Reflections on Mathematics*

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Though the philosophy of mathematics encompasses many kinds of questions, my response to the five questions primarily focuses on the prospects of developing a unified approach to the metaphysical and epistemological issues concerning mathematics. My answers will be framed from within a single conceptual framework. By 'conceptual framework', I mean an explicit and formal listing of primitive notions and first principles, set within a well-understood background logic. In what follows, I shall assume the primitive notions and first principles of the (formalized and) axiomatized theory of abstract objects, which I shall sometimes refer to as 'object theory'.[1] These notions and principles are mathematics-free, consisting only of metaphysical and logical primitives. The first principles assert the existence, and comprehend a domain, of abstract objects, and in this domain we can identify (either by definition or by other means) logical objects, natural mathematical objects, and theoretical mathematical objects. These formal principles and identifications will help us to articulate answers not only to the five questions explicitly before us, but also to some of the other fundamental questions in the philosophy of mathematics raised below.

1. Why were you initially drawn to the foundations of mathematics and/or the philosophy of mathematics?

As a metaphysician, I've always been interested in data that consists of (apparently) true sentences and valid inferences that appear to be about

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[1]This theory was outlined in detail in Zalta 1983 and 1988, and has been applied to issues in the philosophy of mathematics in the works referenced below.

objects and relations other than those studied by the natural sciences. These sentences and inferences often form part of a correct description of the world, and the challenge for the metaphysician is to explain this data, by developing a systematic theory of the truth conditions for the sentences in question that reveal why those sentences have the apparent truth value, and consequences, that they do have. The sentences and inferences deployed in the practice of mathematics are interesting examples of this kind of data, since they appear to reference, or quantify over, special objects and relations that are not studied *per se* by the natural sciences. The data are made even more interesting by the fact that serious scientific investigation employs the language of some segment of mathematics. Thus, as a metaphysician, it is important to develop an overall ontological theory that allows us to assign a significance, or denotation, to the terms and predicates of mathematical sentences in such a way that accounts for the truth of those sentences and for the valid inferences that we may make in terms of them. This is what drew me to the philosophy of mathematics.

As to the foundations of mathematics, I shall adhere to the distinctions, drawn explicitly in Shapiro 2004, between the metaphysical, epistemological, and mathematical foundations for mathematics. Metaphysical foundations for mathematics address the issues outlined in the previous paragraph. Epistemological foundations for mathematics center around the questions: (1) what kind of knowledge is knowledge of mathematics?, and (2) how (by what cognitive mechanisms) do we acquire such knowledge? Mathematical foundations for mathematics address the questions: (1) Is there a mathematical theory distinguished by the fact that all other mathematical theories can be reduced to, or translated into, it? (2) What notions of reducibility and translatability are appropriate for comparing the strength of mathematical theories?

Now given these distinctions, I shall focus primarily in what follows on metaphysical and epistemological foundations for mathematics. As to mathematical foundations for mathematics, I shall assume that although philosophers may have something to say about the nature of the reducibility or translatability relation in play when determining the strength of mathematical theories, it is primarily a mathematical, and not a philosophical, question as to whether there is a foundational mathematical theory. Therefore I shall suppose that the metaphysical and epistemological foundations of mathematics developed by philosophers should not imply whether there is or is not a single foundational mathematical theory. The metaphysics and epistemology of mathematics should be consistent with whatever conclusion mathematicians (including set theorists, category theorists, etc.) draw with respect to the existence of such a theory. Some philosophers might wonder how this is possible, but the theory described below shows that it is. Finally, I should mention that there are many other non-foundational issues in the philosophy of mathematics, but I shall little to say about them here.

