The Systems of *Principia Logico-Metaphysica*
Second-Order Modal Object Theory
and
Typed Object Theory

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Second-Order Modal Object Theory

Language

Standard Definition:

- Object variables and constants: $x, y, z, \ldots \quad a, b, c, \ldots$
- Relation variables and constants: $F^n, G^n, H^n, \ldots \quad P^n, Q^n, R^n, \ldots \quad (n \geq 0)$
  $(p, q, r, \ldots$ when $n = 0)$
- Distinguished unary relation: $E!$ *'being concrete'*
- Basic formulas:
  $F^n x_1 \ldots x_n$ ('$x_1, \ldots, x_n$ exemplify F^n') $(n \geq 0)$
  $x_1 \ldots x_n F^1$ ('$x_1, \ldots, x_n$ encode F^n') $(n \geq 1)$
- Complex Formulas: $\neg\varphi$, $\varphi \rightarrow \psi$, $\forall\alpha\varphi$ (α any variable), $\Box\varphi$, $\mathcal{A}\varphi$ ('Actually φ')
- Complex Terms:
  Descriptions: $\imath\nu\varphi$ (ν any individual variable and $\imath\nu\varphi$ interpreted rigidly)
  λ-expressions $(n \geq 0)$: $[\lambda x_1 \ldots x_n\, \varphi]$

BNF (Optional):

Syntactic Categories:

δ	primitive individual constants
ν	individual variables
Σ^n	primitive n-ary relation constants $(n \geq 0)$
Ω^n	n-ary relation variables $(n \geq 0)$
α	variables
κ	individual terms
Π^n	n-ary relation terms $(n \geq 0)$
φ	formulas
τ	terms

$$
\begin{array}{rrcl}
 & \delta & ::= & a_1, a_2, \ldots \\
 & \nu & ::= & x_1, x_2, \ldots \\
(n \geq 0) & \Sigma^n & ::= & P^n_1, P^n_2, \ldots \quad \text{(with } P^1_1 \text{ distinguished and written as } E!\text{)} \\
(n \geq 0) & \Omega^n & ::= & F^n_1, F^n_2, \ldots \\
 & \alpha & ::= & \nu \mid \Omega^n \ (n \geq 0) \\
 & \kappa & ::= & \delta \mid \nu \mid \imath\nu\varphi \\
(n \geq 1) & \Pi^n & ::= & \Sigma^n \mid \Omega^n \mid [\lambda \nu_1 \ldots \nu_n\ \varphi] \quad (\nu_1, \ldots, \nu_n \text{ are pairwise distinct}) \\
 & \varphi & ::= & \Sigma^0 \mid \Omega^0 \mid \Pi^n \kappa_1 \ldots \kappa_n \ (n \geq 1) \mid \kappa_1 \ldots \kappa_n \Pi^n \ (n \geq 1) \mid \\
 & & & [\lambda\, \varphi] \mid (\neg\varphi) \mid (\varphi \rightarrow \varphi) \mid \forall\alpha\varphi \mid (\Box\varphi) \mid (\mathcal{A}\varphi) \\
 & \Pi^0 & ::= & \varphi \\
 & \tau & ::= & \kappa \mid \Pi^n \quad (n \geq 0)
\end{array}
$$

Definitions

Operators and Terms

$\&$, $\vee$, $\equiv$, $\exists$, and $\Diamond$ are all defined in the usual way

$O! =_{df} [\lambda x\, \Diamond E!x]$ (*'being ordinary'*)

$A! =_{df} [\lambda x\, \neg\Diamond E!x]$ (*'being abstract'*)

Existence ($\downarrow$) (defined by cases)

$x\!\downarrow\, \equiv_{df} \exists F\, Fx$

$F^n\!\downarrow\, \equiv_{df} \exists x_1 \dots \exists x_n (x_1 \dots x_n F^n)$ $\qquad (n \geq 1)$

$p\!\downarrow\, \equiv_{df} [\lambda x\, p]\!\downarrow$

Identity ($=$) (defined by cases)

$x = y \;\equiv_{df}\; (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$

$F^1 = G^1 \;\equiv_{df}\; \Box\forall x(xF^1 \equiv xG^1)$

$F^n = G^n \;\equiv_{df}$ (where $n > 1$)
$\forall x_1 \dots \forall x_{n-1}([\lambda y\, F^n y x_1 \dots x_{n-1}] = [\lambda y\, G^n y x_1 \dots x_{n-1}] \,\&$
$[\lambda y\, F^n x_1 y x_2 \dots x_{n-1}] = [\lambda y\, G^n x_1 y x_2 \dots x_{n-1}] \,\& \dots \&$
$[\lambda y\, F^n x_1 \dots x_{n-1} y] = [\lambda y\, G^n x_1 \dots x_{n-1} y])$

