \equiv Fy) \rightarrow x = y)$

- $=_D$ is an equivalence relation on discernibles.

- Necessarily, identity$_D$ is equivalent, on the discernibles, to identity.
  $(D!x \vee D!y) \rightarrow \Box(x = y \equiv x =_D y)$ (15)

- Haecceities of discernibles exist.
  $D!y \rightarrow [\lambda x\, x = y]\downarrow$ (16)

- Distinct discernibles have distinct haecceities.
  $(D!x \,\&\, D!y) \rightarrow (x \neq y \equiv [\lambda z\, z = x] \neq [\lambda z\, z = y])$

- Any formula φ defines a property on discernibles.
  $[\lambda x\, D!x \,\&\, \varphi]\downarrow$ (by Kirchner's Theorem)

- There is a discernible object.
  $\exists x D!x$ (17)

It is important to observe that to *derive* (17), we make use of the modal axiom (2),[24] but as we shall see in our discussion of non-modal Frege Arithmetic in Section 6.2, one could replace the modal axiom (2) by asserting (17) as an axiom instead, so as to preserve, in a *non-modal setting*, the goal of reconstructing Frege Arithmetic without mathematical primitives.

Given the notion of discernibility, we define *equinumerosity with respect to the discernible objects* ($\approx_D$) as follows, where u, v range over discernibles:[25]

$$F \approx_D G =_{df} F\downarrow \,\&\, G\downarrow \,\&\,$$
$$\exists R[\forall u(Fu \rightarrow \exists! v(Gv \,\&\, Ruv)) \,\&\, \forall v(Gv \rightarrow \exists! u(Fu \,\&\, Ruv))]$$

Now one might expect that we will next replace definition (8) by (18):

$$Numbers(x, G) \equiv_{df} A!x \,\&\, \forall F(xF \equiv F \approx_D G) \tag{18}$$

But we shall *not* adopt this definition because it would still yield distinct numbers at distinct worlds (i.e., it doesn't carve out universal natural cardinals or universal natural numbers).

4.2 Defining Universal Natural Cardinals

The key to defining universal natural cardinals is to use the cardinal numbers from the actual world to count objects at every world.[26] We do this by strategically placing an actuality operator in the definition of x *numbers* G. Instead of (8) or (18), we shall say that x *numbers* G if and only if x is an abstract object that encodes all and only the properties F such that the property *actually exemplifying* F is equinumerous to G with respect to the ordinary objects:

$$Numbers(x, G) \equiv_{df} A!x \,\&\, G\downarrow \,\&\, \forall F(xF \equiv [\lambda z\, \mathcal{A}Fz] \approx_D G) \tag{19}$$

This definition yields the following necessary equivalence:

$$\Box(Numbers(x, G) \equiv A!x \,\&\, \forall F(xF \equiv [\lambda z\, \mathcal{A}Fz] \approx_D G))$$

Let's see how this definition and equivalence yields *universal* cardinals. By including the actual world w_α in our picture, as in Figure 2, things become clearer. In Figure 2, just as in Figure 1, the number of Ps at w_1 should be identical to the number of Ps at w_2, since there are exactly two objects exemplifying P at both w_1 and w_2. We verify this over the next two paragraphs by using our revised definition.

Suppose x numbers P at w_1. By our adjusted Fregean definition (19), x encodes all and only the properties F such that $[\lambda z\, \mathcal{A}Fz]$ is equinumerous$_D$ to P at w_1. Inspection then shows that x encodes both Q and R, since $[\lambda z\, \mathcal{A}Qz]$ and $[\lambda z\, \mathcal{A}Rz]$ are both equinumerous$_D$ to P at w_1, i.e.,

[24] Axiom (2) asserts that there might be a concrete object that's not actually concrete, i.e., $\Diamond\exists x(E!x \,\&\, \neg\mathcal{A}E!x)$. Then by the Barcan formula (1946), $\exists x\Diamond(E!x \,\&\, \neg\mathcal{A}E!x)$. Suppose a is such an object, so that we know $\Diamond(E!a \,\&\, \neg\mathcal{A}E!a)$. *A fortiori*, $\Diamond E!a$, i.e., a is an ordinary object. Then by (14), $D!b$.

[25] We add existence clauses such as $F\downarrow$ to the definiens because we're now working within a language in which relation terms might be empty. This ensures that if the definition is instanced by a non-denoting property term Π, the definiens will be false and the definiedum will fail to hold. The definitions of equinumerosity cited earlier didn't need such clauses since they were constructed in languages where it was assumed that every relation term has a denotation.

[26] Frege didn't have possible worlds and so we'll extend Frege's view by defining the numbers he constructed in such a way that they can serve as universal numbers.

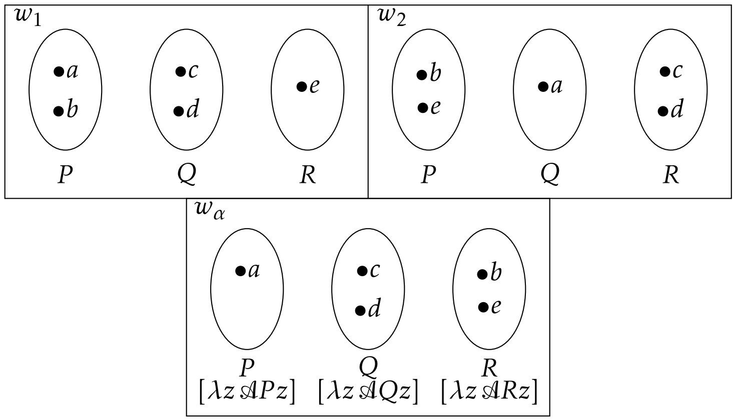

Figure 2: Emergence of universal natural cardinals.

there is a one-to-one correspondence from the discernible objects exemplifying $[\lambda z\, \mathcal{A}Qz]$ to the discernible objects exemplifying P at w_1, and similarly from the discernible objects exemplifying $[\lambda z\, \mathcal{A}Rz]$.

Now suppose y numbers P at w_2. By the adjusted Fregean definition (19), y encodes all and only those Fs such that $[\lambda z\, \mathcal{A}Fz]$ is equinumerous$_D$ to P at w_2. Inspection then shows that y encodes Q and R, since these are still the only Fs such that *actually exemplifying* F is equinumerous$_D$ to P at w_2. Since these are the only encoding facts available, x and y encode, and so necessarily encode, by (5), exactly the same properties; hence $x = y$. Not only is the number of Ps at w_1 identical to the number of Ps at w_2, but the number of Ps at w_1 is identical to the number of Rs at w_2. And so on.

Our definition (19) thus changes the corresponding definition in Zalta 1999 in two important ways: it introduces the actuality operator and it appeals to equinumerous$_D$ properties. The first step is the one that yields universal numbers: if we had used equinumerous$_E$ instead of equinumerous$_D$ in (19), we would have still obtained universal numbers. The numbers are just objects that are abstracted from the specific patterns of numerosity of properties at the actual world. Through the actuality operator, every world has access to these same specific patterns. So by placing the actuality operator in a judicious manner, we preserve the Fregean understanding of numbering in a modal context. The objects that number properties are defined with respect to the actual world

but can be used to number properties at any world.[27]

But the second way in which (19) changes the corresponding definition in Zalta 1999 holds the key to reconstructing Frege's proof that every number has a successor. By using equinumerous$_D$ instead of equinumerous$_E$, we thereby define numbers that have wider applicability, especially when we show that the numbers themselves are discernible. Our new effort thus begins by proving that for every property G, there is a unique abstract object x such that $Numbers(x,G)$, as this latter notion is now defined in (19). This, in turn, grounds the following, explicit definitions of: *the number of* Gs (#G), x is a *natural cardinal*, and *Zero* (0):

$$\#G =_{df} \imath x Numbers(x,G) \tag{20}$$

$$NaturalCardinal(x) \equiv_{df} \exists G(x = \#G) \tag{21}$$

$$0 =_{df} \#[\lambda z\, z \neq_D z] \tag{22}$$

With these definitions, we can improve the derivation of the Dedekind-Peano axioms that was described in Zalta 1999. We won't spend the time here rehearsing the derivations, except to point out where the developments are interesting and new.

4.3 Relations on Discernibles and Their Ancestrals

We start with some key definitions needed to derive the Dedekind-Peano axioms, namely, those of the strong and weak ancestral of a relation. As in Zalta 1999, we define $Hereditary(F,G)$ as $\forall x \forall y(Gxy \rightarrow (Fx \rightarrow Fy))$ and then define the strong ancestral G^* of a binary relation G as follows:[28]

$$G^* =_{df} [\lambda xy\, \forall F(\forall z(Gxz \rightarrow Fz)\, \&\, Hereditary(F,G) \rightarrow Fy)]$$

Now to define weak ancestrals in a way that avoids reintroducing the McMichael-Boolos paradox, we first say that F is a *relation on discernibles* just in case F exists and necessarily, any objects that exemplify F are discernible.[29] Clearly, if G is a relation that relates only discernibles, then

[27] Cf. Cook 2016, which offers a different method for addressing the problem of the world-bound numbers that arise for the Fregean conception in a modal context.