2. What examples from your work (or the work of others) illustrate the use of mathematics for philosophy?

To answer this question, let me distinguish philosophical questions about the foundations of mathematics (i.e., the metaphysical and epistemological foundations discussed above) from other philosophical questions. In my view, philosophical theories about the foundations of mathematics should employ mathematical methods (e.g., the axiomatic method) but not assume any mathematically primitive expressions other than numerical indices to indicate the arity of relations.[2] I see metaphysics as an *a priori* science that is prior to mathematics: whereas mathematical theories are about particular abstract objects (e.g., the natural numbers, the ZF sets, etc.) and particular relations and operations (e.g., successor, membership, group addition, etc.), metaphysics is about abstract objects in general and relations in general. So metaphysics should be free of mathematical primitives, though primitive mathematical terms and predicates might be imported into metaphysics when those primitives are accompanied by principles that identify the denotations of the terms and predicates as entities already found in the background metaphysics. Another reason not to have mathematical primitives in our metaphysical foundations is to avoid ontological danglers. That is why set theory or model theory cannot serve as a metaphysical foundations; the metaphysical and epistemological problems about mathematics cannot be solved by an appeal to set theory or model theory, for that is just more mathematics and therefore part of the data to be explained. Such problems must be solved by an appeal to a more general theory of abstract objects and relations.

Thus, while I endorse and use the axiomatic method to organize a metaphysical foundations for mathematics, that framework (the 'second-order' modal theory of abstract objects),[3] employs only the following metaphysical and logical primitives: *individual* $(x, y, z \ldots)$, *n-place relation* $(F^n, G^n, H^n, \ldots)$, *exemplification* $(F^n x_1 \ldots x_n)$, *encoding* (xF^1),[4] *it is not the case that* $(\neg\phi)$, *if-then* $(\phi \rightarrow \psi)$, *every* $(\forall\alpha\phi)$,[5] and *it is necessarily the case that* $(\Box\phi)$.[6] In the higher-order formulations of the theory of abstract objects, we sometimes also employ the notion of *type*, defined recursively in the usual way. As we utilize this framework for the philosophy of mathematics, the fact that it is mathematics-free should be kept in mind.

So mathematical methods, such as the axiomatic method, may be used in responding to philosophical questions about the foundations of mathematics. But, of course, there are many other philosophical questions that have nothing to do with the foundations of mathematics. Here, the philosopher is free to employ whatever mathematics suits the task at hand. Some early examples are very well known. Leibniz (1690) used an algebraic operation ($\oplus$, for concept addition) and axioms for semilattices (governing the relation $\preceq$ of concept inclusion) to formulate his 'calculus of concepts'. Frege (1891) employed functions and functional application (conceived mathematically) to analyze predication in natural language. The 20th century saw an explosion of such applications of mathematics in philosophy. A noteworthy recent example is Leitgeb's use of the (graph-theoretic) mathematics of similarity relations to understand Carnap's notion of quasianalysis (Leitgeb 2007).

As long as the mathematics employed in these applications is used to *model* the entities, or the structure of the entities, or the reasoning we engage in, etc., I have no qualms. But we should not confuse the mathematical entities in a model with the entities being modeled. To give

[2]This appeal to numerals to indicate the arity of relations doesn't entail, as Frege realized, that we quantify over numbers. One might eliminate the numerals by using tics, i.e., indicating the arity of relation F as F', F'', F''', ... But if it could be shown that there is a ineliminable appeal to the natural numbers in using numerals in this way, then it may be that the best we can do is use this numeralized logic of relations to reconstruct the concept of *number* and the Dedekind-Peano postulates from our metaphysical first principles, as in Zalta 1999.

[3]I put 'second-order' in quotes because while the language of the theory is second-order, the theory doesn't require full second order logic.

[4]See Linsky & Zalta 2006 (80) for a discussion as to why the new form of predication, x encodes F (xF) is not to be conceived as a mathematical primitive.

[5]Here α may be any individual variable or relation variable.

