Natural Numbers and Natural Cardinals as Abstract Objects: A Partial Reconstruction of Frege's *Grundgesetze* in Object Theory*

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In this paper, I derive a theory of numbers from a more general theory of abstract objects. The distinguishing feature of this derivation is that it involves no appeal to mathematical primitives or mathematical theories. In particular, no notions or axioms of set theory are required, nor is the notion 'the number of Fs' taken as a primitive. Instead, entities that we may justifiably call 'natural cardinals' and 'natural numbers' are explicitly defined as species of the abstract objects axiomatized in Zalta [1983], [1988a], and [1993a]. This foundational metaphysical theory is

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supplemented with some plausible assumptions and the resulting system yields the Dedekind/Peano axioms for number theory. The derivations of the Dedekind/Peano axioms should be of interest to those familiar with Frege's work for they invoke patterns of reasoning that he developed in [1884] and [1893]. However, the derivation of the claim that every number has a successor does not follow Frege's plan, but rather exploits the logic of modality that is embedded in the system.

In Section 1, there is a review of the basic theory of abstract objects for those readers not familiar with it. Readers familiar with the theory should note that the simplest logic of actuality (governing the actuality operator $\mathcal{A}\varphi$) is now part of the theory. In Section 2, some important consequences of the theory which affect the development of number theory are described and the standard models of the theory are sketched. In Section 3, the main theorems governing natural cardinals are derived. In Section 4, the definitions and lemmas which underlie the Dedekind/Peano axioms are outlined, and in particular, the definition of 'predecessor' and 'natural number'. In Section 5, the Dedekind/Peano axioms are derived. The final section consists of observations about the work in Sections 2 – 5.

Although there are a myriad of philosophical issues that arise in connection with these results, space limitations constrain me to postpone the full discussion of these issues for another occasion. The issues include: how the present theory relates to the work of philosophers attempting to reconstruct Frege's conception of numbers and logical objects;[1] how the theory supplies an answer to Frege's question 'How do we apprehend numbers given that we have no intuitions of them?'; how the theory avoids 'the Julius Caesar problem'; and how the theory fits into the philosophy of mathematics defended in Linsky and Zalta [1995]. A full discussion of these issues would help to justify the approach taken here when compared to other approaches. However, such a discussion cannot take place without a detailed development of the technical results and it will be sufficient that the present paper is devoted almost exclusively to this development. In the final section, then, there is only a limited discussion of the aforementioned philosophical issues. It includes a brief comparison of the present approach with that in Boolos [1987].

Before we begin, I should emphasize that the word 'natural' in the

[1]See Parsons [1965], Wright [1983], Burgess [1984], Hazen [1985], Boolos [1986], [1987], Parsons [1987], Heck [1993], Hale [1987], Fine [1994], and Rosen [1995], and Burgess [1998].

expressions 'natural cardinal', 'natural number' and 'natural arithmetic' needs to be taken seriously. This is a theory about numbers which are abstracted from the facts about concrete objects in this and other possible worlds.[2] As such, the numbers that we define will not count other abstract objects; for example, we cannot use them to count the natural numbers less than or equal to 2 (though we can, of course, define numerical quantifiers in the usual way and use them to assert that there are three natural numbers less than or equal to 2). This consequence will be discussed and justified in the final section. Though some reductions and systematizations of the natural numbers do identify the numbers as objects which can count the elements falling under number-theoretic properties, those reductions typically appeal to mathematical primitives and mathematical axioms. It should therefore be of interest to see a development of number theory which makes no appeal to mathematical primitives. From the present point of view, one consequence of eliminating mathematical primitives is that the resulting numbers are even more closely tied to their application in counting the objects of the natural world than Frege anticipated. This, however, would be a welcome result in those naturalist circles in which abstract objects are thought to exist *immanently* in the natural world, in some sense dependent on the actual pattern in which ordinary objects exemplify properties.[3]

§1: The Theory of Abstract Objects

The Language: The theory of abstract objects is formulated in a syntactically second-order S5 modal predicate calculus without identity, modified only so as to include xF^1 ('x *encodes* F^1') as an atomic formula along with $F^n x_1 \ldots x_n$. The notion of encoding derives from Mally [1912] and an informal version appears in Rapaport [1978]. Interested readers may find a full discussion of and motivation for this new form of predication in Zalta [1983] (Introduction), [1988a] (Introduction), and [1993a]. It is

[2]It should be emphasized that the following work does *not* constitute an attempt develop an overarching foundations for mathematics. Once the mathematicians decide which, if any, mathematical theory ought to be the foundation for mathematics, I would identify the mathematical objects and relations described by such a theory using the ideas developed in Linsky and Zalta [1995] and in Zalta [2000].

[3]Though, strictly speaking, on the conception developed here, reality includes modal reality, and so the identity of the abstract objects that satisfy the definition of 'natural number' may depend on the patterns in which possibly concrete objects exemplify properties.

not too hard to show that encoding formulas of the form 'xF' embody the same idea as Boolos' η relation, which he uses in formulas of the form '$F\eta x$' in his papers [1986] and [1987].[4] The complex formulas and terms are defined simultaneously. The complex formulas include: $\neg\varphi$, $\varphi \rightarrow \psi$, $\forall\alpha\varphi$ (where α is an object variable or relation variable), $\Box\varphi$, and $\mathcal{A}\varphi$ ('it is actually the case that φ'). There are two kinds of complex terms, one for objects and one for n-place relations. The complex object terms are rigid definite descriptions and they have the form $\imath x\varphi$, for any formula φ. The complex relation terms are λ-predicates and they have the form $[\lambda x_1 \ldots x_n\ \varphi]$, where φ has no encoding subformulas.[5] In previous work, I have included a second restriction on λ-predicates, namely, that φ not contain quantifiers binding relation variables. This restriction was included to simplify the 'algebraic' semantics. But since the semantics of the system will not play a role in what follows, we shall allow impredicative formulas inside λ-predicates. Models demonstrate that the theory remains consistent even in the presence of the new instances of comprehension which assert the existence of relations defined in terms of impredicative formulas.

Definitions and Proper Axioms: The distinguished 1-place relation of *being concrete* ('$E!$') is used to partition the objects into two cells: the ordinary objects ('$O!x$') are possibly concrete, whereas abstract objects ('$A!x$') couldn't be concrete:

$$O!x =_{df} \Diamond E!x$$

$$A!x =_{df} \neg\Diamond E!x$$

Thus, $O!x \vee A!x$ and $\neg\exists x(O!x \;\&\; A!x)$ are both theorems. Though the theory asserts (see below) that ordinary objects do not encode properties, abstract objects both encode and exemplify properties (indeed, some abstract objects exemplify the very properties that they encode). Next we define a well-behaved, distinguished identity symbol $=_E$ that applies to ordinary objects as follows:

$$x =_E y \ =_{df}\ O!x \;\&\; O!y \;\&\; \Box\forall F(Fx \equiv Fy)$$

[4]A full discussion of this would take us too far afield. I hope to discuss the connection at length in another, more appropriate context. However, I'll say a more about this connection in the final section of the paper.

[5]A subformula is defined as follows: every formula is a subformula of itself. If χ is $\neg\varphi$, $\varphi \rightarrow \psi$, $\forall\alpha\varphi$, or $\Box\varphi$, then φ (and ψ) is a subformula of χ. If φ is a subformula of ψ, and ψ is a subformula of χ, then φ is a subformula of χ.

Given this definition, the λ-expression $[\lambda xy\, x\!=_E\! y]$ is well-formed. So $=_E$ denotes a relation.

The five other (proper) axioms and definitions of the theory are:

1. $O!x \rightarrow \Box\neg\exists F xF$
2. $\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$, where φ has no free xs
3. $x\!=\!y \ =_{df}\ x\!=_E\! y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$
4. $F\!=\!G \ =_{df}\ \Box\forall x(xF \equiv xG)$
5. $\alpha = \beta \rightarrow [\varphi(\alpha,\alpha) \equiv \varphi(\alpha,\beta)]$, where α, β are either both object variables or both relation variables and $\varphi(\alpha,\beta)$ is the result of replacing one or more occurrences of α by β in $\varphi(\alpha,\alpha)$, provided β is substitutable for α in the occurrences of α that it replaces

The first principle is an axiom that asserts that ordinary objects necessarily fail to encode properties. The second principle is a proper axiom schema, namely, the comprehension principle for abstract objects. This asserts the existence of an abstract object that encodes just the properties F satisfying formula φ, whenever φ is any formula with no free variables x. The third principle is a definition of a general notion of identity. Objects x and y are said to be 'identical' just in case they are both ordinary objects and necessarily exemplify the same properties or they are both abstract objects and necessarily encode the same properties. The fourth principle, the definition for property identity, asserts that properties are identical whenever they are necessarily encoded by the same objects.[6] Since both the identity of objects ('$x\!=\!y$') and the identity of properties ('$F\!=\!G$') are defined notions, the fifth principle tells us that expressions for identical objects or identical relations can be substituted for one another in any context.

The Logic: The logic that underlies this proper theory is essentially classical. The logical axioms of this system are the modal closures of the instances of axiom schemata of classical propositional logic, classical quantification theory (modified only to admit empty descriptions), and second-order S5 modal logic with Barcan formulas (modified only to admit rigid descriptions and the actuality operator). The logical axioms for encoding are the modal closures of the following axiom:

Logic of Encoding: $\Diamond xF \rightarrow \Box xF$

The logical axioms for the λ-predicates are the modal closures of the following principle of λ-conversion:[7]

λ-Conversion:[8] $[\lambda x_1 \ldots x_n\, \varphi]y_1 \ldots y_n \equiv \varphi^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}$, where φ has no definite descriptions and $\varphi^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}$ is the result of substituting y_i for x_i $(1 \leq i \leq n)$ everywhere in φ.[9]

The rules of inference (see below) will allow us to derive the following comprehension principle for n-place relations $(n \geq 0)$ from λ-conversion:

Relations: $\exists F^n \Box\forall y_1 \ldots \forall y_n(F^n y_1 \ldots y_n \equiv \varphi)$, where φ has no free Fs, no encoding subformulas and no definite descriptions

[6] This definition can be generalized easily to yield a definition of identity for n-place relations $(n \geq 2)$ and propositions $(n = 0)$. The more general formulation may be found in Zalta [1983], p. 69; Zalta [1988a], p. 52; and Zalta [1993a], footnote 21. These definitions of relation identity have been motivated and explained in the cited works. The definition allows one to consistently assert that there are distinct relations that are (necessarily) equivalent.

[7] It is a logical axiom that interchange of bound variables makes no difference to the identity of the property denoted by the λ-expression: $[\lambda x_1 \ldots x_n\, \varphi] = [\lambda y_1 \ldots y_n\, \varphi']$, where φ and φ' differ only by the fact that y_i is substituted for the bound occurrences of x_i. The following is also a logical axiom: $F^n = [\lambda x_1 \ldots x_n\, Fx_1 \ldots x_n]$.

[8] It is important to remember that the formulas φ in λ-expressions may not contain encoding subformulas. This restriction serves to eliminate the paradox which would otherwise arise in connection with the comprehension principle for abstract objects. Were properties of the form $[\lambda z\, \exists F(zF \,\&\, \neg Fz)]$ formulable in the system, one could prove the following contradiction. By comprehension for abstract objects, the following would be an axiom:

$$\exists x(A!x \,\&\, \forall F(xF \equiv F\!=\![\lambda z\, \exists F(zF \,\&\, \neg Fz)]))$$

Call such an object 'a' and ask the question: $[\lambda z\, \exists F(zF \,\&\, \neg Fz)]a$? We leave it as an exercise to show that a exemplifies this property iff it does not.

We remove the threat of this paradox by not allowing encoding subformulas in property comprehension. This still leaves us with a rich theory of properties, namely, all of the predicable and impredicable properties definable in standard second-order exemplification logic.

[9] A definite description $\imath y\psi$ may appear in instances of λ-conversion whenever (it is provable that) $\imath y\psi$ has a denotation. Whenever we assume or prove that $\exists x(x\!=\!\imath y\psi)$, we can prove the instance of λ-conversion that asserts:

$$[\lambda x\, Rx\imath y\psi]z \equiv Rz\imath y\psi$$

by first deriving:

$$\forall y([\lambda x\, Rxy]z \equiv Rzy),$$

and then instantiating the description into the universal claim for the variable y.

The logic of the actuality operator is governed by the idea that $\mathcal{A}\varphi$ is true at a world just in case φ is true at the distinguished actual world.[10] This sets up a situation in which the first group of logical axioms for the actuality operator are logical truths which are not necessary:[11]

Actuality: $\mathcal{A}\varphi \equiv \varphi$

Therefore, only the ordinary non-modal instances of this axiom are asserted as logical axioms of the system (i.e., the modal closures of instances of this axiom are not taken as logical axioms). The second group of logical axioms for actuality reflect the fact that even when $\mathcal{A}\varphi$ occurs in a modal context, its truth depends only on whether φ is actually the case. Thus, the modal closures of the instances of the following principle are to be logical axioms of the system:

$\Box$ Actuality: $\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi$

In other words, if it is actually the case that φ then necessarily, it is actually the case that φ.

Finally, two features of the logic of definite descriptions are relevant: (1) The logic of definite descriptions is free. We may not generalize on $\imath x\varphi$ or instantiate it into universal claims, unless we know that $\exists y(y = \imath x\varphi)$. (2) Definite descriptions are to be understood rigidly. As such, we take the ordinary (non-modal) instances of the following as the logical axioms governing definite descriptions:

Descriptions: $\psi_y^{\imath x\varphi} \equiv \exists x(\varphi \,\&\, \forall z(\varphi_x^z \rightarrow z = x) \,\&\, \psi_y^x)$, for any atomic or identity formula $\psi(y)$ in which y is free.