$p = q \;\equiv_{df}\; [\lambda y\, p] = [\lambda y\, q]$

Axioms

A *closure* of a formula φ is the result of prefacing any string of quantifiers $\forall\alpha$, necessity operators $\Box$, or actuality operators $\mathcal{A}$ to φ. We take, as axioms, the closures (modal, universal, actualizations) of all (the instances of) the following axioms (axiom schemata), with the exception of the axiom schema $\mathcal{A}\varphi \to \varphi$, which we take only the universal closures of the instances:

Axioms for Negations and Conditionals:

- $\varphi \to (\psi \to \varphi)$
- $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
- $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

Axioms for Free Logic of Complex Terms:

- $\forall\alpha\varphi \to (\tau\!\downarrow\, \to \varphi^\tau_\alpha)$, provided τ is substitutable for α in φ
- $\tau\!\downarrow$, provided τ is primitive constant, a variable, or a λ-expression in which the λ does *not* bind a variable that occurs in encoding position in φ.[1]
- $\forall\alpha(\varphi \to \psi) \to (\forall\alpha\varphi \to \forall\alpha\psi)$
- $\varphi \to \forall\alpha\varphi$, provided α doesn't occur free in φ
- $\Pi^n \kappa_1 \dots \kappa_n \to (\Pi^n\!\downarrow \,\&\, \kappa_1\!\downarrow \,\&\, \dots \,\&\, \kappa_n\!\downarrow)$ $\quad (n \geq 0)$
  $\kappa_1 \dots \kappa_n \Pi^n \to (\Pi^n\!\downarrow \,\&\, \kappa_1\!\downarrow \,\&\, \dots \,\&\, \kappa_n\!\downarrow)$ $\quad (n \geq 1)$

[1] Formally, we may define: a variable α occurs in *encoding position* in φ just in case α is one of the primary terms of an encoding formula that occurs as a subterm of φ. For the definitions of *subterm* and *primary* term, see item (7) of *Principia Logico-Metaphysica*, at https://mally.stanford.edu/principia.pdf.

Axioms for the Substitution of Identicals:

- $\alpha = \beta \rightarrow (\varphi \rightarrow \varphi')$, whenever β is substitutable for α in φ, and φ' is the result of replacing zero or more free occurrences of α in φ with occurrences of β

Axioms for Actuality:

- $\mathcal{A}\varphi \rightarrow \varphi$ (only universal closures)
- $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$
- $\mathcal{A}(\varphi \rightarrow \psi) \equiv (\mathcal{A}\varphi \rightarrow \mathcal{A}\psi)$
- $\mathcal{A}\forall\alpha\varphi \equiv \forall\alpha\mathcal{A}\varphi$
- $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$

Axioms for Necessity:

- $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- $\Box\varphi \rightarrow \varphi$
- $\Diamond\varphi \rightarrow \Box\Diamond\varphi$
- $\Diamond\exists x(E!x \,\&\, \neg\mathcal{A}E!x)$

Axioms for Necessity and Actuality:

- $\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi$
- $\Box\varphi \equiv \mathcal{A}\Box\varphi$

Axioms for Definite Descriptions:

- $x = \imath x\varphi \;\equiv\; \forall z(\mathcal{A}\varphi^z_x \equiv z = x)$, provided z is substitutable for x in φ and doesn't occur free in φ

Axioms for Relations (λ-Calculus for Relations):

- $[\lambda\nu_1 \dots \nu_n\, \varphi]\!\downarrow\; \rightarrow [\lambda\nu_1 \dots \nu_n\, \varphi] = [\lambda\nu_1 \dots \nu_n\, \varphi]'$ $(n \geq 0)$
  ($[\lambda\nu_1 \dots \nu_n\, \varphi]'$ an alphabetic variant)
- $[\lambda x_1 \dots x_n\, \varphi]\!\downarrow\; \rightarrow ([\lambda x_1 \dots x_n\, \varphi]x_1 \dots x_n \equiv \varphi)$ $(n \geq 1)$
- $[\lambda x_1 \dots x_n\, F^n x_1 \dots x_n] = F^n$ $(n \geq 0)$
- $([\lambda x_1 \dots x_n\, \varphi]\!\downarrow \,\&\, \Box\forall x_1 \dots \forall x_n(\varphi \equiv \psi)) \;\rightarrow\; [\lambda x_1 \dots x_n\, \psi]\!\downarrow$ $(n \geq 1)$