[6]The theory also employs a distinguished predicate '$E!$'. Formulas of the form '$E!x$' and '$xE!$' are to be read as 'x exemplifies being concrete' and 'x encodes being concrete', respectively.

just a simple example: though set-theoretic models of propositions (i.e., ones that treat them as functions from worlds to truth-values or as sets of truth-values) have some interest in so far as they can represent the truth-conditions of, and inferential relations among, the sentences expressing those propositions, we shouldn't identify the propositions expressed with sets or functions. Instead we should try to develop theories of propositions that have these set-theoretic structures as models.

3. What is the proper role of the philosophy of mathematics in relation to logic, foundations of mathematics, the traditional core areas of mathematics, and science?

This question raises a host of further questions, such as: How do we demarcate logic and mathematics and what is the relationship between them? Are there logical objects and how do they differ from mathematical objects? To what extent do logical and mathematical foundations overlap? To begin to answer these questions, let us focus on the questions of how the metaphysical and epistemological foundations of mathematics relate to those of logic and science.

It is important begin by noting that true, ordinary mathematical claims typically occur either (a) in the context of 'natural' or naive mathematics, such as ordinary, naive geometrical claims, ordinary number statements appealing to the natural numbers, and ordinary, naive statements about sets or classes (i.e., extensions of ordinary properties), or (b) explicitly or implicitly in the context of some mathematical theory T. Thus, whenever we attempt to analyze some true mathematical claim, or consider its relationship to logic and science, we must decide whether we have a case of (a) or (b). We shall assume that (a) and (b) are exclusive possibilities, and that any ambiguity must be resolved in one way or the other.[7]

In the theory of abstract objects, we represent true mathematical claims of type (a) as claims about natural mathematical objects definable from the metaphysical and logical primitives of our background theory. In particular:

1. True ordinary and naive geometrical claims are analyzed claims about Platonic Forms, as these are described in Pelletier and Zalta 2000. For example, ordinary claims about the triangle (not made in a context that assumes Euclid's axioms) are analyzed as claims about the Platonic Form of Triangularity (Φ_T), which in object theory is the abstract object that encodes all and only the properties necessarily implied by the property of being a triangle.

2. True ordinary and naive claims about the (natural) numbers are analyzed as claims about the (Fregean) natural numbers developed in Zalta 1999. In that theory, the natural cardinal, the number of Fs ($\#F$), is explicitly defined in terms of metaphysical and logical primitives of object theory. ($\#F$ is the abstract object that encodes all and only the properties G which are in 1–1 correspondence with F on the ordinary objects.) Moreover, this notion can be used to define zero and the *predecessor* relation, and the axioms of Dedekind-Peano number theory can be derived.[8]

3. True ordinary and naive statements about sets, such as ordinary statements about the class of humans not made in the context of set theory, can be analyzed as statements about abstract objects that are *extensions*, as defined in Anderson and Zalta 2004. The extension of F (ϵF) is the abstract object that encodes all and only the properties materially equivalent to F.

By contrast, true mathematical claims of type (b) are represented in object theory in the manner set out in Zalta 2000a. The theorems of each mathematical theory T are imported into the theory of abstract objects by (i) prefacing the theory operator "In theory T" to each theorem and (ii) indexing the individual terms and predicates used in T to T. The ordinary claim 'In theory T, ...' is analyzed in object theory as: $T[\lambda y\, \phi^*]$. This latter is an encoding claim for which ϕ is the usual translation of '...' into the encoding-free formulas of classical logic and ϕ^* is just ϕ but with all the terms and predicates of T indexed to T. Thus, mathematical theories are identified as abstract objects that encode propositions by encoding propositional properties of the form $[\lambda y\, \phi^*]$. Now for any primitive or defined individual term κ used in theory T, the object κ_T can identified as the abstract object that encodes all and only the properties F satisfying the open formula 'In theory T, $F\kappa_T$'. Similarly, in the context of the

[7]The ideas in this paragraph and the next were first sketched in Zalta 2006.