When descriptions are understood rigidly, this is a logical truth that is not necessary.[12] The following simple consequence of Descriptions plays an important role in the reasoning that is used in what follows:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi_F^G$$

[10]In models of this system, a distinguished actual world is always assumed. This assumption is justified by the metaphysical theory being developed within the object language, for 'worlds' are definable within this theory and it is provable that there is a unique 'actual' world (i.e., a world where all and only the propositions true at that world are true *simpliciter*). See Zalta [1983], [1988a], and [1993a].

[11]See Zalta [1988b].

[12]See Zalta [1988b].

In other words, the abstract object that encodes just the properties satisfying φ encodes a property G iff G satisfies φ.

To complete this logic, we take Modus Ponens and the Rule of Generalization as our two primitive rules of inference. The Rule of Necessitation is derivable, though restricted as follows: if φ is a theorem, and the proof of φ doesn't depend on any instance of the contingent Actuality axiom or the contingent Description axiom, then $\Box\varphi$ is a theorem. Since the logic is classical, it is provable that every object is complete with respect to exemplification: $Fx \vee \bar{F}x$ (where $\bar{F} =_{df} [\lambda x\, \neg Fx]$). However, abstract objects may be incomplete with respect to the properties they encode.

Model-theoretically, the quantifiers $\forall x$ and $\forall F$ of the S5 quantified modal logic range over a fixed domain of objects and a fixed domain of relations, respectively (the domains are mutually exclusive). However, the validity of the first- and second-order Barcan formulas poses no philosophical problem concerning the contingency of ordinary objects. Note that the theory allows for two kinds of contingent ordinary object: (1) those that satisfy the formula $E!x \,\&\, \Diamond\neg E!x$ and (2) those that satisfy the formula $\neg E!x \,\&\, \Diamond E!x$. The former are concrete (spatiotemporal) at our world but may fail to be concrete at other worlds. (Examples are the rocks, tables, trees, planets, etc., of our world.) The latter are 'contingently nonconcrete' objects; these are (actually existing) objects that are nonconcrete in this world but concrete at other worlds. (Examples are things that at other possible worlds are million carat diamonds, talking donkeys, etc., there, but which are not million carat diamonds, talking donkeys, etc., here at our world.) The appeal to both kinds of ordinary objects demonstrates that the Barcan formulas are compatible with the existence of contingent objects.[13]

The Theory of Identity: The treatment of identity is of some interest and merits some discussion. Our strategy has been to: (1) eliminate '=' as a primitive of the language altogether, (2) introduce in its place the special defined relation $=_E$ which is logically well-behaved on ordinary objects and which can be used in Relations and λ-expressions to form complex relations, and (3) define more general notions of identity that can apply to objects and properties. It is trivial to establish that $x =_E y \rightarrow x = y$, and so substitution of identicals applies to identical$_E$ ordinary objects

[13]For a detailed defence of this simplest quantified modal logic, see Linsky and Zalta [1994].

as well. It is easy to establish both that $x = y \rightarrow \Box x = y$ and that $F = G \rightarrow \Box F = G$.

This theory of identity has served well in the various applications of the theory of abstract objects. It is important to note that whereas $=_E$ may be used in both in λ-expressions and in the comprehension principle for relations, the defined identity symbol '=' may not be used in either. The reason for this will be discussed in the next section, where we examine some interesting consequences of the theory and describe the standard model, which helps to picture these consequences.

§2: Important Consequences and the Standard Model

The comprehension principle for abstract objects is a schema that has an infinite number of instances. Each instance involves some condition φ and asserts that there is an abstract object that encodes just the properties satisfying the condition. It turns out, however, that the traditional mode of predication (exemplification) cannot always distinguish the abstract objects assertible by comprehension. Here is why. From a model-theoretic point of view, the comprehension principle for abstract objects attempts to correlate abstract objects with (expressible) sets of properties. Thus, the domain of abstract objects is roughly the size of the power set of the set of properties. Therefore, there cannot be a distinct property of the form $[\lambda z\, z = k]$ for each distinct abstract object k, for otherwise, there would be a one-to-one mapping from the power set of the set of properties to a subset of the set of properties, in violation of Cantor's Theorem.[14] The system avoids paradox because the expression $[\lambda z z{=}k]$ is not well-formed. The matrix '$z{=}k$' is an abbreviation of a longer formula

[14] This is McMichael's Paradox, which was first reported in McMichael and Zalta [1980] (footnote 15) and described further in Zalta [1983] (p. 159). If there were a distinct property of the form $[\lambda z z{=}k]$ for each distinct abstract object k, we could prove a contradiction, as follows. By A-Objects, the following instance of comprehension would be well-formed:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(F{=}[\lambda z\, z{=}y] \,\&\, \neg yF)))$$

Call an arbitrary such object a. Now ask the question, $a[\lambda z\, z = a]$? Suppose so. Then, by definition of a, there exists a y, say b, such that $[\lambda z\, z{=}a]{=}[\lambda z\, z{=}b]$ and $\neg b[\lambda z\, z{=}a]$. But since $a{=}a$, we know $[\lambda z\, z{=}a]a$, and hence that $[\lambda z\, z{=}b]a$. Thus $a{=}b$, and so $\neg a[\lambda z\, z{=}a]$, contrary to our assumption. So suppose $\neg a[\lambda z\, z{=}a]$. Then, by definition of a, every object y is such that if $[\lambda z\, z{=}a]{=}[\lambda z\, z{=}y]$, then $y[\lambda zz{=}a]$. So, in particular, this universal claims holds for a, and hence, if $[\lambda z\, z{=}a]{=}[\lambda z\, z{=}a]$, then $a[\lambda z\, z{=}a]$. But, since the antecedent is clearly provable, $a[\lambda z\, z{=}a]$, which is a contradiction.

containing encoding subformulas and so this matrix cannot appear in λ-expressions. So though there is a well-defined condition that governs the identity of abstract objects, there is no relation of identity on abstract objects analogous to the classical identity relation $=_E$ on the ordinary objects.

The result is even more general. It follows that, given any relation R, there cannot be a distinct property of the form $[\lambda x\, Rxk]$ (or $[\lambda x\, Rkx]$) for each distinct abstract object k. Intuitively, the picture underlying this result is like the above: were there such distinct properties for distinct abstract objects, there would be a one-to-one mapping from the power set of the set of properties (i.e., the domain of abstract objects) to a subset of the set of properties. The theory avoids inconsistency in this case by yielding the following consequences:

1) **Theorems:** Some Non-Classical Abstract Objects.

.1) $\forall R\exists x\exists y(A!x \,\&\, A!y \,\&\, x{\neq}y \,\&\, [\lambda z\, Rzx]{=}[\lambda z\, Rzy])$

.2) $\forall R\exists x\exists y(A!x \,\&\, A!y \,\&\, x{\neq}y \,\&\, [\lambda z\, Rxz]{=}[\lambda z\, Ryz])$

.3) $\forall F\exists x\exists y(A!x \,\&\, A!y \,\&\, x{\neq}y \,\&\, [\lambda\, Fx]{=}[\lambda\, Fy])$

The proofs of (1.1) and (1.3) are in the Appendix.[15] It might prove helpful here to discuss relational properties such as $[\lambda z\, Rzx]$ which appear in (1.1) and propositions such as $[\lambda\, Fx]$ which appear in (1.3). In the former case, the expression '$[\lambda z\, Rzx]$' is to be read: being an object z such that z bears R to x. This λ-expression denotes a one-place relational property. So (1.1) tells us that given any relation R, there is at least one pair of distinct abstract objects such that the relational properties contructed out of those objects become identified. In the case of (1.3),

[15] Anderson [1993] notes (p. 226) that the theory also yields abstract objects x such that every property x exemplifies is exemplified by some other abstract object y. Though Anderson calls such objects 'undistinguished', I point out (Zalta [1993b], pp. 239-240) that such objects can nevertheless be distinguished by the properties they encode. Anderson's point was directed at an earlier version of the present system, in which impredicative conditions were not permitted in relation comprehension. However, since we now allow such impredicative relations, there are not only 'undistinguished' objects, but 'indiscernible' objects, i.e., distinct abstract objects which exemplify exactly the same properties (see below). Again, it is to be remembered that the members of each such 'pair' of 'indiscernible' abstract objects can be distinguished from one another by the fact that one encodes a property that the other one fails to encode.

the expression '$[\lambda\, Fx]$' is to be read: that x exemplifies F. Here the λ operator doesn't bind any variables, and the resulting expression denotes a proposition (i.e., a 0-place relation). So (1.3) tells us that given any property F, there is at least one pair of distinct abstract objects such that the simple atomic exemplification propositions contructed out of those objects become identified.

Given that we are now allowing impredicative formulas in λ-predicates and Relations, an even stronger result follows, namely, some distinct abstract objects exemplify the same properties. Consider:

2) **Theorems:** Some Further Non-Classical Abstract Objects. There are distinct abstract objects which are 'indiscernible' from the point of view of the traditional mode of predication.

$$\exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, \forall F(Fx \equiv Fy))$$

The proof appeals to the theorems in item (1).[16]

The standard models of the theory articulate a structure that helps us to picture these facts. These models also show how abstract objects (conceived in the model as sets of properties) can exemplify the very same properties that they encode (i.e., how sets of properties can exemplify their elements). In Zalta [1997], the standard model for the modal version of the theory was constructed and discussed in detail. It was based on Peter Aczel's model construction for the elementary (non-modal) version of the theory of abstract objects.[17] The leading ideas of Aczel's model are preserved in the modal version and can be described as follows, in which I correct a minor error in [1997].

The standard model assumes that the language has been interpreted in a structure containing several mutually exclusive domains of primitive entities:

1. a domain of *ordinary* objects $\mathbf{O}$ and a domain of *special* objects $\mathbf{S}$; the union of these domains is called the domain of ordinary* objects $\mathbf{O}^*$,

[16]Let R_0 be the relation $[\lambda xy \forall F(Fx \equiv Fy)]$. We know from item (1.1) that, for any relation R, there exist distinct abstract objects a, b such that $[\lambda z\; Rza] = [\lambda z\; Rzb]$. So, in particular, there are distinct abstract objects a, b such that $[\lambda z\; R_0 za] = [\lambda z\; R_0 zb]$. But, by the definition of R_0, it is easily provable that $R_0 aa$, from which it follows that $[\lambda z\, R_0 za]a$. But, then, $[\lambda z\, R_0 zb]a$, from which it follows that $R_0 ab$. Thus, by definition of R_0, $\forall F(Fa \equiv Fb)$.

[17]Aczel sketched models of the nonmodal version of the theory during his stay at Stanford in 1987 and in a personal communication of January 10, 1991.

2. a domain $\mathbf{R}$ of relations, which is a general union of domains of n-place relations $\mathbf{R}_n$ $(n \geq 0)$,

3. a domain of possible worlds, which contains a distinguished actual world.

The domain of relations $\mathbf{R}$ is subject to two conditions: (a) it is closed under logical functions that harness the simple properties and relations into complex properties and relations (these logical functions are the counterparts of Quine's predicate functors, except that they operate on relations instead of predicates), and (b) there are at least as many relations in each $\mathbf{R}_n$ as there are elements of $\mathcal{P}([\mathbf{O}^*]^n)$ (= the power set of the n^{th}-Cartesian product of the domain of ordinary* objects). Relative to each possible world $\mathbf{w}$, each n-place relation $\mathbf{r}^n$ in $\mathbf{R}_n$ is assigned an element of $\mathcal{P}([\mathbf{O}^*]^n)$ as its exemplification extension at $\mathbf{w}$ (when $n = 0$, each proposition $\mathbf{r}^0$ is assigned a truth value at $\mathbf{w}$).[18] In what follows, we refer to the exemplification extension of a relation $\mathbf{r}^n$ at world $\mathbf{w}$ as $\mathbf{ext_w}(\mathbf{r}^n)$.

The standard model is completed by letting the domain of abstract objects $\mathbf{A}$ be the power set of the set of properties (i.e., $\mathbf{A} = \mathcal{P}(\mathbf{R}_1)$). Each abstract object in $\mathbf{A}$ is then mapped to one of the *special* objects in $\mathbf{S}$; the object correlated with abstract object a is called *the proxy of* a. Some distinct abstract objects will therefore get mapped to the same proxy. Finally, the ordinary and abstract objects are combined into one set $\mathbf{D}$ $(= \mathbf{O} \cup \mathbf{A})$.[19] Letting the variable $\mathbf{x}$ range over $\mathbf{D}$, we define a mapping $|\;|$ from $\mathbf{D}$ into the set of ordinary* objects as follows:

$$|\mathbf{x}| = \begin{cases} \mathbf{x}, \text{ if } \mathbf{x} \text{ is ordinary} \\ \text{the proxy of } \mathbf{x}, \text{ if } \mathbf{x} \text{ is abstract} \end{cases}$$

[18]Constraints on the logical functions ensure that the exemplification extension of a complex relation $\mathbf{r}$ meshes in the proper way with the exemplification extensions of the simpler relations $\mathbf{r}$ may have as a part.

[19]This corrects the error in [1997]. In that paper, the domain $\mathbf{D}$ was set to $\mathbf{O}^* \cup \mathbf{A}$ instead of $\mathbf{O} \cup \mathbf{A}$. But this won't yield a model, for the following reason. Consider distinct *special* objects a and b in $\mathbf{S}$. Then there will be some property, say P, such that Pa and $\neg Pb$. But given the definitions below, it will follow that both a and b are abstract, i.e., that $A!a$ and $A!b$ are both true. Moreover, since a and b are special objects, they necessarily fail to encode properties, so $\Box\forall F(aF \equiv bF)$. But, then, by the definition of identity for abstract objects, it follows that $a = b$, and thus $Pa \,\&\, \neg Pa$. By setting $\mathbf{D}$ to $\mathbf{O} \cup \mathbf{A}$, we avoid this result. I am indebted to Tony Roy for pointing this out to me.