Axioms for Encoding:

- $x_1 \dots x_n F^n \;\equiv$
  $x_1[\lambda y\, F^n y x_2 \dots x_n] \,\&\, x_2[\lambda y\, F^n x_1 y x_3 \dots x_n] \,\&\, \dots \,\&\, x_n[\lambda y\, F^n x_1 \dots x_{n-1} y]$
- $xF \rightarrow \Box xF$
- $O!x \rightarrow \neg\exists F\, xF$
- $\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$, provided x doesn't occur free in φ

Deductive Systems

Rule of Inference: Modus Ponens

Derivations and Theoremhood:

- There are two derivability systems: $\Gamma \vdash \varphi$ and $\Gamma \vdash_{\Box} \varphi$.
- $\Gamma \vdash \varphi$ (derivations) and $\vdash \varphi$ (theorems) defined in the usual way: these are derivations (theorems) from inferred from *any* axioms.
- $\Gamma \vdash_{\Box} \varphi$ (*modally strict* derivations) and $\vdash_{\Box} \varphi$ (*modally strict* theorems): these are derivations (theorems) that don't depend on the axiom $\mathcal{A}\varphi \to \varphi$.
  - $\mathcal{A}\varphi \to \varphi$ is a 'modally fragile' axiom and can't be necessitated.
  - We mark non-modally strict derivations and theorems with a $\star$.
  - The system is therefore set-up for additional axioms whose necessitations aren't asserted.
- Derived Metarule GEN:
  - If $\Gamma \vdash \varphi$ and α doesn't occur free in any formula in Γ, then $\Gamma \vdash \forall\alpha\varphi$.
  - If $\Gamma \vdash_{\Box} \varphi$ and α doesn't occur free in any formula in Γ, then $\Gamma \vdash_{\Box} \forall\alpha\varphi$.
- Derived Metarule RN, where $\Box\Gamma$ is $\{\Box\psi \,|\, \psi \in \Gamma\}$:
  - If $\Gamma \vdash_{\Box} \varphi$, then $\Box\Gamma \vdash_{\Box} \Box\varphi$
  - If $\Gamma \vdash_{\Box} \varphi$, then $\Box\Gamma \vdash \Box\varphi$
- Derived Metarule RA, where $\mathcal{A}\Gamma$ is $\{\mathcal{A}\psi \,|\, \psi \in \Gamma\}$:
  - If $\Gamma \vdash \varphi$, then $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$.
  - If $\Gamma \vdash_{\Box} \varphi$, then $\mathcal{A}\Gamma \vdash_{\Box} \mathcal{A}\varphi$.

See https://mally.stanford.edu/principia.pdf.

Some Distinctive Theorems Governing Existence and Identity

The principles (theorems) of classical propositional logic and the principles of predicate logic (with a negative free logic for complex terms) are all preserved. But the following $\vdash_{\Box}$ theorems governing existence and identity are distinctive – the numbers refer to the numbered items the latest version of *Principia Logico-Metaphysica*, at URL in red noted above.

(104.2)	$\varphi\downarrow$	(for any formula φ)
(106)	$\tau\downarrow \to \Box\tau\downarrow$	(logical existence implies necessary logical existence)
(107.1)	$\tau = \sigma \to \tau\downarrow$	
(107.2)	$\tau = \sigma \to \sigma\downarrow$	
(111.2)	$[\lambda\, \varphi] \equiv \varphi$	("that-φ is true iff φ")
(117.1)	$\alpha = \alpha$	
(117.2)	$\alpha = \beta \to \beta = \alpha$	
(117.3)	$(\alpha = \beta \,\&\, \beta = \gamma) \to \alpha = \gamma$	
(121.1)	$\tau\downarrow \equiv \exists\beta(\beta = \tau)$	(provided that β doesn't occur free in τ)
(125.1)	$\alpha = \beta \to \Box\alpha = \beta$	(necessity of identity)

Typed Object Theory

(Latest unpublished version)

Language

Types:

- i is a type.
- If $t_1, \ldots, t_n$ are any types ($n \geq 0$), $\langle t_1, \ldots, t_n \rangle$ is a type.