[8]For the full details, see Zalta 1999. The derivation requires the assumption that Predecessor is a relation, and a modal assumption that guarantees, when n numbers the Gs, that there might have been an concrete object distinct from all the Gs.

third-order theory of abstract objects, for any predicate Π appearing in theory T, the relation Π_T can be identified as the abstract relation that encodes all and only the second-order properties $\boldsymbol{F}$ that satisfy the open formula 'In theory T, $\boldsymbol{F}\Pi_T$'.

To make the analysis of the preceding paragraph maximally explicit, here is an example of a natural mathematical object, a theoretical mathematical object, and a theoretical mathematical relation, where the $T \models \psi$ abbreviates $T[\lambda y\, \psi]$ in the second and third identities:

$$\Phi_T =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv \Box\forall y(Ty \rightarrow Fy))) \qquad (\theta)$$

$$\emptyset_{\mathrm{ZF}} = \imath x(A!x \,\&\, \forall F(xF \equiv \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}})) \qquad (\zeta)$$

$$\in_{\mathrm{ZF}} = \imath \boldsymbol{x}(\boldsymbol{A!x} \,\&\, \forall \boldsymbol{F}(\boldsymbol{xF} \equiv \mathrm{ZF} \models \boldsymbol{F}{\in_{\mathrm{ZF}}})) \qquad (\eta)$$

(θ) is the explicit definition of the Form of the Triangle (Φ_T) described above. (ζ) is not a definition but a derivable principle that is a consequence of the Reduction Axiom (Zalta 2000a, Section 3).[9] (η) is analogous to (ζ); it is a higher-order object-theoretic principle governing the identity of the membership relation of ZF. (In (η), the variable '$\boldsymbol{x}$' ranges over relations among individuals, and '$\boldsymbol{F}$' ranges over properties of such relations.)

This formal analysis, based on the distinction between (a) natural mathematical objects and (b) theoretical mathematical objects and relations, reveals that logic is more closely related to natural mathematics than it is to theoretical mathematics. The objects of natural mathematics described above look very much like logical objects, for when we compare the definitions of Φ_F, $\#F$, and ϵF (introduced in the enumerated paragraphs (1), (2), and (3) above) with the object-theoretic definitions of truth-values, directions, shapes, concepts, possible worlds, impossible worlds, etc., we find that the definitions can all be constructed using logical and metaphysical notions alone.[10] Thus, our metaphysical theory of objects and relations yields both logical objects and natural mathematical objects from its own first principles. By contrast, the analysis of theoretical mathematical objects and relations requires that we import

[9] In a separate work, Zalta 2006, (ζ) is described as an instance of a Theoretical Identification Principle (Section 2.4).

[10] See Anderson and Zalta 2004 for the theory of truth-values, directions, and shapes, Zalta 2000b for the theory of concepts, Zalta 1993 for the theory of possible worlds, and Zalta 1997 for the theory of impossible worlds.

the primitive notions and axioms of (explicit or implicit) mathematical theories into object theory. The synthetic axioms and theorms ϕ of a mathematical theory T are represented in object theory as analytic claims of the form 'In theory T, ϕ^*' (with indexed terms as described above). Each primitive mathematical individual and relation gets associated with a principle that identifies it in terms of an abstract individual or relation that is guaranteed to exist by the principles of the theory of abstract objects. So though the primitive expressions of mathematical theories become imported into object theory, each is accompanied by a principle that offers an analysis of the object or relation it signifies.

This is how I see the philosophical foundations of mathematics as relating to those of logic and the traditional core areas of mathematics. The relationship of the philosophy of mathematics to science is discussed in the answer to the next question.

