Mathematical Pluralism*

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Abstract

Mathematical pluralism can take one of three forms: (1) every consistent mathematical theory consists of truths about its own domain of individuals and relations; (2) every mathematical theory, consistent or inconsistent, consists of truths about its own (possibly uninteresting) domain of individuals and relations; and (3) many of the principal philosophies of mathematics are based upon some insight or truth about the nature of mathematics that can be preserved. (1) includes the multiverse approach to set theory. (2) helps us to understand the significance of the distinguished non-logical individual and relation terms of even inconsistent theories. (3) is a metaphilosophical form of mathematical pluralism and hasn't been discussed in the literature. In what follows, I show how the analysis of theoretical mathematics in object theory exhibits all three forms of mathematical pluralism.

1 Introduction

It is no overstatement to say that mathematics plays a central role in our scientific understanding of the universe. Mathematics is so ubiquitous that it is essential to the formulation of many physical theories. Indeed,

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at least one physicist has argued that the physical universe just *is* an abstract mathematical structure (Tegmark 2008). Similarly, some philosophers of mind suggest that subjective experiences (e.g., color sensations) can be strictly identified with an activation *vector* in a certain kind of mathematically-described space (Churchland 2005). And one doesn't go far in cognitive science before encountering the view that concepts are mathematically-defined regions in a similarity space (Churchland 1995, 83, 90; 1998; Gärdenfors 2000, 2014).[1]

Given the centrality of mathematics, it becomes incumbent upon us to have a philosophical analysis not only of the content of mathematical claims but also of how we can come to know that content. Without such a combined analysis, views such as platonism have an epistemological problem (namely, how to account for knowledge of mathematics) and views such as naturalism have an ontological problem (namely, how to account for the content of mathematical claims). So for a discipline this important, we should, at the very least, be able to (a) say what the terms of mathematical languages signify, (b) state conditions, if there are such, under which the sentences of mathematical languages are true, and (c) give some epistemological account of mathematical knowledge.

In the 20th century, one of the main strategies for addressing these problems was *mathematical foundationalism*, i.e., the view that all of mathematics is reducible to some foundational mathematical theory (set theory, category theory, constructive type theory, etc.), thereby reducing the problem of accounting for the content and epistemology of multiple mathematical theories to the problem of accounting for the content and epistemology of the foundational one. Quine (1948 [1980], 1970) famously committed himself to sets on the grounds that our best scientific theories quantify over set-theoretically-reducible mathematical objects, and took the epistemology of mathematics to be part of the epistemology of natural science.[2]

But Quine's strategy offers no account of unapplied mathematics and leaves one puzzled about how to adjudicate disputes among set theorists arguing the question: which set-theoretic axioms (among those that go beyond the requirements of physical theories) are true? And, at least

[1] For example, in Gärdenfors 2014, we find that "a property is a convex region in some domain" (24), and "properties, as characterized . . . , form a special case of *concepts*" (25).

[2] See Colyvan 2001 for a nice discussion of the indispensability argument in the philosophy of mathematics and a full discussion of the issues regarding Quine's decision.

since Benacerraf 1965, philosophers have been suspicious of theory reduction in mathematics. Even if we can reduce (by some standard of reduction, e.g., relative interpretability) some particular number theory N to some particular set theory, it doesn't follow that the terms of N denote sets. In general, the fact that mathematical theory T can be reduced to theory T' doesn't imply that the quantifiers of T range over what the quantifiers of T' range over.

An alternative philosophical account of mathematics, with roots in Hilbert and Carnap, has been gathering some momentum in recent years, namely, *mathematical pluralism*. The *first* and most common form of mathematical pluralism is the view that every consistent mathematical theory consists of truths about its own domain of individuals and relations. Thus, Hilbert is a pluralist, at least early on in his career, in virtue of his conditional principle that if a mathematical theory T is consistent, then the objects systematized by T exist.[3] Carnap is also an early pluralist, since he took each linguistic framework to be about the objects and relations represented by its primitive notions (1950 [1956]).[4] But Carnap refused to draw any conclusions about the 'external' existence of the objects and relations of *any* framework – at best, such external existence questions were seen as questions about the expediency of adopting one framework rather than another.

In what follows, we shall be construing this first form of mathematical pluralism in complete generality, as the acceptance of any consistent theory stated in terms of generally accepted mathematical primitives, without attempting to distinguish only some of those theories as true. This form of mathematical pluralism is unmoved by Resnik's objection to deductivism when he suggested that mathematicians would say "Sure, set theory with the negation of the pairing axiom is consistent, but it is not true" (1980, 133). Resnik was concerned that deductivism would legitimize random theory development, which seems inconsistent with

[3]The letter from Hilbert to Frege, dated 29 December 1899, is the classical source:

> . . . if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist.

See Gabriel et al. (eds.) 1980 (39). I emphasize that this is a very early statement from Hilbert concerning consistency and existence. See Detlefsen 1993 and Dean 2021 for nuanced approaches to the evolution of Hilbert's views. In any case, this quote from Hilbert nicely captures the position, even if Hilbert later came to hold a different view.

[4]See Kissel m.s., however, where it is denied that Carnap is a *logical* pluralist.

mathematical practice.[5] But a pluralist is not suggesting that the constraints on mathematical practice be abandoned.[6] Set theory without pairing is a legitimate, if unusual, set concept among the many different conceptions of sets. And who is to say that it won't one day prove useful in the development of a natural science?[7]

This first form of mathematical pluralism has been defended, or described as a serious option, by a number of recent authors. Despite his reservations about deductivism, Resnik later (1989) seems to suggest that we can obtain a naturalized epistemology if we suppose that each mathematical theory 'postulates' or 'posits' the relevant mathematical objects. Both Field and Balaguer have argued that if one is going to be a platonist, one should adopt a plenitude principle on which every possible mathematical object exists, so that each mathematical theory describes some part of mathematical reality (Field 1994, 392, 420–422; and Balaguer 1995, 1998a,b). Linsky & Zalta (1995) argued that the nonlogical expressions of arbitrary mathematical theories can be interpreted in terms of well-defined descriptions that denote abstract objects and abstract relations governed by an unrestricted comprehension principle.