Now suppose that an assignment function g to the variables of the language has been extended to a denotation function $\mathbf{d}_g$ on all the terms of the language (so that in the case of the variables x and F, we know $\mathbf{d}_g(x) \in \mathbf{D}$ and $\mathbf{d}_g(F^n) \in \mathbf{R}^n$). We then define *true at* $\mathbf{w}$ (with respect to g) for the atomic formulas as follows:[20] (a) '$F^n x_1 \ldots x_n$' is true$_g$ at $\mathbf{w}$ iff $\langle |\mathbf{d}_g(x_1)|, \ldots, |\mathbf{d}_g(x_n)| \rangle \in \mathbf{ext_w}(\mathbf{d}_g(F^n))$, and (b) '$xF$' is true$_g$ at world $\mathbf{w}$ iff $\mathbf{d}_g(F) \in \mathbf{d}_g(x)$.[21] It is easy to constrain standard models so that $\mathbf{ext_w}(\mathbf{d}_g(O!))$ is simply the subdomain of ordinary objects and that $\mathbf{ext_w}(\mathbf{d}_g(A!))$ is the subdomain of special objects.[22]

These definitions have the following consequences: (1) the comprehension principle for abstract objects is true in these standard models;[23] (2) λ-Conversion and Relations are both true in such models;[24] (3) an abstract object x (i.e., set of properties) will exemplify (according to the model) a property F just in case the proxy of x exemplifies F in the traditional way; and (4) whenever distinct abstract objects x and y get mapped to the same proxy, x will exemplify a property F iff y exemplifies F.

This standard model construction, therefore, helps to picture the earlier results about the absence of distinct 'haecceities' for distinct abstract objects and about the existence of distinct but 'indiscernible' abstract objects.[25] It is worth emphasizing at this point, however, that it would be a

[20]For simplicity, we are using representative atomic formulas containing only variables.

[21]Note that since the truth of encoding formulas at a world is defined independently of a world, an encoding formula will be true at all worlds if true at any. This validates the Logical Axiom of Encoding.

[22]We simply require that $\mathbf{ext_w}(\mathbf{d}(E!))$ be some subset of the domain of ordinary objects and that the domain of ordinary objects be the union, for every $\mathbf{w}$, of all the sets $\mathbf{ext_w}(\mathbf{d}(E!))$. Since $O!$ is defined as $[\lambda x\, \Diamond E!x]$ and $A!$ is defined as $[\lambda x\, \neg\Diamond E!x]$, this guarantees that all the ordinary objects are in the exemplification extension of $O!$ and all the special objects are in the exemplification extension of $A!$. It should now be straightforward to see that the proper axiom, $O!x \rightarrow \Box\neg\exists F\, xF$, is true in such a model.

[23]For each condition φ on properties, there is a set of properties a such that: (a) the proxy of a is a special object and (b) a property F is an element of a iff F satisfies φ.

[24]The logical functions ensure that for any condition φ on objects $x_1, \ldots, x_n$ without encoding subformulas, there is a relation F^n whose exemplification extension, at any given possible world, contains all and only those n-tuples of objects satisfying φ. See Zalta [1983] and [1988a] for further details.

[25]This picture is preserved in the 'minimal' *extensional* model of the theory, in which the following all hold: (1) there is one possible world, (2) there are no ordinary objects, (3) there is one special object a^*, (4) there are exactly two properties, one of which has the set $\{a^*\}$ as its extension at the single world, the other having $\emptyset$ its extension at the single world; the latter is assigned as the extension of the property $E!$ (since this is an extensional model, we simply identify properties with their extensions for the next and final clause), and (5) there are exactly four abstract objects: $\emptyset$, $\{\{a^*\}\}$, $\{\emptyset\}$, and $\{\{a^*\}, \emptyset\}$. However, as both Thomas Hofweber and Tony Roy have emphasized in recent discussions, there are non-standard models in which the domains of special objects, properties, and abstract objects are all countably infinite.

mistake to construe abstract objects in what follows as sets of properties. With the consistency of the theory secure, we shall assume that the world is just the way that the theory says it is. Thus, sets and set membership are not presupposed, nor is any other notion of set theory.

Our standard models also demonstrate that the exemplification mode of predication can always discriminate among the ordinary objects. The following logical theorems, concerning the classical nature of $=_E$, govern ordinary objects:[26]

3) Theorems: $=_E$ is an Equivalence Relation on Ordinary Objects. Here are some facts about the $=_E$ relation.

$$O!x \rightarrow x =_E x$$
$$O!x \;\&\; O!y \rightarrow (x =_E y \rightarrow y =_E x)$$
$$O!x \;\&\; O!y \;\&\; O!z \rightarrow (x =_E y \;\&\; y =_E z \rightarrow x =_E z)$$

4) Theorems: Ordinary Objects Obey Leibniz's Identity$_E$ of Indiscernables:

$$O!x \;\&\; O!y \;\&\; \forall F(Fx \equiv Fy) \rightarrow x =_E y$$

5) Theorems: Distinct Ordinary Objects x, y Have Distinct Haecceities:

$$O!x \;\&\; O!y \;\&\; x \neq_E y \rightarrow [\lambda z\, z =_E x] \neq [\lambda z\, z =_E y]$$

In what follows, we exploit the classical nature of identity$_E$ by considering a certain equivalence condition on properties F and G, namely, the equinumerosity of F and G *with respect to* the ordinary objects. Using this equivalence condition, we can identify, for each property G, an abstract object that serves as the *natural cardinal* numbering the Gs, namely, the abstract object that encodes all and only the properties F that are equinumerous to G with respect to the ordinary objects. To develop the natural numbers, however, we define 'predecessor' and follow Frege's general plan for deriving the Dedekind/Peano axioms. As we shall see in the coming sections, two additional principles must be added to our system, namely,

[26]The proofs of some of these claims are given in the Appendix.

(i) if there is a natural number that numbers the Fs, then there might have been a concrete object distinct from all the actual Fs, and (ii) predecessor and its weak ancestral are relations. The first is required to prove that every number has a successor; the second is required to prove the principle of induction. These additional principles appear as items (39) and (42) in the following sections.

§3: Natural Cardinals

Throughout the following, we use u, v as variables ranging only over ordinary objects. Where α is any variable, we employ the abbreviation '$\exists!\alpha\varphi$' to assert 'there exists a unique thing α such that φ'. It is defined in the usual way: $\exists\alpha\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$. The proofs of most of the theorems that follow are given in the Appendix.

6) Definition: Equinumerosity with Respect to the Ordinary Objects. We say that properties F and G are *equinumerous with respect to the ordinary objects* ('$F \approx_E G$') just in case there is a relation R that constitutes a one- to- one and onto function from the ordinary objects in the exemplification extension of F to the ordinary objects in the exemplification extension of G:[27]

$$F \approx_E G \;=_{df}\; \exists R[\forall u(Fu \rightarrow \exists! v(Gv \,\&\, Ruv)) \,\&\, \forall u(Gu \rightarrow \exists! v(Fv \,\&\, Rvu))]$$

So F and G are equinumerous$_E$ just in case there is a relation R such that: (a) every ordinary object that exemplifies F bears R to a unique ordinary object exemplifying G (i.e., R is a function from the ordinary objects of F to the ordinary objects of G), and (b) every ordinary object that exemplifies G is such that a unique ordinary object exemplifying F bears R to it (i.e., R is a one-to-one function from the ordinary objects of F onto the ordinary objects of G). In the proofs of what follows, we say that such a relation R is a *witness* to the equinumerosity$_E$ of F and G.

7) Theorems: Equinumerosity$_E$ Partitions the Domain of Properties.

.1) $F \approx_E F$

.2) $F \approx_E G \rightarrow G \approx_E F$

.3) $F \approx_E G \,\&\, G \approx_E H \rightarrow F \approx_E H$

[27] cf. Frege, *Grundlagen*, §71 and §72.

It is important to give the reason for employing the condition '$F \approx_E G$' instead of the more traditional notion of one-to-one correspondence between properties, '$F \approx G$', which is defined without the restriction to ordinary objects. It is a consequence of the theorems concerning the existence of non-classical abstract objects (items (1) and (2)) that '$F \approx G$' does not define an equivalence condition on properties.[28] Although $F \approx G$ is not an equivalence relation, $F \approx_E G$ proves to be a useful substitute.

8) Definitions: Numbering a Property. We may appeal to our definition of equinumerosity$_E$ to say when an object numbers a property: *x numbers* (the ordinary objects exemplifying) G iff x is an abstract object that encodes just the properties equinumerous$_E$ with G:

$$\mathit{Numbers}(x, G) \;=_{df}\; A!x \,\&\, \forall F(xF \equiv F \approx_E G)$$

9) Theorem: Every Property is Uniquely Numbered. It is an immediate consequence of the previous definition and the comprehension and identity conditions for abstract objects that for every property G, there is a unique object which numbers G:

$$\forall G \exists! x \mathit{Numbers}(x, G)$$

10) Definition: The Number of (Ordinary) Gs. Since there is a unique number of Gs, we may introduce the notation '$\#_G$' to refer to *the* number of Gs:

$$\#_G \;=_{df}\; \imath x \mathit{Numbers}(x, G)$$

11) Theorem: The Number of Gs Exists. It is an immediate consequence of the logic of descriptions that for every property G, the number of Gs exists:

[28] Here is the argument that establishes this. By (2), we know that there are at least two distinct abstract objects, say a and b, which are 'indiscernible' (i.e., which exemplify the same properties). Now consider any property that both a and b exemplify, say, P. Then there won't be any property to which P is equinumerous. For suppose, for some property, say Q, that $P \approx Q$. Then there would be a relation R which is a one-one and onto function from the Ps to the Qs. Since R maps each object exemplifying P to some unique object exemplifying Q, R maps a to some object, say c, that exemplifies Q. So a exemplifies the property $[\lambda z\, Rzc]$. But, since a and b are indiscernible, b exemplifies the property $[\lambda z\, Rzc]$, i.e., Rbc. But this contradicts the one-one character of R, for both Rac and Rbc and yet a and b are distinct. Thus, P can't be equinumerous to any property, including itself! Since $F \approx G$ is not a reflexive condition on properties, it is not an equivalence condition.

$$\forall G \exists y (y = \#_G)$$

12) Lemmas: Equinumerosity$_E$ and The Number of Gs. It now follows that: (.1) the number of Gs encodes F iff F is equinumerous$_E$ with G, and (.2) the number of Gs encodes G. In the following, formal renditions, note that '$\#_G F$' asserts that the number of Gs encodes F.

.1) $\#_G F \equiv F \approx_E G$

.2) $\#_G G$

13) Theorem: Hume's Principle. The following claim has now become known as 'Hume's Principle': The number of Fs is identical to the number of Gs if and only if F and G are equinumerous$_E$.[29]

$$\#_F = \#_G \equiv F \approx_E G$$

14) Definition: Natural Cardinals. We may now define: x is a *natural cardinal* iff there is some property F such that x is the number of Fs.[30]

$$NaturalCardinal(x) \ =_{df} \ \exists F(x = \#_F)$$

15) Theorem: Encoding and Numbering F. A natural cardinal encodes a property F just in case it is the number of Fs:

$$NaturalCardinal(x) \rightarrow (xF \equiv x = \#_F)$$

16) Definition: Zero.[31]

$$0 \ =_{df} \ \#_{[\lambda z\, z \neq_E z]}$$

17) Theorem: 0 is a Natural Cardinal.

$$NaturalCardinal(0)$$

18) Theorem: 0 Encodes the Properties Unexemplified by Ordinary Objects. 0 encodes all and only the properties which no ordinary object exemplifies:

$$0F \equiv \neg \exists u F u$$

[29] cf. Frege, *Grundlagen*, §72.
[30] cf. Frege, *Grundlagen*, §72; and *Grundgesetze I*, §42.
[31] cf. Frege, *Grundlagen*, §74; and *Grundgesetze I*, §41.

19) Corollary: Empty Properties Numbered 0. It is a simple consequence of the previous theorems and definitions that F fails to be exemplified by ordinary objects iff the number of Fs is zero:[32]

$$\neg \exists u F u \equiv \#_F = 0$$

20) Definition: Materially Equivalent$_E$ Properties. We say that properties F and G are *materially equivalent with respect to the ordinary objects* ('$F \equiv_E G$') iff the same ordinary objects exemplify F and G:

$$F \equiv_E G \ =_{df} \ \forall u (Fu \equiv Gu)$$

21) Lemmas: Equinumerous$_E$ and Equivalent$_E$ Properties. The following consequences concerning equinumerous$_E$ and materially equivalent$_E$ properties are easily provable: (.1) if F and G are materially equivalent$_E$, then they are equinumerous$_E$; (.2) if F and G are materially equivalent$_E$, then the number of Fs is identical to the number of Gs; and (.3) if F is equinumerous$_E$ to G and G is materially equivalent$_E$ to H, then F is equinumerous$_E$ to H:

.1) $F \equiv_E G \rightarrow F \approx_E G$

.2) $F \equiv_E G \rightarrow \#_F = \#_G$

.3) $F \approx_E G \;\&\; G \equiv_E H \rightarrow F \approx_E H$

§4. Predecessor, Ancestrals, and Natural Numbers[33]

22) Definition: Predecessor. We say that x *precedes* y iff there is a property F and ordinary(!) object u such that (a) u exemplifies F, (b) y is the number of Fs, and (c) x is the number of (the property) *exemplifying-F- but- not- identical$_E$- to-u*:[34]

$$Precedes(x, y) \ =_{df} \ \exists F \exists u (Fu \;\&\; y = \#_F \;\&\; x = \#_{[\lambda z\, Fz \,\&\, z \neq_E u]})$$

Note that the definition of $Precedes(x, y)$ contains the identity sign '=', which is defined in terms of encoding subformulas. As such, there is no guarantee as yet that $Precedes(x, y)$ is a relation,[35] though objects x and

[32] cf. Frege, *Grundlagen*, §75; and *Grundgesetze I*, Theorem 97.
[33] I am greatly indebted to Bernard Linsky for suggesting that I try to prove the Dedekind/Peano Axioms using just the machinery of object theory. He pointed out that the definition of Predecessor was formulable in the language of the theory. I have also benefited from reading Heck [1993].
[34] See Frege, *Grundlagen*, §76; and *Grundgesetze I*, §43.
[35] That is, the Comprehension Principle for Relations does not ensure that it is a relation.

y may satisfy the condition nonetheless.