4. What do you consider the most neglected topics and/or contributions in late 20th century philosophy of mathematics?

I focus here on two of the most neglected topics from my point of view. The first is the applicability of mathematics to the natural world. There are lots of interesting issues that fall within this topic, such as those that trace back to Wigner's (1960) question about the 'unreasonable effectiveness' of mathematics in science (see Steiner 1998, Colyvan 2001). The basic question I am concerned about is the proper analysis of the language of science. By the 'language of science', I refer not only to the language used in scientific theories (which is often simply mathematical in nature), but also to the language used by scientists themselves as they consider and formulate hypothesis, design experiments, etc. The language of science contains both (1) expressions that refer to concrete, spatiotemporal objects and to the relations among them, as well as (2) expressions for mathematical objects and relations. The philosophers of science who think that science is primarily about building models and representations of objects and processes in the natural world won't face much of a problem when analyzing the language of science, since models for them are essentially set-theoretic structures and they accept set theory (and model theory) as part of the philosophical foundations of science. But I don't accept set theory or model theory as part of the philosophical foundations of science, unless these theories are analyzed in the manner described in the previous sections. The language of science should not be interpreted in terms of (pure or impure) set-theoretic models unless it is explicitly intended to be language about models of the natural world instead of directly about the world itelf. With these distinctions, I can say that my interest lies in the question of *how* to analyze the mathematical expressions in language about the natural world used by scientists in their theories and in their everyday work.

The problem here is this: it may be that a proper understanding of abstract objects and relations (of which mathematical objects and relations form a subdomain) entails that they are entirely defined by our theories of them. We should not conceive of them as non-natural entities made of some Platonic substance, accessible by some special faculty of intuition. We should not use the model of physical objects to understand the mind-independence and objectivity of abstract objects (or mathematical objects), as Linsky and I have argued (1995). But if mathematical objects are defined by our theories of them, what is the relationship between the objects defined by pure mathematical theories and the objects defined by applied versions of those theories (assuming that the applied theories are non-conservative extensions of the pure ones)? To take a simple example, pure ZF is a different theory from ZF + Urelements (say, for example, ZF + {Socrates} or ZF + {Concrete Objects}). So, does the expression '$\{\emptyset\}$' as it appears in pure ZF denote the same object as the expression '$\{\emptyset\}$' in ZF + {Socrates}? If you think that $\{\emptyset\}$ is an object independent of our theories of sets and accessible to some special faculty of intuition, then you will answer this last question with a 'Yes'. But I don't think that a principled metaphysics and systematic epistemology for this conception of abstract and mathematical objects can be sustained. So I see the general question, of how the objects of our pure mathematical theories relate to the objects of those same theories when applied, as defining an issue that needs to be explored. Clearly, it has direct bearing on how we are to understand the applied mathematics appearing in the language of science. And the issue, as we've described it, just touches the tip of the iceberg, since the problems become even harder when we consider the fact that *relations* sometimes used in (the axioms of) scientific theories are essentially just mathematical relations.

A second neglected topic concerns one of the deepest insights that Frege had concerning the way we apprehend mathematical objects, namely, that we apprehend a mathematical object x when we can extract an identity claim about x from general truths about x. Few philosophers have

developed an epistemological mechanism for moving from general mathematical truths to identity claims about mathematical objects. But Frege wrote:

> If there are logical objects at all—and the objects of arithmetic are such objects—then there must also be a means of apprehending, or recognizing them. This ... is performed ... by the fundamental law of logic that permits the transformation of an equality holding generally into an equation. (1903, §147)

The fundamental law Frege was referring to here is Basic Law V, and he assumed that it legitimately turned equalities holding generally (of the form $\forall x[f(x) = g(x)]$ into an equation of the form $\epsilon f = \epsilon g$. (Here I am taking liberties with Frege's notation by assuming that 'ϵ' is an operator on functions f so that ϵf is the course-of-values of the function f.)