[28] Note that the matrix of the λ-expression that constitutes the definiens has no encoding subformulas and so denotes a relation.

[29] Formally:

$$OnDiscernibles(F) \equiv_{df} F\!\downarrow\ \&\ \Box\forall x_1 \ldots \forall x_n(Fx_1 \ldots x_n \rightarrow (D!x_1\ \&\ \ldots\ \&\ D!x_n)) \qquad (n \geq 1)$$

G^* is also a relation on discernibles. Now since Frege's understanding of numbers presupposes that the domain of objects consisted solely of discernible objects, we can focus our attention just on relations on discernibles. Indeed, the weak ancestral of a relation on discernibles becomes definable in such a way that all of the facts concerning the weak ancestral are preserved.

To see this, let $\underline{G}$ be a variable ranging over relations on discernibles. We then define the weak ancestral $\underline{G}^+$ of $\underline{G}$ as follows:

$$\underline{G}^+ =_{df} [\lambda xy\, \underline{G}^*xy \vee x =_D y] \tag{23}$$

The definiens here is significant and denotes a relation, since it is a disjunction of two existing relations. It is now straightforward to prove such facts about weak ancestrals, such as $\underline{G}^*xy \rightarrow \underline{G}^+xy$, etc.[30]

4.4 Derivation of the Dedekind-Peano Axioms

Though these definitions will help us to define *predecessor*, its ancestrals, and *natural number*, one important additional refinement we've introduced is that, instead of defining *predecessor* and asserting that it is a relation, we now assert that the λ-expression needed to define *predecessor* is significant. We do this in two steps. Where u ranges over discernible objects, we first define:

$$F^{-u} =_{df} [\lambda z\, Fz \,\&\, z \neq_D u] \tag{24}$$

I.e., F^{-u} is the property *being an F distinct*$_D$ *from u*. Then we assert the following existence claim:

$$[\lambda xy\, \exists F \exists u(Fu \,\&\, Numbers(y,F) \,\&\, Numbers(x,F^{-u}))]\downarrow \tag{25}$$

This asserts the existence of a logical ordering relation on certain abstract objects that number properties. It should be emphasized that this

[30]In reproving these facts, we corrected another error in Zalta 1999: it was incorrectly asserted and 'proved' that the principle $G^*xy \rightarrow \exists z(G^+xz \,\&\, Gzy)$ holds for any relation G, whereas in fact it holds only for relations on discernibles. (The 'proof' in Zalta 1999 incorrectly referenced a λ-expression that contained the formula G^+xz; this λ-expression was ill-formed, given the syntactic restrictions on λ-expressions used in that paper.) Fortunately, nothing relied upon that theorem in Zalta 1999. Nevertheless, it is important to note that this principle must be restricted to relations on discernibles, for it will play an important role in our new derivation of the Dedekind-Peano axioms and PA2. So the principle $\underline{G}^*xy \rightarrow \exists z(\underline{G}^+xz \,\&\, \underline{G}zy)$ is valid and provable, for any relation on discernibles $\underline{G}$.

relation is a non-mathematical one. The condition $Numbers(x,F)$ is not primitive – it is defined in terms of the primitive notions of object theory, none of which are mathematical. The matrix of the λ-expression is a version of the condition Frege used to define predecessor, and since the above is an axiom that asserts that the λ-expression denotes a relation, we then define *immediate predecessor* ($\mathbb{P}$) in terms of this relation:

$$\mathbb{P} =_{df} [\lambda xy\, \exists F \exists u(Fu \,\&\, Numbers(y,F) \,\&\, Numbers(x,F^{-u}))]$$

One can now *derive* that $\mathbb{P}$ is a rigid, one-to-one relation on discernibles.[31]

Some key pieces of reasoning now follow from the fact that we can use (2) to prove (17), i.e., $\exists x D!x$, as described in footnote 24. From (17), it follows that there are objects that stand in the *predecessor* relation, i.e., that $\exists x \exists y \mathbb{P}xy$.[32] Next we may use the 1-1 and functional character of $\mathbb{P}$ to show that natural cardinals are discernible. That is, from the following facts:

$$\mathbb{P}xy \,\&\, \mathbb{P}zy \rightarrow x = z \tag{$\mathbb{P}$ is 1-1}$$

$$\mathbb{P}xy \,\&\, \mathbb{P}xz \rightarrow y = z \tag{$\mathbb{P}$ is functional}$$

it follows that:

$$NaturalCardinal(x) \rightarrow D!x \tag{26}$$

The proof is by cases.[33]

[31]A *rigid* binary relation G is one such that $\Box\forall x\forall y(Gxy \rightarrow \Box Gxy)$. A one-to-one binary relation is defined in the usual way: $\forall x\forall y\forall z(Gxz \,\&\, Gyz \rightarrow x = y)$.

[32]Let a be any discernible object (we've seen that there is at least one). Consider $[\lambda x\, x = a]$, which exists by (16). Now let b be such that $Numbers(b,[\lambda x\, x = a])$. Then 0 and b are witnesses to $\exists x\exists y Pxy$. To see this, we need to show $\exists F\exists u(Fu \,\&\, Numbers(b,F) \,\&\, Numbers(0,F^{-u}))$. So we need witnesses for $\exists F$ and $\exists u$. But this is easy, since $[\lambda x\, x = a]$ and a will do. Clearly $[\lambda x\, x = a]a$, and by hypothesis, b numbers $[\lambda x\, x = a]$. Then it only remains to show that 0 numbers $[\lambda x\, x = a]^{-a}$, which is straightforward, given definition (24).

[33]In the proof in footnote 32, we established $\exists y \mathbb{P}0y$. Call this object b. Now the two cases are $x = 0$ and $x \neq 0$. If $x = 0$, then since $\mathbb{P}$ is 1-1, one can show that 0 uniquely exemplifies $[\lambda y\, Pyb]$. So, by (13), $D!x$. If $x \neq 0$, then first we need to know that there is a z such that $\mathbb{P}zx$. But this is straightforward because x is a natural cardinal, so there is some property G such that $Numbers(x,G)$. And since $x \neq 0$, there is some discernible c that exemplifies G. (Otherwise, $x = 0$, contradiction.) So let d be such that $Numbers(d,G^{-c})$. Then it follows that $\mathbb{P}dx$. Now, since $\mathbb{P}$ is functional, $[\lambda y\, Pdy]$ is uniquely exemplified by x. So, by (13), $D!x$.

Finally, after we establish that the strong ($\mathbb{P}^*$) and weak ($\mathbb{P}^+$) ancestrals of predecessor are rigid relations on discernibles,[34] the (rigid) property *being a natural number* becomes definable as: *being an object to which 0 bears the weak ancestral of predecessor*:

$$\mathbb{N} =_{df} [\lambda x\, \mathbb{P}^+ 0x] \tag{27}$$

This definition, together with facts about the weak ancestral of *predecessor*, implies that natural numbers are natural cardinals and, hence, discernible.[35]

With these definitions and facts, the derivations of the Dedekind-Peano postulates now proceed along Fregean lines. The postulates are:

- $\mathbb{N}0$
- $\neg\exists n \mathbb{P}n0$
- $\forall n \forall m \forall k (\mathbb{P}nk \,\&\, \mathbb{P}mk \rightarrow n = m)$
- $\forall n \exists m \mathbb{P}nm$
- $F0 \,\&\, \forall n \forall m (\mathbb{P}nm \rightarrow (Fn \rightarrow Fm)) \rightarrow \forall n Fn$

Familiarity with Frege's derivations of these postulates (e.g., as reconstructed in Zalta 1998 [2023b]), will be assumed in what follows. The first three postulates follow immediately from the definitions and theorems above.

Knowledge of Frege's derivations puts us in a position to understand how to prove the fourth postulate with with the simpler modal axiom (2) rather than with (10) (i.e., without the modal axiom used in Zalta 1999); natural numbers can now be used to count the discernible objects, including the natural numbers, that fall under properties.

So the proof that every number has a successor now proceeds along the lines of Frege's proof, namely, by appealing to the theorem that n immediately precedes the number of the concept *being less than or equal to*

[34]The first of these theorems is non-trivial. The proof relies on constructing a variant of the relation $\mathbb{P}^*$, namely, $\mathbb{P}^\star$, which is defined over *rigid* properties F that are hereditary with respect to the predecessor relation (i.e., $\mathbb{P}^\star$ is the strong ancestral of predecessor restricted to rigid properties). It then can more easily be shown that $\mathbb{P}^\star$ is rigid and that it is necessarily equivalent to $\mathbb{P}^*$. It follows that $\mathbb{P}^*$ is rigid. From this and the rigidity of $=_D$ it follows directly that $\mathbb{P}^+$ is a rigid relation on discernibles.

