The Modal Object Calculus and its Interpretation*

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The modal object calculus is the system of logic which houses the (proper) axiomatic theory of abstract objects.[1] This calculus has some rather interesting features in and of itself, independent of the proper theory. The most sophisticated, type-theoretic incarnation of the calculus can be used to analyze the intensional contexts of natural language and so constitutes an intensional logic. However, the simpler second-order version of the calculus couches a theory of fine-grained properties, relations and propositions and serves as a framework for defining situations, possible worlds, stories, and fictional characters, among other things. In the present paper, we focus on the second-order calculus.

The second-order modal object calculus is so-called to distinguish it from the second-order modal predicate calculus. Though the differences are slight, the extra expressive power of the object calculus significantly enhances its ability to resolve logical and philosophical concepts and problems. Our primary goal in this paper is to describe a new interpretation of the modal object calculus. As a secondary and preliminary goal, we shall

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[1]The proper theory of abstract objects is developed in Zalta [1983] and [1988] and extended in [1987] and [1993a]. This system was developed within the tradition of 'object theory', which has recently been revived by Parsons [1980], Routley [1979], Castañeda [1974], and Rapaport [1978], and which originated in the naive theories of Meinong [1904] and Mally [1912].

recast its intended interpretation in a new and interesting way, based on a reconception of logic and model theory that we take to be philosophically more perspicuous than the standard conception. But in order to accomplish these goals, we shall first need to motivate and define the calculus and then sketch some of its applications. Then we will be in a position to redescribe its intended interpretation and develop the new interpretation. In §1, then, we explain the ideas underlying the basic definitions that construct the calculus. In §2, we examine some of the applications of the calculus which do not require the proper theory of abstract objects. In §3, we describe the intended interpretation of the calculus in terms of the reconception of logic and model theory. In §4, we construct the new interpretation of the calculus by developing, in a modal setting, a suggestion due to Peter Aczel. Finally, §5 contains some observations about the ideas that have been presented and offers a brief outline of how to extend those ideas to produce a model of the theory of abstract objects. Readers who are familiar with the definition and applications of the object calculus may wish to skip ahead to §3, where the intended interpretation of the calculus is recast in a new theoretical setting.

§1: The Second Order Modal Object Calculus

The (modal) object calculus is the logical system for asserting and proving facts about *abstract* objects. Abstract objects are to be distinguished from ordinary spatiotemporal objects such as you, me, this computer, electrons, planets, etc., and from ordinary objects that are spatiotemporally located at other possible worlds. The predicate calculus is the background logic in which we assert facts and draw inferences about ordinary objects because such objects *exemplify* the properties we discover them to have. Exemplification is the mode of predication upon which the predicate calculus is based, for the philosophical claim that objects $x_1, \ldots, x_n$ *exemplify* relation F^n is represented by ordinary atomic formulas of the form $F^n x_1 \ldots x_n$. But *abstract* objects are completely different in kind from ordinary objects. They are not the kind of thing that could have a location in spacetime; they are not contingent and they are not sparsely distributed in their domain. They are not discovered by sensory perception or postulated in scientific theories. Rather, abstract objects are *constituted* by the properties that we use to conceptualize, define, and individuate them. We conceptualize and define the number 2, for example, in terms of its number-theoretic properties; we identify the null set in terms of its set-theoretic properties; we individuate the actual world in terms of its propositional properties; we conceptualize Sherlock Holmes in terms of the properties attributed to him in the Conan Doyle novels. The groups of properties just mentioned are even more important to the identity of their respective objects than some of the properties these objects necessarily exemplify. For example, all of these objects necessarily fail to be buildings, necessarily fail to be spoons, etc., and from this it follows that they necessarily exemplify the negations of these properties. But these necessarily exemplified negations are not properties by which we conceive the objects in question.

Every abstract object is characterized by some group of constitutive properties. We need a way of predicating properties of abstract objects that is even stronger than necessary exemplification; i.e., we need to be able to predicate the property F that is constitutive of the abstract object x in a way that is stronger than the assertion that $\Box Fx$. To do this, we have introduced a second mode of predication and incorporated it into our logic. If property F is constitutive of, or intrinsic to, abstract object x, we say that x *encodes* F. We shall say that the number two encodes rather than exemplifies the property of being the successor of 1, for this is one of its constitutive number-theoretic properties. Similarly, since the property of *being such that Clinton is married* is one of the properties we use to differentiate our world from other possible worlds, this property will be one that is encoded by the actual world. And we shall suppose that Sherlock Holmes encodes rather than exemplifies the property of being a detective, for this is one of the properties attributed to him in the novels. Notice that our distinction allows us to explain the following asymmetry between Socrates and the unit set of Socrates: the property of having Socrates as an element is constitutive of the unit set of Socrates, but it is not constitutive of Socrates that he is an element of that unit set. To explain this fact, we say that the unit set of Socrates encodes the property of having Socrates as an element, but that Socrates is an ordinary object and so does not encode properties—he is not therefore 'constituted' by the property of being an element of his unit set.

We incorporate the notion of encoding into our logic by introducing an atomic formula 'xF' to express: x encodes F. We shall assume that F is a 1-place relation term and that encoding is a monadic form of predication. The second-order modal object calculus can now be described simply as

a second-order modal predicate calculus that is based on both atomic exemplification and atomic encoding formulas. In what follows, we shall include complex n-place relation terms in our presentation of the calculus. With the aid of metavariables o (ranging over all object terms), ν (ranging over object variables), ρ (ranging over relation terms), α (ranging over all variables), τ (ranging over all terms), and φ, ψ (ranging over all formulas), the calculus is defined as follows:

(a) Two kinds of primitive terms: (1) object terms (constants $a_1, a_2, \ldots$, and variables $x_1, x_2, \ldots$), and (2) n-place relation terms (predicates $P^n_1, P^n_2, \ldots$, and variables $F^n_1, F^n_2, \ldots$), for $n \geq 0$.

(b) Two kinds of atomic formulas: (1) exemplification formulas of the form $\rho^n o_1 \ldots o_n$ ($n \geq 0$), and (2) encoding formulas of the form $o\rho^1$. An exemplification formula such as $F^n x_1 \ldots x_n$ should be read: objects $x_1, \ldots, x_n$ *exemplify* relation F^n. An encoding formula such as xF^1 should be read: object x *encodes* property F^1.

(c) Molecular, quantified, and modal formulas of the form $\neg\varphi$ ('it is not the case that φ'), $\varphi \rightarrow \psi$ ('if φ then ψ'), $\forall\alpha\varphi$ ('every α is such that φ'), and $\Box\varphi$ ('necessarily φ'), where α is any object or relation variable.

(d) Complex n-place relation terms ($n \geq 0$) of the form: $[\lambda\nu_1 \ldots \nu_n\, \varphi]$, where φ has neither encoding subformulas nor quantifiers binding relation variables, and none of the object variables $\nu_1, \ldots, \nu_n$ in $[\lambda\nu_1 \ldots \nu_n\, \varphi]$ appear free in any other λ-expression occurring in φ. A λ-predicate such as $[\lambda y_1 \ldots y_n\, \varphi]$ should be read: being objects $y_1, \ldots, y_n$ such that φ.

Intuitively, the object constants denote, and object variables range over, elements of a primitive domain of objects; similarly, the predicates denote, and the predicate variables range over, elements of a primitive domain of relations. In addition to calling the 1-place relations *properties*, we call the 0-place relations *propositions*. The n-place λ-expressions ($n \geq 1$) allow us to name complex relations, and 0-place λ-expressions of the form $[\lambda\, \varphi]$ allow us to name complex propositions. Of the three restrictions on the formation of λ-expressions, only the first is really necessary—encoding formulas are banished from λ-expressions in order to avoid paradox when the comprehension schema for abstract objects is added to the system.[2]

[2]See Zalta [1983], pp. 158–60.

The other two restrictions are for convenience—λ-expressions may not have relation quantifiers (such as $[\lambda x\, \exists F Fx]$) or free variables in nested λ-expressions (such as $[\lambda x\, [\lambda y\, Ryx]a]$). This allows us to simplify the presentation of the interpretations of the calculus.[3] The restrictions on the formation of λ-expressions simply guarantee that only the familiar first-order definable relations and propositions will be found in the system. The main difference, then, between the languages of the object calculus and predicate calculus is the presence of a second kind of atomic formula in the former. In what follows, we employ the usual conventions for introducing formulas containing & (and), $\vee$ (or), $\equiv$ (iff), $\exists$ (some), and $\Diamond$ (possibly).

If we add to this language the primitive predicate '$E!$' to denote the property of being spatiotemporally located, we can formally define the notions of *ordinary* and *abstract* object and assert things about them:

x is an *ordinary* object ('$O!x$') $=_{df}$ $[\lambda y\, \Diamond E!y]x$

x is an *abstract* object ('$A!x$') $=_{df}$ $[\lambda y\, \neg\Diamond E!y]x$

Intuitively, these definitions tell us that ordinary objects have a location in spacetime in some possible world, whereas abstract objects are not the kind of thing that could have a location in spacetime. The usual conception of ordinary objects is captured by the following two principles: (A) ordinary objects x and y are identical iff they necessarily exemplify the same properties, and (B) ordinary objects are not the kind of thing that could encode properties:

(A) $O!x \,\&\, O!y \rightarrow (x = y \equiv \Box\forall F(Fx \equiv Fy))$

(B) $O!x \rightarrow \Box\neg\exists F xF$

By contrast, the two basic principles governing abstract objects are:

(C) $\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$, where x is not free in φ

(D) $A!x \,\&\, A!y \rightarrow (x = y \equiv \Box\forall F(xF \equiv yF))$

[3]One can develop interpretations in which these restrictions are removed. To remove the 'no relations quantifiers' restriction, one would have to extend the algebraic semantics developed in §3 to higher order operations such as those developed in Došen [1988] or those utilized in the type-theoretic version of the calculus. To remove the 'no free nested λ-variables' restriction, one would have to incorporate into the algebraic semantics a *relativized* predication operation of the kind used in Bealer [1982] and Menzel [1986].

Principle C is a comprehension principle for abstract objects; it guarantees that for every condition φ on properties F, there is an abstract individual that *encodes* exactly the properties satisfying the condition. If we think model-theoretically for the moment, and allow ourselves talk about sets, then this principle correlates an abstract object with every (expressible) set of properties. Principle D is the identity principle for abstract objects; it says that abstract objects x and y are identical iff they necessarily encode the same properties. It is a simple consequence of Principles C and D that for every condition on properties φ, there is a *unique* abstract object that encodes just the properties satisfying the condition; there couldn't be two distinct abstract objects encoding exactly the properties satisfying a given condition if distinct abstract objects have to differ by at least one encoded property.

Principle C asserts the existence of a wide variety of abstract objects, some of which are complete with respect to the properties they encode, while others are incomplete in this respect. For example, one instance of Principle C asserts there exists an abstract object that encodes just the properties Gorbachev exemplifies. This object is complete because Gorbachev either exemplifies F or exemplifies the negation of F, for every property F. Another instance of comprehension asserts that there is an abstract object that encodes just the three properties: *being golden*, *being a mountain*, and *having a spatiotemporal location* ($E!$). This object is incomplete because for every *other* property F, it encodes neither F nor the negation of F.[4] But though abstract objects may be partial with respect to their encoded properties, they are all complete with respect to the properties they *exemplify*. In other words, the following principle of classical logic is preserved: for every object x and property F, either x exemplifies F or x exemplifies the negation of F. We can express this formally if we use our λ-notation to define the negation of F ('$\bar{F}$') as: $[\lambda y\, \neg Fy]$. So we preserve the following formal principle of classical logic: $Fx \vee \bar{F}x$.[5]

Since xF does not entail Fx, abstract objects may encode incompatible properties without contradiction, for incompatible properties are defined as properties that couldn't be *exemplified* by the same objects.