23) **Theorem**: Nothing is a Precedessor of Zero.[36]

$$\neg\exists x\, \mathit{Precedes}(x, 0)$$

Since nothing precedes zero, it follows that no cardinal number precedes zero.

24) **Lemma**: Let F^{-u} designate $[\lambda z\, Fz \,\&\, z \neq_E u]$ and G^{-v} designate $[\lambda z\, Gz \,\&\, z \neq_E v]$. Then if F is equinumerous$_E$ with G, u exemplifies F, and v exemplifies G, F^{-u} is equinumerous$_E$ with G^{-v}:[37]

$$F \approx_E G \,\&\, Fu \,\&\, Gv \rightarrow F^{-u} \approx_E G^{-v}$$

25) **Theorem:** Predecessor is One-to-One. If x and y precede z, then $x = y$:[38]

$$\mathit{Precedes}(x, z) \,\&\, \mathit{Precedes}(y, z) \rightarrow x = y$$

26) **Lemma**: Let F^{-u} designate $[\lambda z\, Fz \,\&\, z \neq_E u]$ and G^{-v} designate $[\lambda z\, Gz \,\&\, z \neq_E v]$. Then if F^{-u} is equinumerous$_E$ with G^{-v}, u exemplifies F, and v exemplifies G, then F is equinumerous$_E$ with G:[39]

$$F^{-u} \approx_E G^{-v} \,\&\, Fu \,\&\, Gv \rightarrow F \approx_E G$$

27) **Theorem**: Predecessor is Functional. If z precedes both x and y, then x is y:[40]

$$\mathit{Precedes}(z, x) \,\&\, \mathit{Precedes}(z, y) \rightarrow x = y$$

[36] cf. *Grundgesetze I*, Theorem 108.

[37] cf. *Grundgesetze I*, Theorem 87ϑ. This is the line on p. 126 of *Grundgesetze I* which occurs *during* the proof of Theorem 87. Notice that Frege proves the contrapositive. Notice also that this theorem differs from Frege's theorem only by two applications of Hume's Principle: in Frege's theorem, $\#_F = \#_G$ is substituted for $F \approx_E G$ in the antecedent and $\#_{F^{-u}} = \#_{G^{-v}}$ is substituted for $F^{-u} \approx_E G^{-v}$ in the consequent.

[38] cf. Frege, *Grundlagen*, §78; and *Grundgesetze I*, Theorem 89.

[39] cf. Frege, *Grundgesetze I*, Theorem 66. This theorem differs from Frege's Theorem 66 only by two applications of Hume's Principle: in Frege's Theorem, $\#_{F^{-u}} = \#_{G^{-v}}$ is substituted for $F^{-u} \approx_E G^{-v}$ in the antecedent, and $\#_F = \#_G$ is substituted for $F \approx_E G$ in the consequent.

[40] cf. Frege, *Grundgesetze I*, Theorem 71.

28) **Definition**. Properties Hereditary with Respect to Relation R. We say that a property F *is hereditary with respect to* R iff every pair of R-related objects are such that if the first exemplifies F then so does the second:

$$\mathit{Hereditary}(F, R) \;=_{df}\; \forall x, y(Rxy \rightarrow (Fx \rightarrow Fy))$$

Hereafter, whenever $\mathit{Hereditary}(F, R)$, we sometimes say that F is R-hereditary.

29) **Definition:** The Ancestral of a Relation R. We define: x comes before y in the R-series iff y exemplifies every R-hereditary property F which is exemplified by every object to which x is R-related:[41]

$$R^*(x, y) \;=_{df}\; \forall F[\forall z(Rxz \rightarrow Fz) \,\&\, \mathit{Hereditary}(F, R) \rightarrow Fy]$$

So if we are given a genuine relation R, it follows by comprehension for relations that $R^*(x, y)$ is a genuine relation as well (the quantifier over relations in the definition of $R^*(x, y)$ is permitted by the comprehension principle for relations).

30) **Lemmas**: The following are immediate consequences of the two previous definitions: (.1) if x bears R to y, then x comes before y in the R-series; (.2) if x comes before y in the R-series, F is exemplified by every object to which x bears R, and F is R-hereditary, then y exemplifies F; (.3) if x exemplifies F, x comes before y in the R-series, and F is R-hereditary, then y exemplifies F; (.4) if x bears R to y and y comes before z in the R series, then x comes before z in the R series; and (.5) if x comes before y in the R series, then something bears R to y:

.1) $Rxy \rightarrow R^*(x, y)$

.2) $R^*(x, y) \,\&\, \forall z(Rxz \rightarrow Fz) \,\&\, \mathit{Hereditary}(F, R) \rightarrow Fy$[42]

.3) $Fx \,\&\, R^*(x, y) \,\&\, \mathit{Hereditary}(F, R) \rightarrow Fy$[43]

.4) $Rxy \,\&\, R^*(y, z) \rightarrow R^*(x, z)$[44]

.5) $R^*(x, y) \rightarrow \exists z Rzy$[45]

[41] cf. Frege, *Begriffsschrift*, Proposition 76; *Grundlagen*, §79; and *Grundgesetze I*, §45.

[42] cf. Frege, *Grundgesetze I*, Theorem 123.

[43] cf. Frege, *Grundgesetze I*, Theorem 128.

[44] cf. Frege, *Grundgesetze I*, Theorem 129.

[45] cf. Frege, *Grundgesetze I*, Theorem 124.

31) **Definition:** Weak Ancestral. We say that y *is a member of the R-series beginning with* x iff either x comes before y in the R-series or $x = y$.[46]

$$R^+(x,y) \ =_{df} \ R^*(x,y) \vee x = y$$

The definition of $R^+(x,y)$ involves the identity sign, which is defined in terms of encoding subformulas. So though x and y may satisfy the condition $R^+(x,y)$, there is no guarantee as yet that they stand in a relation in virtue of doing so.

32) **Lemmas**: The following are immediate consequences of the previous three definitions: (.1) if x bears R to y, then y is a member of the R-series beginning with x; (.2) if x exemplifies F, y is a member of the R-series beginning with x, and F is R-hereditary, then y exemplifies F; (.3) if y is a member of the R series beginning with x, and y bears R to z, then x comes before z in the R-series; (.4) if x comes before y in the R-series and y bears R to z, then z is a member of the R-series beginning with x; (.5) if x bears R to y, and z is a member of the R series beginning with y, then x comes before z in the R series; and (.6) if x comes before y in the R series, then some member of the R-series beginning with x bears R to y:

.1) $Rxy \rightarrow R^+(x,y)$

.2) $Fx \ \& \ R^+(x,y) \ \& \ \mathit{Hereditary}(F,R) \rightarrow Fy$[47]

.3) $R^+(x,y) \ \& \ Ryz \rightarrow R^*(x,z)$[48]

.4) $R^*(x,y) \ \& \ Ryz \rightarrow R^+(x,z)$

.5) $Rxy \ \& \ R^+(y,z) \rightarrow R^*(x,z)$[49]

.6) $R^*(x,y) \rightarrow \exists z(R^+(x,z) \ \& \ Rzy)$[50]

[46] cf. Frege, *Grundlagen*, §81; and *Grundgesetze I*, §46.
[47] cf. Frege, *Grundgesetze I*, Theorem 144.
[48] cf. Frege, *Grundgesetze I*, Theorem 134.
[49] cf. Frege, *Grundgesetze I*, Theorem 132.
[50] cf. Frege, *Grundgesetze I*, Theorem 141.

§5. The Dedekind/Peano Axioms

33) **Definition:** Natural Numbers. We may now define:[51]

$$\mathit{NaturalNumber}(x) \ =_{df} \ \mathit{Precedes}^+(0,x)$$

We sometimes use 'm', 'n', and 'o' as restricted variables ranging over natural numbers.

34) **Theorem**: Natural Numbers are Natural Cardinals. It is a relatively straightforward consequence of the previous definition that natural numbers are natural cardinals:

$$\mathit{NaturalNumber}(x) \rightarrow \mathit{NaturalCardinal}(x)$$

35) **Theorem:** 0 is a Natural Number.[52]

$$\mathit{NaturalNumber}(0)$$

With this theorem, we have derived the 'first' Dedekind/Peano axiom.

36) **Theorems:** 0 Is Not the Successor of Any Natural Number. It now follows that: (.1) 0 does not ancestrally precede itself, and (.2) no natural number precedes 0.[53]

.1) $\neg \mathit{Precedes}^*(0,0)$

.2) $\neg \exists n \mathit{Precedes}(n,0)$

With (36.2), we have derived the 'second' Dedekind/Peano axiom.

37) **Theorems:** No Two Natural Numbers Have the Same Successor. From (25), it follows that no two natural numbers have the same successor.

$$\forall n, m, o(\mathit{Precedes}(n,o) \ \& \ \mathit{Precedes}(m,o) \rightarrow m = n)$$

With (37), we have derived the 'third' Dedekind/Peano axiom. We now work our way towards a proof that for every natural number, there is a unique natural number which is its successor.

38) **Lemma**: Successors of Natural Numbers are Natural Numbers. If a natural number n precedes an object y, then y is itself a natural number:

[51] cf. Frege, *Grundlagen*, §83; and *Grundgesetze I*, §46. In the latter section, Frege informally reads the formula $\mathit{Precedes}^+(0,x)$ as 'x is a finite number', though he doesn't officially introduce new notation for this notion.
[52] cf. Frege, *Grundgesetze I*, Theorem 140. Frege here proves only the general theorem that $\forall x \mathit{Precedes}^+(x,x)$, but doesn't seem to label the result of instantiating the universal quantifier to the number zero as a separate theorem.
[53] cf. Frege, *Grundgesetze I*, Theorem 126.

$Precedes(n, y) \to NaturalNumber(y)$

39) Modal Axiom. Richness of Possible Objects. The following modal claim is true *a priori*: if there is a natural number which numbers the Gs, then there might have been a concrete object y which is distinct$_E$ from every ordinary object that *actually* exemplifies G. We may formalize this *a priori* truth as the following modal axiom, using u to range over ordinary objects:

$$\exists x(NaturalNumber(x) \,\&\, x = \#_G) \to \\ \quad \Diamond \exists y(E!y \,\&\, \forall u(\mathcal{A}Gu \to u \neq_E y))$$

40) Modal Lemma. Distinctness of Possible Objects. It is a consequence of the logic of actuality and the logic of the identity$_E$ relation that if it is possible that ordinary object v is distinct$_E$ from every ordinary object which actually exemplifies G, then in fact v is distinct$_E$ from every ordinary object which actually exemplifies G:

$$\Diamond \forall u(\mathcal{A}Gu \to u \neq_E v) \to \forall u(\mathcal{A}Gu \to u \neq_E v)$$

41) Theorem: Every Natural Number Has a Unique Successor. It now follows from (39) and (40) that for every natural number n, there exists a unique natural number m which is the successor of n:

$$\forall n \exists ! m Precedes(n, m)$$

With this theorem, we have derived the 'fourth' Dedekind/Peano Axiom.

42) Axioms: Predecessor and Its (Weak) Ancestral Are Relations. The definitions of *Predecessor* and its weak ancestral involve encoding subformulas, and so they are not automatically guaranteed to be relations. In what follows, we assume that these conditions do in fact define relations:

.1) $\exists F \forall x \forall y(Fxy \equiv Precedes(x, y))$

It follows from this by the comprehension principle for relations that $Precedes^*(x, y)$ is a relation:

.2) $\exists F \forall x \forall y(Fxy \equiv Precedes^*(x, y))$

However, since the definition of $Precedes^+(x, y)$ involves an encoding formula, we explicitly assume the following:

.3) $\exists F \forall x \forall y(Fxy \equiv Precedes^+(x, y))$

In the final section of the paper, we sketch an extension of the standard model which demonstrates that we may consistently add these assumptions.

43) Theorem: Generalized Induction. If R^+ is a relation, then if an object a exemplifies F and F is hereditary with respect to R when R is restricted to the members of the R-series beginning with a, then every member of the R-series beginning with a exemplifies F:[54]

$$\exists G \forall x, y(Gxy \equiv R^+(x, y)) \to \\ \quad \forall F[Fa \,\&\, \forall x, y(R^+(a, x) \,\&\, R^+(a, y) \,\&\, Rxy \to (Fx \to Fy)) \to \\ \qquad \forall x(R^+(a, x) \to Fx)]$$

44) Corollary: Principle of Induction. The Principle of Induction falls out as a corollary to the previous theorem and the assumption that $Predecessor^+$ is a relation:

$$F0 \,\& \\ \forall x, y[NaturalNumber(x) \,\&\, NaturalNumber(y) \,\&\, Precedes(x, y) \to \\ \qquad (Fx \to Fy)] \to \\ \forall x(NaturalNumber(x) \to Fx)$$

We may put this even more simply by using our restricted variables n, m which range over numbers:

$$F0 \,\&\, \forall n, m(Precedes(n, m) \to (Fn \to Fm)) \to \forall n Fn$$

With the Principle of Induction, we have derived the 'fifth' and final Dedekind/Peano axiom.