[35]Assume $\mathbb{N}x$, that is, $\mathbb{P}^+0x$. We reason from two cases. If $x = 0$, then *NaturalCardinal*(x), by (21) and (22). If $x \neq 0$, then by (15), $x \neq_D 0$. Hence, by (23), $\mathbb{P}^*0x$. Now it is a fact about the strong ancestral of a relation G that $G^*xy \rightarrow \exists z Gzy$ (since, intuitively, the ancestral is the transitive closure of G). It then follows *a fortiori* that $\exists z \mathbb{P}zx$. Let a be such an object, so that we know $\mathbb{P}ax$. Then by the definition of $\mathbb{P}$ (25), $\exists F(\mathit{Numbers}(x,F))$, which is provably equivalent to $\exists F(x = \#F)$. So *NaturalCardinal*(x), by (21). Thus, $D!x$, by (26).

n. Of course, this last property is defined in terms of the weak ancestral $\mathbb{P}^+$ of predecessor (rather than by defining $\leq$). The key theorem in our revised proof takes one of two forms:

- Every natural number n immediately precedes the number of *being a weak predecessor-ancestor of n*.
  $$\forall n \mathbb{P}n\#[\lambda z\, \mathbb{P}^+ zn] \tag{28}$$

Frege proves the base case of the second form as Theorem 154 (1893 [2013, 147]) and proves the inductive case as Theorem 155 (1893 [2013, 149]). With these theorems in place, it is then straightforward to prove that every number has a successor.

The fifth, and final, postulate (Mathematical Induction) is derived as an instance of Frege's more general theorem:

$$[Fz \,\&\, \forall x \forall y((\underline{G}^+ zx \,\&\, \underline{G}^+ zy) \rightarrow (\underline{G}xy \rightarrow (Fx \rightarrow Fy)))] \rightarrow \forall x(\underline{G}^+ zx \rightarrow Fx)$$

This is derived from the definitions of the weak and strong ancestrals of a binary relation (on discernibles).

Note here that no mathematical primitives are used in the derivation of the Dedekind-Peano axioms. The primitive notions used are those of 2nd-order QML extended with encoding, an actuality operator and two kinds of complex terms (descriptions and λ-expressions.). The notions defined in terms of these primitives are: identity ($=$), identity$_D$ ($=_D$), equinumerosity$_D$ ($\approx_D$), x numbers G (*Numbers*(x,G)), the number of Gs ($\#G$), Zero (0), predecessor ($\mathbb{P}$), the strong ancestral of predecessor ($\mathbb{P}^*$), the weak ancestral of predecessor ($\mathbb{P}^+$), and natural number ($\mathbb{N}$). Later, in Section 6.2, we'll see that in a non-modal setting (which eliminates the actuality operator), the Dedekind-Peano axioms are still derivable if one replaces (2) by an even weaker, non-modal claim that asserts the existence of at least one discernible.

5 Extending the New Analysis

Now that we've addressed the issues and improved Zalta 1999 in various ways, we next extend the work there in the following three ways:

- Section 5.1 sketches the derivation of the Recursion Theorem.
- Section 5.2 then completes the derivation of the remaining axioms of PA2.

- Section 5.3 develops a new derivation of the existence of an infinite cardinal and a derivation of an infinite class.

5.1 Derivation of the Recursion Theorem

At present we have only three means of proving the existence of complex relations: (a) by an instance of the comprehension principle for relations, (b) by formulating a λ-expression for which $[\lambda x_1 \ldots x_n\, \varphi]\downarrow$ is axiomatic, or (c) by the 'safe extension' axiom (4) of object theory that tells us that the candidate λ-expression is equivalent to one that has already been shown to have a denotation. But recursive definitions have 2 parts: a base clause, and a recursive clause in which the definiendum itself may occur within the definiens. None of the methods (a) – (c) allow one to assert the existence of relations defined by such apparently circular means. So we have to prove that functions and operations defined by recursion do, in fact, provably exist as relations by one of methods (a) – (c) and thus without circularity. One way to do this is by proving a Recursion Theorem which asserts that a correctly specified recursive definition implies the existence of a relation that satisfies both clauses of that definition.[36]

For example, it is easy enough to define addition recursively with base clause $n + 0 = n$ and the recursive clause $n + m' = (n + m)'$. But this is not sufficient to show that there exists a *relation* of addition signified by the + symbol. We can't directly apply any of (a) – (c) to the 2-part recursive definition. But once we've proved a Recursion Theorem, we can rest assured that a well-formed recursive definition guarantees the existence of the recursively defined relation.

To prove the Recursion Theorem, the following notions need to be defined: functions (defined as relations of a certain kind), rigid functions, function application, numerical operations, constant functions, the successor function (which, as we shall see, is the predecessor *relation*), projection functions, and function composition. These definitions are straightforward in a set-theoretic environment that takes functions to be sets of n-tuples, but may not be as familiar when working solely with primitive relations of second-order logic. In what follows, we describe only the highlights and leave many of the details to footnotes or rely on standard definitions. For example, we often gloss over the fact that numerical operations are *rigid* relations on the numbers; their rigidity derives from the fact that $\mathbb{N}$ is a rigid property.[37]

Let $R_{\restriction F}$ be the restriction of the 2-place relation R to the domain of objects that exemplify the property F:

$$R_{\restriction F} =_{df} [\lambda xy\, Fx \,\&\, Rxy] \tag{29}$$

So, for example, we can apply (29) to obtain:

$$< \;=_{df}\; \mathbb{P}^{*}_{\restriction \mathbb{N}} \tag{30}$$

$$\leq \;=_{df}\; \mathbb{P}^{+}_{\restriction \mathbb{N}}$$

Moreover, we can apply (29) to obtain *number identity* ('$\doteq$') as the restriction of the relation of identity$_D$ ($=_D$) to the natural numbers ($\mathbb{N}$), i.e.,

$$\doteq \;=_{df}\; =_{D\restriction \mathbb{N}} \tag{31}$$

It doesn't take much reasoning to show that number-identity is an equivalence relation on the natural numbers and is also a rigid relation.

In what follows, we focus on relations that are *functions*. We adopt the well-known definition of an n-ary function as any n+1-ary relation R such that $\forall x_1 \ldots \forall x_n \forall y \forall z(Rx_1 \ldots x_n y \,\&\, Rx_1 \ldots x_n z \rightarrow y = z)$. We may then rewrite, in functional notation, formulas that assert that an n+1-ary functional relation holds. So for example, the $Rx_1 \ldots x_n y$ may be rewritten as $R(x_1, \ldots, x_n) = y$. So where f is an n+1-ary functional relation, we define the functional notation $f(x_1, \ldots, x_n)$ as: $\imath y f x_1 \ldots x_n y$. When we are describing functions on the numbers, we can use $\doteq$ interchangeably with $=$, since $x = y \equiv x \doteq y$ when x and y are numbers.

The predecessor relation $\mathbb{P}$ is easily shown to be a functional relation, but since $\mathbb{P}nm$ means that n precedes m, when we view the relation $\mathbb{P}$ as a function, it maps a number n to it successor m. So we use it to define the *numerical successor* function $\boldsymbol{s}$; that is, the successor of n is the number m such that n bears the predecessor relation (restricted to the natural numbers) to m:

[36] Another way to do this is to introduce addition, for example, as a relation via comprehension by the following instance:

$$\exists F \forall x \forall y \forall z(Fxyz \equiv \exists G \exists H(x{=_D}\#G \,\&\, y{=_D}\#H \,\&\, \neg\exists x(Gx \,\&\, Hx) \,\&\, z{=_D}\#[\lambda w\, Gw \vee Hw]))$$

It can be shown that for any such F and any numbers n and m, $n = 0 = n$ and $n{+}m' = (n{+}m)'$. Similarly, one can introduce multiplication as an instance of comprehension – see Heck 2014, 288–289. This is a somewhat complicated series of definitions.

However, we prefer to prove a Recursion Theorem, since it more elegant, more closely follows mathematical practice, and offers a clear path for justifying all of the basic recursive functions using classical recursive definitions.

[37] This, in turns, derives from the fact that $\mathbb{P}^+$ is rigid, as discussed in footnote 34.

$$\boldsymbol{s}(n) =_{df} \imath m \mathbb{P}_{\restriction\mathbb{N}} nm \tag{32}$$

Henceforth, we regard the standard notation for successor, n', to be defined as $\boldsymbol{s}(n)$.