[4]Notice that this object is consistent with the contingent fact that nothing *exemplifies* all three properties.

[5]Note that encoding satisfies classical bivalence: $\forall F\forall x(xF \vee \neg xF)$. But the incompleteness of abstract objects is captured by the fact that the following is not in general true: $xF \vee x\bar{F}$.

The following, for example, are jointly consistent: x encodes roundness (xR), x encodes squareness (xS), and necessarily everything that *exemplifies* being round fails to *exemplify* being square ($\Box\forall y(Ry \rightarrow \neg Sy)$). Thus, the notorious 'round square' may simply be the abstract object that encodes just being round and being square. Since there are no restrictions on the comprehension principle, it guarantees that no matter what properties one brings to mind to conceive of a thing, there is something that encodes just the properties involved in that conception.

Given that they they necessarily fail to have a location in spacetime, abstract objects necessarily fail to exemplify the properties of ordinary objects that entail spatiotemporality. For example, they necessarily fail to have a shape, they necessarily fail to be subject to the laws of generation and decay, they necessarily fail to be buildings, people, planets, etc. Consequently, by the classical laws of complex properties, abstract objects necessarily exemplify the negations of these properties. But notice that the properties abstract objects encode are more important than the properties they necessarily exemplify, since the former are the ones by which we individuate them.[6] And it is important to mention that abstract objects may contingently exemplify certain relations to ordinary objects, such as being sought by y, being worshipped by z, inspiring u to action, etc.

Principles A – D are part of the proper theory of (abstract) objects. In this paper, however, we shall be more concerned with the underlying modal object calculus. In addition to being a vehicle for expressing the basic principles about ordinary and abstract objects, the calculus offers a framework for asserting the basic existence and identity principles governing n-place relations ($n \geq 0$). We shall treat these basic existence and identity principles of relations as a part of the calculus, for we conceive of these entities as logical objects. So the calculus consists of the language described above together with the principles of classical quantified modal logic, the logic of encoding, and the logic of relations. In the next section, we shall sketch some of the applications of this logical framework, for even without a commitment to Principle C, the system can be used to define a wide range of important philosophical notions and prove basic logical

[6]In the logic that we shall describe shortly, it will be a logical axiom that $\Diamond xF \rightarrow \Box xF$. So properties that are encoded by an abstract object at some world are encoded by that object at every world. This also helps to capture the idea that encoded properties are intrinsic to abstract objects.

facts concerning them. We conclude this section, however, with a more precise specification of the logical portion of the calculus.

The logic consists of: (a) the axioms and rules of classical propositional logic, (b) the axioms and rules for classical quantification theory, (c) the axioms and rules for the classical modal logic $S5$, (d) an axiom for the logic of encoding, and (e) axioms and definitions for the logic of relations. This logical basis can be explicitly identified as follows:

(a) Classical Propositional Logic:

Axioms: Tautologies of Propositional Logic

Rule of Modus Ponens (MP): if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(b) Classical Quantification Theory:[7]

Axioms: $\forall\alpha\varphi \rightarrow \varphi^{\tau}_{\alpha}$, provided τ substitutable for α

Axioms: $\forall\alpha(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall\alpha\psi)$, provided α not free in φ

Rule of Generalization (Gen): if $\vdash \varphi$, then $\vdash \forall\alpha\varphi$

(c) Classical Modal Logic $S5$:

Axioms: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

Axioms: $\Box\varphi \rightarrow \varphi$

Axioms: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$

Rule of Necessitation (RN): if $\vdash \varphi$, then $\vdash \Box\varphi$

(d) Logic of Encoding: $\Diamond xF \rightarrow \Box xF$

(e) Logic of Relations:[8]

λ-Conversion: $[\lambda y_1 \ldots y_n\ \varphi]x_1 \ldots x_n \equiv \varphi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$,
where x_i is substitutable for y_i in φ ($1 \leq i \leq n$).

Property Identity: $F^1 = G^1 =_{df} \Box\forall x(xF \equiv xG)$

[7]In these axioms, τ may be any object term or relation term, and α any object variable or relation variable, with the proviso that for τ to be substitutable for α, they must both be object terms or both relation terms.

[8]To make the notation easier to read, we state the axioms and definitions using $x, x_1, \ldots$ and $y, y_1, \ldots$ as typical variables in place of the metavariables $\nu_1, \nu_2, \ldots$. The notation $\varphi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$ stands for the result of replacing, respectively, x_i for y_i in φ, and the requirement that x_i be substitutable for y_i guarantees that x_i will not be 'captured' by a quantifier when the substitution is carried out.

Relation Identity: $F^n = G^n =_{df}$ (for $n > 1$) $(\forall x_1)\ldots(\forall x_{n-1})$:
$[\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, G^n y x_1 \ldots x_{n-1}]$ &
$[\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, G^n x_1 y x_2 \ldots x_{n-1}]$ & ... &
$[\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, G^n x_1 \ldots x_{n-1} y]$

Proposition Identity: $F^0 = G^0 =_{df} [\lambda y\, F^0] = [\lambda y\, G^0]$

λ-Identity$_1$: $[\lambda y_1 \ldots y_n\, F^n y_1 \ldots y_n] = F^n$

λ-Identity$_2$: $[\lambda y_1 \ldots y_n\, \varphi] = [\lambda y'_1 \ldots y'_n\, \varphi']$ (alphabetic variants)

This completes the definition of the calculus. With the exception of the logic of encoding and the logic of relations, the deductive apparatus should be familiar. These additional groups of axioms will be discussed in the next section.

§2: Applications of the Calculus

In previous work, we have not distinguished the results that require an appeal to Principle C from those that are derivable solely as logical theorems of the modal object calculus. In this section, we distinguish the latter for those who may be hesitant about committing themselves to a proper metaphysical theory. We discuss the following topics: relations, situations and possible worlds, modality and the Barcan formula, and the distinction between fact and fiction.

Relations

From the logic of relations we may derive a precise theory of relations. The cornerstone of the theory is the comprehension principle for relations that follows from the λ-Conversion principle by applications of (Universal) Generalization, the Rule of Necessitation, and the derived rule of Existential Generalization:

> $\exists F^n \Box \forall x_1 \ldots \forall x_n (F^n x_1 \ldots x_n \equiv \varphi)$, provided φ has no free F^ns, no encoding formulas, no relation quantifiers, and no free occurrences of the x_i within λ-expressions in φ.

When $n = 1$ and $n = 0$, this becomes a comprehension principle for properties and propositions, respectively. Using variables p, q, ... to go proxy for F^0, G^0, ... , we may formulate the comprehension schema for propositions as follows:

> $\exists p \Box (p \equiv \varphi)$, provided φ has no free ps, no encoding formulas, and no relation quantifiers.

These comprehension principles yield a wide range of complex properties, relations, and propositions, and we presume familiarity with the typical examples of such.

The definitions that stipulate when relations F and G are identical complete the theory. These definitions were stated as part of the logic of relations, and there are three cases to consider: when $n = 1$, $n > 1$, and $n = 0$. Look again at the definition of property identity ($n = 1$): properties F and G are identical iff necessarily, they are encoded by the same objects. It is useful to know at this point that in the semantics for our language, every property receives two extensions—an exemplification extension and an encoding extension. Thus, properties can be logically equivalent in one of two ways: by having the same exemplification extension at every possible world or by having the same encoding extension at every possible world. We can express this in our language as two ways in which properties F and G can be *necessarily* equivalent: $\Box\forall x(Fx \equiv Gx)$ and $\Box\forall x(xF \equiv xG)$. Property identity is equated with the latter. So properties F and G can be 'distinct' even though they are necessarily equivalent in the traditional sense. By stipulating that properties having the same encoding extension throughout a fixed domain of worlds are 'identical', we offer an extensional theory of intensional entities. In the next section, it becomes clear that whereas the exemplification extension of a property may vary from world to world, its encoding extension does not. The fact that the encoding extension of a property is fixed across possible worlds explains the Axiom of Encoding—if an object is in the encoding extension of a property at some world, it is in the encoding extension of that property at every world. Thus, if F and G have the same encoding extension at one world, they have the same encoding extension at all worlds, and so to prove $F = G$, it suffices to prove $\forall x(xF \equiv xG)$.[9]

[9]We employ the modality in the definition of $F = G$ because, from a philosophical point of view, identity is a modal notion. If we think model-theoretically in terms of a primitive notion of identity on properties and assume Principles C and D, then we can see why properties encoded by the same objects would be the same. For suppose not, i.e., suppose F and G are properties that are encoded by the same objects, but that F and G are distinct. If F and G are distinct, there are sets of properties containing F and not G (and vice versa). But Principles C and D ensure that there is a distinct abstract object for each set of properties. So there will be lots of distinct abstract objects encoding F and not G (and vice versa), contradicting the assumption that F

The definition of property identity holds the key to the definition of relation and proposition identity. Relations F^n and G^n, for $n > 1$, are defined to be identical just in case, intuitively, no matter how you plug $n-1$ objects into F^n and G^n (provided you plug them up in the same order), the resulting properties are always identical. Again, this definition of relation identity allows us to assert that $F^n \neq G^n$ even though $\Box\forall x_1 \ldots \forall x_n(F^n x_1 \ldots x_n \equiv G^n x_1 \ldots x_n)$. Necessarily equivalent relations, therefore, may be distinct.

If we continue to use the variables p and q as substitutes for F^0 and G^0, then the definition of proposition identity tells us that propositions p and q are identical iff the properties $[\lambda y\, p]$ and $[\lambda y\, q]$ are identical. This reduces the identity of propositions to the identity of the 'propositional properties' one can construct in terms of them. And this, in turn, is a matter of encoding. Since there is no logical connection between encoding and exemplifying a property, one can assert $p \neq q$ even though $\Box(p \equiv q)$. So necessarily equivalent propositions may be distinct.

Situations and Worlds

This analysis of relations provides the foundation for the theories of situations and worlds. To define the basic notions of situation theory, we first define truth for propositions by elimination: a proposition p *is true* iff p. We say that a *propositional* property is any property F such that for some proposition p, F is the property of being such that p; i.e.,

$$\mathit{Propositional}(F) =_{df} \exists p(F = [\lambda y\, p])$$

Then, we say that a *situation* is any (abstract) object x that encodes only propositional properties; i.e., x is a *situation* iff x is abstract and for every F, if xF, then F is propositional. So each situation encodes a group of propositional properties. The propositions p encoded in a situation via $[\lambda y\, p]$ are the ones true in that situation. More specifically, where s is a variable ranging over situations, we can define the idea that p is *true in* s as: s encodes $[\lambda y\, p]$ (i.e., $s[\lambda y\, p]$). In what follows, we represent the notion that p is true in s more picturesquely as: $s \models p$. We also say that a situation s is *part-of* situation s' iff every proposition true in s is true in s'. An *actual* situation s is one such that every proposition true in s is true (simpliciter), and a situation is *possible* iff it is possible that it is actual. A situation s is *consistent* iff there are no propositions p and q such that (a) the conjunction of p and q is impossible, and (b) p and q are both true in s.

and G are encoded by the same objects.

In Zalta [1993a], we have shown that from these definitions, one can derive a rather large group of basic theorems in situation theory. Though some of these theorems appeal to Principle C, most are simply logical theorems of the modal object calculus. Among the theorems that are simple consequences of the above definitions we find: that situations s and s' are identical iff the same propositions are true in both; that every part of a situation is a situation; that a situation s is a part of situation s' iff every proposition true in s is true in s'; that situations s and s' are identical iff each is part of the the other; that part-of is reflexive, antisymmetric, and transitive on the situations (should there be any); that no proposition and its negation are both true in any actual situation; that some propositions are not true in any actual situation; that if p is true in actual situation s, then s exemplifies $[\lambda y\, p]$, and that all possible situations are consistent.