[5]In Resnik (1980, 132), we find:

> According to the deductivist, it would be perfectly legitimate for mathematicians to make up axiom sets through some random method and then proceed to investigate their logical properties. But . . . we would not develop a set theory with the negation of the pair-set axiom, although it is possible . . . .

[6]Bueno 2011, in explaining why *relativism* in mathematics doesn't imply 'anything goes', nicely puts the point as follows (555):

> . . . mathematicians do meet various constraints while doing mathematics: they adopt an underlying logic (even if only implicitly); they embrace standards of rigor (even if the latter change in time); they work within certain frameworks, which constrain the acceptable definitions, allowable moves in a proof, and specify the suitable language they can use.

And yet he subsequently notes that:

> . . . everything in mathematics is in principle revisable. Definitions of mathematical concepts, fundamental mathematical principles, even the underlying logic are all revisable, and have been revised, in mathematical practice.

And, finally, Bueno correctly points out (558) that not every theory is equally fruitful, equally rich, or equally acceptable.

[7]So the enhanced form of deductivism in Maddy (forthcoming), won't count as mathematical pluralism, since it takes only some mathematical theories to be true, as determined on the basis of mathematical practice, which includes "the mathematically rational process of selecting concepts and assumptions" (forthcoming, 23).

Structuralists (Shapiro 1997, Resnik 1997) could be seen as endorsing mathematical pluralism in so far as they take arbitrary mathematical theories to be about structures (Nodelman & Zalta 2014 explicitly do so). Inferentialists (Wittgenstein 1956; Sellars 1953 [1980], 1974) are pluralists as well, in so far as they take the meaning of the terms of arbitrary mathematical theories to be captured by their inferential roles. And we shall count, as a final example of this first form of pluralism, the multiverse approach to set theory, as described in Hamkins 2012. Hamkins claims (216) that "there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths". These views will be discussed in some detail in Section 2. Though, for reasons that won't be developed until Section 4.2, we shall not count modal structuralism (or deductivism generally) as a kind of mathematical pluralism, notwithstanding the argument in Hellman & Bell (2006).

It should be noted that this first form of mathematical pluralism extends beyond classical mathematics to constructive, intuitionistic, finitist, and other types of consistent mathematical theories. For example, Davies (2005) argues for the 'validity' of both classical and constructive mathematics (253):

> The debate about which is the 'right' way to do mathematics is sterile and counterproductive. Each of the frameworks is valid and has advantages in appropriate circumstances. As we explain below, this pluralistic viewpoint is not our invention, but we believe that some of the arguments that we marshal in favour of it are novel.

He ends up with the view that mathematical statements are not true simpliciter but only relative to a theory. In 2005 (257) we find:

> We accept Kreisel's dictum that the important issue is not the existence of mathematical objects, but rather the objectivity of mathematical statements ... We define a theorem to be a statement made within a particular mathematical framework together with some proof of that statement. The same statement in a different framework is regarded as a different theorem. ... When talking about mathematics, as opposed to the philosophy of mathematics, one does not have to discuss truth, epistemology, transcendence, etc. A mathematician might say that 'a theorem X is true', but this means exactly the same as 'X is a theorem' as defined above, and does not refer to any theory of truth ... . When mathematicians say as mathematicians that they do not know whether Goldbach's conjecture is true, they mean exactly the same as when they say that nobody has yet found a proof of Goldbach's conjecture. If (in 2004) they say that they believe that Goldbach's conjecture is actually true even though no proof exists, they are not discussing the nature of truth, but speculating about what theorems might be proved in the future.

I quote Davies at length because I suspect he captures the sentiments of a significant cross-section of mathematicians.

A *second* form of mathematical pluralism extends the first form to inconsistent mathematical theories. Beall (1999) argues that every mathematical theory—consistent and inconsistent alike—truly describes some part of the mathematical realm. Beall calls the view 'Real Full Blooded Platonism' (RFBP) and to ensure that it doesn't degenerate into triviality, he assumes paraconsistent logic as a background for inconsistent theories such as those investigated in Mortensen 1995 and 2009. The theories described by Mortensen are not trivial and for each such theory, there is a reason why some of its sentences are theorems and others are not (having partly to do with the axioms and partly to do with the significance of the terms and the way in which they are arranged in the sentence). Beall concludes (1999, 325):

> ... if we really are going to expand platonic heaven in an effort to ensure our epistemic footing, then we need to explore the option of expanding heaven to its nontrivial limits. If this option is to be rejected, then we need good reason for rejecting it.[5] For now, no such reason seems to exist.

Bueno's (2011) mathematical *relativism* is closely related to this second form of mathematical pluralism, though without any commitment to the existence of mathematical objects and relations. Friend (2013, 2014) argues explicitly for the second form of mathematical pluralism as a new position in the philosophy of mathematics. Warren defends an unrestricted inferentialism (2015, 1353ff; 2020, 55ff) on which (a) the rules that implicitly define an expression are automatically valid and (b) any collections of rules can be used to implicitly a meaning for an expression. This leads to a logical pluralism that results in an inferentialist version of the second form of mathematical pluralism (2020, 199ff), on which mathematical theories consist of conventional truths (he isn't committed to the further claim that mathematical theories are about their own domain of objects and relations). Priest (2019, §11) also explicitly defends the second form of mathematical pluralism.