We next say that an n'-ary relation R ($n \geq 1$) is an *n-ary numerical operation*, written $Op^n(R)$, if and only if R is rigid and R is an n-ary function that relates n natural numbers to a unique natural number:[38]

$$Op^n(R) \equiv_{df} Rigid(R) \,\&\, \forall x_1 \ldots x_n\big((\mathbb{N}x_1 \,\&\, \ldots \,\&\, \mathbb{N}x_n) \rightarrow \exists! y(\mathbb{N}y \,\&\, Rx_1 \ldots x_n y)\big)$$

We now discuss the definition of the *basic* or *initial* functions that help us to build recursively-defined functions. These are the successor function, constant functions, and projection functions. Since the successor function $\boldsymbol{s}$ is rigid numerical function on the natural numbers, it follows that *numerical successor* is a unary numerical operation:[39]

$$Op^1(\boldsymbol{s})$$

To define *constant* functions, let $n, m \geq 0$ be any natural numbers. Then we define the constant n'-ary relation $\mathcal{C}^{n'}_m$ as *being an* $x_1,\ldots,x_n$ *and* y *such that* $x_1,\ldots,x_n$ *are natural numbers and* y *is number-identical to* m:

$$\mathcal{C}^{n'}_m =_{df} [\lambda x_1 \ldots x_n y\; \mathbb{N}x_1 \,\&\, \ldots \,\&\, \mathbb{N}x_n \,\&\, y \doteq m] \qquad (n, m \geq 0)$$

Note that $\mathcal{C}^1_m$ is simply $[\lambda y\; y \doteq m]$. It now follows that $\mathcal{C}^{n'}_m$ is an n-ary operation:

$$Op^n(\mathcal{C}^{n'}_m)$$

Next, for each $i \geq 1$, we introduce i-ary functions that *project* the k^{th} argument as the value. In the present case, we will define these as i'-ary *relations* ($i \geq 1$) indexed by k ($1 \leq k \leq i$) that hold among i' numbers whenever the final argument matches the k^{th} argument. That is, these projection relations can be seen as selecting the k^{th} argument. In the simplest case, π^2_1 relates any two numbers n and m whenever its final (i.e, second) argument m is numerically identical its first argument n. In the general case, $\pi^{i'}_k$ is an i'-ary *projection* relation that relates a string of i' arguments whenever the final argument is numerically identical to the k^{th} argument in the string, for $1 \leq k \leq i$:

[38] For the definition of *rigid*, see footnote 31.

[39] The rigidity of $\boldsymbol{s}$ (i.e., $\mathbb{P}_{\restriction\mathbb{N}}$) follows from the facts that (a) $\mathbb{P}$ is rigid, (b) its ancestrals are rigid, (c) $\mathbb{N}$ is rigid, and (d) the restriction of a rigid n'-ary relation to a rigid n-ary relation is rigid. For more about (a) – (c), see footnote 37.

$$\pi^{i'}_k =_{df} [\lambda x_1 \ldots x_i y\; \mathbb{N}x_1 \,\&\, \ldots \,\&\, \mathbb{N}x_i \,\&\, y \doteq x_k] \qquad (1 \leq k \leq i)$$

It follows that:

$$Op^i(\pi^{i'}_k) \qquad (1 \leq k \leq i)$$

We next define composed functions as relations. Suppose G and H are both binary relations. Then G *composed with* H (written '$G \circ H$' or, when delimiters are needed, '$[G \circ H]$') is the binary relation *being an* x *and* y *such that for some* z, x *bears* H *to* z *and* z *bears* G *to* y:

$$G \circ H =_{df} [\lambda xy\, \exists z(Hxz \,\&\, Gzy)]$$

Now let G be an m'-ary relation ($m \geq 1$) and $H_1,\ldots,H_m$ be n'-ary relations ($n \geq 0$). Then G *composed with* $H_1,\ldots,H_m$ (written '$G \circ (H_1,\ldots,H_m)$' or, when delimiters are needed, '$[G \circ (H_1,\ldots,H_m)]$') is the n+1-ary relation *being* $x_1,\ldots,x_n$ *and* y *such that for some* $z_1,\ldots,z_m$, $x_1,\ldots,x_n$ *bear* H_1 *to* z_1 *and* $\ldots$ *and* $x_1,\ldots,x_n$ *bear* H_m *to* z_m *and* $z_1,\ldots,z_m$ *bear* G *to* y:

$$G \circ (H_1,\ldots,H_m) =_{df} [\lambda x_1 \ldots x_n y \exists z_1 \ldots \exists z_m (H_1 x_1 \ldots x_n z_1 \& \ldots \& H_m x_1 \ldots x_n z_m \& G z_1 \ldots z_m y)]$$

These definitions allow us to define operations by composition. It is then provable that if H and G are unary operations, then $G \circ H$ is a unary operation such that for any x, $[G \circ H](x) = G(H(x))$:

$$Op^1(H) \,\&\, Op^1(G) \rightarrow \Big(Op^1(G \circ H) \,\&\, \forall x\big([G \circ H](x) = G(H(x))\big)\Big)$$

Moreover, if $H_1,\ldots,H_m$ are n-ary operations ($n \geq 0$) and G is an m-ary operation ($m \geq 1$), then the result of applying G composed with $H_1,\ldots,H_m$ to the arguments $x_1,\ldots,x_n$ is identical to the result of applying G to the arguments $H_1(x_1,\ldots,x_n),\ldots,H_m(x_1,\ldots,x_n)$:

$$(Op^n(H_1) \,\&\, \ldots \,\&\, Op^n(H_m) \,\&\, Op^m(G)) \rightarrow \Big(Op^n(G \circ (H_1,\ldots,H_m)) \,\&\, \forall x_1 \ldots \forall x_n\big([G \circ (H_1,\ldots,H_m)](x_1,\ldots,x_n) = G(H_1(x_1,\ldots,x_n),\ldots,H_m(x_1,\ldots,x_n))\big)\Big)$$

Now that we have the basic functions and composition, we can complete the definition of the class of primitive recursive functions by showing how to recontruct recursively-defined functions as relations, culminating in a proof of the Recursion Theorem. To simplify the presentation, we focus on binary operations only; the following discussion generalizes to n-ary operations.

Our goal is to build a ternary relation $\boldsymbol{F}$ that satisfies the conditions of a binary numerical operation defined recursively in terms of a unary operation H and a ternary operation G:

$$\boldsymbol{F}(n,0) = H(n)$$

$$\boldsymbol{F}(n,m') = G(n,m,\boldsymbol{F}(n,m))$$

Note that using our notation for projection functions and composition, we can rewrite $G(n,m,\boldsymbol{F}(n,m))$ as $[G \circ (\pi^3_1,\pi^3_2,\boldsymbol{F})](n,m)$. Our strategy for constructing $\boldsymbol{F}$ is as follows:

- start with a given unary numerical operation H and a ternary numerical operation G,
- define an inductive sequence of binary relations $\boldsymbol{F}_m$ relative to H and G,
- show, by induction, that each $\boldsymbol{F}_m$ is a unary numerical operation,
- define the ternary relation $\boldsymbol{F}$, relative to H and G, in terms of the sequence of relations $\boldsymbol{F}_m$,
- show that $\boldsymbol{F}$ is a binary numerical operation, and
- show that $\boldsymbol{F}$ satisfies the conditions of a numerical operation recursively defined in terms of H and G.

In the definitions and theorems used to introduce $\boldsymbol{F}_m$ and $\boldsymbol{F}$, we *suppress*, for readability, the indices that relativize the relations $\boldsymbol{F}_m$ and the relation $\boldsymbol{F}$ to H and G. But we introduce the indices later, since they are needed to state the Recursion Theorem.

We implement the above strategy with an initial 'base' definition and theorem. The definition sets $\boldsymbol{F}_0$ to H, and the theorem asserts that $\boldsymbol{F}_0$ is a unary numerical operation, which follows from the fact that H is unary:

$$\boldsymbol{F}_0 =_{df} H$$

$$Op^1(\boldsymbol{F}_0)$$

We then show that given any m, if $\boldsymbol{F}_m$ is a unary numerical operation, then the composition of G with three unary numerical operations, namely the projection function π^2_1 (which always returns its argument as the value), the constant function $\mathcal{C}^2_m$ (which ignores its argument and returns m), and $\boldsymbol{F}_m$, is also a unary numerical operation. Then we define $\boldsymbol{F}_{m'}$ as that composition.

$$Op^1(\boldsymbol{F}_m) \to Op^1(G \circ (\pi^2_1,\mathcal{C}^2_m,\boldsymbol{F}_m))$$

$$\boldsymbol{F}_{m'} =_{df} G \circ (\pi^2_1,\mathcal{C}^2_m,\boldsymbol{F}_m)$$

At this point it should be clear that $\boldsymbol{F}_m$ has been defined relative to H and G and, strictly speaking, should be indexed to these initial relations.