The basic notions of world theory can also be defined. A *world* is a situation s such that it is possible that all and only true propositions p are true in s; i.e.,

$$\mathit{World}(s) =_{df} \Diamond\forall p(s \models p \equiv p)$$

Truth at a world ($w \models p$) may therefore be defined as the same notion as truth in a situation. It is provable that worlds w are maximal in the sense that for every proposition p, either $w \models p$ or $w \models \neg p$. It also follows that all worlds are possible and consistent, and that all the necessary consequences of propositions true at a world are also true at that world. By combining the definitions of 'world' and 'actual', we obtain a notion of an actual world, and though Principle C is needed to prove the existence of such an object, it is not needed to to prove that if there is one, there is a unique one. Under the assumption that there is an actual world, say w_α, it also follows that w_α is nonwellfounded in the following sense: all the facts about w_α are *true in* w_α; i.e., Fw_α iff $w_\alpha \models Fw_\alpha$, for any property F. So, in particular, if w_α exists, it exemplifies exactly the propositional properties that it encodes.

Modality and the Barcan Formula

Notice that the modal object calculus employs the simplest quantified modal logic. This is the modal logic that results from combining classical quantification theory with the axioms and rules of S5. The Barcan formula and the converse Barcan formula are both derivable in such a logic:[10]

(BF) $\forall x \Box \varphi \to \Box \forall x \varphi$

(CBF) $\Box \forall x \varphi \to \forall x \Box \varphi$

We know from the study of modal logic that in such a simple system, the quantifier $\forall x$ ranges over a single, fixed domain of objects. The reader will be able to verify this in the next section. But for now, it is useful to point out that by having a single fixed domain instead of the variable domains used in Kripke [1963], we don't have to address the question of whether an object in one domain is the same as an object in another domain. That is, the problem of trans-world identification does not arise.

We conclude this subsection with a brief account of the nature of possible objects. Recall the equivalent formulation of the Barcan formula:

(BF) $\Diamond \exists x Fx \to \exists x \Diamond Fx$

To take a simple example, consider the claim that person x (who doesn't have a brother) might have had a brother. We might represent this claim as: $\Diamond \exists y Byx$. From the Barcan formula it then follows that $\exists y \Diamond Byx$. So our logic, in conjunction with possibility claims, entails that there is something which possibly is x's brother. Any such object will be treated as an *ordinary* object which isn't spatiotemporally located but which might have been. That is because the relation of brotherhood is such that necessarily, any two things exemplifying this relation both exemplify $E!$. That is, our logic is consistent with the following non-logical principle:

$\Box \forall x \forall y (Bxy \to E!x \;\&\; E!y)$

Since any object that possibly is x's brother will exemplify $E!$ in any world where it is x's brother, we know that it possibly exemplifies $E!$, and so by definition, it is an ordinary object. It is important to stress that such an object is not x's brother (recall that x doesn't have a brother). We may consistently suppose that any ordinary object that might have been

[10]The second order Barcan Formulas are also derivable, but we shall not discuss these formulas in what follows.

x's brother exemplifies many of the same properties that abstract objects exemplify, namely, the *negations* of such properties as being spatiotemporally located, having a shape, having a texture, being a building, being a person, etc. However, such ordinary objects exemplify certain modal properties that *bona fide* abstract objects lack (should there be any). An abstract object, by definition, couldn't possibly exemplify $E!$ and so given the non-logical principle displayed above, couldn't possibly be someone's brother. These consequences indicate the range of judgements we can consistently add to our calculus in connection with common sense claims such as 'x might have had a brother'.

Fact and Fiction

Finally, the distinction between fact and fiction can be analyzed in the calculus. Using a primitive 2-place relation of authorship, we may say that a *story* is any situation s authored by some ordinary object. Since stories are identified as situations, one can define 'According to the story s, p' as $s \models p$. Intuitively, the true English sentences of the form 'According to s, p' constitute the data of fiction. We presuppose that for each story s, there is a group of propositions that satisfy the sentence 'According to s, p'. Just as one cannot drop the story operator 'According to s' and preserve truth, one cannot validly move from $s \models p$ to p. Notice that stories, unlike worlds, need not be maximal nor even consistent. Of course, two people may disagree about which propositions p are true in a given story, but that doesn't matter because both persons could nevertheless accept the identification of the story s with the abstract object that encodes all the properties of the form $[\lambda y\ p]$ such that p is a proposition true according to the story. That is, despite disagreement about exactly which propositions are true in a story, our analysis of stories is a precise philosophical characterization of what stories are *in principle*. Finally, a *character* x of story s can be analyzed as any object x such that for some F, $s \models Fx$. Not all characters of a story are fictional. We allow that real objects such as London, the Prince of Wales, the planet Jupiter, etc., can appear as characters in a story. But the fictional characters are the ones that 'originate' in the story. If the logic of fiction can be simplified by identifying such 'native characters' as abstract objects, then we have a reason for accepting Principle C. But without this principle, a definite description like 'the abstract object that encodes just the properties F

that, according to story s, x exemplifies F' ('$\imath x(A!x \& \forall F(xF \equiv s \models Fx))$') is not well-defined.

This brief review of the applications of the modal object calculus should have drawn attention to one point, namely, that even if one has metaphysical scruples about accepting Principle C, the modal object calculus can still serve as a vehicle for defining interesting philosophical notions and proving basic facts about them. Though the calculus has other applications, we shall not describe these in any detail here.

§3: The Intended Interpretation of the Calculus

Before we describe the new interpretation of the calculus (in §4), we must first review the intended interpretation. However, in the years since the intended interpretation of the calculus was first developed, Etchemendy [1990] has developed a more critical view of Tarski's formal definition of truth and logical consequence. To address some of Etchemendy's concerns, we shall recast our original semantic definitions to make it clear that the various models of our language do not constitute different interpretations of the constants and predicates. This results in a more perspicuous definition of truth and logical truth. To this end, we take the domains required for the interpretation of the terms, quantifiers, and modal operators *out* of the models! The domain of objects, the domain of relations, and the domain of possible worlds shall be grouped together and specified antecedently as part of an interpretation of the language. So an *interpretation* of the language will be distinguished from the *models* that can be defined for that interpretation. This way, the domains of the interpretation will not vary from model to model. The models for an interpretation **I** simply represent the various ways that the objects of **I** can be assigned to the extensions of the relations of **I** (at the various possible worlds of **I**). On this picture, the only thing that varies from model to model is: (a) the function that indicates which objects exemplify which relations at each world, and (b) the function that indicates which objects encode which properties.

On this picture of modal language, the notion of 'truth at a world in a model' is defined for an an *interpreted* sentence. We do *not* evaluate a purely formal sentence at world **w** in model **M** by considering the extension of the terms and predicates at world **w** in **M**. Instead, we consider an interpretation **I** of the sentence and find out what objects and relations the terms and predicates of the sentence denote with respect to **I**. Then once we have these, we may ask, with respect to a world **w** in the interpretation **I**, whether model **M** structures the relations and objects in the appropriate way at **w**. In other words, we interpret formulas first and then discover whether they are true. Thus, truth (and logical truth) will be defined relative to a fixed interpretation of the formulas.[11]

Consider, as an example, the formula 'Pa'. Before we can decide whether it is true, we have to know what it means. So let us fix an interpretation **I** by supposing that 'a' denotes Socrates (a member of the domain of objects in **I**), that 'P' denotes the property of being snub-nosed (a member of the domain of properties in **I**), and that the domain of possible worlds in **I** contains three possible worlds $\mathbf{w}_1$, $\mathbf{w}_2$, and a distinguished actual world $\mathbf{w}_\alpha$. Now a given model **M** for this interpretation will tell us at which worlds Socrates is an element of the exemplification extension of the property of being snub-nosed, for **M** will be defined, in part, by a function that assigns ordered sets of **I**-objects as the exemplification extensions of the **I**-relations at each **I**-world. The formula 'Pa' under interpretation **I** will be true at world **w** in model **M** iff **M** assigns Socrates to the exemplification extension of the property of being snub-nosed at world **w**. The formula 'Pa' under interpretation **I** will be true (*simpliciter*) in a model **M** iff **M** assigns Socrates to the exemplification extension of the property of being snub-nosed at the distinguished world $\mathbf{w}_\alpha$.

Now consider the modal formula '$\Box Pa$' under interpretation **I**. This interpreted formula will be true at world **w** in model **M** iff **M** assigns Socrates to the exemplification extension of the property of being snub-nosed at all the worlds accessible from **w** (suppose that **I** also specifies the accessibility relation on worlds) . It will be true (*simpliciter*) in model **M** iff **M** assigns Socrates to the exemplification extension of the property of being snub-nosed at all the worlds accessible from the actual world $\mathbf{w}_\alpha$. Since we have adopted an S5 modal logic, all worlds are accessible to each other and so we shall simplify our semantics by eliminating the accessibility relation altogether.

Since extensions are *not* directly assigned to the predicates of the language, our semantical treatment differs from Tarski [1936] and [1944], though we shall still employ his notion of satisfaction (see below). Nor do we assign extensions to predicates *relative* to worlds, and so our treat-

[11]For an extended discussion motivating this conception of modal logic, see Zalta [1993b].

ment also differs from that of Kripke [1959], [1963] and Hintikka [1961]. In general, we do not use the method of intension and extension developed by Carnap [1947] and extended by Montague [1974]. That is, we do not recover an 'intension' function from the assignment of extensions to constants, predicates, and sentences at each possible world. Instead, we take relations as primitive, fine-grained entities and in the course of defining models, something analogous to a Montagovian 'intension' function is used to indicate what extensions these primitive relations have at various worlds. This semantic method, therefore, takes up Frege's [1892] idea that a predicate denotes a 'concept' and that an 'extension' is something associated, in the first instance, with a concept rather than with a predicate.

These explanatory remarks should help motivate the following formal definitions of an interpretation, intended interpretation, model, assignments and denotation, satisfaction, truth, and logical truth.

Interpretations

An interpretation of the modal object calculus is a quadruple $\mathbf{I} = \langle \mathbf{W}, \mathbf{D}, \mathbf{R}, \mathbf{F} \rangle$, the members of which are as follows:

1. $\mathbf{W}$ is the nonempty set of *possible worlds* with a distinguished *actual world* $\mathbf{w}_\alpha$ (the S5 modal operators will be construed as quantifiers over $\mathbf{W}$ in the simplest possible way, namely, without an accessibility relation).

2. $\mathbf{D}$ is the nonempty domain of *objects* (which includes both abstract and ordinary objects).

3. $\mathbf{R}$ is the domain of primitive *relations* (defined as the union of a sequence of nonempty, pairwise disjoint sets $\mathbf{R}_0, \mathbf{R}_1, \ldots$) which is closed under the following algebraic logical operations (discussed below): $\mathbf{PLUG}_i$ (plug-into-the-i^{th}-place), **NEG** (negate), **NEC** (necessitate), **COND** (conditionalize), $\mathbf{UNIV}_i$ (universalize-on-the-i^{th}-place), $\mathbf{REFL}_{i,j}$ (reflect-the-i^{th}-and-j^{th}-places), $\mathbf{CONV}_{i,j}$ (convert-the-i^{th}-and-j^{th}-places), and $\mathbf{VAC}_i$ (vacuously-expand-the-i^{th}-place).

4. $\mathbf{F}$ is an interpretation function that maps each object constant of the language to an element of $\mathbf{D}$, and maps each n-place relation constant of the language to an element of $\mathbf{R}_n$.[12]

To complete this definition, we must define the algebraic operations in clause 3.