From this position it seems no great leap to extend the second form of mathematical pluralism one step further. For suppose there were a *consistent* theory of 'impossible' objects, some of which are 'trivial' (i.e., that 'have', in some sense, every property) and some of which are 'mathematically trivial but not trivial *simpliciter*' (i.e., that 'have', in some sense, every property expressible in the language of some mathematical theory but don't have every property whatsoever). I'll describe a theory of this kind in Section 3. But for now, note that such a theory would allow us to extend the second form of mathematical pluralism to inconsistent mathematical theories that are developed within the context of *classical* logic. For then one could claim that such mathematical theories are about impossible objects that are mathematically trivial (and thus relatively uninteresting) without being simply trivial. Such a view has one thing going for it: we do in fact understand the language and 'proofs' of Frege's *Grundgesetze* (1893/1903); its formal sentences have content. And we understand why Frege thought, however mistakenly, that its theorems can be validly derived from the axioms. It may be that the best way to explain this is to analyze the *denotation* of the terms in *Grundgesetze* as mathematically trivial objects and relations but regard the *sense* of the mathematical terms, relative to any person unaware of the paradox, as objects that don't involve incompatible properties. In any case, the second form of mathematical pluralism, taken to its limit, is the view that every mathematical theory, whether consistent or inconsistent, is about its own domain of individuals and relations, even if it is inconsistent and expressed in a classical logic.

The *third* form of mathematical pluralism is metaphilosophical and new; it is the view that the various philosophies of mathematics are often based upon some truth or insight about mathematics that can be preserved. Of course, most philosophers don't subscribe to this view. Most believe that if platonism is true, then nominalism and fictionalism are false, or vice versa. Similarly, many would claim that if structuralism is true, then inferentialism is false; either the terms of mathematical language refer to elements of an abstract structures or their significance is constituted by their inferential role within a theory, but not both. The standard view is that structuralism is 'realist' and referential, whereas inferentialism is 'anti-realist' and non-referential. But I plan to show how the basic insights from these and other philosophies of mathematics can be preserved.

Specifically, I argue in what follows that the analysis of mathematics in *object theory* (Nodelman & Zalta 2014, Zalta 2000a, Linsky & Zalta 1995, and elsewhere) embraces all three forms of mathematical pluralism. Object theory ('OT') exhibits the first form of mathematical pluralism because its methodology specifies, for any consistent mathematical theory T, the denotations of the terms and the truth conditions of the sentences of T. Thus, each theory T is about its own domain of mathematical individuals and relations. We'll see that this form of pluralism applies to well-known non-classical forms of mathematics, such as constructivism, intuitionism, finitism, etc. OT also exhibits the second form of mathematical pluralism since its analysis can be extended to inconsistent mathematical theories, expressed either in paraconsistent logic or classical logic. Even here, the terms and sentences of these mathematical theories can be assigned a precise meaning, as we'll see in Section 3. Finally, OT exhibits the third form of mathematical pluralism: the formalism in which OT is couched has a number of different interpretations, each of which validate a central element in one of the main philosophies of mathematics. We'll see, in Section 4, how this unifies a significant number of philosophical positions in the philosophy of mathematics.

2 The First Form of Mathematical Pluralism

I begin with a brief review of OT and its analysis of mathematical language and theories. Readers already familiar with the general theory can skip to Section 2.1, and readers already familiar with OT's analysis of classical mathematics can skip to Section 2.2, where I show how OT can be extended to the multiverse conception of sets and to consistent but non-classical mathematics. Finally, Section 2.3 contains an examination of how OT supplies theoretical components that are missing from other attempts to develop this form of mathematical pluralism.

Since OT has been presented in a number of publications, some familiarity with one of the presentations of the theory and its application to mathematics is presumed, though for the purpose of making this paper self-contained, we describe, in a footnote, the first principles of

OT, as expressed in a 2nd-order, quantified modal language.[8] To state the full analysis of mathematical theories, however, we use the type-theoretic version of OT, using only the simplest form of type theory. Relational type theory utilizes one primitive type i for individuals, and derived types of the form $\langle t_1,\ldots,t_n\rangle$ for n-place relations, where $t_1,\ldots,t_n$ are any types, $n \geq 0$.[9] When the language and axioms of OT are all typed according to this scheme, a comprehension principle asserts the existence of abstract entities at each type t. Where 'x' is a variable of any given type t, '$A!$' denotes the property of *being abstract* having type $\langle t\rangle$, and 'F' is a variable of type $\langle t\rangle$, the comprehension schema of typed OT asserts:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)), \tag{1}$$
where φ is any condition in which x doesn't occur free.

This asserts that there is an abstract object of type t that encodes just the properties F such that φ.[10] As we'll see below, mathematical individuals will be identified as abstracta of type i and mathematical properties and relations will be identified as abstracta of type $\langle i\rangle$, $\langle i,i\rangle$, $\langle i,i,i\rangle$, etc.[11]

[8]Start with a 2nd-order quantified modal logic *without* identity, which has atomic formulas of the form $F^n x_1 \ldots x_n$, i.e., $x_1,\ldots,x_n$ *exemplify* F^n. OT extends this system by adding new atomic formulas of the form xF, which represent a new mode of predication that can be read as x *encodes* F, where F is a 1-place relation (i.e., property) variable. OT includes primitive definite descriptions of the form $\imath x\varphi$ for any φ, and primitive n-place relation terms of the form $[\lambda x_1 \ldots x_n\, \varphi]$ when φ has no encoding subformulas.

OT also uses a primitive 1-place relation term $E!$ for *being concrete*, and defines *ordinary* objects ($O!x$) as objects x that are possibly concrete, and defines *abstract* objects ($A!x$) as objects x that couldn't be concrete. It is axiomatic that ordinary objects necessarily fail to encode properties, though abstract objects can both exemplify and encode properties. It is also axiomatic that if x encodes a property, it necessarily does so. But the key principle about abstract objects is the comprehension principle that asserts, for any condition φ in which x doesn't occur free, that $\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$. Readers can consult the works on OT to see how this principle is applied.