It now follows that for every natural number m, $\boldsymbol{F}_m$ is a unary numerical operation:

$$\forall m Op^1(\boldsymbol{F}_m)$$

We next define $\boldsymbol{F}$ as the relation *being natural numbers n, m, and j such that (the value of) $\boldsymbol{F}_m$, when applied to (the argument) n, is j*:

$$\boldsymbol{F} =_{df} [\lambda nmj\, \boldsymbol{F}_m(n) \doteq j] \tag{33}$$

It now follows that $\boldsymbol{F}$ is a binary numerical operation:

$$Op^2(\boldsymbol{F})$$

Finally, note that $\boldsymbol{F}$ satisfies the conditions of a binary numerical operation defined recursively in terms of H and G, for the following are now theorems:

$$\boldsymbol{F}(n,0) = H(n)$$

$$\boldsymbol{F}(n,m') = G(n,m,\boldsymbol{F}(n,m))$$

Since definition (33) introduces $\boldsymbol{F}$ relative to a given H and G, we shall henceforth write $\boldsymbol{F}$ as $\boldsymbol{F}_{H,G}$. We now have the Recursion Theorem for Recursive Binary Numerical Operations:

> **Theorem:** Let H be any unary numerical operation, and let G be any ternary numerical operation. Then $\boldsymbol{F}_{H,G}$ is a binary numerical operation that satisfies the conditions:
>
> $$Op^2(\boldsymbol{F}_{H,G}) \,\&\, \boldsymbol{F}_{H,G}(n,0) = H(n) \,\&\, \boldsymbol{F}_{H,G}(n,m') = G(n,m,\boldsymbol{F}_{H,G}(n,m))$$

With this theorem, the usual recursively stated axioms for addition, multiplication, exponentiation, factorialization, etc., all fall out as theorems once these operations are defined.

For example, where H is π^2_1, and G is $\boldsymbol{s} \circ \pi^4_3$, addition may be defined as a binary operation (= ternary relation):

$$\boldsymbol{A} =_{df} \boldsymbol{F}_{\pi^2_1,\boldsymbol{s}\circ\pi^4_3}$$

To see that the two standard axioms for recursive addition become derivable, note that $H(n) = \pi^2_1(n) = n$ and $G(n,m,j) = [\boldsymbol{s} \circ \pi^4_3](n,m,j) = \boldsymbol{s}(j) = j'$. It now follows that:

$$\boldsymbol{A}(n,0) = n$$
$$\boldsymbol{A}(n,m') = (\boldsymbol{A}(n,m))'$$

i.e., where we use + for $\boldsymbol{A}$ and infix notation:

$$n + 0 = n$$
$$n + m' = (n+m)' \tag{34}$$

And where H is $\mathcal{C}^2_0$, and G is $\boldsymbol{A} \circ (\pi^4_1, \pi^4_3)$, multiplication becomes defined as:

$$\boldsymbol{M} =_{df} \boldsymbol{F}_{\mathcal{C}^2_0, \boldsymbol{A} \circ (\pi^4_1, \pi^4_3)}$$

With the above definition, the following two axioms for recursive multiplication are derivable in the standard way:

$$\boldsymbol{M}(n,0) = 0$$
$$\boldsymbol{M}(n,m) = n + \boldsymbol{M}(n,m)$$

i.e., where we use × for $\boldsymbol{M}$ and infix notation:

$$n \times 0 = 0$$
$$n \times m' = n + (n \times m) \tag{35}$$

And so on, for the other recursively defined functions.

Since we've now (a) defined the basic or *initial* functions, namely, the successor function ($\boldsymbol{s}$), the constant functions ($\mathcal{C}^{n'}_m$), and the projection functions ($\pi^{i'}_k$), and (b) shown that we can derive new functions by composition ($G \circ (H_1, \ldots, H_m)$) and recursion ($\boldsymbol{F}_{H,G}$), we have reconstructed the entire class of *primitive recursive functions*. In order to reconstruct the class of *general* recursive functions, we need only show that the minimization operator can be defined in our system.

Where f is any j+1 numerical operation ($j \geq 0$), we define the (restricted) variable-binding μ operator as follows. *The least* natural number n *such that* f maps $m_1, \ldots, m_j, n$ to Zero is the natural number n such that (a) f maps $m_1, \ldots, m_j, n$ to Zero and (b) for any number i less than n, $f(m_1, \ldots, m_j, i)$ is defined and equal to a value other than Zero:

$$\mu n(f(m_1, \ldots, m_j, n) = 0) =_{df}$$
$$\imath n\big(f(m_1, \ldots, m_j, n) = 0 \;\&\; \forall i(i < n \rightarrow \exists k(f(m_1, \ldots, m_j, i) = k \;\&\; k \neq 0))\big)$$

Note that if there is no minimal n such that $f(m_1, \ldots, m_j, n) = 0$, or if there is some i such that $i < n$ and $\neg f(m_1, \ldots, m_j, i)\!\downarrow$, then the description will fail, in which case $\mu n f(m_1, \ldots, m_j, n)$ will not be significant, i.e., our logic will guarantee that $\neg \mu n f(m_1, \ldots, m_j, n)\!\downarrow$.[40]

5.2 Derivation of Second-Order Peano Arithmetic

We've now assembled all of the parts needed for the derivation of PA2 in object theory.

5.2.1 Formulation of PA2

PA2 is classically formulated (e.g., Simpson 1999 [2009], 2–3) in a language containing number constants 0 and 1, number variables $i, j, k, m, n, \ldots$, set variables $X, Y, Z, \ldots$ (intended to range over all subsets of ω), and complex number terms $\tau_1 + \tau_2$ and $\tau_1 \cdot \tau_2$, whenever τ_1 and τ_2 are any number terms. (Here + and · are binary numerical operation symbols intended to denote addition and multiplication of natural numbers and the numerical terms are intended to denote natural numbers.) The formation rules of the language of PA2 are:

Atomic formulas

- Where τ_1 and τ_2 are any number terms and X is any set variable, $\tau_1 = \tau_2$, $\tau_1 < \tau_2$, and $\tau_1 \in X$ are formulas.

Complex Formulas

- Whenever φ and ψ are formulas, n is a number variable and X is a set variable, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$, $\forall n\varphi$, $\exists n\varphi$, $\forall X\varphi$, $\exists X\varphi$ are formulas.

Finally, the axioms of PA2 are:

[40] This is a consequence of the inferential role of definite definitions that's operative in object theory. A definition of the form $\tau =_{df} \sigma$ introduces an axiom that is a conjunction of two conditional claims, one of which asserts that if σ is significant, then the identity $\tau = \sigma$ holds, and the other of which asserts that if σ is not significant, then neither is τ:

$$(\sigma\!\downarrow \rightarrow (\tau = \sigma)) \;\&\; (\neg\sigma\!\downarrow \rightarrow \neg\tau\!\downarrow)$$

So this inferential role for definitions allows for the case where the definiens fails to denote anything, e.g., as in the case of division by Zero. Cf. Grzegorczyk, Mostowski, and Ryll-Nardzewski 1958, where descriptions $\imath x\varphi$ are used in a system of arithmetic but which denote Zero if φ is not uniquely satisfied.

- Basic Axioms:

$$n+1 \neq 0$$
$$n+1 = m+1 \rightarrow n=m$$
$$n+0 = n$$
$$n+(m+1) = (n+m)+1$$
$$n \cdot 0 = 0$$
$$n \cdot (m+1) = n+(n \cdot m)$$
$$\neg(n < 0)$$
$$n < m+1 \leftrightarrow (n < m \vee n=m)$$

- Induction Axiom:

$$(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X)$$

- Comprehension Scheme:

$\exists X \forall n(n \in X \leftrightarrow \varphi)$, where φ is any formula of the language of PA2 in which X doesn't occur free.

5.2.2 Reduction of PA2 to OT

We begin by translating the terms and formulas of PA2 into object theory as follows:

Simple Terms

- Number constants: The symbol '0' of PA2 is to be translated as the defined term '0' of OT (22), and the symbol '1' of PA2 is to be translated as the defined term '1' of OT, where this latter is defined via (32) as '$\boldsymbol{s}(0)$'.
- Number variables $n, m, \ldots$ in PA2 are to be translated as the restricted variables $n, m, \ldots$, introduced in OT to range over the objects exemplifying the property $\mathbb{N}$, as the latter is defined in (27).
- Set variables $X, Y, \ldots$ are to be translated as the primitive unary relation (i.e., property) variables $F, G, H, \ldots$.

If we use the decorated metavariables τ_1^* and τ_2^* to designate the individual terms of object theory that serve as the translation of the number terms τ_1 and τ_2 of PA2, respectively, and use φ^* and ψ^*, respectively, to designate the translations of the formulas φ and ψ of PA2, then we complete the translation as follows:

Complex Terms for Numbers

- The term $\tau_1 + \tau_2$ is to be translated as $\tau_1^* + \tau_2^*$, where $+$ is defined as in (34)
- The term $\tau_1 \cdot \tau_2$ is to be translated as $\tau_1^* \times \tau_2^*$, where $\times$ is defined as in (35)

Atomic formulas

- $\tau_1 = \tau_2$ is to be translated as $\tau_1^* \doteq \tau_2^*$, where $\doteq$ is defined as in (31)
- $\tau_1 < \tau_2$ is to be translated as $\tau_1^* < \tau_2^*$, where $<$ is defined as in (30)
- $\tau_1 \in X$ is to be translated as $F\tau_1^*$, where F is the translation of X and $F\tau_1^*$ is a primitive exemplification formula.