Intuitively, the algebraic operations harness relations into complex, structured relations that serve as the denotations of the λ-expressions. These operations are analogues of Quine's predicate functors in [1960], but whereas Quine's functors operate on predicates to eliminate variables, our operations are defined on the relations denoted by the predicates. Here are some examples:

1. The operation $\mathbf{PLUG}_1$ takes the 3-place relation $\mathbf{r}$ (in $\mathbf{R}_3$) and an object $\mathbf{o}$ (in $\mathbf{D}$) and plugs $\mathbf{o}$ into the first place of $\mathbf{r}$ to produce the complex relation 2-place $\mathbf{PLUG}_1(\mathbf{r}, \mathbf{o})$ (in $\mathbf{R}_2$).[13] If $\mathbf{r}$ is the denotation of the 3-place predicate R, and $\mathbf{o}$ is the denotation of the constant a, then $\mathbf{PLUG}_1(\mathbf{r}, \mathbf{o})$ serves as the denotation of the 2-place λ-expression $[\lambda xy\, Raxy]$. $\mathbf{PLUG}_3(\mathbf{r}, \mathbf{o})$, on the other hand, serves as the denotation of $[\lambda xy\, Rxya]$. $\mathbf{PLUG}_3$ is not defined for relations having fewer than three places, whereas $\mathbf{PLUG}_1$ is defined for all except the 0-place relations.

2. The operation **NEG** produces an n-place relation that is the negation of a given an n-place relation. So, building on a previous example, the 2-place relation $\mathbf{NEG}(\mathbf{PLUG}_1(\mathbf{r}, \mathbf{o}))$ serves as the denotation of the predicate $[\lambda xy\, \neg Raxy]$.

3. The operation $\mathbf{UNIV}_2$ maps the 3-place relation $\mathbf{r}$ to the 2-place relation $\mathbf{UNIV}_2(\mathbf{r})$, which serves as the denotation of the 2-place expression $[\lambda xy\, \forall z Rxzy]$.

These examples should give the reader some indication of how the operations work.

We shall not reproduce the definitions of the algebraic operations here, for they have been characterized precisely in such places as Zalta [1983] and [1988], Bealer [1982], and Menzel [1986]. But it should be mentioned

[12] This function is not yet a full-blooded denotation function because it is not defined on the complex λ-expressions. In what follows, a denotation function, relative to $\mathbf{I}$ and a variable assignment $\mathbf{f}$, will *extend* $\mathbf{F_I}$ to all terms by assigning a denotation to the variables and to the complex terms.

[13] Quine has no need for the **PLUG** operation in his [1960] because he eliminates constants in favor of predicates.

that various constraints must be placed on the (exemplification) extension functions and denotation functions for the algebraic operations to work properly. To see why, recall that each model will assign exemplification extensions to the relations (at possible worlds) in different ways. However, in each model, the extensions of the complex relations produced by the algebraic operations must mesh properly with the extensions of the relations they may have as parts, no matter how the latter extensions are assigned. For example (from the previous paragraph), these constraints ensure that the exemplification extension of the complex relation $\mathbf{PLUG}_1(\mathbf{r},\mathbf{o})$ at world $\mathbf{w}$ consists of all those pairs $\langle \mathbf{o}_1, \mathbf{o}_2\rangle$ such that the triple $\langle \mathbf{o}, \mathbf{o}_1, \mathbf{o}_2\rangle$ is in the exemplication extension of the relation $\mathbf{r}$ at $\mathbf{w}$. The constraints also ensure that the exemplification extension of the complex relation $\mathbf{NEG}(\mathbf{PLUG}_1(\mathbf{r},\mathbf{o}))$ at world $\mathbf{w}$ consists of all those pairs $\langle \mathbf{o}_1, \mathbf{o}_2\rangle$ that fail to be in the exemplification extension of the relation $\mathbf{PLUG}_1(\mathbf{r},\mathbf{o})$ at $\mathbf{w}$. An appropriate constraint is therefore defined for each algebraic function, and as a group, the constraints guarantee that the interpreted instances of the logical axiom λ-Conversion are true in every model, since the exemplification extension functions of every model will be constrained in these ways.

There is one other subtlety to the logic of the algebraic operations. This concerns the fact that they 'overgenerate' denotations. For example, if we let the constant 'b' denote object $\mathbf{o}'$, then both $\mathbf{PLUG}_2(\mathbf{PLUG}_3(\mathbf{r}, \mathbf{o}), \mathbf{o}')$ and $\mathbf{PLUG}_2(\mathbf{PLUG}_2(\mathbf{r}, \mathbf{o}'), \mathbf{o})$ could equally well serve as the denotation of $[\lambda x\ Rxba]$. So we need some means of ensuring that the predicate $[\lambda x\, Rxba]$ receives a unique denotation, i.e., that the denotation function is well-defined on the λ-expressions. In our previous work, we have partitioned the λ-expressions into syntactic categories that correspond with the algebraic logical operations. Then the denotation function, which maps each term of the language to an entity in the relevant domain, is defined so that it maps each λ-expression to an appropriately structured relation based on the expression's syntactic category. This yields a mechanical procedure that selects a unique denotation for the λ-expression. We'll say more about this when we turn to the precise definition of denotation in what follows.

The Intended Interpretation

The *intended interpretation* of our calculus will simply be that interpretation such that: (a) the domains $\mathbf{W}$, $\mathbf{D}$, and $\mathbf{R}$ contain, respectively, all the possible worlds there in fact are, all the relations that there are, and all the objects that there are, and (b) whenever any constant (predicate) is treated as an abbreviation of an English name (predicate), the function $\mathbf{F}$ assigns the object (relation) denoted by the English name (predicate) as the denotation of the constant (predicate). In the remainder of this section, we suppose that the intended interpretation is fixed and we shall simply call it '$\mathbf{I}$'. The interpretation function $\mathbf{F}$ of the intended interpretation will be referred to as $\mathbf{F_I}$.

Models

Before we actually state the definition of a model, let us informally anticipate the definition with some examples, to see how the definitions of truth and logical truth for our *interpreted* language will eventually work. A model $\mathbf{M}$ for the intended interpretation $\mathbf{I}$ will be defined so that: (a) all of the relations in $\mathbf{R}$ receive *exemplification* extensions at each possible world, and (b) all of the properties in $\mathbf{R}_1$ receive, in addition, an *encoding* extension. The definition of truth will then tell us that the formula 'Pa' under the interpretation $\mathbf{I}$ is true at world $\mathbf{w}$ relative to $\mathbf{M}$ iff $\mathbf{F_I}(a)$ is an element of the $\mathbf{M}$-exemplification extension at $\mathbf{w}$ of the property $\mathbf{F_I}(P)$. And the formula 'aP' under $\mathbf{I}$ is true at $\mathbf{w}$ relative to $\mathbf{M}$ iff $\mathbf{F_I}(a)$ is a member of the $\mathbf{M}$-encoding extension of $\mathbf{F_I}(P)$. Of course, the definition of truth won't be stated in terms of $\mathbf{F_I}$, since that function only interprets the primitive constants and predicates. We shall need to define a full-blooded denotation function that extends $\mathbf{F_I}$ and assigns a denotation to every term in the language, including the variables and the complex expressions.

The definition of logical truth will be cast in terms of the definition of truth: a formula φ under the interpretation $\mathbf{I}$ is logically true iff φ is true at the actual world in every model for $\mathbf{I}$. For example, '$Pa \vee \neg Pa$' (under $\mathbf{I}$) will be a logical truth, *not* because it turns out true under every interpretation of 'a' and 'P', but rather because it is true (at $\mathbf{w}_\alpha$) no matter how models for $\mathbf{I}$ assign (exemplification) extensions to $\mathbf{F_I}(P)$. To make this more vivid, suppose the object constant 'a' abbreviates the proper name 'Bill Clinton' and that $\mathbf{F_I}(a)$ is the man Bill Clinton (who is an element of **D**). And assume that the primitive relation constant 'P' abbreviates the property-name 'being a U. S. president' and that $\mathbf{F_I}(P)$ is the property of being a U. S. president (this is an element of $\mathbf{R}_1$). Then the sentence 'Pa' under the interpretation **I** *asserts* that Clinton exemplifies being a U. S. president, and the reason '$Pa \vee \neg Pa$' (under **I**) is a logical truth is that it remains true at $\mathbf{w}_\alpha$ no matter how models for **I** assign an exemplification extension to the property of being a U. S. president. As we mentioned earlier, this formulation of the definition of truth and logical truth has been influenced by the ideas in Etchemendy [1990], from which one might conclude that a philosophically proper model-theoretic definition of logical truth should not depend on alternative interpretations of the constants 'a' and 'P'.[14]

To make these ideas precise and more general, we have to define models and extend the definition of truth and logical truth to all the formulas of the language, including those involving variables and/or complex terms. Thus, given our intended interpretation **I**, we can define a model **M** for **I** as consisting of two functions:

(a) **ext**, for $n \geq 1$, is binary function that maps each pair $\langle \mathbf{r}^n, \mathbf{w}\rangle$ consisting of an n-place relation $\mathbf{r}^n$ ($\in \mathbf{R}_n$) and world $\mathbf{w}$ ($\in \mathbf{W}$) to a set of n-tuples of objects drawn from **D**, and, for $n=0$, maps each pair $\langle \mathbf{r}^0, \mathbf{w}\rangle$ consisting of a 0-place relation $\mathbf{r}^0$ and world **w** to an element of $\{\mathbf{T}, \mathbf{F}\}$. We hereafter index **ext** to its second argument. The function $\mathbf{ext_w}$ must satisfy a separate constraint for each of the algebraic operations mentioned above (see below).

(b) $\mathbf{ext_A}$ is a function that maps each property in $\mathbf{R}_1$ to a subset of **D**.

We call $\mathbf{ext_w}(\mathbf{r^n})$ the *exemplification* extension of $\mathbf{r}^n$ at **w**. The constraints that $\mathbf{ext_w}$ must satisfy ensure that the complex relations generated by the algebraic operations have exemplification extensions at a world **w** which are defined in terms of the exemplification extensions at **w** of the simpler relations the complex relations may have as parts. For example, here are three constraints on $\mathbf{ext_w}$, governing the operations $\mathbf{PLUG}_i$, **NEG**, and **NEC**, respecitvely, for any relation $\mathbf{r} \in \mathbf{R}_n$ and objects $\mathbf{o}, \mathbf{o}_1, \ldots, \mathbf{o}_n \in \mathbf{D}$:

1. $n > 1$: $\mathbf{ext_w}(\mathbf{PLUG}_i(\mathbf{r}^n, \mathbf{o})) = \{\langle \mathbf{o}_1, \ldots, \mathbf{o}_{i-1}, \mathbf{o}_{i+1}, \ldots, \mathbf{o}_n\rangle \mid$

$\langle \mathbf{o}_1, \ldots, \mathbf{o}_{i-1}, \mathbf{o}, \mathbf{o}_{i+1}, \ldots, \mathbf{o}_n\rangle \in \mathbf{ext_w}(\mathbf{r}^n)\}$

$n = 1$: $\mathbf{ext_w}(\mathbf{PLUG}_1(\mathbf{r}^1, \mathbf{o})) = \mathbf{T}$ iff $\mathbf{o} \in \mathbf{ext_w}(\mathbf{r}^1)$

2. $n \geq 1$: $\mathbf{ext_w}(\mathbf{NEG}(\mathbf{r}^n)) = \{\langle \mathbf{o}_1, \ldots, \mathbf{o}_n\rangle \mid \langle \mathbf{o}_1, \ldots, \mathbf{o}_n\rangle \notin \mathbf{ext_w}(\mathbf{r}^n)\}$