BNF:

$$\begin{array}{ll} \delta^t & \text{primitive constants of type } t \\ \alpha^t & \text{variables of type } t \\ \tau^t & \text{terms of type } t \\ \varphi & \text{formulas} \end{array}$$

$$\boxed{\begin{array}{rrl} & \delta^t & ::= a^t_1, a^t_2, \ldots \quad (E!^{\langle t \rangle} \text{ a distinguished constant, for every } t) \\ & \alpha^t & ::= x^t_1, x^t_2, \ldots \\ & \mathit{Base}^t & ::= \delta^t \mid \alpha^t \mid \iota\alpha^t\varphi \\ & \tau^i & ::= \mathit{Base}^i \\ (n \geq 1) & \tau^{\langle t_1,\ldots,t_n\rangle} & ::= \mathit{Base}^{\langle t_1,\ldots,t_n\rangle} \mid [\lambda\alpha^{t_1}\ldots\alpha^{t_n}\,\varphi] \quad (\alpha^{t_1}\ldots\alpha^{t_n} \text{ pairwise distinct}) \\ & \varphi & ::= \mathit{Base}^{\langle\rangle} \mid \tau^{\langle t_1,\ldots,t_n\rangle}\tau^{t_1}\ldots\tau^{t_n}\ (n \geq 1) \mid \tau^{t_1}\ldots\tau^{t_n}\tau^{\langle t_1,\ldots,t_n\rangle}\ (n \geq 1) \mid \\ & & \quad [\lambda\,\varphi] \mid (\neg\varphi) \mid (\varphi \rightarrow \varphi) \mid \forall\alpha^t\varphi \mid (\Box\varphi) \mid (\mathcal{A}\varphi) \\ & \tau^{\langle\rangle} & ::= \varphi \end{array}}$$

Definitions

(.1) $\varphi \,\&\, \psi \ \equiv_{df}\ \neg(\varphi \rightarrow \neg\psi)$

(.2) $\varphi \vee \psi \ \equiv_{df}\ \neg\varphi \rightarrow \psi$

(.3) $\varphi \equiv \psi \ \equiv_{df}\ (\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)$

(.4) $\exists x\varphi \ \equiv_{df}\ \neg\forall x\neg\varphi$ — x any type

(.5) $\Diamond\varphi \ \equiv_{df}\ \neg\Box\neg\varphi$

(.6.a) $x\!\downarrow\, \equiv_{df} \exists F\, Fx$ — x has type i

(.6.b) $p\!\downarrow\, \equiv_{df} \exists F\, Fp$ — p has type $\langle\,\rangle$

(.6.c) $F\!\downarrow\, \equiv_{df} \exists x_1 \ldots \exists x_n(x_1 \ldots x_n F)$ — F has type $t_1, \ldots, t_n$ ($n \geq 1$)

(.7) $O! =_{df} [\lambda x\, \Diamond E!x]$ — x has any type

(.8) $A! =_{df} [\lambda x\, \neg\Diamond E!x]$ — x has any type

(.9) $x = y \ \equiv_{df}\ (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$ — x, y have type i

(.10) $F = G \ \equiv_{df}\ (O!F \,\&\, O!G \,\&\, \Box\forall x(xF \equiv xG)) \vee (A!F \,\&\, A!G \,\&\, \Box\forall\mathcal{H}(F\mathcal{H} \equiv G\mathcal{H}))$ — F, G have type $\langle t \rangle$

(.11) $F = G \ \equiv_{df}$ — F, G have type $\langle t_1, \ldots, t_n \rangle$

$$\begin{array}{l} O!F \,\&\, O!G \,\&\, \forall x_2 \ldots \forall x_n([\lambda x_1\ Fx_1 \ldots x_n] = [\lambda x_1\ Gx_1 \ldots x_n]) \,\&\, \\ \forall x_1 \forall x_3 \ldots \forall x_n([\lambda x_2\ Fx_1 \ldots x_n] = [\lambda x_2\ Gx_1 \ldots x_n]) \,\&\ldots\&\, \\ \forall x_1 \ldots \forall x_{n-1}([\lambda x_n\ Fx_1 \ldots x_n] = [\lambda x_n\ Gx_1 \ldots x_n]) \quad \vee \\ A!F \,\&\, A!G \,\&\, \Box\forall\mathcal{H}(F\mathcal{H} \equiv G\mathcal{H}) \end{array}$$

(.12) $p = q \ \equiv_{df}\ (O!p \,\&\, O!q \,\&\, [\lambda x\ p] = [\lambda x\ q]) \vee (A!p \,\&\, A!q \,\&\, \Box\forall\mathcal{H}(p\mathcal{H} \equiv q\mathcal{H}))$ — p, q have type $\langle\,\rangle$

Axioms

Negations and Conditionals.

(.1) $\varphi \to (\psi \to \varphi)$

(.2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(.3) $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

Quantification and Logical Existence.