Frege was putting his finger on something important here. The question is: how do we transform fundamental logical and mathematical truths into metaphysical (or not purely mathematical) identities in which the expressions denoting mathematical objects and relations constitute one of the terms flanking the identity sign? Such identities are required no matter what our background philosophy of mathematics is, for every philosophy of mathematics needs to precisely state the semantic significance of mathematical terms and predicates. We are obliged to say what the significance is of terms like 'the triangle', 'the number of planets', 'the class of humans', '3', 'π', '$\emptyset$', 'ω', '$\aleph_0$', etc., and predicates and operations such as $<$, $\leq$, $\in$, $\oplus$, etc. And one must give some account of how the significance of these expressions is related to the significance of the sentences in which they appear, no matter whether one is a platonist, structuralist, inferentialist, etc. It is even encumbent on nominalists and fictionalists to give an account of the semantic significance of these expressions in so far as they contribute to the meanings of (false) mathematical sentences.

In the theory of abstract objects, the identities in question result either by explicit definitions or by way of reduction principles, depending on whether the entity in question is a natural mathematical object or a theoretical mathematical object or relation, respectively. Consider our examples (θ), (ζ), and (η) above. These instances of definitions and principles show how general claims and theorems can be transformed into identities of the objects in question. (θ) asserts the identity of the Form of the Triangle, which is defined in terms of the properties F necessarily implied by being a triangle (T). This latter generality ($\Box\forall y(Ty \rightarrow Fy)$) is incorporated into the description of the abstract object on the right side of the identity claim. By contrast, (ζ) asserts the identity of the empty set of ZF, which is specified in terms of the properties attributed to the empty set in the theorems of ZF (the theorems of ZF become imported into object theory prefaced by the operator 'In ZF'). Thus, all and only the properties satisfying the formula '$\mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}$' are encoded in the emptyset of ZF. And similarly for (η), which asserts the identity of the membership relation of ZF: all and only the second-order properties $\boldsymbol{F}$ satisfying the formula '$\mathrm{ZF} \models \boldsymbol{F}\in_{\mathrm{ZF}}$' are encoded in the membership relation of ZF.

So this is the means by which the identities of particular mathematical objects and relations can be extracted from the general mathematical claims that govern them. The epistemological significance of this cannot be overstated, for now we simply need the faculties of the understanding and reasoning to become acquainted with mathematical objects. When coupled with the version of neologicism defended in Linsky and Zalta 2006 (on which mathematics becomes reducible to weak third-order logic and analytic truths),[11] it becomes clear that no special faculty (such as intuition) is needed to apprehend mathematical objects and relations or to recognize the truth of mathematical claims. Our work suggests that theoretical mathematical truths are reducible to (1) the analytic principles of weak third-order logic, (2) an analytic abstraction principle for abstract objects, and (3) analytic truths of the form 'In theory T, ϕ'. As such, only our faculties for understanding language and drawing inferences are required for having knowledge of mathematics. This result, I suggest, forms part of the epistemological foundations of mathematics.

5. What are the most important open problems in the philosophy of mathematics and what are the prospects for progress?

In the answer to this final question, I describe one important open problem and propose its solution. The open question is: how can we *unify* the apparently divergent views in the philosophy of mathematics

[11] By 'weak' third-order logic, we mean a logic that is no more powerful than first-order logic with separate domains for first-order relations and for second-order relations, and which assumes nothing more than weak comprehension principles for those domains, the smallest models of which require that there are only two first-order relations and four second-order relations.

such as platonism, nominalism, fictionalism, structuralism, and inferentialism? The point behind this question is to suggest that the philosophy of mathematics would do well to have a theory that unifies these positions to whatever extent possible. Such a unification should explain why there is *disagreement* about the data, for example, over the truth values of mathematical sentences. Some philosophers say that the sentences of mathematics are true, while others say they are false, while still others say they are not truth-apt and so not candidates for truth or falsity, etc. So is it possible to unify the different traditions in the philosophy of mathematics and explain (away) these differences?