Complex Formulas

- The formulas $\neg\varphi$ and $\varphi \rightarrow \psi$ are to be translated using the primitive connectives $\neg$ and $\rightarrow$ of OT, respectively, and so are to be translated as the formulas $\neg\varphi^*$ and $\varphi^* \rightarrow \psi^*$.
- The formulas $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \leftrightarrow \psi$ are to be translated into the defined formulas $\varphi^* \,\&\, \psi^*$, $\varphi^* \vee \psi^*$, and $\varphi^* \equiv \psi^*$, as these are standardly defined, respectively.
- The formulas $\forall n\varphi$ and $\exists n\varphi$ are to be translated using both (a) the primitive quantifier $\forall$, and the standardly-defined existential quantifier $\exists$, respectively, and (b) the restricted variables ranging over numbers, so that $\forall n\varphi$ and $\exists n\varphi$ become translated as $\forall n\varphi^*$ and $\exists n\varphi^*$, respectively. (Note that these formulas can be expanded, by eliminating the restricted variables, to $\forall x(\mathbb{N}x \rightarrow \varphi^{*x}_{\ n})$ and $\exists x(\mathbb{N}x \,\&\, \varphi^{*x}_{\ n})$, respectively.)
- The formulas $\forall X\varphi$ and $\exists X\varphi$ are to be translated similarly but with the variable F replacing X (so that the quantifiers bind the property variable F), with the result that $\forall X\varphi$ and $\exists X\varphi$ become translated as $\forall F\varphi^*$ and $\exists F\varphi^*$.

It is a simple consequence of our work in the preceding sections that, under this translation scheme, the axioms of PA2 are derivable as theorems of OT.[41] To take one example, consider how Simpson formulates the axiom of induction:

$$\forall X((0 \in X \,\&\, \forall n(n \in X \rightarrow n{+}1 \in X)) \rightarrow \forall n(n \in X))$$

Given the above translation scheme, this becomes:

$$\forall F(F0 \,\&\, \forall n(Fn \rightarrow Fn') \rightarrow \forall nFn)$$

where successor notation n' is used for the translation of $n{+}1$. If we apply definitions and expand $\forall n(Fn \rightarrow Fn')$ to $\forall n \forall m(\mathbb{P}nm \rightarrow (Fn \rightarrow Fm))$, then we obtain:

$$F0 \,\&\, \forall n \forall m(\mathbb{P}nm \rightarrow (Fn \rightarrow Fm)) \rightarrow \forall nFn$$

which is the principle of Mathematical Induction derived in Section 4.4 above.

5.3 Derivation of the Existence of an Infinite Number and Infinite Class

We first establish two important facts, namely that every object that bears the weak ancestral of *predecessor* to a natural number is itself a natural number, and that no natural number numbers (the property) *being a natural number*:

$$\forall x(\mathbb{P}^{+}xm \rightarrow \mathbb{N}x)$$

$$\neg\exists n Numbers(n, \mathbb{N})$$

The first is proved by induction on m. The second is proved using the first, along with a lemma used to prove that every number has a successor and other facts about the natural numbers. Now let us say that κ is *finite* if and only if κ is a natural number, and that κ is *infinite* if and only if κ is not finite:

$$Finite(\kappa) \;\equiv_{df}\; \mathbb{N}\kappa$$

[41] For verification, the proof of this claim can be examined in the online manuscript Nodelman & Zalta m.s.; see the proof of theorem labeled "The Axioms of PA2 Are Theorems".

$$Infinite(\kappa) \;\equiv_{df}\; \neg Finite(\kappa)$$

Then it follows that:

$$Infinite(\#\mathbb{N})$$

Of course, we know that $NaturalCardinal(\#\mathbb{N})$ and so if we define:

$$\aleph_0 \;=_{df}\; \#\mathbb{N}$$

then we have established that $\aleph_0$ is an infinite natural cardinal.[42]

We can extend this result to a proof that an infinite class exists by using the object-theoretic definition of a "class of G".[43] Thus, the notions of an infinite cardinal and an infinite class can be defined without reference to any mathematical primitives, and the proof that such objects exist requires no mathematical axioms. Though one may wish to ask the question about the set-theoretic strength of this subtheory of OT, that question is orthogonal to the goals we've set in this paper.

[42] This significantly improves upon the object-theoretic proof that an infinite cardinal exists described in Linsky & Zalta 2006, 87–88. Note that the proof in the 2006 paper made use of the modal axiom that played a significant role in Zalta 1999 (see the discussion of the modal axiom in Section 4.4 above). Indeed, in the 2006 paper, the infinite cardinal was defined as the number of ordinary objects and not as $\aleph_0$ (since the numbers in Zalta 1999 couldn't count abstract objects and so couldn't count the numbers themselves). So (a) the modal axiom from Zalta 1999 is not used in the present proof of the existence of a infinite cardinal (and, given our work in Section 6.2 below, no modal axiom is needed, strictly speaking), and (b) $\aleph_0$ has been properly defined as the number of natural numbers.

[43] In object theory, one defines:

$$ClassOf(x, G) \equiv_{df} A!x \,\&\, G{\downarrow} \,\&\, \forall F(xF \equiv \forall z(Fz \equiv Gz))$$

That is, a class of Gs (or, an extension of G) is an abstract object that encodes just the properties materially equivalent to G. Thus, by object comprehension, for any property G, there is a unique abstract object that is class of of Gs, and so we define ϵG as that object. Thus, a *class* is any object x such that for some G, x is a class of Gs, and we can define:

$$InfiniteClassOf(x, G) \equiv_{df} G{\downarrow} \,\&\, ClassOf(x, G) \,\&\, \exists\kappa(Infinite(\kappa) \,\&\, Numbers(\kappa, G))$$

$$InfiniteClass(x) \;\equiv_{df}\; \exists G(InfiniteClassOf(x, G))$$

It then follows that $\epsilon\mathbb{N}$ is an infinite class.

One may also define membership in the usual way:

$$y \in x \;\equiv_{df}\; \exists G(ClassOf(x, G) \,\&\, Gy)$$

The Russell paradox is avoided by the facts that classes can only be constructed by reference to some property and $[\lambda yx\, y \in x]$ doesn't define a relation (nor can $y \in x$ be used in Comprehension for Relations to define a relation). Thus, one can't form the property $[\lambda x\, \neg(x \in x)]$ or build a class thereof.

6 The Answers to our Opening Questions

Since we addressed the second of our opening questions in Section 4, we return finally to the first and third of our opening questions, and discuss the consistency of our system, as follows:

- Section 6.1 isolates the existence claims needed to prove the theorems of 'ordinary' mathematics.
- Section 6.2 shows how one can reconstruct Frege's theory of numbers in a non-modal setting without mathematical primitives.
- Section 6.3 describes the basic features of the model sketched in the Appendix, and responds to a potential objection concerning categoricity.

6.1 Isolating the Existence Claims

We are now in a position to answer the first question posed at the outset: what existence axioms are needed to prove the theorems of 'ordinary' mathematics? Our initial answer is that, if one is interested in developing mathematics in a modal setting so that there is a unique group of natural numbers that can count objects in any modal context, we need the following principles:

$$\Diamond\exists x(E!x \,\&\, \neg\mathcal{A}E!x) \tag{2}$$

$$[\lambda x_1 \ldots x_n\, \varphi]\!\downarrow \rightarrow ([\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi) \tag{3}$$

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)) \tag{6}$$

$$[\lambda xy\, \exists F\exists u(Fu \,\&\, Numbers(y,F) \,\&\, Numbers(x,F^{-u}))]\!\downarrow \tag{25}$$

(6) and (25) are explicit existence claims. (6) is a schema that asserts the existence of an abstract object that encodes the properties that satisfy any formula φ without free xs. (25) asserts the existence of an ordering relation, formulable solely in terms of non-mathematical exemplification and encoding formulas. (3) implies the classical 2nd-order comprehension principle for relations governing 2nd-order QML, and so entails an existence claim.[44] And (2) has an existence claim within the scope of a possibility operator.

The least familiar of these principles is no doubt the modal existence principle (2). We should note that this principle is stronger than it needs to be in order to achieve the results of this paper. We could have replaced this principle with the weaker claim: *it is possible that something is discernible*, i.e.,

$$\Diamond\exists x D!x \tag{2$'$}$$

This is indeed weaker because (2) implies (2′), but not vice versa. (2′) is sufficient to show that *Zero* precedes something, and this then allows us to bootstrap the claim that every number has a successor using Frege's techniques. (2′) is justifiable because it makes a presupposition of logic clear, namely, that the domain isn't comprised solely on indiscernibles. Without some such axiom as (2′), OT would only be committed to a domain consisting of abstract objects that are all indiscernible;[45] there would be no distinctive exemplification predications, i.e., one couldn't prove anything (using exemplification predications) that distinguishes any object in the domain from another. Any theory that adopted the negation of (2′) would be unable to express anything distinctive about anything.