[9]So entities of type $\langle i\rangle$ are properties (i.e., 1-place relations) of individuals, and entities of type $\langle i,i\rangle$ are 2-place relations among individuals. Entities of type $\langle\langle i\rangle\rangle$ are properties of properties of individuals, and $\langle\langle i,i\rangle\rangle$ are properties of relations among individuals. When $n=0$, the empty type $\langle\,\rangle$ is the type for propositions, i.e., 0-place relations.

[10]Thus, when x is a variable of type i, F is a variable of type $\langle i\rangle$, and φ is supplied, the principle (1) asserts the existence of an abstract individual that encodes just the properties F such that φ. When x is a variable of type $\langle i\rangle$, F is a variable of type $\langle\langle i\rangle\rangle$, and φ is supplied, (1) asserts that there is an abstract property that encodes just the properties of properties of individuals such that φ. When x is a variable of type $\langle i,i\rangle$, F is a variable of type $\langle\langle i,i\rangle\rangle$, and φ is supplied, (1) asserts that there is an abstract relation that encodes just the properties of relations among individuals such that φ. And so on.

[11]See Zalta 2020 for the most recent discussion of typed OT and its applications.

Note that principle (1) is an *unrestricted* comprehension principle and, as such, is a plenitude principle – no matter what properties are used to define an abstract object of some type t, the principle guarantees that there is an object of type t that encodes just those properties and no others.

Moreover, when x and y are variables of type t, identity is defined so that $x=y$ just in case either x and y are both ordinary objects of type t and necessarily exemplify the same properties, or x and y are both abstract objects of type t and necessarily encode the same properties. Given the 2nd disjunct of this definiens for $x=y$, each instance of (1) yields a *unique* abstract object that encodes just the properties such that φ – there couldn't be two distinct abstract objects that encode exactly the properties such that φ, since distinct abstract objects have to differ by one of their encoded properties. So where x is a variable of type t, descriptions of the form $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ become *canonical* – no matter what φ (with no free occurrences of x) you pick, the description is well-defined, i.e., has a denotation.

2.1 OT's Analysis of Mathematics[12]

OT uses such canonical descriptions to provide an analysis of the objects and relations described by *arbitrary* mathematical theories and thereby exhibits the first form of mathematical pluralism. The analysis proceeds by assigning, for each mathematical theory T, (i) a unique denotation to the distinguished non-logical terms (individual terms and relation terms) of T and (ii) truth conditions to the sentences of T.

The methodology for assigning denotations to the terms of mathematical theories has improved incrementally since Zalta 1983, but the current best practice goes as follows. The first step is to extend the notion of encoding by saying that an abstract object x *encodes* a proposition p just in case $x[\lambda y\, p]$, i.e., just in case x encodes the property $[\lambda y\, p]$ ("*being a y such that p*"). The definiens $x[\lambda y\, p]$ has the form xF, where $[\lambda y\, p]$ has been substituted for F. Then we analyze mathematical theories as abstract individuals that encode propositions. We say that a proposition

[12]Readers familiar with OT's analysis of mathematics can skip this subsection. The analysis has been refined over the years, and some of the more recent presentations on the analysis of mathematics serve as a better review than older ones. In addition to the works mentioned in the final paragraph of Section 1, see Linsky & Zalta 2019, Linsky & Zalta 2006, Bueno & Zalta 2005, Colyvan & Zalta 1999, and Zalta 1983, 147–153.

p is *true in* theory T ('$T \models p$') just in case T encodes p. Formally, this definition is stated as:

$$T \models p \equiv_{df} T[\lambda y\, p] \tag{2}$$

We may also read $T \models p$ as: In theory T, p.

The next step is to consider any classical mathematical theory T and formalize it in a non-modal, higher-order logic without function terms (or definite descriptions) but with relational λ-expressions. The λ-expressions allow one to represent complex properties; for example, in 2nd-order Peano Arithmetic (henceforth 'PA'), we use $[\lambda x\, Px \,\&\, x{<}4]3$ to represent the claim that 3 exemplifies the property *being prime and less than* 4. Then (a) for each non-logical term τ (constant or predicate) of T, we add τ_T to OT, and (b) whenever φ is any closed truth or theorem of T, we add to OT the *analytic* truth $T \models \varphi^*$, where φ^* is just like φ except that every non-logical term τ in φ has been replaced by τ_T. For example, "0 is a number" is asserted in PA and so becomes imported into OT as the claim PA $\models N_{\mathrm{PA}} 0_{\mathrm{PA}}$. This formal claim was defined in the previous paragraph and can be read as the analytic truth "In PA, 0_{PA} exemplifies being a PA-number". We thereby fill out our analysis of a mathematical theory T as an abstract object that encodes all of the truths of T.[13] In general, for theories presented axiomatically, facts of the form $T \vdash \varphi$ become imported as facts of the form $T \models \varphi^*$. But if, for example, one were to identify a theory (i.e., a body of truths) non-axiomatically, then we can introduce a proper name, say '$\mathfrak{T}$', for that body of truths and extend object theory with analytic truths of the form $\mathfrak{T} \models \varphi^*$ for each such truth φ in $\mathfrak{T}$.