$n = 0$: $\mathbf{ext_w}(\mathbf{NEG}(\mathbf{r}^0)) = \mathbf{T}$ iff $\mathbf{ext_w}(\mathbf{r}^0) = \mathbf{F}$

3. $n \geq 1$: $\mathbf{ext_w}(\mathbf{NEC}(\mathbf{r}^n)) =$

$\{\langle \mathbf{o}_1, \ldots, \mathbf{o}_n\rangle \mid \forall \mathbf{w}'(\langle \mathbf{o}_1, \ldots, \mathbf{o}_n\rangle \in \mathbf{ext_{w'}}(\mathbf{r}^n))\}$

$n = 0$: $\mathbf{ext_w}(\mathbf{NEC}(\mathbf{r}^0)) = \mathbf{T}$ iff $\forall \mathbf{w}'(\mathbf{ext_{w'}}(\mathbf{r}^0) = \mathbf{T})$

The constraints on the other logical functions were developed in Zalta [1983] (pp. 62-63) and won't be repeated here. But we can now redescribe two of the examples discussed above in more formal terms. In these examples, **r** is a 3-place relation and **o** is some object in **D**. The constraint on **PLUG** ensures that $\mathbf{ext_w}(\mathbf{PLUG}_1(\mathbf{r}, \mathbf{o}))$ consists of all those pairs $\langle \mathbf{o}_1, \mathbf{o}_2\rangle$ such that the triple $\langle \mathbf{o}, \mathbf{o}_1, \mathbf{o}_2\rangle$ is an element of $\mathbf{ext_w}(\mathbf{r})$. The constraint on **NEG** ensures that $\mathbf{ext_w}(\mathbf{NEG}(\mathbf{PLUG}_1(\mathbf{r}, \mathbf{o})))$ consists of all those pairs $\langle \mathbf{o}_1, \mathbf{o}_2\rangle$ that fail to elements of $\mathbf{ext_w}(\mathbf{PLUG}_1(\mathbf{r}, \mathbf{o}))$.

We call $\mathbf{ext_A}(\mathbf{r})$ the *encoding* extension of **r**. Note that $\mathbf{ext_A}$ is not defined relative to a world, and so, in a given model, stays fixed from world to world. If one were to abandon the Axiom of Encoding, one could allow this function to vary with worlds. Notice also that we have not explicitly asserted, as a *logical* axiom, that ordinary objects can't encode properties, and so we need not, at this time, constrain $\mathbf{ext_A}(\mathbf{r})$ to contain only abstract objects. It is an axiom of the proper theory of abstract objects that ordinary objects can't encode properties, and so this constraint will be satisfied by any model of the proper theory.[15]

[14] I have also been influenced by a different attempt to accomodate Etchemendy's ideas, in Menzel [1990].

[15] We should note that not only will ordinary objects fail to encode properties in models of the proper theory, but $\mathbf{ext_A}$ will stay fixed among such models. The reason is that the comprehension principle for abstract objects guarantees there is an abstract object corresponding to each (expressible) set of properties. In the intended interpretation, $\mathbf{R}_1$ is fixed and contains all the properties there are, and so the subdomain of abstract objects and the encoding extension of properties will not vary across models in which the comprehension principle is true. This gives us a two senses in which the properties an abstract object encodes are 'essential' to it. Not only are predications 'xF' necessary if possibly true, but the proper (non-logical) facts of encoding do not vary across different models of the possibluum (i.e., across different ways of assigning exemplification extensions to relations at worlds).

Though models $\mathbf{M}$ ($= \langle \mathbf{ext_w}, \mathbf{ext_A} \rangle$) are defined relative to (the intended) interpretation $\mathbf{I}$, we hereafter omit the index on $\mathbf{M}$. The main thing to remember is that models can vary depending on how they assign extensions to the relations (at worlds).

Assignments and Denotation

Now relative to our fixed interpretation $\mathbf{I}$, we define an assignment function $\mathbf{f}$ to the variables of the language as any function mapping object variables to elements of $\mathbf{D}$ and mapping n-place relation variables to elements of $\mathbf{R}_n$. Then we may define, for *each* term τ of the language, a denotation function $\boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(\tau)$ relative to interpretation $\mathbf{I}$ and assignment $\mathbf{f}$. Suppressing the indices '$\mathbf{I}$' and '$\mathbf{f}$' for the moment, we can explain how $\boldsymbol{\delta}$ works. The denotation of a constant (predicate) is what the interpretation function $\mathbf{F_I}$ assigns to that constant (predicate). The denotation of a variable is what the assignment function $\mathbf{f}$ assigns to that variable. And the denotation of a λ-expression is defined recursively, depending on the structure of the expression. As we mentioned earlier, the λ-expressions may be partitioned into one of ten mutually exclusive classes (for the complete definition, see Zalta [1983], pp. 64–65). Eight of these classes contain expressions having a syntactic structure that corresponds to an algebraic logical operation; the ninth class is the repository of the elementary λ-expressions of the form $[\lambda y_1 \ldots y_n\, F^y_1 \ldots y_n]$; the tenth class is the repository of the 0-place λ-expressions of the form $[\lambda\, \varphi]$. The denotation function $\boldsymbol{\delta}$ works as follows:

1. To each λ-expression in the first eight classes, $\boldsymbol{\delta}$ assigns a structured relation involving the algebraic operation that corresponds to the syntactic category of the λ-expression. For example, if $\boldsymbol{\delta}(R)$ and $\boldsymbol{\delta}(a)$ are the denotations of the terms R and a, respectively, then $\boldsymbol{\delta}([\lambda x\, Rxa])$ is the property $\mathbf{PLUG}_2(\boldsymbol{\delta}(R), \boldsymbol{\delta}(a))$, because the λ-expression has been syntactically defined to be the '2^{nd}-plugging' of the elementary expression $[\lambda xy\, Rxy]$ by the term a. Similarly, if $\boldsymbol{\delta}(P)$ is the denotation of the expression P, then $\boldsymbol{\delta}([\lambda x\, \Box Px])$ is the property $\mathbf{NEC}(\boldsymbol{\delta}(P))$, because the λ-expression has been syntactically defined to be the 'necessitation' of the elementary expression $[\lambda x\, Px]$.

2. To each elementary λ-expression of the form $[\lambda y_1 \ldots y_n\, F^n y_1 \ldots y_n]$, $\boldsymbol{\delta}$ assigns $\mathbf{F_I}(F^n)$ as its denotation.

3. To each 0-place λ-expression $[\lambda\, \varphi]$, $\boldsymbol{\delta}$ recursively assigns a proposition in $\mathbf{R}_0$—the structure of the proposition it assigns corresponds to the complexity of φ. For example, if $\boldsymbol{\delta}(P)$ and $\boldsymbol{\delta}(a)$ are the denotations of the terms P and a respectively, $\boldsymbol{\delta}([\lambda Pa])$ is the proposition $\mathbf{PLUG}_1(\boldsymbol{\delta}(P), \boldsymbol{\delta}(a))$. Similarly, $\boldsymbol{\delta}([\lambda\, \Box Pa])$ is $\mathbf{NEC}(\mathbf{PLUG}_1(\boldsymbol{\delta}(P), \boldsymbol{\delta}(a)))$.

The full definition of the denotation function won't be repeated here (see Zalta [1983], pp. 65–67), but the foregoing discussion should give the reader a good idea of how that definition works. Now that we can specify the denotation of all the terms of the language under $\mathbf{I}$ and $\mathbf{f}$, we can define satisfaction.

Satisfaction

Satisfaction is defined for interpred formulas and we shall use the notation $[\varphi]_{\mathbf{I}}$ to indicate the formula φ under the interpretation $\mathbf{I}$. If given a model $\mathbf{M}$ and an assignment $\mathbf{f}$, we define $\mathbf{f}$ *satisfies*$_{\mathbf{M}}$ $[\varphi]_{\mathbf{I}}$ *with respect to* world $\mathbf{w}$ as follows:[16]

1. If φ is an atomic exemplification formula of the form $\rho^n o_1 \ldots o_n$,

   $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[\rho^n o_1 \ldots o_n]_{\mathbf{I}}$ with respect to $\mathbf{w}$ iff
   $\langle \boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(o_1), \ldots, \boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(o_n) \rangle \in \mathbf{M\text{-}ext_w}(\boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(\rho^n))$

2. If φ is a atomic exemplification formula of the form ρ^0,

   $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[\rho^0]_{\mathbf{I}}$ with respect to $\mathbf{w}$ iff
   $\mathbf{M\text{-}ext_w}(\boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(\rho^0)) = \mathbf{T}$

3. If φ is an atomic encoding formula of the form $o\rho^1$,

   $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[o\rho^1]_{\mathbf{I}}$ with respect to $\mathbf{w}$ iff
   $\boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(o) \in \mathbf{M\text{-}ext_A}(\boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(\rho^1))$

4. Satisfaction for molecular and quantified φ is classical

[16] Readers familiar with Zalta [1983] and [1988] should realize that we do not include definite description in the present system, and thus we need not account for terms that may fail to denote. So we can formulate the definition of satisfaction in the usual, simple way. Had there been terms that might fail to denote, we would have to use the more general formulation in the earlier works, in which satisfaction conditions explicitly ensure that each term of the formula has a denotation.

5. If φ is a modal formula of the form $\Box\psi$,

$$\mathbf{f}\text{ satisfies}_{\mathbf{M}}\ [\Box\psi]_{\mathbf{I}} \text{ with respect to } \mathbf{w} \text{ iff}$$
$$\text{for all } \mathbf{w}' \in \mathbf{W},\ \mathbf{f}\text{ satisfies}_{\mathbf{M}}\ [\psi]_{\mathbf{I}} \text{ with respect to } \mathbf{w}'$$

Note that because (a) our algebraic operations assign exemplification extensions to propositions in the correct way, and (b) atomic exemplification formulas φ can be turned into terms $[\lambda\ \varphi]$ that denote propositions, we could have collapsed clauses (1) and (2) as follows:

If φ is any atomic exemplification formula,

$$\mathbf{f}\text{ satisfies}_{\mathbf{M}}\ [\varphi]_{\mathbf{I}} \text{ with respect to } \mathbf{w} \text{ iff}$$
$$\mathbf{M}\text{-}\mathbf{ext}_{\mathbf{w}}(\boldsymbol{\delta}([\lambda\ \varphi])) = \mathbf{T}$$

In languages where every formula φ can be turned into a 0-place term $[\lambda\ \varphi]$, this clause would be the only clause necessary for the definition of satisfaction. But the language of the modal object calculus is not such a language.

Truth and Logical Truth

Finally we turn to truth at a world, truth, and logical truth. We define $[\varphi]_{\mathbf{I}}$ is *true at* world $\mathbf{w}$ *in* model $\mathbf{M}$ (in symbols: $\mathbf{M}, \mathbf{w} \models [\varphi]_{\mathbf{I}}$) as follows:

$$\mathbf{M}, \mathbf{w} \models [\varphi]_{\mathbf{I}} \ =_{df}\ \text{Every } \mathbf{f} \text{ satisfies}_{\mathbf{M}}\ [\varphi]_{\mathbf{I}} \text{ with respect to } \mathbf{w}$$

We define $[\varphi]_{\mathbf{I}}$ is *true in* model $\mathbf{M}$ (in symbols: $\mathbf{M} \models [\varphi]_{\mathbf{I}}$) as follows:

$$\mathbf{M} \models [\varphi]_{\mathbf{I}} \ =_{df}\ \mathbf{M}, \mathbf{w}_\alpha \models [\varphi]_{\mathbf{I}}$$

And we define $[\varphi]_{\mathbf{I}}$ is logically true (in symbols: $\models [\varphi]_{\mathbf{I}}$) as follows:

$$\models [\varphi]_{\mathbf{I}} \ =_{df}\ \text{For every model } \mathbf{M},\ \mathbf{M} \models [\varphi]_{\mathbf{I}}$$

Thus, we have defined truth and logical truth for the language of the modal object calculus. As we have set things up, these notions apply to interpreted formulas. Logical truth is not defined by considering variations in the interpretations of the terms, but rather by considering various ways extensions can be assigned (at possible worlds) to the relations denoted by the predicates. The domains of objects, relations, and worlds do not vary from model to model. Rather, given a fixed interpretation of the language in the domain of objects and relations, and given a fixed set of possible worlds over which the modal operators quantify, the models are distinguished as total ways in which extensions to the relations are distributed at all possible worlds. Moreover, the denotation function is *not* relativized to a world. This means that the truth of a modal sentence such as '$\Box Pa$' is not evaluated by examining the denotation of the terms 'a' and 'P' at other possible worlds! Rather, it is evaluated by examining whether the object 'a' in fact denotes and the extension of the relation 'P' in fact denotes are structured in the right way at all possible worlds.