But we prefer (2) to (2′), since (2) does more than just guarantee that there is at least one, provably discernible object and that *Zero* precedes something. In addition, it also guarantees that there are 4 propositions (one necessarily true, one necessarily false, one contingently true, and one contingently false) and that there is at least one non-actual possible

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[44] Any λ-expressions expressed solely in terms of the exemplification language of classical 2nd-order logic is such that $[\lambda x_1 \ldots x_n\, \varphi]\!\downarrow$ is axiomatic in OT. So for such λ-expressions, we immediately obtain:

$$[\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi$$

By universal generalization, it follows that:

$$\forall x_1 \ldots \forall x_n([\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi)$$

By applying Rule RN and then Existential Introduction, we obtain:

$$\exists F^n \Box\forall x_1 \ldots \forall x_n(Fx_1 \ldots x_n \equiv \varphi)$$

And this existential claim provides comprehension conditions for relations in OT.

[45] In the smallest (Aczel) models of unmodalized OT, there are no concrete objects and only one special object, 2 properties, and 4 abstract objects, all of which are mapped to the one special object that serves as their proxy for the purpose of exemplification predications. See Section 6.3 for more on Aczel models.

world (this is provable within the theory!). Note that (2) doesn't commit us to concrete objects and so can be asserted *a priori*. Indeed, it is a rather weak way of minimally capturing the logician's modal intuition that the domain might have been different (which few admit to but all hold dear). But if none of this is convincing, or one is investigating the weakest axioms, that would suffice for the results in this paper, we can preserve our work by adopting (2′).

6.2 Non-modal Frege Arithmetic

There are a number of philosophers and logicians who have concerns about modality. Some Humeans deny that modal claims are true, while others assert that such claims have no business in the logical foundations for ordinary mathematics or even natural science. Though we are comfortable with modality and our modal reconstruction of Frege Arithmetic, there is a way to define the numbers in OT that should satisfy those who abjure modality and would prefer to see PA2 developed *without* mathematical primitives in a non-modal setting. In this section we show how to (a) eliminate the modality, and (b) preserve Frege's original plan for both (i) analyzing the numbers as logical objects and (ii) answering the question "How do we apprehend the numbers?". Here is how.

First, re-formulate OT without the modal operator and without the actuality operator. Frege didn't use the notions these express and, strictly speaking, they aren't needed for constructing the numbers in a non-modal context. Indeed, we may take $A!$ as a primitive, instead of attempting to define it. Then comprehension for abstract objects can be preserved, as stated by (6). Next, redefine discernibility ($D!x$) without the modal operator, so that we have:

$$D! =_{df} [\lambda x\, \forall y(y \neq x \rightarrow \exists F \neg(Fx \equiv Fy))]$$

Clearly, the definiens is a significant λ-expression, by the same reasoning used to establish (11). Note we can now redefine $=_D$ in terms of the redefined $D!$. Moreover, when we strip any modal operators from the facts pertaining to $D!$ and $=_D$ in the bulleted list in Section 4.1, the resulting facts all remain theorems. Given the non-modal definitions of, and facts about, $D!$ and $=_D$, we don't need an actuality operator in the definition of $Numbers(x, G)$ and so we can use (18) instead of (19) as the definition.

Then, the final change is to replace the modal axiom (2), or the weaker axiom (2′), by the existence claim (17), now taken as an axiom:[46]

$$\exists x D!x \tag{17}$$

Given these changes, we can take on board the definitions of $\#G$ (20), $NaturalCardinal(x)$ (21), and Zero (22), and then redo the work in Sections 4.3 – 5.3, which is made simpler by ignoring the modalities (such as the notion of a rigid relation). That is, all of the definitions and theorems in these sections are preserved under this formulation of non-modal OT. In particular, we've seen that the existence of a discernible object implies that there are natural cardinals that stand in the *predecessor* relation (footnote 32), and that this relation makes the natural cardinals discernible (26). So when the natural numbers are defined as objects to which Zero stands in the weak *predecessor* relation, they can be used to count properties whose instances are natural numbers, such as *being a number standing in the weak predecessor relation to n* (i.e., *being a number less than or equal to n*). The number of the latter property is the successor to n (28).

So if you are a modal skeptic, replace (2) with (17). This not only preserves Frege's original plan for deriving the Dedekind-Peano postulates, but also preserves the rest of the derivation of PA2. Moreover, we can answer Frege's epistemological question, "How do we apprehend the numbers?", by uniquely identifying them as discernible abstract objects.

We conclude with the following observations. First, we now have two ways to address our opening question, "Which existence claims are needed for ordinary mathematics?" We can either (A) assert the four existence claims listed at the beginning of Section 6.1, namely, (2), (3), (6), and (25), or (B) use the nonmodal version OT and replace (2) in this list with (17). (Clearly, (A) is not a conservation extension of (B), since (A) requires the existence of at least two possible worlds, whereas (B) doesn't.) In either case, OT's analysis of the natural numbers doesn't require mathematical primitives or mathematical axioms.

[46] This is easily justifiable as an axiom, along lines analogous to the justification of (2′). If (17) weren't true and it were the case that $\neg\exists x D!x$, then no individual, ordinary or abstract, can be picked out by any group of properties. There would be no predications that uniquely distinguish any object. Indeed, there would be no uniquely distinguishable predications. We wouldn't, in general, be able to discern whether two arbitrary objects a and b were distinct and, hence, whether Pa was a different predication than Pb.

Second, given how powerful the non-mathematical theory OT seems to be, a series of questions for future research present themselves. Is there a predicative fragment of OT that will accomplish some part of the reduction, e.g., that implements ACA_0?[47] Is there a reduction of OT to PA2? We leave these questions for future research.

6.3 Consistency and The Irrelevance of Categoricity

In this final section, we (a) motivate the basic construction of the Aczel model described in the Appendix (this model shows that the foregoing theory is consistent), and (b) respond to a possible objection about our methods based on the notion of categoricity.

The key feature of an Aczel model is the way it solves the following problem: if abstract objects are modeled as sets of properties, where properties are modeled as sets of *Urelemente*, then how can a model validate the fact that there are true claims in object theory of the form $xF \,\&\, Fx$. For example, one can prove a claim of this form from the following instance of (6):

$$\exists x(A!x \,\&\, \forall F(xF \equiv F = A!))$$

This asserts the existence of an object that exemplifies $A!$ and that encodes just one property, namely, $A!$. So if we model abstract objects as sets of properties, then although it is clear that the truth of 'xF' would be represented in the model by claims of the form '$F \in x$', how would a set of properties exemplify one of the properties that is among its members, i.e., how would we represent the truth of claims of the form 'Fx'? Aczel solved this problem by introducing a special subdomain of *Urelemente*, the *special* objects, which serve as the *proxies* of abstract objects for the purposes of modeling exemplification (ordinary objects serve as their own proxies). Thus, 'Fx' is true in the model just in case the proxy of x is an element of F.[48]

So Aczel models start with a domain of *Urelemente* divided into subdomains of *ordinary* objects and *special* objects. And no matter whether properties are modeled extensionally as sets of *Urelemente*, or as primitive entities whose extensions are sets of *Urelemente*, abstract objects are then modeled as sets of properties.

[47] See Ebels-Duggan & Boccuni forthcoming for possible answers to these questions.

[48] It is important to remember that this set-theoretic model doesn't imply that abstract objects *just are* sets of properties. According to OT, abstract objects may exemplify and encode the very same properties, but in the model, no property F which is an element of x has x as an element. In such cases, the proxy of x is an element of F.

Note that this basic structure of an Aczel model suggests a way to understand theorem (7), discussed in Section 3.1, which asserts that there are distinct but indiscernible abstract objects. To see why, note that given the model structure we've described so far, not every distinct abstract object x (modeled as a set of properties) can be correlated with a distinct property $[\lambda y\, y = x]$, since that would constitute a 1-to-1 mapping from the power set of the set of properties into a subset of the set of properties, in violation of Cantor's Theorem. Indeed, if OT included a general relation of identity that gives rise to such properties, one could prove a contradiction in the system; this is known as the McMichael-Boolos Paradox.[49] The contradiction is avoided within OT itself because it is a theorem that for every relation R, there are distinct abstract objects x and y (they are distinct because they encode different properties) such that $[\lambda z\, Rzx] = [\lambda z\, Rzy]$. And this theorem is used in the proof that there are indiscernible abstract objects (Zalta 1999, footnote 16).

Since abstract objects are represented as sets of properties in Aczel models and the proxies are *Urelemente*, at least some distinct abstract objects must have the same proxy. Whenever abstract objects x and y have by the same proxy, they exemplify the same properties and are therefore indiscernible with respect to exemplification. Moreover, Aczel models help explain why one can't have full second-order models of OT, i.e., why the domain of properties can't be the full power set of the domain of individuals. The explanation is that the domain of individuals, i.e., the domain over which the variable x of the language ranges, is defined to be the union of the abstract objects (i.e., sets of properties) and the ordinary objects. Since the domain of properties is just the power set of all the *Urelemente* (or at least equivalent in size), it is not the full power set of the set of individuals. Thus, the domain over which the variable F ranges is not the power set of the domain over which the variable x ranges. So Aczel models show why OT doesn't have a full second-order consequence relation. Note, finally, that the special *Urelemente* that serve as proxies aren't in the range of the quantifier '$\forall x$'; no piece of language

[49] This was first reported in McMichael & Zalta 1980 (310, fn. 15). It was also described independently by Boolos in 1987 (17; 1998, 198–199), where he derived the contradiction from the claim he labelled 'SuperRussell'.

denotes or ranges over the special *Urelemente*.