It remains to assign denotations to the terms of T and truth conditions to the sentences of T. For any well-defined individual constant κ of T, we may identify what κ denotes in T by using the following definite description, where x is a variable of type i and the other expressions are appropriately typed:

$$\kappa_T = \imath x(A!x \,\&\, \forall F(xF \equiv T \models F\kappa_T)) \tag{3}$$

[13] That is, when we judge pretheoretically that T is a mathematical theory and import T into object theory as described above, we assert that the following identity holds:

$$T = \imath x(A!x \,\&\, \forall F(xF \equiv \exists p(T \models p \,\&\, F = [\lambda y\, p])))$$

I.e., T is the abstract object that encodes all and only the properties F of the form $[\lambda y\, p]$ when p is some proposition true in T. This is *not* a definition, but rather a principle that identifies mathematical theories in OT.

In other words, (3) identifies the individual κ of theory T as the abstract individual that encodes exactly the properties F exemplified by κ in T. This is *not a definition* of κ_T (since κ_T occurs on both the left and right side of the identity symbol) but rather a principle asserting an identity that is the part of the analysis of mathematics in OT. The principle gets it purchase from, and is grounded in, data of the form $T \models F\kappa_T$.

For example, let T be Zermelo-Fraenkel set theory (ZF) and consider the term '$\emptyset$' in ZF. Then the following is an instance of (3):

$$\emptyset_{\mathrm{ZF}} = \imath x(A!x \,\&\, \forall F(xF \equiv \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}})) \tag{4}$$

This same analysis can be generalized to the *relation* terms of a mathematical theory. Suppose Π is a 2-place relation term of T. We may identify what Π denotes relative to T by using the following definite description, where x is now a variable of type $\langle i,i\rangle$, and $A!$ and F have type $\langle\langle i,i\rangle\rangle$:

$$\Pi_T = \imath x(A!x \,\&\, \forall F(xF \equiv T \models F\Pi_T)) \tag{5}$$

(5) identifies the relation Π of theory T as the abstract relation that encodes exactly the properties F of relations exemplified by Π in T. The following is an example of (5):

$$\in_{\mathrm{ZF}} = \imath x(A!x \,\&\, \forall F(xF \equiv T \models F{\in_{\mathrm{ZF}}})) \tag{6}$$

That is, the membership relation $\in$ of ZF is the abstract relation that encodes exactly the properties F of relations exemplified by $\in_{\mathrm{ZF}}$ in ZF. For example, this abstract relation encodes the property *being a relation R such that the empty set bears R to the unit set of the empty set*, a property that we can represent using the λ-expression $[\lambda R\, \emptyset R\{\emptyset\}]$ (the indices have been suppressed for readability).

And principles analogous to (5) hold when Π is an n-place relation term of T for $n \neq 2$. For example, 'being a number' ('N') is a 1-place relation term of PA and would be subject to an identification similar to (6), though expressed using the identity principle for 1-place mathematical relations. This analysis makes it clear that OT's pluralism extends to both the individual and relation terms of a theory T; few mathematical pluralists make this explicit in their accounts.

Now that we have denotations for the terms of mathematical theories, we can state the truth conditions for mathematical sentences. In the first instance, the data (i.e., the theory-relative sentences – recall the 2nd quote from Davies 2005) are parsed just as one might expect, though with *truth in* a theory defined in terms of encoding. For example, the truth conditions for:

In ZF, no set is a member of the null set. (7)

can be represented as follows:

$\mathrm{ZF} \models \neg\exists x(S_{\mathrm{ZF}}x \,\&\, x \in_{\mathrm{ZF}} \emptyset_{\mathrm{ZF}})$, i.e., (8)
ZF encodes (the proposition): nothing that exemplifies the property of being a ZF-set bears the ZF-membership relation to the emptyset of ZF.

This states the truth conditions of the target sentence in terms of encoding predications, exemplification predications, and mathematical objects and relations that have been antecedently identified as abstract entities. In other words, the truth conditions have been stated entirely in terms of the background ontology of OT.

But now consider the result of *removing* the 'In ZF' operator from (7). We obtain the bare ('unprefixed') mathematical sentence "No set is a member of the null set". OT treats this sentence, stated in the context of some set theory, as ambiguous; it has both true and false readings. If we take the context to be ZF, then the false reading is the pure exemplification formula $\neg\exists x(Sx \,\&\, x \in \emptyset)$, in which the index on '0', 'S', and '$\in$' to ZF has been suppressed. OT stipulates that such unprefixed *exemplification readings* of theoretical mathematics are not true – this is a key to OT's form of pluralism.[14]

OT also assigns true readings to "No set is a member of the empty set". For example, in the context of ZF, this claim can be seen as a fact about $\emptyset_{\mathrm{ZF}}$, or a fact about the property *being a set*$_{\mathrm{ZF}}$ (S_{ZF}), or a fact about the membership relation of ZF ($\in_{\mathrm{ZF}}$) – indeed it can be analyzed as a conjunction of all three facts. That is, the unprefixed mathematical claim has the following true readings (suppressing indices for readability):[15]

[14]This is to be contrasted with the unprefixed statements of *natural* mathematics made in the context of non-technical, natural language. OT analyzes "the number of planets is eight", or "the class of insects is larger than the class of humans", or "the direction of lines a and b are parallel", by applying its subtheory of natural mathematical objects, which doesn't assume any *theoretical* mathematical principles. As such, OT analyzes such statements differently and regards them as true. See Zalta 1999 and Anderson & Zalta 2004 for details.

[15]Strictly speaking, the λ-expressions in the following representations should be indexed to ZF as well and we've suppressed the index here as well. The indexed λ-expressions denote abstract properties. For example, where t is any type and α a variable of type t, the expression $[\lambda\alpha^t\, \varphi^*]_{\mathrm{ZF}}$ denotes the abstract property of type $\langle t\rangle$ that encodes just the higher-order properties F (i.e., having type $\langle\langle t\rangle\rangle$) such that in ZF, $[\lambda\alpha^t\, \varphi^*]_{\mathrm{ZF}}$ exemplifies F. The formalization is straightforward, but again it should be remembered that this is not a definition but a principle of identity that is part of the OT analysis of mathematics.