§4: A New Interpretation of the Calculus

Since the modal object calculus has been designed to serve as the background logic for the proper theory of abstract objects, the intended interpretation yields truth conditions for the non-logical axioms of the theory. When the calculus and proper axioms together are used to analyze philosophically puzzling data, explanatory success often depends on the fact that some abstract objects encode the very same properties that they exemplify.[17] Given our semantics, this means that in the models of the proper theory, there are properties $\mathbf{r}$ such that $\mathbf{ext}_{\mathbf{w}_\alpha}(\mathbf{r})$ and $\mathbf{ext}_{\mathbf{A}}(\mathbf{r})$ contain the same abstract objects. But recall that while abstract objects are not sets of properties (not, at least, in the intended interpretation), intuitively, they are correlated with sets of properties, and for the purpose of building a set-theoretic interpretation and set-theoretic model in which the proper axioms are true, it is natural to think of them as such. The problem is, however, in what model-theoretic sense can a (well-founded) set of properties exemplify the very properties it has as members?[18] The intended interpretation gives us no further model-theoretic understanding of the very important primitive fact that an abstract object can exemplify the very same properties that it encodes. The principal logical question concerning the set-theoretic representation of the calculus, therefore, is how to explain this primitive fact.

[17]For example, all abstract objects exemplify the property of *nonsquareness*, so consider the abstract objects that encode this property. As another example, note that Principle C, for which much of the explanatory success of the overall system depends, asserts that there are objects that encode just the property $\bar{E}!$ (i.e., the negation of the property $E!$), and so by the definition of 'abstract' and λ-Conversion, it follows that any such object exemplifies this property as well.

[18]If you try to represent xF as $F \in x$ and represent Fx as $x \in F$, then the presence of objects such that $xF \,\&\, Fx$ violates the Foundation Axiom of Zermelo-Fraenkel set theory.

Peter Aczel has offered the following suggestion for modeling the calculus and proper theory in standard Zermelo-Fraenkel set theory:

> ... start with a somewhat larger domain of concrete individuals, some of them *ordinary* and the others *special*, and assume suitable domains of ordinary n-place relations ($n \geq 0$) over the *concrete* individuals. So an ordinary proposition $Ra_1 \ldots a_n$ can be formed by an exemplification whenever R is an ordinary n-place relation and $a_1, \ldots, a_n$ are concrete, possibly special, individuals. Now form the abstract objects (i.e., sets of ordinary properties) ... . To get over the problem of ordinary exemplification of abstract objects we take the following steps. Choose, in whatever way you wish, an assignment of a concrete individual $|a|$ to each abstract individual a. (It might be best that $|a|$ should always be special, but this does not seem necessary.) On the grounds of cardinality, many abstract objects will be assigned the same concrete object. ... Extend the assignment to all objects by putting $|a| = a$ if a is already concrete. We can now take ordinary exemplification $Ra_1 \ldots a_n$ to stand for the ordinary proposition $R|a_1| \ldots |a_n|$, even when some of the $a_1, ..., a_n$ are abstract.[19]

In the remainder of this section, we follow up on Aczel's suggestion and recast it in our modal setting.

Although the 'Aczel-interpretations' are rich enough to demonstrate the consistency of the proper theory of abstract objects as well as of the modal object calculus, such interpretations will help us to visualize the idea that abstract objects, though correlated with sets of properties, can exemplify the very same properties that they encode. The reader is cautioned, however, not to think that the *intended* interpretation of the modal object calculus is an Aczel-interpretation. In Aczel-interpretations, abstract objects are identified as sets of properties. But in the intended interpretation, abstract objects are not sets of properties. In fact, sets are not included as elements of any domain, except when the theory is applied to the analysis of mathematical objects as abstract objects of a certain sort.[20] To think that abstract objects are sets is to mistake

[19]Personal communication; letter of January 10, 1991.

[20]And even then, 'the Zermelo-Fraenkel sets' are objects that encode rather than exemplify the property of being a set, since that is a property attributed to them in the theory we use to conceptualize them.

a metaphysical entity for a mathematical one. The properties that an abstract object encodes, in an important new sense, characterize that object; they are predicable of that object. But the properties that are members of a set of properties do not characterize that set in any way. Encoding predication is introduced to describe a phenomena that sets do *not* exhibit. In the technical study of the formal properties of the object calculus, sets prove useful as 'models' of abstract objects, but it doesn't follow from this that abstract objects just are sets.

Aczel-interpretations

An Aczel-interpretation $\mathbf{I}$ is formalizable as a 9-tuple $\langle \mathbf{W}, \mathbf{O}^*, \mathcal{P}([\mathbf{O}^*]^n), \mathbf{R}, \mathbf{A}, \|a\|, \mathbf{D}, |\mathbf{o}|, \mathbf{F} \rangle$ the elements of which satisfy the following conditions:

1. $\mathbf{W}$ is a nonempty set of possible worlds, and contains the distinguished element $\mathbf{w}_\alpha$.

2. $\mathbf{O}^*$ is a nonempty primitive domain of *ordinary** objects consisting of two nonempty, disjoint subsets, a set $\mathcal{O}$ of *ordinary* objects and a set $\mathcal{S}$ of *special* objects.[21]

3. $\mathcal{P}([\mathbf{O}^*]^n)$ is the power set of the n^{th} Cartesian product of $\mathbf{O}^*$, for each $n \geq 1$. That is, for $n \geq 1$,

$$\mathcal{P}([\mathbf{O}^*]^n) = \mathcal{P}(\underbrace{\mathbf{O}^* \times \ldots \times \mathbf{O}^*}_{n \text{ times}})$$

   In the definition of a model, we will take the exemplification extension of an n-place relation $\mathbf{r}^n$ ($n \geq 1$) at world $\mathbf{w}$ to be a member of this set.

4. $\mathbf{R}$ is a nonempty, primitive domain of *relations* satisfying the following two conditions:

[21]In the above quotation, Aczel called the members of $\mathbf{O}^*$ the 'concrete' objects. However, we are calling them *ordinary** objects. The ordinary objects (i.e., the members of $\mathcal{O}$) constitute a subset of the ordinary* objects, and intuitively, the 'concrete' objects constitute a subset of the ordinary objects, namely, the ones that exemplify spatiotemporal location at the actual world. So the ordinary objects divide up into the concrete objects and the objects that have spatiotemporal locations at other possible worlds. We should also mention that, intuitively, $\mathcal{S}$ should outstrip $\mathcal{O}$ in size by a couple of orders of magnitude, but for the present purposes, we need not make this explicit.

(a) $\mathbf{R}$ is the union of a sequence of nonempty, pairwise disjoint sets, $\mathbf{R}_0, \mathbf{R}_1, \ldots$ (i.e., $\mathbf{R} = \bigcup_{n \geq 0} \mathbf{R}_n$) such that each member $\mathbf{R}_n$ (the set of *n-place relations*), for $n \geq 1$, is greater in size than the set of all functions from $\mathbf{W}$ into $\mathcal{P}([\mathbf{O}^*]^n)$.

(b) $\mathbf{R}$ is closed under the operations $\mathbf{PLUG}_i$, $\mathbf{NEG}$, $\mathbf{COND}$, $\mathbf{UNIV}_i$, $\mathbf{REFL}_{i,j}$, $\mathbf{CONV}_{i,j}$, $\mathbf{NEC}$, and $\mathbf{VAC}_i$.

Condition (a) will guarantee that there are more relations than there are Montagovian intensions (Montague [1974]). This will ensure that some distinct relations have the same Montagovian intension.

5. $\mathbf{A} = \mathcal{P}(\mathbf{R}_1)$; i.e., the set $\mathbf{A}$ of *abstract* objects is simply the power set of the set of properties.

6. $\|\mathbf{a}\|$ is a mapping which is defined for $\mathbf{a} \in \mathbf{A}$ and which takes values in $\mathcal{S}$. Recall that $\mathcal{S}$ is the set of special objects and is a subset of $\mathbf{O}^*$. The special object $\|\mathbf{a}\|$ will serve as a proxy for $\mathbf{a}$

7. $\mathbf{D} = \mathbf{A} \cup \mathbf{O}^*$; i.e., the domain $\mathbf{D}$ of *all* objects is the union of $\mathbf{A}$ and $\mathbf{O}^*$.

8. $|\mathbf{o}|$ is a mapping which is defined for $\mathbf{o} \in \mathbf{D}$ and which takes values in $\mathbf{O}^*$. It must satisfy the following conditions:

   if $\mathbf{o} = \mathbf{a}$, for some $\mathbf{a} \in \mathbf{A}$, $|\mathbf{o}| = \|\mathbf{a}\|$
   if $\mathbf{o} = \mathbf{o}^*$, for some $\mathbf{o}^* \in \mathbf{O}^*$, $|\mathbf{o}| = \mathbf{o}^*$

   This extends the function $\|\mathbf{a}\|$ to a function $|\mathbf{o}|$ defined on all the members of $\mathbf{D}$. $|\mathbf{o}|$ agrees with $\|\mathbf{a}\|$ on the abstract objects and maps each ordinary* object to itself.

9. $\mathbf{F}$ is an interpretation function that maps the object constants of the language to an element of $\mathbf{D}$ and the n-place relation constants to an element of $\mathbf{R}_n$.

Models for Aczel-interpretations

A model $\mathbf{M}$ for an Aczel-interpretation $\mathbf{I}$ consists of two functions:

1. **ext** is a function that meets the following conditions:

   (a) for $n \geq 1$, $\mathbf{ext}\colon \mathbf{R}_n \times \mathbf{W} \to \mathcal{P}([\mathbf{O}^*]^n)$.

   (b) for $n = 0$, $\mathbf{ext}\colon \mathbf{R}_0 \times \mathbf{W} \to \{\mathbf{T}, \mathbf{F}\}$.

   (c) **ext** satisfies the constraints on the operations $\mathbf{PLUG}_i$, $\mathbf{NEG}$, $\mathbf{NEC}$, $\mathbf{COND}$, $\mathbf{UNIV}_i$, $\mathbf{REFL}_{i,j}$, $\mathbf{CONV}_{i,j}$, and $\mathbf{VAC}_i$.

   We henceforth index **ext** to its second argument, and so $\mathbf{ext_w}$ maps each $\mathbf{r}^n \in \mathbf{R}_n$ to its exemplification extension at world $\mathbf{w}$.