Given these facts, one might object to our results as follows:

> The virtue of moving to PA2 from PA1 is that when PA2 is understood in the context of full second-order logic (where the domain of properties is the full power set of the set of individuals), it is categorical (i.e., its models are isomorphic) and so singles out a unique number-theoretic structure.[50] But OT requires only Henkin models and doesn't have full second-order models; the domain of properties can't be the full power set of the domain of individuals. Thus, OT's reconstruction of PA2 isn't categorical and so fails to have the virtue of singling out a unique structure for the natural numbers.

Our response is in part motivated by the passage in Väänänen (2001, 505) quoted at the end of Section 1. We reject the presupposition that model theory is needed to tell us whether OT defines a single structure of the numbers. We are justified in rejecting this presupposition because numbers are not taken to be primitive objects or represented/modeled by other mathematical objects. Rather they are *identified* as particular abstract objects governed by more general background principles. When dealing with an axiom system in which the numbers are primitive objects (e.g., as in the Dedekind-Peano axioms), or represented as objects in some model, one might suppose that any class of objects that satisfy the axioms are rightfully called the numbers. In such a situation, categoricity provides a check on the uniqueness and well-definedness of the objects called numbers. The point was made by Veblen (1904) in connection with undefined terms in geometry.[51]

OT doesn't axiomatize the natural numbers nor does it take them to be the elements of some (possibly distinguished) model. Rather, it defines them and shows that the standard axioms are theorems. Thus, it

[50] Dedekind 1888, ¶132, [1893, XX; 1939, 33]; Enderton 1972 [2001, 287]; Väänänen 2015, 469; Väänänen 2021, 987–988.

[51] Veblen says (1904, 346):

> Inasmuch as the terms *point* and *order* are undefined one has a right, in thinking of the propositions, to apply the terms in connection with any class of objects of which the axioms are valid propositions.

Veblen's point applies to any attempt to represent numbers as sets as long as the axioms of number theory have a valid interpretation on those sets.

doesn't require model theory or the notion of categoricity to prove the existence of uniquely defined natural numbers. The non-standard Henkin models of PA2 are irrelevant, since we identify the numbers not as primitive mathematical objects but as distinctive objects. Since it is a mistake to suppose that the natural numbers are the elements of some model of OT, the fact that the consistency of OT is established by a Henkin model is irrelevant. OT is expressed in the conceptual framework of the predicate calculus, and we would argue that that framework is even more fundamental than that of set theory, since set theory is typically formulated as a non-logical theory within the conceptual framework of the predicate calculus. So non-standard models of OT in set theory are irrelevant if OT offers a more fundamental theory of what numbers *are*; models only give you numbers reconstructed as something else and they don't constitute theories of numbers. Thus, the model in the Appendix that follows is presented solely to demonstrate that the current framework is consistent.

7 Appendix: A Modal Aczel Model

The following constitute the key elements of the smallest Aczel model of the *modal* version of the theory described in the main part of the text, though the model can be simplified if one reconstructs number theory along the lines described in Section 6.2:

- A domain **W** of possible worlds with 2 possible worlds, $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$.
- A domain of *Urelements* divided into two subdomains:
  - The set **O** of *ordinary* Urelements contains one ordinary object a; the exemplification extension of the property $E!$ contains a only at $\boldsymbol{w}_1$, and this will make the modal axiom, namely (2), true.
  - The set **S** of *special* Urelements contains the following objects: b, c, and $0^*, 1^*, 2^*, \ldots$, where b is the proxy for all the indiscernibles, c is the proxy for $\aleph_0$, and $0^*, 1^*, 2^*, \ldots$ are the proxies for the natural numbers.
- $\mathbf{R}_1$ is the domain of properties, and it has to be at least as large as the power set of $\mathbf{O} \cup \mathbf{S}$. If we take the elements of $\mathbf{R}_1$ to be primitive

properties, then the $\mathbf{R}_1$ must include the properties $D!$ and $\mathbb{N}$. The exemplification extension of $D!$ at both worlds is $\{a,c,0^*,1^*,2^*,\ldots\}$, and the exemplification extension of $\mathbb{N}$ at both worlds is $\{0^*,1^*,2^*,\ldots\}$. If one construes properties extensionally for the purposes of the model, then $D!$ and $\mathbb{N}$ can be identified, respectively, as functions that map each possible world to the corresponding set just described.

- $\mathbf{R}_2$ is the domain of binary relations, and it has to be at least as large as the power set of $(\mathbf{O}\cup\mathbf{S})\times(\mathbf{O}\cup\mathbf{S})$. If we take the elements of $\mathbf{R}_2$ to be primitive relations, then $\mathbf{R}_2$ must include the relations $\mathbb{P}$, $\mathbb{P}^*$, and $\mathbb{P}^+$. The exemplification extension of $\mathbb{P}$ at both worlds is:

  $$\{\langle c,c\rangle,\langle 0^*,1^*\rangle,\langle 1^*,2^*\rangle,\langle 2^*,3^*\rangle,\ldots\}$$

  The exemplification extension of $\mathbb{P}^*$ at both worlds is the transitive closure of the exemplification extension of $\mathbb{P}$, and the exemplification extension of $\mathbb{P}^+$ at both worlds is the reflexive transitive closure of the exemplification extension of $\mathbb{P}$. If one construes relations extensionally for the purposes of the model, then $\mathbb{P}$, $\mathbb{P}^*$, and $\mathbb{P}^+$ can be identified, respectively, as functions that map each possible world to the corresponding set of pairs just described.

- $\mathbf{R}_3$ is the domain of ternary relations, and it has to be at least as large as the power set of $(\mathbf{O}\cup\mathbf{S})\times(\mathbf{O}\cup\mathbf{S})\times(\mathbf{O}\cup\mathbf{S})$. The relations of *addition* and *multiplication* are found here.

- The domain of abstract objects $\mathbf{A}$ is the power set of $\mathbf{R}_1$, where the numbers $0,1,2,\ldots$, and $\aleph_0$ are identified as follows:

  - 0 is $\{F \mid \boldsymbol{w}_0 \models F \approx_D [\lambda x\, D!x \,\&\, x\neq x]\}$
  - n' is $\{F \mid \boldsymbol{w}_0 \models F \approx_D [\lambda m\, \mathbb{P}^+ mn]\}$
  - $\aleph_0$ is $\{F \mid \boldsymbol{w}_0 \models F \approx_D \mathbb{N}\}$

  Strictly speaking, the 2 possible worlds in $\mathbf{W}$ are 'indiscernible' but distinct members of $\mathbf{A}$, i.e., they are modeled as distinct sets of propositional properties and, being indiscernible, both have b as their proxy.

- The proxy function is set according to the indications in the second bullet.

The object terms of the language take values only in $\mathbf{A}\cup\mathbf{O}$ (i.e., none of the object terms denote one of the special objects in $\mathbf{S}$ that serve as proxies). The property variables range over $\mathbf{R}_1$, the binary relation variables range over $\mathbf{R}_2$, and so on. Then we say that an encoding formula of the form 'xF' is true at a possible worlds just in case the property assigned to F is an element of the object assigned to x (so if x is assigned the ordinary object a, 'xF' is false, since only abstract objects are sets of properties).[52] And we say that an exemplification formula of the form 'Fx' is true at a possible world just in case the proxy of the object assigned to x is an element of the exemplification extension, at that world, of the property assigned to F; that an exemplification formula of the form 'Rxy' is true at a possible world just in case the ordered pair consisting of the proxies of the objects assigned to x and y is an element of the exemplification extension, at that world, of the relation assigned to R; and so on for ternary formulas of the form $Rxyz$. So, for example, '$\mathbb{P}\aleph_0\aleph_0$' is true at $\boldsymbol{w}_0$ in virtue of the fact that $\langle c,c\rangle$ is in the exemplification extension of $\mathbb{P}$ at $\boldsymbol{w}_0$.

Given this model and definition of truth, the axioms of object theory are all true.

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[52]The truth of an n-ary encoding formula of the form '$x_1\ldots x_nF^n$' follows from the truth of n unary encoding formulas as governed by the following axiom:

$$x_1\ldots x_nF^n \equiv x_1[\lambda y\, F^n y x_2\ldots x_n] \,\&\, x_2[\lambda y\, F^n x_1 y x_3\ldots x_n] \,\&\, \ldots \,\&\, x_n[\lambda y\, F^n x_1\ldots x_{n-1} y]$$

Since this axiom wasn't used directly in the paper, we omitted it from the presentation.

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