- $\emptyset[\lambda z\, \neg\exists x(Sx \,\&\, x \in z)]$, i.e., (9)
  $\emptyset$ encodes the property: being an individual z such that no set is a member of z.

- $S[\lambda F\, \neg\exists x(Fx \,\&\, x \in \emptyset)]$, i.e., (10)
  The property S of being a set encodes the property: being an property F such that nothing exemplifying F is a member of $\emptyset$.

- $\in[\lambda R\, \neg\exists x(Sx \,\&\, xR\emptyset)]$, i.e., (11)
  The relation $\in$ encodes the property: being a relation R such that no set bears R to $\emptyset$.

These readings are *provable* in OT, given our methodology. For example, (9) follows from (4) and the result of importing the proof-theoretic fact that $\mathrm{ZF} \vdash [\lambda z\, \neg\exists x(Sx \,\&\, x \in z)]\emptyset$. (11) follows from (6) and the result of importing the proof-theoretic fact that $\mathrm{ZF} \vdash [\lambda R\, \neg\exists x(Sx \,\&\, xR\emptyset)]\in$. (The proof-theoretic facts just cited are themselves derivable from the proof-theoretic fact that $\mathrm{ZF} \vdash \neg\exists x(Sx \,\&\, x \in 0)$].) So the conjunction of (9) – (11) is derivable, and if we use the conjunction to define a single, tertiary encoding claim, then we have fully represented the true reading of the unprefixed claim "No set is a member of the null set" when considered relative to ZF.[16]

[16]To make this precise, use the conjunction of (9) – (11) as the definiens of the following 3-place encoding claim:

> The emptyset ($\emptyset$), the property of being a set (S), and the membership relation ($\in$) encode the 3-place relation: being an individual z, a property F, and a relation R such that no individual that exemplifies F bears R to z.

Formally:

$$\emptyset S{\in}[\lambda zFR\, \neg\exists x(Fx \,\&\, xRz)]$$

This contains all the information in the bare statement "No set is an element of the emptyset". Current research into OT takes these n-ary encoding claims as primitive and axiomatizes them, so that n-ary encoding predications can be directly used to represent the data.

2.2 The Multiverse and Non-Classical Mathematics

The foregoing analysis of theoretical mathematics in OT easily extends to the multiverse conception of set theory and to non-classical mathematical theories. To see how it captures the multiverse conception, consider Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). Then we have the following instances of (3) and (5):

$$\emptyset_{\mathrm{ZFC}} = \imath x(A!x \,\&\, \forall F(xF \equiv \mathrm{ZFC} \models F\emptyset_{\mathrm{ZFC}})) \tag{12}$$

$$\in_{\mathrm{ZFC}} = \imath x(A!x \,\&\, \forall F(xF \equiv \mathrm{ZFC} \models F{\in_{\mathrm{ZFC}}})) \tag{13}$$

A similar identification can be given for the property *being a set*$_{\mathrm{ZFC}}$ (S_{ZFC}). Clearly, the membership relation of ZF is different from that of ZFC—the latter supports the truth of the Axiom of Choice while the former does not.

Thus, whenever we formulate a distinctive set theory, we end up with a distinct universe of sets and a distinct membership relation. As long as the theorems of set theories T and T' are distinct (and aren't mere alphabetic variants), the notion of 'set' each implicitly defines is distinct. OT doesn't assume that there is one *Urconcept* of membership out there, waiting to be discovered by the one true set theory. There isn't any justification for this; rather, we obtain a different 'membership' relation, and different 'sets', depending on which principles we take as our background set theory.[17]

This captures the multiverse view in Hamkins (2012) quoted earlier: "there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths" (416). But our analysis also makes it clear that the distinct universes embody distinct conceptions of 'membership'. Hamkins contrasts his view with the *universe* view, namely, that "there is a unique absolute background concept of set, instantiated in the corresponding absolute set-theoretic universe, the cumulative universe of all sets, in which every set-theoretic assertion has a definite truth-value" (416). Hamkins argues that, on the multiverse view, each set-theoretic universe "exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist" (416–17).

There are, of course, points of difference between the present view and Hamkins' multiverse view. Some are minor differences, while others are more significant. Hamkins regards the multiverse view as a 'higher-order realism' and a platonism about universes (417), though clearly the OT analysis extends this to realism and platonism about abstract objects generally, at least in the interpretation of the formalism we've assumed for the purposes of exposition. (We'll see other interpretations of the OT formalism in Section 4.) And Hamkins sees the claim that there are diverse concepts of set as a metamathematical claim (417), whereas here, the diversity of set concepts is a philosophical claim that *follows from* the analysis of mathematical theories – given our analysis, it is provable that $S_{\mathrm{ZF}} \neq S_{\mathrm{ZFC}}$. A more significant difference is that Hamkins says that "the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and ... I shall simply identify a set concept with the model of set theory to which it gives rise" (417). OT does identify set concepts by description but not with models of set theory. Model theory already assumes set theory and so it constitutes part of the data OT attempts to explain. We'll return to this issue in Section 2.3, where we investigate whether one can, as Hamkins suggests, identify a set concept with the model of set theory to which it gives rise.