45) Definition: Notation for Successors. We introduce the functional notation n' to abbreviate the definite description 'the successor of n' as follows:

$$n' =_{df} \imath y(Precedes(n, y))$$

By (41), we know that every natural number has a unique successor. So n' is always well-defined.

46) Definitions: Introduction of the Integer Numerals. We introduce the integer numerals '1', '2', '3', ... , as abbreviations, respectively, for the descriptions 'the successor of 0', 'the successor of 1', 'the successor of 2', etc.:

[54] cf. Frege, *Grundgesetze I*, Theorem 152.

$1 =_{df} 0'$

$2 =_{df} 1'$

$3 =_{df} 2'$

$\vdots$

Note that the definite descriptions being abbreviated are well-defined terms of our formal language. So it is provable that the numerals have denotations.

47) Definitions. Natural Arithmetic. Finally, we note that the development of natural arithmetic is straightforward. We may define, in the usual way:

$$n + 0 =_{df} n$$
$$n + m' =_{df} (n + m)'$$

And we may define:

$$n < m =_{df} \mathit{Precedes}^*(n, m)$$
$$n \leq m =_{df} \mathit{Precedes}^+(n, m)$$

From these definitions, much can be done.

§6: Observations

Since the fourth and fifth Dedekind/Peano postulates are a consequence of the theory of abstract objects together with (39) and (42), respectively, the question of consistency arises. Peter Aczel describes a standard model of the extended theory:

> ... use my model construction with an infinite set of urelements and have among the special objects a copy of the natural numbers. For each natural number n let α_n be the set of those ordinary properties that are exemplified by exactly n ordinary objects. Now in choosing proxies just make sure that the special object that is the copy of the natural number n is chosen as the proxy of α_n for each n. The copy of the Predecessor relation will be an ordinary relation, as are its (weak) ancestral.[55]

Given this suggestion, we can extend the modal version of Aczel models developed in Section 2. First, we start with a denumerably infinite domain **O** and we include a copy of the natural numbers $0^*, 1^*, 2^*, \ldots$ in **S**. Then we identify the natural numbers $0, 1, 2, \ldots$ as those abstract objects (i.e., sets of properties) which are sets of equinumerous properties. Next, we set the proxy function so that $|n| = n^*$. We then stipulate that the domain of relations $\mathbf{R}_2$ contains the *Predecessor* relation, and that its extension at the actual world is the distinguished set of ordered pairs of proxies of predecessors: $\{\langle 0^*, 1^*\rangle, \langle 1^*, 2^*\rangle, \langle 2^*, 3^*\rangle, \ldots\}$. To constrain the model so that the modal axiom (39) is true, we simply require that the domain of worlds includes an ω-sequence of possible worlds, $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$ and stipulate that at $\mathbf{w}_n$, there are n ordinary objects in $\mathbf{ext}_{\mathbf{w}_n}(\mathbf{d}_g(E!))$. Thus, no matter what the characteristics of the distinguished actual world are and no matter which property G is chosen, whenever natural number n is the number of Gs at the actual world, there is a world $\mathbf{w}_{n+1}$ where there is an ordinary object y that is distinct$_E$ from all the objects that are actually G.

I shall not attempt to justify (39) and (42) in the present work. It is perfectly reasonable to add a special axiom or two to develop some special science. However, a few remarks about (39) are in order. This modal axiom is *not* contingent. It does not assert the existence of concrete objects. Rather, it merely asserts the possible existence of concrete objects (whenever a certain condition holds). The difference is vast. The claim that concrete objects exist is an empirical claim, but the claim that it is possible that concrete objects exist is not. Indeed, by the Rule of Necessitation, (39) is a necessary truth. Moreover, this is the kind of fact that logicians appeal to when defending the view that logic should have no existence assumptions and make no claims about the size of the domain of objects.[56]

[55] This is quoted from his personal communication of November 11, 1996, with the symbol 'α_n' replacing his symbol 'G_n' (since I have been using 'G' for other purposes) and with 'Predecessor' replacing 'proceeds'.

[56] I suspect that it is not hard to find passages where logicians have argued that the domain of objects 'might be of any size' and that 'logic therefore ought not imply anything about the size of the domain'. The following quotation from Boolos [1987]

The fact that every number has a successor does not imply that there are an infinite number of concrete objects. Rather, it implies only that there are an infinite number of possibly concrete objects.[57] These possible concrete objects are all 'ordinary' objects (by definition) and so can be counted by our natural numbers. So the fourth Dedekind/Peano axiom has no contingent consequences. Moreover, we have employed no axiom of infinity such as the one asserted in Russell and Whitehead [1910] or in Zermelo-Fraenkel set theory. Although the infinity of natural numbers falls out as a consequence of our system as a whole, the principal forces underlying this consequence are (39) and the Barcan Formula working together. Both can be justified independently.

It is important to indicate why it is that we cannot follow Frege's proof that every number has a successor. Frege's strategy was to prove by induction that every number n immediately precedes the number of members in the *Predecessor* series ending with n, i.e.,[58]

(a) $\forall n\, Precedes(n, \#_{[\lambda x\; Precedes^{+}(x,n)]})$

However, given our definition of $\#_F$, (a) fails to be true; indeed, no number n is the number of members of the predecessor series ending with n. For $\#_{[\lambda x\; Precedes^{+}(x,n)]}$ is defined to be the number of *ordinary* objects that are members of the predecessor series ending with n. That natural

(p. 18/199) is an example that may be sufficient:

> In logic, we ban the empty domain as a concession to technical convenience but draw the line there: We firmly believe that the existence of even two objects, let alone infinitely many, cannot be guaranteed by logic alone. ... Since there might be fewer than two items that we happen to be talking about, we cannot take even $\exists x \exists y(x \neq y)$ to be valid.

It seems clear from the antecedent of his last sentence, that Boolos takes claims of the form 'there might have been fewer than n objects' to be true *a priori*. It seems clear that he would equally accept the claim 'there might have been more than n objects' to be true *a priori*. No doubt one could find other logicians who are even more explicit about this point. Our modal axiom is simply one way of formalizing this assumption. Since our system includes proper axioms of metaphysics, it is not logic alone that is guaranteeing the infinity of possibly concrete objects.

[57]Indeed, there is a natural cardinal that numbers the ordinary objects, namely, $\#[\lambda z\, z =_E z]$. It is easy to see that that this natural cardinal is not a natural number. For suppose it is a natural number. Then by our modal axiom (39), it is possible for there to be a concrete object distinct from all the objects actually exemplifying $[\lambda z\, z =_E z]$. But this is provably not possible, for such an object would be an ordinary object distinct$_E$ from itself.

[58]cf. *Grundgesetze I*, Theorem 155.

cardinal number will always be zero, for any n. So (a) is false in our system because the natural numbers and natural cardinals only count the ordinary (i.e., possibly concrete) objects that fall under a concept.

A discussion of the significance of this last fact must begin with the observation that the devices of quantification and identity are still available for the development of 'natural arithmetic'. We can still formulate and prove such claims as 'There are three (natural) numbers in the predecessor series ending with 2', i.e.,

$$\exists x \exists y \exists z [Precedes^{+}(x,2)\ \&\ Precedes^{+}(y,2)\ \&\ Precedes^{+}(z,2)\ \&\ x \neq y\ \&\ y \neq z\ \&\ x \neq z\ \&\ \forall w(Precedes^{+}(w,2) \rightarrow w = x \vee w = y \vee w = z)]$$

We just cannot infer from such claims identity statements of the form:

$$\#_{[\lambda x\; Precedes^{+}(x,2)]} = 3$$

At least, we cannot infer such claims within the present application of object theory (however, see the discussion below of the two-stage philosophy of mathematics described in Linsky and Zalta [1995]).

I think it is a mistake to judge this limitation on our reconstruction of the natural numbers without having a wide perspective on the problems involved in formulating a foundational metaphysical theory of abstract objects, in constructing a theory of natural numbers without using mathematical primitives, and in reconstructing a general theory of Fregean logical objects. The theory should be judged not on the basis of a single issue, but on its overall success in dealing with a myriad of philosophical issues, many of which are tangled and thorny. Although full discussion of these issues would occupy far more space than available in this concluding section, the following series of observations may prove useful. To focus our attention, let us just compare the present theory of natural numbers with 'Frege Arithmetic' ('FA'), as described in Boolos [1987] and [1986].[59]

[59]For the purposes of the following discussion, I will put aside the suggestion of adding Hume's Principle to second order logic. Hume's Principle obviously employs primitive mathematical objects ('the number of Fs'). Moreover, it collapses comprehension and identity principles into a single principle; modern logicians typically now separate these two kinds of principles. Hume's principle also obscures the fact that some non-logical existence assertions concerning objects have to be added to logic in order to prove the existence of the natural numbers. Finally, Hume's Principle is subject to the Julius Caesar problem, namely, it doesn't establish conditions under which $\#_F = x$, for an arbitrary object x. This makes it difficult to *apply* the system of

In [1987] (p. 5/186), Boolos suggests that with (a simplified version of) second-order logic as background, the sole non-logical axiom of FA is the following principle:

Numbers: $\forall G \exists ! x \forall F(F \eta x \equiv F \approx G)$

In this principle, equinumerosity among properties ($\approx$) is not restricted in any way. Moreover, Boolos allows formulas of the form '$F\eta x$' to appear in comprehension conditions for complex properties.

Although Boolos' formulation of FA has elements in common with the present theory,[60] its most striking feature is that it requires no other special axioms (such as our (39) and (42)) for the derivation of the Peano/Dedekind axioms. Moreover, the numbers postulated by FA can be used to count such number-theoretic properties as being a natural number less than or equal to 2. Finally, it seems that FA requires no mathematical primitives; it requires only the syntactic resources of second-order logic, the η relation, and the definable notion of one-to-one correspondence. So, on first appearance, it would seem that this approach has clear virtues that make it preferable as a theory of numbers and as a reconstruction of Frege's views about them.

A deeper look at the matter, however, seems to show otherwise. It seems plausible to suggest that the proper philosophical formulation of FA requires a mathematical primitive.[61] Strictly speaking, the *label* 'Numbers' for the non-logical axiom of FA needs to be introduced as a predicate and made part of the statement of the axiom. To apply this theory (i.e., use it to count ordinary objects and properties, by adding names for ordinary objects and ordinary predicates), Frege Arithmetic has to be properly reformulated as follows:

$$\forall G \exists ! x(Number(x) \,\&\, \forall F(F \eta x \equiv F \approx G)),$$

second-order logic with Hume's Principle (i.e., add to the formalism names of ordinary objects).

[60]It is not too difficult to show that '$F\eta x$' and 'xF' are notational variants; one simply has to compare the paradoxes of $F\eta x$, described in Boolos [1987] (p. 17/198-199), with the paradoxes of 'xF', described in Zalta [1983] (pp. 158-159). The two paradoxes discussed in these works can be traced back further. One can be traced to Clark [1978] (p. 184) and Rapaport [1978]; the other to McMichael and Zalta [1980] (footnote 15).

[61]I am putting aside the question of whether 'second-order logic' is logic. I'll assume that the language of second-order logic involves only logical notions.

just as ZF has to be reformulated with the predicate '$Set(x)$' when urelements are added. Otherwise, there is no way to distinguish the special objects asserted to exist from ordinary objects. Indeed, it is natural to supplement this revised formulation with the following identity conditions for numbers:

$$Number(x) \,\&\, Number(y) \rightarrow (x = y \equiv \forall F(F \eta x \equiv F \eta y))$$

These identity conditions properly individuate the numbers axiomatized by FA and they are stated in terms of the distinctive feature of such numbers, namely, that properties bear the η relation to such entities. It is now possible to formulate completely general identity conditions on objects as follows: x and y are identical iff either (1) x and y are both numbers and $\forall F(F\eta x \equiv F\eta y)$ or (2) x are y are both ordinary objects (i.e., not numbers) and $\forall F(Fx \equiv Fy)$. Given the fact that Julius Caesar is an ordinary object, it then follows that he is not identical with the number of planets.

A proper, philosophical formulation of FA, then, seems to require a primitive mathematical notion, namely, the non-logical predicate '*Number*(x)'. From this apparent fact, and the fact that the non-logical axiom of FA limits comprehension in terms of η to *equinumerosity* conditions, it seems reasonable to think that FA axiomatizes a primitive domain of mathematical objects. It was no accident that Boolos used the label 'Numbers' for the non-logical axiom of FA, for this axioms allows one to define '$\#_F$' and to derive Hume's Principle.[62]

By contrast, the present theory does not axiomatize a primitive kind of mathematical object. The non-logical predicate '$A!x$' in the present reconstruction is a metaphysical rather than mathematical notion and object comprehension involving encoding predication is not limited to equinumerosity conditions. *Any* formula without free xs is allowed in the comprehension schema for abstract objects, and restrictions on property comprehension are required so as to avoid paradox. Consequently, if we have an interest in developing a theory of natural numbers that presupposes no mathematical primitives and which is formulated within the context of a general metaphysical theory of abstract objects, it is not clear

[62]I think it would be a mistake, for example, to suggest replacing the non-logical predicate '*Number*(x)' by '*LogicalObject*(x)'. The resulting theory would be a rather weak theory of logical objects, and it would hardly correspond to Frege's conception of 'logical object'.

that Boolos's formulation of FA is preferable. The costs of developing a theory of natural numbers without appeal to mathematical primitives are: (1) the addition of two metaphysical axioms (i.e., (39) and (42)), and (2) the consequences of the fact that unrestricted equinumerosity ($\approx$), as opposed to equinumerosity with respect to the ordinary objects ($\approx_E$), is not an equivalence condition on properties.