2. $\mathbf{ext_A}$ is a function defined on the elements of $\mathbf{R}_1$ as follows:

   $$\mathbf{ext_A}(\mathbf{r}^1) = \{\mathbf{o} \in \mathbf{D} \,|\, \mathbf{o} \in \mathbf{A} \text{ and } \mathbf{r}^1 \in \mathbf{o}\}.$$

Two observations are in order here. The first is that the function $\mathbf{ext_w}$ is essentially the same function as its counterpart in the models for the intended interpretation, with the exception that it takes values in $\mathcal{P}([\mathbf{O}^*]^n)$ rather than $\mathcal{P}([\mathbf{D}]^n)$. Since the abstract objects (i.e., sets of properties) in $\mathbf{A}$ are not elements of $\mathbf{O}^*$, they are not officially in the exemplification extension of any relation. But we will exploit the fact that their *proxies* are in the exemplification extensions of relations: the definition of satisfaction will allow an abstract object $\mathbf{a}$ to 'exemplify' a relation if its proxy $\|\mathbf{a}\|$ is an element of the exemplification extension of that relation. The second observation is that the function $\mathbf{ext_A}$ is more specific than its counterpart in the models for the intended interpretation. It requires that the encoding extension of a property contain as elements all those abstract objects of which it is a member. Notice that the encoding extension of a property neither varies from world to world nor varies from model to model.[22] The different models for a given Aczel-interpretations can therefore only vary in the way they assign exemplification extensions to relations at the various possible worlds.

The Denotation Function

If given an Aczel-interpretation $\mathbf{I}$, we next fix an assignment $\mathbf{f}$ to the variables of the object language, as before. Then we may define a denotation function $\boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(\tau)$, relative to interpretation $\mathbf{I}$ and function $\mathbf{f}$, for all the terms τ of the language, as follows (suppressing the subscripts on $\boldsymbol{\delta}$):

[22]The reason is that in Aczel-interpretations, the domain of abstract objects $\mathbf{A}$ is fixed as the power set of the set properties. Since the set of properties $\mathbf{R}_1$ is an element of the Aczel-interpretation, its power set is independent of models $\mathbf{M}$. So whether an object $\mathbf{o} \in \mathbf{D}$ encodes a property $\mathbf{r}$ is simply a matter of whether $\mathbf{o}$ is an element of the subdomain $\mathbf{A}$ and $\mathbf{r} \in \mathbf{o}$. So the $\mathbf{ext_A}$ function is fixed in all models for Aczel-interpretations.

(a) where o is an object constant, $\boldsymbol{\delta}(o) = \mathbf{F_I}(o)$

(b) where o is an object variable, $\boldsymbol{\delta}(o) = \mathbf{f}(o)$

(c) where ρ^n is an n-place predicate, $\boldsymbol{\delta}(\rho^n) = \mathbf{F_I}(\rho^n)$

(d) where ρ^n is an n-place predicate variable, $\boldsymbol{\delta}(\rho^n) = \mathbf{f}(\rho^n)$

(e) where μ is a λ-expression $[\lambda\nu_1 \ldots \nu_n\, \varphi]$, $\boldsymbol{\delta}(\mu)$ is defined as on pp. 65–66 of Zalta [1983], with the following substitution:

where μ is the i^{th}-plugging of λ-expression ξ by term o,
$\boldsymbol{\delta}(\mu) = \mathbf{PLUG}_i(\boldsymbol{\delta}(\xi), |\boldsymbol{\delta}(o)|)$

(f) where $[\lambda\, \varphi]$ is a 0-place λ-expression, $\boldsymbol{\delta}([\lambda\, \varphi])$ is defined as on p. 67 of Zalta [1983], with the following substitution:[23]

if $\varphi = \rho^n o_1 \ldots o_n$, $\boldsymbol{\delta}([\lambda\, \varphi]) =$

$$\mathbf{PLUG}_1(\ldots(\mathbf{PLUG}_n(\boldsymbol{\delta}(\rho^n), |\boldsymbol{\delta}(o_n)|), \ldots), |\boldsymbol{\delta}(o_1)|)$$

The denotation function, therefore, works in the same manner as its counterpart in the previous section, except for the changes described in clauses (e) and (f). If we let $\boldsymbol{\delta}(o)$ be the object $\mathbf{o}$, then these clauses tell us that the complex relational properties and propositions denoted by λ-expressions containing the object term o have the object $|\mathbf{o}|$ as a constituent instead of the object $\mathbf{o}$. This makes a difference only for the complex relations and propositions denoted by λ-expressions containing terms that denote an abstract object. In such cases, the relations and propositions in question have the proxies rather than the abstract objects themselves as constituents.

Satisfaction for Interpreted Formulas

If given an arbitrary formula φ, we may now define, assignment $\mathbf{f}$ *satisfies*$_{\mathbf{M}}$ $[\varphi]_{\mathbf{I}}$ *with respect to* world $\mathbf{w}$, as follows:

1. If φ is an atomic exemplification formula of the form $\rho^n o_1 \ldots o_n$,

$\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[\rho^n o_1 \ldots o_n]_{\mathbf{I}}$ with respect to $\mathbf{w}$ iff

$$\langle|\boldsymbol{\delta}_{\mathbf{I,f}}(o_1)|, \ldots, |\boldsymbol{\delta}_{\mathbf{I,f}}(o_n)|\rangle \in \mathbf{M\text{-}ext_w}(\boldsymbol{\delta}_{\mathbf{I,f}}(\rho^n))$$

2. If φ is a atomic exemplification formula of the form ρ^0,

$\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[\rho^0]_{\mathbf{I}}$ with respect to $\mathbf{w}$ iff

$$\mathbf{M\text{-}ext_w}(\boldsymbol{\delta}_{\mathbf{I,f}}(\rho^0)) = \mathbf{T}$$

3. If φ is an atomic encoding formula of the form $o\rho^1$,

$\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[o\rho^1]_{\mathbf{I}}$ with respect to $\mathbf{w}$ iff

$$\boldsymbol{\delta}_{\mathbf{I,f}}(o) \in \mathbf{M\text{-}ext_A}(\boldsymbol{\delta}_{\mathbf{I,f}}(\rho^1))$$

4. Satisfaction for molecular and quantified φ is classical

5. If φ is a modal formula of the form $\Box\psi$,

$\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[\Box\psi]_{\mathbf{I}}$ with respect to $\mathbf{w}$ iff

for all $\mathbf{w}' \in \mathbf{W}$, $\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[\psi]_{\mathbf{I}}$ with respect to $\mathbf{w}'$

The only real difference between the definition of satisfaction for Aczel-interpretations and its counterpart for intended interpretations concerns clause 1. It defines the sense in which an abstract object can exemplify a relation, namely, by proxy. Consider the formula 'Pa' under $\mathbf{I}$, and suppose that $\mathbf{F_I}(a) = \mathbf{o}$ and $\mathbf{F_I}(P) = \mathbf{r}$. Then $\mathbf{f}$ satisfies$_{\mathbf{M}}$ 'Pa' at $\mathbf{w}$ iff either (1) $\mathbf{o}$ is ordinary* and an element of $\mathbf{M\text{-}ext_w}(\mathbf{r})$ (since $|\mathbf{o}|$ is just $\mathbf{o}$ itself when $\mathbf{o}$ is ordinary*), or (2) $\mathbf{o}$ is some abstract object $\mathbf{a}$ and $\|\mathbf{a}\|$ is an element of $\mathbf{M\text{-}ext_w}(\mathbf{r})$ (since $|\mathbf{o}|$ is $\|\mathbf{o}\|$ when $\mathbf{o}$ is an abstract object). We should also remark that it would have been somewhat more direct to define clause 3 in the definition of satisfaction as:

$\mathbf{f}$ satisfies$_{\mathbf{M}}$ $[o\rho^1]_{\mathbf{I}}$ with respect to $\mathbf{w}$ iff

$$\boldsymbol{\delta}_{\mathbf{I,f}}(o) \in \mathbf{A} \text{ and } \boldsymbol{\delta}_{\mathbf{I,f}}(\rho^1) \in \boldsymbol{\delta}_{\mathbf{I,f}}(o)$$

As it stands, however, clause 3 demonstrates that the satisfaction of atomic encoding formulas appeals to an *extension* of the property denoted.

[23] In Zalta [1983], we let φ itself go proxy for the 0-place relation term $[\lambda\, \varphi]$. But it now seems more perspicuous to avoid such an abbreviation.

Truth and Logical Truth

With satisfaction defined, we may define $[\varphi]_{\mathbf{I}}$ is *true at* world $\mathbf{w}$ *in* model $\mathbf{M}$ in the usual way:

$$\mathbf{M}, \mathbf{w} \models [\varphi]_{\mathbf{I}} \;=_{df}\; \text{Every } \mathbf{f} \text{ satisfies}_{\mathbf{M}}\ [\varphi]_{\mathbf{I}} \text{ with respect to } \mathbf{w}$$

And, finally, we define $[\varphi]_{\mathbf{I}}$ is *true in* model $\mathbf{M}$ and define $[\varphi]_{\mathbf{I}}$ is *logically true*, as follows:

$$\mathbf{M} \models [\varphi]_{\mathbf{I}} \;=_{df}\; \mathbf{M}, \mathbf{w}_\alpha \models [\varphi]_{\mathbf{I}}$$

$$\models [\varphi]_{\mathbf{I}} \;=_{df}\; \text{For every model } \mathbf{M}, \mathbf{M} \models [\varphi]_{\mathbf{I}}$$

§5: Conclusion

Aczel-interpretations demonstrate the consistency of the modal object calculus as well as the consistency of the proper theory of abstract objects. Though we shall not develop the proofs in detail here, it is relatively straightforward to show that, under Aczel-interpretations, the five groups of logical axioms for the calculus described in §1 are logically true and the three rules of inference preserve logical truth. In particular, the proof that the axioms of propositional logic, quantificational logic, and S5 modal logic are logically true are essentially classical, as are the proofs that the rules MP, Gen, and RN preserve logical truth. To see that the axiom of encoding (i.e., $\Diamond xF \rightarrow \Box xF$) is logically true, pick an arbitrary model $\mathbf{M}$ and reason from the point of view of the actual world: if at some possible world $\boldsymbol{\delta}(x) \in \mathbf{ext_A}(\boldsymbol{\delta}(F))$, then since the encoding extension of a property does not vary from world to world, $\boldsymbol{\delta}(x) \in \mathbf{ext_A}(\boldsymbol{\delta}(F))$ at all worlds.

To see that the axioms for the logic of relations are logically true, consider first the λ-Conversion principle with respect to an arbitrarily chosen model $\mathbf{M}$. Given the facts that the denotation function assigns each λ-expression an appropriately structured relation and that $\mathbf{ext_w}$ is appropriately constrained for each logical function, it is relatively straightforward to show, by induction on the complexity of φ, that for every world $\mathbf{w}$, every assignment function $\mathbf{f}$ satisfies the formula $[\lambda y_1 \ldots y_n\, \varphi]x_1 \ldots x_n$ iff $\mathbf{f}$ satisfies $\varphi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$ with respect to $\mathbf{w}$. For no matter what n-tuple of objects $\langle \mathbf{o}_1, \ldots, \mathbf{o}_n \rangle$ that $\mathbf{f}$ assigns to the variables $x_1, \ldots, x_n$, that n-tuple is an element of $\mathbf{ext_w}(\boldsymbol{\delta}([\lambda y_1 \ldots y_n\, \varphi]))$ iff $\mathbf{f}$ satisfies (at $\mathbf{w}$) the formula that results by replacing the y_i in φ with the x_i, that is, iff the objects $\mathbf{o}_1, \ldots, \mathbf{o}_n$ are in the exemplification extensions of relations at $\mathbf{w}$ in just the way $\varphi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$ says.[24] The λ-Identity$_1$ principle ($[\lambda y_1 \ldots y_n\, F^n y_1 \ldots y_n] = F^n$) is logically true, notwithstanding the special definition of relation identity (in which F^n and G^n are said to be 'identical' iff no matter how you 'plug' them up with $n-1$ objects the two resulting properties are encoded by the same objects). This principle holds because the denotation function guarantees that the denotation of $[\lambda y_1 \ldots y_n\, F^n y_1 \ldots y_n]$ is the same relation as the denotation of F^n. Thus, in an arbitrary model, the *defined* conditions for relation identity between $[\lambda y_1 \ldots y_n\, F^n y_1 \ldots y_n]$ and F^n will be trivially true at the actual world, given that these expressions denote the same relation in the semantics. Finally, the λ-Identity$_2$ principle ($[\lambda y_1 \ldots y_n\, \varphi] = [\lambda y'_1 \ldots y'_n\, \varphi']$) is logically true, since mere alphabetic changes don't affect the denotation of the λ-expression. Again, the special definition of relation identity is trivially true at the actual world (in any arbitrarily chosen model) if $[\lambda y_1 \ldots y_n\, \varphi]$ and $[\lambda y'_1 \ldots y'_n\, \varphi']$ denote the same relation.