(.4) $\forall x\varphi \to (\tau{\downarrow} \to \varphi^{\tau}_{x})$, provided τ is substitutable for x in φ — x, τ have type t

(.5) $\tau{\downarrow}$, whenever τ is either a primitive constant, a variable, or a core λ-expression

(.6) $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$ — x any type

(.7) $\varphi \to \forall x\varphi$, provided x doesn't occur free in φ — x any type

(.8) (a) $\Pi\tau_1 \dots \tau_n \to (\Pi{\downarrow}\ \&\ \tau_1{\downarrow}\ \&\ \dots\ \&\ \tau_n{\downarrow})$ — $(n \geq 0)$
(b) $\tau_1 \dots \tau_n\Pi \to (\Pi{\downarrow}\ \&\ \tau_1{\downarrow}\ \&\ \dots\ \&\ \tau_n{\downarrow})$ — $(n \geq 1)$

Substitution of Identicals.

(.9) $x = y \to (\varphi \to \varphi')$ — x, y have type t

$\star$**Actuality** (only universal closures).

(.10) $\mathcal{A}\varphi \to \varphi$

Actuality (all closures).

(.11) $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$

(.12) $\mathcal{A}(\varphi \to \psi) \equiv (\mathcal{A}\varphi \to \mathcal{A}\psi)$

(.13) $\mathcal{A}\forall x\varphi \equiv \forall x\mathcal{A}\varphi$ — x any type

(.14) $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$

Necessity.

(.15) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(.16) $\Box\varphi \to \varphi$

(.17) $\Diamond\varphi \to \Box\Diamond\varphi$

(.18) $\Diamond\exists x(E!x\ \&\ \neg\mathcal{A}E!x)$ — x has type i and $E!$ has type $\langle i\rangle$

Necessity and Actuality.

(.19) $\mathcal{A}\varphi \to \Box\mathcal{A}\varphi$

(.20) $\Box\varphi \equiv \mathcal{A}\Box\varphi$

Descriptions.

(.21) $x = \imath x\varphi \ \equiv\ \forall y(\mathcal{A}\varphi^{y}_{x} \equiv y = x)$ — x, y have type t

Relations.

(.22) $[\lambda x_1 \ldots x_n\, \varphi]{\downarrow} \to O!_{[\lambda x_1 \ldots x_n\, \varphi]}$ $\qquad x_1, \ldots, x_n$ have types $t_1, \ldots, t_n$, $O!$ has type $\langle\langle t_1, \ldots, t_n\rangle\rangle$

(.23) $O!\varphi$, provided φ is not in $Base^{\langle\rangle}$, i.e., provided φ is not a constant of type $\langle\rangle$, a variable of type $\langle\rangle$, or a description of type $\langle\rangle$

(.24) $A!_F \to \neg\exists x_1 \ldots \exists x_n F x_1 \ldots x_n$ $\qquad x_1, \ldots, x_n$ have types $t_1, \ldots, t_n$

(.25) $[\lambda x_1 \ldots x_n\, \varphi]{\downarrow} \to [\lambda x_1 \ldots x_n\, \varphi] = [\lambda x_1 \ldots x_n\, \varphi]'$ $\qquad$ (α-Conversion)

(.26) $[\lambda x_1 \ldots x_n\, \varphi]{\downarrow} \to ([\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi)$ $\qquad$ (β-Conversion)

(.27) $O!_F \to ([\lambda x_1 \ldots x_n\, F x_1 \ldots x_n] = F)$ $\qquad$ (η-Conversion)

(.28) $([\lambda x_1 \ldots x_n\, \varphi]{\downarrow} \,\&\, \Box\forall x_1 \ldots \forall x_n(\varphi \equiv \psi)) \to [\lambda x_1 \ldots x_n\, \psi]{\downarrow}$ $\qquad n \geq 1$, $x_1, \ldots, x_n$ any types,

Encoding.

(.29) $x_1 \ldots x_n F \equiv x_1[\lambda y_1\, F y_1 x_2 \ldots x_n] \,\&\, x_2[\lambda y_2\, F x_1 y_2 x_3 \ldots x_n] \,\&\, \ldots \,\&\, x_n[\lambda y_n\, F x_1 \ldots x_{n-1} y_n]$
(x_i, y_i have the same type)

(.30) $xF \to \Box xF$ $\qquad x$ any type

(.31) $O!x \to \neg\exists F xF$ $\qquad x$ any type

(.32) $\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$, where φ has no free xs $\qquad x$ any type