From the present perspective, this price may simply reflect one of the ways in which Frege may have overextended his conception of numbers. When Frege developed his insight that 'a statement of number is an assertion about a concept' (*Grundlagen*, §46), all of his examples were of ordinary concepts, such as 'moon of Venus', 'horse that draws the King's carriage', 'inhabitant of Germany', etc. These are concepts whose instances are concrete objects. Frege *assumed* that it was unproblematic to extend this insight to cover such concepts as 'natural number less than or equal to 2', 'prime number between π and 6', etc. That is, Frege assumed that the natural numbers could count everything whatsoever, including any domain of logical objects and/or abstract objects that might be the subject of an *a priori* investigation. It is precisely this assumption that the present reconstruction questions. One cannot automatically assume that an insight that unifies our conception of the natural world *extends* to the domain of logical and abstract objects without having a prior theory of what abstract objects and properties there are.

Further, in contrast to FA, the present theory can offer an account of many of the other kinds of logical objects that interested Frege. To give just one example, we are now in a position to define 'the truth value of proposition p' as that abstract object that encodes all and only the properties F of the form $[\lambda y\, q]$ which are constructed out of propositions q materially equivalent to p. An abstract object will be a 'truth-value' just in case it is the truth-value of some proposition p and one can derive as a theorem: the truth-value of p is identical to the truth-value of q iff p is materially equivalent to q.[63] The abstract objects The True and the The False can be precisely identified and it can be shown not only that these are both truth-values, but that there are exactly two truth-values. This is only a sketch of one application; others are available (e.g., natural sets, directions, shapes, etc.). Though a full treatment has to be reserved for another occasion, any comparison of FA with the present approach

[63]Compare Boolos [1986] (p. 148/180), who identifies the claim $Vp = Vq \equiv p \equiv q$ *as an axiom*.

must consider the other kinds of logical objects that might be definable in terms of our comprehension schema.

One also has to consider whether FA can be justified epistemologically as easily as the present theory. The work in Linsky and Zalta [1995] suggests that the comprehension principle for abstract objects can be epistemologically justified. The particular strategy used there, of showing that the comprehension principle is required for our understanding of any possible scientific theory, can *not* be applied to the non-logical axiom of FA. Suppose this is right, and further suppose that the present theory offers not only an answer to Frege's epistemological question 'In what way are we to conceive logical objects, in particular, numbers?', but also has a ready solution to 'the Julius Caesar problem' (which is not unlike the one suggested several paragraphs back, when we 'reformulated' FA and added an identity principle for numbers). Finally, assume that the fact, that our natural numbers can't count the abstracta in the exemplification extension of properties, provides a partial key to the naturalization of these particular abstract objects (i.e., makes it easier to reconcile their existence with our naturalized conception of the world—it may be that such objects are somehow dependent or supervenient on natural patterns of properties). If the present theory fares better in terms of these epistemological considerations, then the significance of the virtues of FA may start to fade in comparison.

One last observation about the relative merits of FA and the present theory derives from the fact that the present theory offers a two-stage approach to the philosophy of mathematics. The second-stage analysis of *theoretical* mathematics, as opposed to the first-stage analysis of *natural* mathematics (the focus in the present paper), may recapture the idea that numbers can apparently be used to count the objects falling under distinctively mathematical properties. The difference between natural and theoretical mathematics is simply this: natural mathematics is the mathematics derivable from our comprehension principle alone without any mathematical primitives; theoretical mathematics is the mathematics formulated in terms of distinctive mathematical notions. The work in Linsky and Zalta [1995] and in Zalta (2000) establishes that the present metaphysical system offers a way of *interpreting* the language of arbitrary mathematical theories T, once analytic truths asserting *that such and such theorems are true in* T are added and analyzed in terms of encoding predications. That work has an important consequence for the present essay once we consider those mathematical *theories* which postulate numbers of various kinds and in which it can be proved that numbers of one kind can count numbers of the same or other kinds. For the cited papers then show us the way to interpret such claims as "In Peano Number Theory, the number of numbers between 1 and 4 is identical to the number 2", and "In the theory of positive and negative integers, the number of roots to the equation $x^2 - 4 = 0$ is identical to the number 2". Given such an interpretation, the fact that our *natural* numbers can't count other numbers may not be that significant.

Let me conclude the present essay with a final observation on a different topic. In [1984] and [1990], Hodes argues that numbers are "fictions created to encode cardinality quantifiers, thereby clothing a certain higher-order logic in the attractive garments of lower-order logic." ([1990], p. 350). Our work in Section 5 validates this idea. To see how, consider the following inductive definition of the *exact* numerical quantifiers 'there are exactly n ordinary F-things' ('$\exists!_n uFu$'):

$$\exists!_0 uFu \ =_{df} \ \neg\exists uFu$$

$$\exists!_n uFu \ =_{df} \ \exists u(Fu \,\&\, \exists!_n v[\lambda z\, Fz \,\&\, z \neq_E u]v)$$

Note that from the point of view of higher-order logic, the condition $\exists!_n uFu$ defines a property of properties; it defines a different property of properties for each cardinal number. Our natural numbers, in effect, encode the first-order properties satisfying these higher-order properties and they do so in just the way Hodes claims. This is revealed by the following metatheorem:

48) Metatheorem: Numbers 'Encode' Numerical Quantifiers. For each numeral n, it is provable that n is the abstract object that encodes just the properties F such that there are exactly n ordinary objects which exemplify F, i.e.,

$$\vdash n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n uFu))$$

(A sketch of the proof may be found in the Appendix.) The fact that Hodes thinks of numbers as 'fictions' does not necessarily imply that our definition of the numbers does not capture his view, for a proper analysis of fictions might identify them as abstracta. Moreover, the work in Linsky and Zalta [1995] meets his challenge to the platonist to provide an explanation of the 'microstructure of reference'.[64]

[64] In [1984], Hodes asks (p. 126):

Appendix: Proofs of Selected Theorems

In this Appendix, we prove some even quite obvious theorems. This demonstrates that the theorems can be derived in the system as it has been developed (i.e., that nothing has been overlooked). The reader may, of course, skip such proofs. In what follows, we use the variables x, y, z as variables for any kind of object and the variables u, v, w as *restricted* variables ranging over ordinary objects. We use $a, b, \ldots, l$ as constants for any kind of object. We use F, G, H as variables for properties, and P, Q as constants for properties. R is used both as a constant and a variable for two-place relations.

(1.1): Pick an arbitrary relation R. Consider the following instance of abstraction:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(F = [\lambda z\, Rzy] \,\&\, \neg yF)))$$

Call such an object k. So we know the following about k:

$$\forall F(kF \equiv \exists y(F = [\lambda z Rzy] \,\&\, \neg yF))$$

Now consider the property $[\lambda z\, Rzk]$ and ask the question whether k encodes this property. Assume, for reductio, $\neg k[\lambda z Rzk]$. Then, by definition of k, for any object y, if the property $[\lambda z\, Rzk]$ is identical with the property $[\lambda z\, Rzy]$, then y encodes $[\lambda z\, Rzk]$. Instantiate this universal claim to k. Since the property $[\lambda z\, Rzk]$ is self-identical, it follows that k encodes $[\lambda z\, Rzk]$, contrary to assumption. So $k[\lambda z\, Rzk]$. So by the definition of k, there is an object, say l, such that the property $[\lambda z\, Rzk]$ is identical to the property $[\lambda z\, Rzl]$ and such that l doesn't encode $[\lambda z\, Rzk]$. But since k encodes, and l does not encode, $[\lambda z\, Rzk]$, $k \neq l$. So there are objects x and y such that $x \neq y$, yet such that $[\lambda z\, Rzx] = [\lambda z\, Rzy]$. ⋈

(1.3): Pick an arbitrary property P. Consider the following instance of abstraction:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(F = [\lambda z\, Py] \,\&\, \neg yF)))$$

> The challenge to the mathematical-object theorist [Fregean] is: Tell us about the microstructure of reference to, e.g., cardinal numbers. In what does our ability to refer to such objects consist? What are the facts about our linguistic practice by virtue of which expressions in our language designate such objects and the concepts under which they fall or fail to fall?

This challenge is met once we note that the natural cardinals and natural numbers described here are subject to the epistemology described in Linsky and Zalta [1995].

By reasoning analogous to the above, it is straightforward to establish that there are distinct abstract objects k, l such that $[\lambda z\, Pk]$ is identical to $[\lambda z\, Pl]$. But, then by the definition of proposition identity ($p\!=\!q \;\; =_{df} \; [\lambda z\; p] = [\lambda z\; q]$), it follows that the proposition $[\lambda\, Pk]$ is identical to the proposition $[\lambda\, Pl]$. $\bowtie$

(4): Suppose $O!x$, $O!y$, and $\forall F(Fx \equiv Fy)$. To show $x =_E y$, we simply have to show that $\Box\forall F(Fx \equiv Fy)$. But, for reductio, suppose not, i.e., suppose $\Diamond\neg\forall F(Fx \equiv Fy)$. Without loss of generality, suppose $\Diamond\exists F(Fx \,\&\, \neg Fy)$. Then, by the Barcan formula, $\exists F\Diamond(Fx \,\&\, \neg Fy)$. Say P, for example, is our property such that $\Diamond(Px \,\&\, \neg Py)$. Now, consider the property: $[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]$. We know by λ-Conversion that:

$$[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]x \equiv \Diamond(Px \,\&\, \neg Py)]$$

But we know the right hand side of this biconditional, and so it follows that: $[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]x$. But it is also a consequence of λ-Conversion that:

$$[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]y \equiv \Diamond(Py \,\&\, \neg Py)]$$

But clearly, by propositional modal logic, $\neg\Diamond(Py \,\&\, \neg Py)$. So we may conclude: $\neg[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]y$. So we have established:

$$[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]x \,\&\, \neg[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]y$$

So, by EG, $\exists F(Fx \,\&\, \neg Fy)$, which contradicts our hypothesis $\forall F(Fx \equiv Fy)$. $\bowtie$

(7.1): Pick an arbitrary property P. To show that equinumerosity$_E$ is reflexive, we must find a relation that is a one-to-one function from the ordinary objects of P onto the ordinary objects of P. However, we need look no further than the relation $=_E$. We have to show: (a) that $=_E$ is a function from the ordinary objects of P to the ordinary objects of P, and (b) that $=_E$ is a one-to-one function from the ordinary objects of P onto the ordinary objects of P. To show (a), pick an arbitrary ordinary object, say b, such that Pb. We need to show that there is an ordinary object v which is such that $\forall w(Pw \,\&\, b\!=_E\!w \equiv w\!=_E\!v)$. But b is such a v, for pick an arbitrary ordinary object, say c. ($\rightarrow$) If $Pc \,\&\, b\!=_E\!c$, then $c\!=_E\!b$. ($\leftarrow$) If $c\!=_E\!b$, then since Pb by assumption, we know Pc. So $Pc \,\&\, b\!=_E\!c$. Therefore, since c was arbitrary, we know $\forall w(Pw \,\&\, b =_E w \equiv w =_E b)$, and so there is an object v such that $\forall w(Pw \,\&\, b\!=_E\!w \equiv w\!=_E\!v)$. This

demonstrates (a). To demonstrate (b), we need only show that $=_E$ is a one-to-one function from the ordinary objects of P to the ordinary objects of P, for by previous reasoning, we know that $=_E$ is a function from the ordinary objects of P onto the ordinary objects of P (i.e., we already know that every ordinary object exemplifying P bears $=_E$ to an ordinary object exemplifying P). For reductio, suppose that $=_E$ is not one-to-one, i.e., that there are distinct ordinary objects P which bear $=_E$ to some third P-object. But this is impossible, given that $=_E$ is a classical equivalence relation. $\bowtie$

(7.2): To show that that equinumerosity$_E$ is symmetric, assume that $P \approx_E Q$ and call the relation that is witness to this fact R. We want to show that there is a relation R' from Q to P such that (a) $\forall u(Qu \rightarrow \exists!v(Pv \,\&\, R'uv))$, and (b) $\forall u(Pu \rightarrow \exists!v(Qv \,\&\, R'vu))$. Consider the converse of R: $[\lambda xy\, Ryx]$, which we may call R^{-1}. We need to show that (a) and (b) hold for R^{-1}. To show (a) holds for R^{-1}, pick an arbitrary ordinary object, say b, such that Qb. We want to show that there is a unique ordinary object exemplifying P to which b bears R^{-1}. By the definition of R and the fact that Qb, we know that there is a unique ordinary object that exemplifies P and bears R to b. But such an object bears R to b iff b bears R^{-1} to it. So there is a unique ordinary object exemplifying P to which b bears R^{-1}. To prove that (b) holds for R^{-1}, the reasoning is analogous: consider an arbitrary object, say a, that exemplifies P. We want to show that there is a unique object exemplifying Q that bears R^{-1} to a. But by the definition of R and the fact that a exemplifies P, we know that there is a unique object that exemplifies Q and to which a bears R. But then, by the definition of R^{-1}, there is a unique object exemplifying Q that bears R^{-1} to a. $\bowtie$

(7.3): To show that equinumerosity$_E$ is transitive, assume both that $P \approx_E Q$ and $Q \approx_E S$. Call the relations that bear witness to these facts R_1 and R_2, respectively. Consider the relation: $[\lambda xy \exists z(Qz \& R_1xz \& R_2zy)]$. Call this relation R. To show that R bears witness to the equinumerosity$_E$ of P and S, we must show: (a) R is a function from the ordinary objects of P to the ordinary objects of S, and (b) R is a one-to-one function from the ordinary objects of P onto the ordinary objects of S. To show (a), consider an arbitrary ordinary object, say a, such that Pa. We want to find a unique ordinary object exemplifying S to which a bears R. To find such an object, note that given the equinumerosity$_E$ of P and Q, it is a fact about R_1 that there is a unique ordinary object exemplifying Q, say b, to which a bears R_1. And from the equinumerosity$_E$ of Q and S, it is a fact about R_2 that there is a unique ordinary object exemplifying S, say c, to which b bears R_2. So if we can show that c is a unique ordinary object exemplifying S to which a bears R, we are done. Well, by definition, c exemplifies S. By the definition of R, we can establish Rac if we can show $\exists z(Qz \,\&\, R_1az \,\&\, R_2zc)$. But since b is such a z, it follows that Rac. So it remains to prove that any object exemplifying S to which a bears R just is (identical$_E$ to) c. So pick an arbitrary ordinary object, say d, such that both d exemplifies S and Rad. We argue that $d =_E c$ as follows. Since Rad, we know by the definition of R that there is an object, say e, such that $Qe \,\&\, R_1ae \,\&\, R_2ed$. But recall that a bears R to a unique object exemplifying Q, namely b. So $b =_E e$. But since R_2ed, it then follows that R_2bd. So we know $Sd \,\&\, R_2bd$. But recall that b bears R to a unique object exemplifying S, namely c. So $c =_E d$.