It shouldn't be difficult to see how OT's analysis extends to constructive, intuitionistic, finitistic, etc., mathematical theories. Some of these theories (e.g., finitist theories) are expressed in classical logic but their non-logical axioms are just weaker than classical mathematical theories, while others (e.g., intuitionistic, constructive theories) use non-classical logic. In the former case, we use the analysis described above. In the latter case, we consider the deductive system as a whole, i.e., the system that results by combining the non-logical axioms and the logic. Some non-classical theories use the same non-logical axioms as classical theories but are just formulated within a non-classical logic. So, for example, Heyting Arithmetic (HA) uses the same language and non-logical axioms as PA but asserts the latter in the context of intuitionistic predicate logic (IQC). So, although we could regard the proof-theoretic claim HA $\vdash \varphi$ as having the form $T \vdash_L \varphi$, where T = PA and L = IQC, we can equally well regard HA as a single deductive system comprising the log-

[17] Again, the exception to this is the natural or logical conception of set, which isn't based on set-theoretic axioms. Instead, one may consistently take the extension of a property G, i.e., ϵG, to be the abstract object that encodes all and only the properties F materially equivalent to G. One can then define $y \in x$ as: $\exists G(x = \epsilon G \,\&\, Gy)$. Thus, from 'Socrates is a human' ('Hs'), it follows that $s \in \epsilon H$. Moreover, one can prove Extensionality, and a number of other set theoretic principles, thereby obtaining a 'flat' theory of property extensions. See Anderson & Zalta 2004.

ical axioms and rules of IQC and the non-logical axioms of PA. So the claim HA $\vdash \varphi$ becomes a claim of the form $T_L \vdash \varphi$. Then we can use the methods outlined above to analyze the terms and truth conditions of HA. And if the consistent theory T in question asserts non-classical axioms within a non-classical logic L, we again consider the theory to be the body of theorems as a whole system T_L and use the same method to analyze sentences φ such that $T_L \vdash \varphi$.

2.3 What's Missing From Other Accounts of Pluralism

Given this understanding of OT's analysis of consistent mathematical theories, we can theoretically describe components that are missing from the other accounts of the first form of mathematical pluralism. OT supplies a precisely formulated plenitude principle in support of the pluralism of Hilbert, Carnap, and Resnik. Hilbert's early view is an informal conditional (roughly, "if the theory is consistent, its objects and relations exist"). We now have an account that explains why the consequent follows from the antecedent and that tells us about the nature of mathematical objects and relations that exist. Given Carnap's interest in semantics, one might expect his work (1950 [1956]) to contain an explicit statement of the principle that guarantees the 'internal existence' of the appropriate objects for each logical framework. OT provides such a principle; without it, we lack a semantic interpretation of the terms for *arbitrary* frameworks and can't therefore say why the 'internal' questions of existence for arbitrary frameworks are always answerable in the affirmative.[18] Resnik's postulational view isn't unrelated to Carnap's view, since a mathematical language is needed to posit the objects in question. But Resnik admits that his view "raises many questions concerning how positing can generate knowledge about preexisting entities – especially how it can do this when the entities are mathematical ones" (1989, 8). The comprehension principle of OT connects postulation with existence and provides denotations for mathematical terms, and addresses the open problem of 'aboutness' stated at the end of Resnik's 1989 paper (26).[19]

Field and Balaguer both agree that mathematical platonism needs an explicit plenitude principle, and Balaguer (1998a, 7) attempts to formulate one. His 'full-blooded platonism' (FBP) is the thesis that every mathematical object that could possibly exist does exist. So FBP is clearly pluralistic. But the FBP plenitude principle faces the 'non-uniqueness problem', namely, it doesn't provide unique denotations to the individual constants and relation terms of a mathematical theory. If we only have recourse to possible mathematical objects, and not to objects that are 'partial' (e.g., in the sense of encoding *only* the properties attributed them in a theory), FBP is subject to questions such as: what does the symbol '$\emptyset$' of ZF denote? Does it denote (a) a set that has no members and such that the Axiom of Choice (AC) is true, or (b) a set that has no members and such AC is false, or (c) a set that has no members and such that the Continuum Hypothesis (CH) is true, or ... ? Indeed, analogous questions apply to the symbol '$\in$' of ZF, but we'll discuss this issue further below.

The non-uniqueness problem becomes even more important when we consider the truth conditions Balaguer offers for unprefixed mathematical claims. He says (1998a, 89–90):

> In order for it to be the case that '3 is prime' is true, it needs to be the case that (a) there is at least one object that satisfies all of the desiderata for being 3, and (b) all the objects that satisfy all of these desiderata are prime. Or more simply, it needs to be the case that (a) there is at least one standard model of arithmetic, and (b) '3 is prime' is true in all of the standard models of arithmetic.

This immediately raises the questions, what does '3' contribute to the expression 'being 3' and how could 'being 3' denote a unique property if '3' doesn't uniquely denote. In a paper directly addressing the

[18] Note that Carnap puts aside his method of intension and extension in 1950 [1956]. In footnote 7 (= footnote 2 of the original 1950), he says:

> The distinction I have drawn in the latter book [Carnap 1947] between the method of the name-relation and the method of intension and extension is not essential for our present discussion. The term "designation" is used in the present article in a neutral way; it may be understood as referring to the name-relation or to the intension relation or to the extension-relation or to any similar relations used in other semantical methods.

So the problem we're stating here is independent of the semantic method in play.

[19] Resnik asks (1989, 26):

> A more subtle problem concerns the *aboutness* of our mathematical beliefs. What makes them about mathematical objects? And in what sense are they about them? ... A related problem concerns the apparent lack of "epistemic contact" with mathematical objects which positing does not seem to provide.

These are all questions and problems answered in the previous section.

non-uniqueness problem (1998b), the proffered truth conditions change slightly (80):

> In order for '3 is prime' to be true, it needs to be the case that there is an object that (a) satisfies all of the desiderata for being 3 and (b) is prime. This, of course, is virtually identical to what traditional U-platonists would say about the truth conditions of '3 is prime'. The only difference is that FBP-NUP-ists allow that it may be that there are numerous objects here that make '3 is prime' true.