I believe it is. If we bring together the results of several pieces of prior research, then platonism, nominalism, fictionalism, and structuralism can be unified. Basically the idea is to interpret the formalism for the theory of abstract objects in different ways. One interpretation yields a form of platonism, another a form of nominalism, yet another a form of fictionalism, and yet another a form of structuralism. The interpretation of the theory of abstract objects as a version of platonism is to be found in Linsky & Zalta 1995 (536–541); as a version of nominalism in Bueno & Zalta 2005 (299–305); as a version of fictionalism in Zalta 1983 (Chapter VI), Colyvan & Zalta 1999 (346–348), and Zalta 2000a (255–256); and as as a version of structuralism in Linsky & Zalta 1995 (545–546). To complete this picture, we sketch how to interpret the formalism as a version of inferentialism.

To see how our analysis of mathematical objects and relations becomes a form of inferentialism, it is important to mention first that inferentialism is the view that the meaning of a mathematical term is to be identified with its inferential role in mathematical discourse. Now reconsider the formal claims (described above) that identify of theoretical mathematical objects in object theory. For example, the empty set of ZF is identified above on line (ζ) as:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}))$$

In other words, the empty set of ZF encodes all and only the properties F that satisfy the condition $\mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}$. As we saw earlier, conditions of this form arise when we import all the theorems of ZF into object theory under the theory operator 'In ZF' and index the terms and predicates to ZF. The particular properties of $\emptyset$ satisfying these conditions are therefore keyed to the theorems of ZF involving the term '$\emptyset$', i.e., for any formula

$\phi(\emptyset)$ such that $\vdash_{\text{ZF}} \phi(\emptyset)$, one can use λ-abstraction to produce a formula of the form $\vdash_{\text{ZF}} [\lambda z\, \phi(z)]\emptyset$. This latter then picks out a property that satisfies the condition on properties used in the identification of $\emptyset_{\text{ZF}}$. But the ZF-formulas of the form $\phi(\emptyset)$ such that $\vdash_{\text{ZF}} \phi(\emptyset)$ jointly constitute the inferential role of the term '$\emptyset$' in ZF. Thus, our analysis uses a double abstraction process to abstract out the inferential role of '$\emptyset$' in ZF and objectify it: λ-abstraction on the theorems of ZF involving '$\emptyset$' yield theorems identifying the derivable properties of $\emptyset$ according to the theory, and object abstraction reifies those properties as an object. On this conception, (ζ) identifies the significance of '$\emptyset$' in the theory ZF as its inferential role in that theory.

If we generally apply this conception to all the other theoretical identifications of the objects and relations of mathematical theories, we have an inferentialist interpretation of mathematics. For each term or predicate of mathematical theory T becomes identified with nothing other than the inferential role of that expression in T, something that can be precisely described in the theory of abstract objects. This, then, completes our answer to the open question about the unification of the divergent views in the philosophy of mathematics. The different philosophies of mathematics can now all be seen as different interpretations of the same underlying formalism! Moreover, the disagreement about the data is explained by the fact that ordinary theoretical claims in mathematics become ambiguous. Unadorned claims of mathematics such as '2 is prime', '$\emptyset$ is an element of $\{\emptyset\}$', etc., have exemplification readings (which are false, but which become true when prefaced by the theory operator) as well as encoding readings (which are true).[12] The platonists focus on the true readings to the exclusion of the false, while the nominalists and fictionalists focus on the false readings to the exclusion of the true. The structuralists and inferentialists, meanwhile, focus on the 'incompleteness' of the structures and inferential roles. Heretofore, it was widely thought that no form of classical logic could treat such incomplete objects as the indeterminate elements of mathematical structures (objects defined only by their mathematical properties) and the inferential roles of the terms and predicates of mathematical theories (defined only by the theorems in which they play a part). But it would be an invalid inference to draw such a conclusion

[12]See Zalta 2000a (Section 6) for a thorough description of how to formulate the exemplification and encoding readings of ordinary mathematical claims. For a more accessible account, see Zalta 2006, 674–678, and 688-691.

in the context of the logic of encoding and the theory of objects.

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