To show (b), pick an arbitrary ordinary object b such that b exemplifies S. We want to show that there is a unique ordinary object exemplifying P that bears R to b. Since R_2 is, by hypothesis, a one-to-one function from the ordinary objects of Q onto the ordinary objects of S, there is a unique object, say c, such that c exemplifies Q and R_2cb. And since R_1 is, by hypothesis, a one-to-one function from the ordinary objects of P onto the ordinary objects of Q, there is a unique object, say d, such that d exemplifies P and R_1dc. We now establish that d is a unique object exemplifying P that bears R to b. Clearly, d is an object that exemplifies P. Moreover, d bears R to b, for there is an object, namely c, that exemplifies Q and is such that both R_1dc and R_2cb. To show that d is unique, suppose, for reductio, that there is an object e, $e \neq_E d$, such that Pe and Reb. Then by the definition of R, there is an object, say f such that Qf and R_1ef and R_2fb. Since $e \neq_E d$, we know by the functionality of R_1, that $f \neq_E c$. But we now have that Qc, R_2cb, Qf, R_2fb, and $f \neq_E c$, and this contradicts the fact that c is the unique object exemplifying Q that bears R_2 to b. ⋈

(12.1): This is immediate from the definition of $\#_G$ and the definition of *Numbers*(x, G). ⋈

(12.2): This follows from (12.1) and the fact that equinumerosity$_E$ is reflexive. ⋈

(13): ($\rightarrow$) Assume that the number of Ps is identical to the number of Qs.

Then, by the definition of identity for abstract objects, we know that $\#_P$ and $\#_Q$ encode the same properties. By (12.2), we know that $\#_P$ encodes P. So $\#_Q$ encodes P. But, by (12.1), it follows that $P \approx_E Q$. ($\leftarrow$) Assume $P \approx_E Q$. We want to show that $\#_P = \#_Q$, i.e., that they encode the same properties. ($\rightarrow$) Assume $\#_P$ encodes S (to show: $\#_Q$ encodes S). Then by (12.1), $S \approx_E P$. So by the transitivity of equinumerosity$_E$, $S \approx_E Q$. But, then, by (12.1), it follows that $\#_Q$ encodes S. ($\leftarrow$) Assume $\#_Q$ encodes S (to show: $\#_P$ encodes S). Then by (12.1), $S \approx_E Q$. By the symmetry of equinumerosity$_E$, it follows that $Q \approx_E S$. So, given our hypothesis that $P \approx_E Q$, it follows by the transitivity of equinumerosity$_E$ that $P \approx_E S$. Again, by symmetry, we have: $S \approx_E P$. And, thus, by (12.1), it follows that $\#_P$ encodes S. ⋈

(15): Assume k is a natural cardinal. Then, by definition, there is a property, say P such that $k = \#_P$. ($\rightarrow$) Assume k encodes Q. Then $\#_P$ encodes Q. So by (12.1), it follows that $Q \approx_E P$. And by Hume's Principle, it follows that $\#_Q = \#_P$. So, $k = \#_Q$. ($\leftarrow$) Assume $k = \#_Q$. By (12.2), we know that $\#_Q$ encodes Q. So, k encodes Q. ⋈

(18): ($\rightarrow$) Assume 0 encodes P. Then P is equinumerous$_E$ to $[\lambda z\, z \neq_E z]$, by (12.1). So there is an R that is a one-to-one function from the ordinary objects of P onto the ordinary objects of $[\lambda z z \neq_E z]$. So, for every ordinary object x such that Px, there is an (unique) ordinary object y such that $[\lambda z\, z \neq_E z]y$ and Rxy. Suppose, for reductio, that $\exists u Pu$, say Pa. Then there is an ordinary object, say b, such that Rab and $[\lambda z\, z \neq_E z]b$. But this contradicts the fact that no ordinary object exemplifies this property. ($\leftarrow$) Suppose $\neg\exists u Pu$. It is also a fact about $[\lambda z\, z \neq_E z]$ that no ordinary object exemplifies it. But then P is equinumerous$_E$ with $[\lambda z\, z \neq_E z]$, for any relation R you pick bears witness to this fact: (a) every ordinary object exemplifying P bears R to a unique ordinary object exemplifying $[\lambda z\, z \neq_E z]$ (since there are no ordinary objects exemplifying P), and (b) every ordinary object exemplifying $[\lambda z\, z \neq_E z]$ is such that there is a unique ordinary object exemplifying P that bears R to it (since there are no ordinary objects exemplifying $[\lambda z\, z \neq_E z]$). Since $P \approx_E [\lambda z\, z \neq_E z]$, it follows by (12.1), that $\#_{[\lambda z\, z \neq_E z]}$ encodes P. So 0 encodes P. ⋈

(19): By (17), 0 is a natural cardinal, and so by (15), $0P$ iff $0 = \#_P$. But by (18), $0P$ iff $\neg\exists u Pu$. So $\neg\exists u Pu$ iff $0 = \#_P$. ⋈

(23): Suppose, for reductio, that something, say a, is a precedessor of 0. Then, by the definition of predecessor, it follows that there is an property, say Q, and an ordinary object, say b, such that Qb, $0 = \#_Q$, and $a = \#_{[\lambda z\, Qz \,\&\, z \neq_E b]}$. But if $0 = \#_Q$, then by (19), $\neg\exists u Qu$, which contradicts the fact that Qb. $\bowtie$

(24): Assume that $P \approx_E Q$, Pa, and Qb. So there is a relation, say R, that is a one-to-one function from the ordinary objects of P onto the ordinary objects of Q. Now we use P^{-a} to designate $[\lambda z\, Pz \,\&\, z \neq_E a]$, and we use Q^{-b} to designate $[\lambda z\, Qz \,\&\, z \neq_E b]$. We want to show that $P^{-a} \approx_E Q^{-b}$. By the definition of equinumerosity$_E$, we have to show that there is a relation R' which is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}. We prove this by cases.

Case 1: Suppose Rab. Then we choose R' to be R itself. Clearly, then, R' is a one-to-one function from the ordinary objects of P^{-a} to the ordinary objects of Q^{-b}.

Case 2: Suppose $\neg Rab$. Then we choose R' to be the relation:

$$[\lambda xy\ (x \neq_E a \,\&\, y \neq_E b \,\&\, Rxy) \vee (x =_E \imath u(Pu \,\&\, Rub) \,\&\, y =_E \imath u(Qu \,\&\, Rau))]$$

To see that there is such a relation, note that the following is an instance of the comprehension principle for Relations, where u, w are *free* variables:

$$\exists F \forall x \forall y (Fxy \equiv (x \neq_E a \,\&\, y \neq_E b \,\&\, Rxy)\ \vee\ (x =_E u \,\&\, y =_E w))$$

By two applications of the Rule of Generalization, we know:

$$\forall u \forall w \exists F \forall x \forall y (Fxy \equiv (x \neq_E a \,\&\, y \neq_E b \,\&\, Rxy)\ \vee\ (x =_E u \,\&\, y =_E w))$$

Now by the assumptions of the lemma, we know that the descriptions $\imath u(Pu \,\&\, Rub)$ and $\imath u(Qu \,\&\, Rau)$ are well-defined and have denotations (if R is a one-to-one function from the ordinary objects of P onto the ordinary objects of Q, and Pa and Qb, then there is a unique ordinary object that exemplifies P that bears R to b and there is a unique ordinary object that exemplifies Q to which a bears R). So we may instantiate these descriptions for universally quantified variables u and w, respectively, to establish that our relation R' exists.

We now leave it as a straightforward exercise to show: (A) that R' is a function from the ordinary objects of P^{-a} to the ordinary objects of Q^{-b}, and (B) that R' is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}. $\bowtie$

(25): Assume that both a and b are precedessors of c. By the definition of predecessor, we know that there are properties and ordinary objects P, Q, d, e such that:

$$Pd \,\&\, c = \#_P \,\&\, a = \#_{P^{-d}}$$

$$Qe \,\&\, c = \#_Q \,\&\, b = \#_{Q^{-e}}$$

But if both $c = \#_P$ and $c = \#_Q$, then $\#_P = \#_Q$. So, by Hume's Principle, $P \approx_E Q$. And by (24), it follows that $P^{-d} \approx_E Q^{-e}$. If so, then by Hume's Principle, $\#_{P^{-d}} = \#_{Q^{-e}}$. But then, $a = b$. $\bowtie$

(26): Assume that $P^{-a} \approx_E Q^{-b}$, Pa, and Qb. So there is a relation, say R, that is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}. We want to show that there is a function R' which is a one-to-one function from the ordinary objects of P onto the ordinary objects of Q. Let us choose R' to be the following relation:

$$[\lambda xy\ (P^{-a}x \,\&\, Q^{-b}y \,\&\, Rxy)\ \vee\ (x =_E a \,\&\, y =_E b)]$$

We know such a relation exists, by the comprehension principle for relations. We leave it as a straightforward exercise to show: (A) that R' is a function from the ordinary objects of P to the ordinary objects of Q, and (B) that R' is a one-to-one function from the ordinary objects of P onto the ordinary objects of Q. $\bowtie$

(27): Assume that a is a precedessor of both b and c. By the definition of predecessor, we know that there are properties and ordinary objects P, Q, d, e such that:

$$Pd \,\&\, b = \#_P \,\&\, a = \#_{P^{-d}}$$

$$Qe \,\&\, c = \#_Q \,\&\, a = \#_{Q^{-e}}$$

But if both $a = \#_{P^{-d}}$ and $a = \#_{Q^{-e}}$, then $\#_{P^{-d}} = \#_{Q^{-e}}$. So, by Hume's Principle, $P^{-d} \approx_E Q^{-e}$. And by (26), it follows that $P \approx_E Q$. Now, by Hume's Principle, $\#_P = \#_Q$. But then, $b = c$. $\bowtie$

(30.1): Assume Rab. Pick an arbitrary property, say P and assume $\forall z(Raz \rightarrow Pz)$ and *Hereditary*(P, R). Then Pb, by the first two of our three assumptions. $\bowtie$

(30.2): This follows immediately from the definition of R^*. $\bowtie$

(30.3): Assume Pa, $R^*(a, b)$, and that *Hereditary*(P, R). Then by the lemma we just proved (30.2), all we have to do to show that b exemplifies P is show that P is exemplified by every object to which R relates a. So suppose R relates a to some arbitrarily chosen object c (to show Pc). Then by the fact that P is hereditary with respect to R and our assumption that Pa, it follows that Pc. ⋈

(30.4): Assume Rab and $R^*(b,c)$. To prove $R^*(a,c)$, further assume $\forall z(Raz \rightarrow Pz)$ and $Hereditary(P,R)$ (to show Pc). So Pb. But from Pb, $R^*(b,c)$, and $Hereditary(P,R)$, it follows that Pc, by (30.3). ⋈

(30.5): Assume $R^*(a,b)$ to show $\exists zRzb$. If we instantiate the variables x,y in (30.2) to the relevant objects, and instantiate the variable F to $[\lambda w\, \exists zRzw]$, the we know the following fact (after λ-conversion):

$$[R^*(a,b) \& \forall x(Rax \rightarrow \exists zRzx) \& \forall x,y(Rxy \rightarrow (\exists zRzx \rightarrow \exists zRzy))] \rightarrow \exists zRzb$$

So we simply have to prove the second and third conjuncts of the antecedent. But these are immediate. For an arbitrarily chosen object c, $Rac \rightarrow \exists zRzc$. So $\forall x(Rax \rightarrow \exists zRzx)$. Similarly, for arbitrarily chosen c,d, the assumptions that Rcd and $\exists zRzc$ immediately imply $\exists zRzd$. So $\forall x,y(Rxy \rightarrow (\exists zRzx \rightarrow \exists zRzy))$. ⋈

(32.1): This is immediate from (30.1). ⋈

(32.2): Assume Pa, $R^+(a,b)$, and $Hereditary(P,R)$. Then by the definition of weak ancestral, either $R^*(a,b)$ or $a = b$. If the former, then Pb, by (30.3). If the latter, then Pb, from the assumption that Pa. ⋈

(32.3): Assume $R^+(a,b)$ and Rbc. Then either (I) $R^*(a,b)$ and Rbc or (II) $a = b$ and Rbc. We want to show, in both cases, $R^*(a,c)$:

Case I: $R^*(a,b)$ and Rbc. Pick an arbitrary property P. To show R^*ac, we assume that $\forall z(Raz \rightarrow Pz)$ and $Hereditary(P,R)$. We now try to show: Pc. But from the fact that $R^*(a,b)$, it then follows that Pb, by the definition of R^*. But from the facts that $Hereditary(P,R)$, Rbc, and Pb, it follows that Pc.

Case II: $a = b$ and Rbc. Then Rac, and so by (30.1), it follows that $R^*(a,c)$.