Finally, readers familiar with the proper theory of abstract objects may also wish to consider the constraints that must be placed on the models for Aczel-interpretations if the proper axioms of the theory of abstract objects are to turn out true. We simply require that models assign an extension to the property denoted by the distinguished 1-place relation constant $E!$ (having a spatiotemporal location) in the right way. Consider those models $\mathbf{M}$ for an Aczel-interpretation $\mathbf{I}$ that satisfy the following two conditions:

1. $\mathbf{M}\text{-}\mathbf{ext_w}(\mathbf{F_I}(E!)) \subseteq \mathcal{O}$, for each $\mathbf{w}$
2. $\mathcal{O} = \bigcup_{\mathbf{w} \in \mathbf{W}} \mathbf{M}\text{-}\mathbf{ext_w}(\mathbf{F_I}(E!))$

[24]As we remarked earlier, a given λ-expression is syntactically categorized either as the plugging of another λ-expression in the i^{th} place by a certain term, or as the conditionalization of two λ-expressions, or as the negation of another λ-expression, or as the conversion of another expression about the i^{th} and j^{th} places, etc. Each syntactic category corresponds to an algebraic logical operation. Complex λ-expressions can therefore be thought of as structural transformations of simpler ones, and the denotation of the whole expression will be built up from the denotations of the simpler expressions in a way that mirrors the transformation process. The constraint on $\mathbf{ext_w}$ of the complex relation denoted at the final stage of transformation can then be decomposed into a variety of constraints on the simpler relations denoted by the simpler expressions involved in the transformation process. So the semantic reason why objects exemplify the complex relation at a world iff they stand in the simpler relations at that world is that the constraints on the exemplification extension of the complex relation decompose into the right constraints on the exemplification extensions of the simpler relations it may have as parts. That is why the λ-conversion principle is logically true.

In such models, the property denoted by $E!$ (having spatiotemporal location) has only ordinary objects in its exemplification extension at each world (recall that $\mathcal{O}$ is the set of ordinary objects and constitutes a subset of the ordinary* objects in $\mathbf{O}^*$). Note that these conditions allow the same ordinary object to appear in the exemplification extension of $\mathbf{F_I}(E!)$ at more than one world. Therefore, the property denoted by the defined 1-place relation constant $O!$ (i.e., $[\lambda x \Diamond E!x]$), which identifies the 'ordinary' objects, will have an exemplification extension at each world $\mathbf{w}$ that consists of all the objects that exemplify existence at some world or other.[25] So at each $\mathbf{w}$ (including $\mathbf{w}_\alpha$), $\mathbf{ext_w}(\boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(O!)) = \mathcal{O}$, no matter what the assignment $\mathbf{f}$. Moreover, the property denoted by the defined 1-place relation constant $A!$ (i.e., $[\lambda x\, \neg\Diamond E!x]$), which identifies the 'abstract' objects, will have an exemplification extension at each world that consists of all the special objects in $\mathcal{S}$. Consequently, each set $\mathbf{a}$ in the power set of the set of properties (i.e., each $\mathbf{a} \in \mathcal{P}(\mathbf{R}_1)$) will be in the exemplification extension of the property denoted by $A!$ in each world, since $|\mathbf{a}|$ is an element of $\mathcal{S}$.[26] These facts guarantee that Principle B turns out true: the assertion that ordinary objects necessarily fail to encode properties (i.e., $O!x \rightarrow \Box\neg\exists F\, xF$) is true in models satisfying the above constraints, since no ordinary objects encode properties at any world. If $\boldsymbol{\delta}(x) \in \mathcal{O}$, then $\boldsymbol{\delta}(x) \notin \mathbf{A}$, and so (at every world $\mathbf{w}$) no property $\mathbf{r}$ is such that $\mathbf{r} \in \boldsymbol{\delta}(x)$.

In any model $\mathbf{M}$ for an Aczel-interpretation satisfying the above constraints, the comprehension principle for abstract objects is true. Recall Principle C:

[25]To see why, suppose for simplicity that $\Diamond$ is a primitive modal operator and that there is a primitive algebraic logical operation **POS** which is the dual of **NEC**. Then the denotation of $[\lambda x \Diamond E!x]$ would be defined as: $\mathbf{POS}(\mathbf{F_I}(E!))$. Constraints on $\mathbf{ext_w}$ would guarantee that:

$$\mathbf{ext_w}(\mathbf{POS}(\mathbf{F_I}(E!))) \;=\; \{\mathbf{o} \,|\, \exists \mathbf{w}'(\mathbf{o} \in \mathbf{ext_{w'}}(\mathbf{F_I}(E!)))\}$$

So at the actual world, the exemplification extension of the property denoted by $O!$ consists of all those objects that exemplify existence at some world or other.

[26]For those readers intimately familiar with the theory, we also need to give an extension to the primitive relation of identity for the ordinary objects. In more precise formulations of the proper theory, we employ the distinguished 2-place relation constant $=_E$ to denote this relation and Principle A (described in §1) is cast as an axiom that gives necessary and sufficient conditions for $x =_E y$, namely, that x and y are both ordinary objects which necessarily exemplify the same properties. To validate this axiom, we must constrain models $\mathbf{M}$ as follows:

$$\mathbf{M\text{-}ext_w}(\mathbf{F_I}(=_E)) = \{\langle \mathbf{o}, \mathbf{o}\rangle \,|\, \mathbf{o} \in \mathcal{O}\}, \text{ for each } \mathbf{w}$$

This validates the axiom of identity for ordinary objects.

$$\exists x(A!x \;\&\; \forall F(xF \equiv \varphi)), \text{ where } \varphi \text{ has no free } x\text{s.}$$

The reason this axiom is true in a model $\mathbf{M}$ satisfying the above constraints is that, for any given φ, there is an abstract object (i.e., a set of properties) $\mathbf{a}$ in $\mathbf{A}$ such that:

(a) $|\mathbf{a}| \in \mathcal{S}$, and

(b) $\boldsymbol{\delta}_{\mathbf{I},\mathbf{f}}(F) \in \mathbf{a}$ iff $\mathbf{f}$ satisfies$_{\mathbf{M}}$ φ w.r.t. $\mathbf{w}_\alpha$ (for any assignment $\mathbf{f}$).

Since we have the power set of the set of properties to choose from, we know that there is a set of properties having as members precisely those properties that 'satisfy' the formula φ. Notice that the definition for the identity of abstract objects (Principle D) is also justified. The definition says that abstract objects are 'identical' iff necessarily, they encode the same properties. But clearly, in Aczel-interpretations, abstract objects $\mathbf{a}$ and $\mathbf{b}$ are sets of properties, and so they are identical iff they have the same properties as elements. And if $\mathbf{a}$ and $\mathbf{b}$ have the same properties as elements, this fact is true at every possible world. So, the semantic fact that $\mathbf{a} = \mathbf{b}$ iff $\forall \mathbf{w} \forall \mathbf{r}(\mathbf{r} \in \mathbf{a}$ iff $\mathbf{r} \in \mathbf{b})$ justifies the object-language definition that two abstract objects x and y are 'identical' iff necessarily, they encode the same properties.

Bibliography

Bealer, G., 1982, *Quality and Concept*, Oxford: Clarendon

Carnap, R., 1947, *Meaning and Necessity*, Chicago: University of Chicago Press

Castañeda, H., 1974, 'Thinking and the Structure of the World', *Philosophia* **4**: 3–40

Došen, K., 1988, 'Second-order Logic Without Variables', in *Categorial Grammar*, edited by W. Buszkowski, W. Marciszewski, and J. van Benthem, Amsterdam: John Benjamins

Etchemendy, J., 1990, *The Concept of Logical Consequence*, Cambridge, MA: Harvard University Press

Frege, G., 1892, 'On Sense and Reference', in *Translations from the Philosophical Writings of Gottlob Frege*, P. Geach and M. Black (eds.), Oxford: Basil Blackwell, 1970, 56–78

Hintikka, J., 1961, 'Modality and Quantification', *Theoria* **27**: 110–128

Kripke, S., 1959, 'A Completeness Theorem in Modal Logic', *Journal of Symbolic Logic* **24**: 1–14

Kripke, S., 1963, 'Semantical Considerations on Modal Logic', *Acta Philosophica Fennica* **16**: 83–94

Mally, E., 1912, *Gegenstandstheoretische Grundlagen der Logik und Logistik*, Leipzig: Barth

Meinong, A., 1904, 'Über Gegenstandstheorie', in *Untersuchungen zur Gegenstandstheorie und Psychologie*, A. Meinong (ed.), Leipzig: Barth; English translation, 'On the Theory of Objects', in R. Chisholm (ed.), *Realism and the Background of Phenomenology*, Glencoe: The Free Press, 1960: 76–117

Menzel, C., 1990, 'Actualism, Ontological Commitment, and Possible World Semantics', *Synthese* **85**: 355–89

Menzel, C., 1986, 'A Complete, Type-free "Second-order" Logic and Its Philosophical Foundations', Technical Report No. CSLI–86–40, Stanford, CA: Center for the Study of Language and Information Publications

Montague, R., 1974, *Formal Philosophy: Selected Papers of Richard Montague*, Richmond Thomason (ed.), New Haven: Yale University Press

Parsons, T., 1980, *Nonexistent Objects*, New Haven: Yale University Press

Quine, W., 1960, 'Variables Explained Away', in *Selected Logical Papers*, New York: Random House, 227–235

Rapaport, W., 'Meinongian Theories and a Russellian Paradox', *Nous* **12**: 153–180

Routley, R., 1979, *Exploring Meinong's Jungle and Beyond*, Departmental Monograph #3, Philosophy Program, Research School of the Social Sciences, Australian National University, Canberra

Tarski, A., 1944, 'The Semantical Conception of Truth and the Foundations of Semantics', *Philosophy and Phenomenological Research* **4**: 341–376

Tarski, A., 1936, 'On the Concept of Logical Consequence', translated from the Polish and published in *Logic, Semantics, and Metamathematics*, Oxford: Clarendon, 1956

———, 1993a, 'Twenty-Five Basic Theorems in Situation and World Theory', *Journal of Philosophical Logic* **22** (1993): 385–428;

Zalta, E., 1993b, 'A Philosophical Conception of Propositional Modal Logic', *Philosophical Topics* **21**/2 (Fall 1993): 263-281

———, 1988, *Intensional Logic and the Metaphysics of Intentionality*, Cambridge, MA: MIT/Bradford

———, 1987, 'On the Structural Similiarities Between Worlds and Times', *Philosophical Studies* **51**: 213–239

———, 1983, *Abstract Objects: An Introduction to Axiomatic Metaphysics*, Dordrecht: D. Reidel