⋈

(32.4): Assume $R^*(a,b)$ and Rbc (to show $R^+(a,c)$). Then by the first assumption and the definition of R^+, it follows that $R^+(a,b)$. So by (32.3), it follows that $R^*(a,c)$. So $R^+(a,c)$, by the definition of R^+. ⋈

(32.5): Assume Rab and $R^+(b,c)$ (to show: $R^*(a,c)$). By definition of the weak ancestral, either $R^*(b,c)$ or $b = c$. If $R^*(b,c)$, then $R^*(a,c)$, by (30.4). If $b = c$, then Rac, in which case, $R^*(a,c)$, by (30.1). ⋈

(32.6): Assume $R^*(a,b)$ (to show: $\exists z(R^+(a,z) \& Rzb)$). The following is an instance of (30.2):

$$R^*(a,b) \& \forall x(Rax \rightarrow Fx) \& Hereditary(F,R) \rightarrow Fb$$

Now let F be the property $[\lambda w\, \exists z(R^+(a,z) \& Rzw)]$. So, by expanding definitions and using λ-conversion, we know:

$$R^*(a,b) \& \forall x(Rax \rightarrow \exists z(R^+(a,z) \& Rzx)) \& \forall x,y[Rxy \rightarrow (\exists z(R^+(a,z) \& Rzx) \rightarrow \exists z(R^+(a,z) \& Rzy))] \rightarrow \exists z(R^+(a,z) \& Rzb)$$

Since the consequent is what we have to show, we need only establish the three conjuncts of the antecedent. The first is true by assumption. For the second, assume Rac, where c is an arbitrarily chosen object (to show: $\exists z(R^+(a,z) \& Rzc)$). But, by definition of R^+, we know that $R^+(a,a)$. So, from $R^+(a,a) \& Rac$, it follows that $\exists z(R^+(a,z) \& Rzc)$. For the third conjunct, assume Rcd and $\exists z(R^+(a,z) \& Rzc)$ (to show: $\exists z(R^+(a,z) \& Rzd)$). Since we know Rcd, we simply have to show $R^+(a,c)$ and we're done. But we know that for some object, say e, $R^+(a,e) \& Rec$. So by (32.3), it follows that $R^*(a,c)$. But, then $R^+(a,c)$, by definition of R^+. ⋈

(34): Let n be a natural number. Then, by definition, $Precedes^+(0,n)$. By definition of R^+, it follows that either $Precedes^*(0,n)$ or $0 = n$. (I) If the former, then by (30.5), there is an object, say a, such that $Precedes(a,n)$. So by the definition of *Predecessor* it follows that there is a property, say P, and an ordinary object, say b, such that:

$$Pb \& n = \#_P \& a = \#_{P^{-b}},$$

(where P^{-b} is defined as in the proof of (24)). Since n is the number of some property, n is a natural cardinal number. (II) If the latter, then since 0 is a natural cardinal, by (17), it follows that n is a natural cardinal. ⋈

(36.1): By (30.5), we know that if x ancestrally precedes y, then there is something that precedes y. But, by (23), we know that nothing precedes zero. So nothing ancestrally precedes zero, and in particular, zero doesn't ancestrally precede itself. ⋈

(36.2): By (23), nothing precedes zero. So no natural number precedes zero. $\bowtie$

(37): By (25), *Predecessor* is one-one. *A fortiori*, it is a one-one when restricted to the members of the *Predecessor* series beginning with 0. $\bowtie$

(38): Assume $Precedes(n, a)$. Since n is a number, $Precedes^+(0, n)$. So by (32.3), it follows that $Precedes^*(0, a)$, and so by the definition of weak ancestral, it follows that $Precedes^+(0, a)$; i.e., $NaturalNumber(a)$. $\bowtie$

(40): Suppose, for an arbitrary ordinary object b, that $\Diamond\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$. We want to show that $\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$. So assume, for an arbitrary ordinary object c, that $\mathcal{A}Gc$ (to show: $c \neq_E b$). Since, $\mathcal{A}Gc$, it follows that $\Box\mathcal{A}Gc$, by $\Box$ Actuality. Since we know that $\Diamond\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$, we know there is a world where $\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$ is true. Let us, for the moment, reason with respect to that world. Since $\Box\mathcal{A}Gc$ is true at our world, we know that $\mathcal{A}Gc$ is true at the world where $\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$ is true. So $c \neq_E b$ is true at that world. So, from the point of view of our world, we know that $\Diamond c \neq_E b$ (since $c \neq_E b$ is true at some world). But, by the logic of $=_E$, we know that $x =_E y \rightarrow \Box x =_E y$. That is, by modal duality, we know that that $\Diamond x \neq_E y \rightarrow x \neq_E y$. So since $\Diamond c \neq_E b$, it follows that $c \neq_E b$, which is what we had to show. [NOTE: This proof involved the natural deduction version of the modal axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$.] $\bowtie$

(41): Suppose $NaturalNumber(a)$. We want to show $\exists! m Precedes(a, m)$. But in virtue of (27), we simply have to show that $\exists m Precedes(a, m)$, and in virtue of (38), it suffices to show that $\exists y Precedes(a, y)$. Since $NaturalNumber(a)$, it follows that $NaturalCardinal(a)$, by (34). So $\exists F(a = \#_F)$. Suppose $a = \#_Q$. Then we know that there is a natural number which is $\#_Q$ and so we may apply our modal axiom (39) to conclude the following:

$$\Diamond\exists y(E!y \,\&\, \forall z(\mathcal{A}Qz \rightarrow z \neq_E y))$$

By the Barcan formula, this implies:

$$\exists y\Diamond(E!y \,\&\, \forall z(\mathcal{A}Qz \rightarrow z \neq_E y))$$

Let c be an arbitrary such object. So we know:

$$\Diamond(E!c \,\&\, \forall z(\mathcal{A}Qz \rightarrow z \neq_E c))$$

By the laws of possibility, it follows that:

$$\Diamond E!c \,\&\, \Diamond\forall z(\mathcal{A}Qz \rightarrow z \neq_E c)$$

From the first conjunct, it follows that $O!c$, and by (40), the second conjunct implies:

$$\forall z(\mathcal{A}Qz \rightarrow z \neq_E c) \qquad \text{(I)}$$

Now the property $[\lambda z\, Qz \vee z =_E c]$ exists by comprehension. Call this property Q^{+c}. By (11), it follows that $\#_{Q^{+c}}$ exists. Now if we can show $Precedes(a, \#_{Q^{+c}})$, we are done. So we have to show:

$$\exists F\exists u(Fu \,\&\, \#_{Q^{+c}} = \#_F \,\&\, a = \#_{[\lambda z\, Fz \,\&\, z \neq_E u]})$$

So we now show that Q^{+c} and c are such a property and object. c exemplifies Q^{+c}, by the definition of Q^{+c} and the fact that c is an ordinary object. $\#_{Q^{+c}} = \#_{Q^{+c}}$ is true by the laws of identity. So it simply remains to show:

$$a = \#_{[\lambda z\, Q^{+c}z \,\&\, z \neq_E c]}$$

Given that, by definition of Q, $a = \#_Q$, we have to show:

$$\#_Q = \#_{[\lambda z\, Q^{+c}z \,\&\, z \neq_E c]}$$

By Hume's Principle, it suffices to show:

$$Q \approx_E [\lambda z\, Q^{+c}z \,\&\, z \neq_E c]$$

But given (21.3), we need only establish the following two facts:

(a) $Q \approx_E [\lambda z\, Qz \,\&\, z \neq_E c]$

(b) $[\lambda z\, Qz \,\&\, z \neq_E c] \equiv_E [\lambda z\, Q^{+c}z \,\&\, z \neq_E c]$

Now to show (a), we simply need to prove that Q and $[\lambda z\, Qz \,\&\, z \neq_E c]$ are materially equivalent$_E$, in virtue of (21.1). ($\rightarrow$) Assume, for some arbitrary ordinary object d, that Qd. Then by the logical axiom Actuality, it follows that $\mathcal{A}Qd$. But then by fact (I) proved above, it follows that $d \neq_E c$. Since $Qd \,\&\, d \neq_E c$, it follows that $[\lambda z\, Qz \,\&\, z \neq_E c]d$. ($\leftarrow$) Trivial. Finally, we leave (b) as an exercise. $\bowtie$

(43): Assume that R^+ is a relation and:

$$Pa \,\&\, \forall x, y(R^+(a, x) \,\&\, R^+(a, y) \,\&\, Rxy \rightarrow (Px \rightarrow Py)).$$

We want to show, for an arbitrary object b, that if $R^+(a,b)$ then Pb. So assume $R^+(a,b)$. To show Pb, we appeal to Lemma (32.2):

$$Fx \,\&\, R^+(x,y) \,\&\, \textit{Hereditary}(F,R) \rightarrow Fy$$

Instantiate the variable F in this lemma to the property $[\lambda z\, Pz \,\&\, R^+(a,z)]$ (that there is such a property is guaranteed by the comprehension principle for relations and the assumption that R^+ is a relation), and instantiate the variables x and y to the objects a and b, respectively. The result is, therefore, something that we know to be true (after λ-conversion):

$$Pa \,\&\, R^+(a,a) \,\&\, R^+(a,b) \,\&\, \textit{Hereditary}([\lambda z\, Pz \,\&\, R^+(a,z)],R) \rightarrow$$
$$Pb \,\&\, R^+(a,b)$$

So if we can establish the antecedent of this fact, we establish Pb. But we know the first conjunct is true, by assumption. We know that the second conjunct is true, by the definition of R^+. We know that the third conjunct is true, by further assumption. So if we can establish:

$$\textit{Hereditary}([\lambda z\, Pz \,\&\, R^+(a,z)],R),$$

we are done. But, by the definition of heredity, this just means:

$$\forall x,y[Rxy \rightarrow ((Px \,\&\, R^+(a,x)) \rightarrow (Py \,\&\, R^+(a,y))].$$

To prove this claim, we assume Rxy, Px, and $R^+(a,x)$ (to show: $Py \,\&$ $R^+(a,y)$). But from the facts that $R^+(a,x)$ and Rxy, it follows from (32.3) that $R^*(a,y)$, and this implies $R^+(a,y)$, by the definition of R^+. But since we now have $R^+(a,x)$, $R^+(a,y)$, Rxy, and Px, it follows from the first assumption in the proof that Py. ⋈

(44): By assumption (42), *Predecessor*$^+$ is a relation. So by (43), it follows that:

$$Fa \,\&$$
$$\forall x,y[\textit{Precedes}^+(a,x) \,\&\, \textit{Precedes}^+(a,y) \,\&\, \textit{Precedes}(x,y) \rightarrow$$
$$(Fx \rightarrow Fy)] \rightarrow$$
$$\forall x(\textit{Precedes}^+(a,x) \rightarrow Fx)$$

Now substitute 0 for a, *NaturalNumber*(x) for *Precedes*$^+(0,x)$, and *NaturalNumber*(y) for *Precedes*$^+(0,y)$. ⋈

(48): We prove this for $n=0$ and then we give a proof schema for any numeral n' which assumes that a proof for n has been given. This proof schema has an instance which constitutes a proof of:

$$\vdash n' = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))$$

from the assumption:

$$\vdash n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n} uFu))$$

Base case. $n=0$. We want to show:

$$\vdash 0 = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{0} uFu))$$

That is, we want to show:

$$\vdash 0 = \imath x(A!x \,\&\, \forall F(xF \equiv \neg\exists uFu))$$

But this is an immediate corollary of (18).

Inductive case. Our Inductive Hypothesis is:

$$\vdash n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n} uFu))$$

We want to show that the following holds for the numeral n':

$$\vdash n' = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))$$

To do this, we have to show that there is a proof that the objects flanking the identity sign encode the same properties; i.e.,

$$\vdash \forall G[n'G \leftrightarrow \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))G]$$

($\rightarrow$) Assume that $n'P$, where P is an arbitrary property. We want to show that:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))P$$

By the laws of description, we have to show

$$\exists!_{n'} uPu,$$

i.e.,

$$\exists u(Pu \,\&\, \exists!_{n} v[\lambda z\, Pz \,\&\, z \neq_E u]v)$$

In other words, we have to show:

$$\exists u(Pu \,\&\, \exists!_{n} vP^{-u}v),$$

where P^{-u} stands for $[\lambda z\, Pz \,\&\, z \neq_E u]$.

Since *Precedes*(n,n'), there is some property, say Q and some ordinary object, say a, such that:

$$Qa \,\&\, n' = \#_Q \,\&\, n = \#_{Q^{-a}},$$

where Q^{-a} denotes $[\lambda z\, Qz \,\&\, z \neq_E a]$. Note that since $n'P$ (our initial assumption) and $n' = \#_Q$, we know that $\#_Q P$, and thus that P is equinumerous$_E$ to Q and vice versa. So there is a relation R which is a one-to-one and onto function from Q to P. Since Qa, we know that $\imath u(Pu \,\&\, Rau)$ exists. Call this object b. If we can show:

$$Pb \,\&\, \exists!_n v P^{-b} v$$

then we are done. But Pb follows by definition of b and the laws of description. To see that $\exists!_n v P^{-b} v$, note that $n = \#_{Q^{-a}}$. And by (24), we may appeal to the facts that $Q \approx_E P$, Qa, and Pb to conclude that $Q^{-a} \approx_E P^{-b}$. So by Hume's Principle, $\#_{Q^{-a}} = \#_{P^{-b}}$. So $n = \#_{P^{-b}}$ and by (15) and the fact that n is a natural cardinal, it follows that nP^{-b}. But we are assuming that the theorem holds for the numeral n:

$$n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n u F u))$$

This entails, by the laws of descriptions, that:

$$nF \equiv \exists!_n v F v$$

So since nP^{-b}, it follows that $\exists!_n v P^{-b} v$, which is what we had to show.

($\leftarrow$) Exercise.

$\bowtie$

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