Here, the 'U-platonists' are those who claim that mathematical theories describe *unique* collections of abstract mathematical objects and the 'FBP-NUP-ists' are full-blooded platonists who adopt non-uniqueness platonism. But the suggested truth conditions are not virtually identical to the compositional ones a U-platonist would give for '3 is prime'. The contrast with OT is clear – assuming the background theory of numbers PA, OT analyzes the denotation of '3' as *the* abstract individual 3_{PA}, analyzes the denotation of 'is prime' ('P') as *the* abstract property P_{PA}, and resolves the ambiguous predication in terms of two truth conditions, one on which 3_{PA} exemplifies P_{PA} (false) and one on which 3_{PA} encodes P_{PA} (true). Moreover, OT doesn't appeal to a quantifier "there is an object such that" (which doesn't appear in the target sentence '3 is prime'), nor to property expressions like 'being 3' or 'desiderata for being 3'. And OT treats 'is prime' in the same way as '3' – as denoting something abstract. Balaguer essentially abandons the idea of *de re* truth conditions and *de re* knowledge for mathematical claims. Jonas (m.s., 25–26) notes that such a result leaves it unclear as to "which one of the countless copies of the numbers 13 and 17 are involved in scientific explanation".

This brings us to the final missing component of FBP, namely, the theoretical treatment of mathematical relations. Here we have a dilemma. Either FBP extends to the claim "Every possible mathematical relation that could exist does exist" or it does not.

- If FBP does extend to this claim, then the non-uniqueness problem arises for every mathematical relation term in every mathematical theory. We can state the problem for ZF: there are just too many possible relations having the properties of relations attributed to $\in$ in ZF. If there is no dimension like encoding on which such entities can be identified in terms of a partial group of higher-order properties, then we can't suppose that the incomplete description that ZF provides for $\in_{ZF}$ does in fact characterize a unique relation. So it isn't at all clear what FBP takes the content of the relation symbol '$\in$' in ZF (or any other set theory) to be.[20] A defender of FBP can't say that it is a 'distinguished' non-logical relation symbol.

- If FBP doesn't extend to this claim, then how could the very same mathematical relation $\in$ support the truth of the theorems of ZFC as well as the theorems of ZF+not-C, both of which are accepted by FBP? More importantly, without a plenitude principle for mathematical relations, FBP would no longer offer the epistemological virtues it claims: we would have to suppose that there is a single, mathematical relation $\in$ that is some how 'out there', independent of our theories about it. How would we obtain knowledge of such a relation?

This dilemma also applies to the discussion of platonism based on a plenitude found in Field 1994 (420–422) and Field 1998 (293).

To see how OT supplies components missing from both structuralist and inferentialist accounts of mathematics, we begin with structuralism, i.e., the view that mathematics is about structures.[21] If a structuralist

[20]So it is not clear why Balaguer, for example, can say (1995, 315):

> This might be expressed by saying that ZFC describes the universe of sets_1, while ZF+not-C describes the universe of sets_2, where sets_1 and sets_2 are different kinds of things.

Cf. Balaguer 1998a, 59. This seems to imply that both ZFC and ZF+not-C respectively describe uniquely distinctive bodies of sets. Adding an index on '*set*' to produce set_1, set_2, etc., suggests that these indexed terms pick out unique domains. If that is what is meant, the indexing isn't justified even on this extended version of FBP, for there are many possible *set* properties for 'set_1' to uniquely denote. (Does set_1 pick out a property whose instances are such that CH is true or whose instances are such that CH is false?) By contrast, the property terms like S_{ZFC} (being a set_{ZFC}) that we introduced above into OT are well-defined.

[21]Hellman (1989, vii) says that "mathematics is concerned principally with the investigation of structures ... in complete abstraction from the nature of individual objects making up those structures." And Parsons (1990, 303) says that "by the 'structuralist view' of mathematical objects, I mean the view that reference to mathematical objects is always in the context of some background structure, and that the objects have no more to them than can be expressed in terms of the basic relations of the structure." And Shapiro writes (1997, 5), "The subject matter of arithmetic is the *natural number structure*, the pattern common to any system of objects that has a distinguished initial object and a successor relation that satisfies the induction principle." So our question now concerns, what are these structures that are being referenced by Hellman, Parsons, and Shapiro?

philosophy of mathematics is to be free of 'ontological danglers', then it must supply a mathematics-free theory of both structures and the elements of structures. We cannot rest with set theory or category theory as our background theory of structures, as that simply turns mathematical pluralism into mathematical foundationalism and leaves us with the question, what is our philosophical account of the foundational theory?[22] So, what are structures? Nodelman & Zalta (2014, 49–53) answer: (a) intuitively, structures are defined by a partial body of propositions that assert which mathematical objects stand in which mathematical relations, and (b) in OT, the structure T can be identified as T itself, since T encodes the partial group of propositions that are true in T. This analysis of structures is mathematics-free.

Moreover, OT has something to say about the 'indeterminate elements' of structures.[23] The abstract objects of OT that serve as mathematical individuals and relations have a dimension (namely, encoding) on which their only properties are their mathematical properties. The $\emptyset$ of ZF, as identified in (4) above, encodes only its mathematical properties according to ZF, and the relation $\in$ of ZF, as identified in (6), encodes only the properties of relations ascribed to it in ZF. Since every mathematical entity is thereby identified entirely by its encoded properties, its 'special character', as given by its *exemplified* properties, is 'entirely ignored'. The classical structuralist philosophies of mathematics lack an

[22]The question has been recognized by the structuralists themselves. For example, in Hellman 1989 (7), we find:

> The second problem ... is that ... it is difficult to see in structuralism any genuine alternative to objects-platonism. This is most obvious when the structures are taken as set-theoretic models, i.e. when the structuralist theory is just set theory (perhaps with urelements), or as members of a category (the theory being category theory, taken literally as quantifying over abstract objects called categories). But this worry also pertains to other attempts (e.g. mathematics as a science of "patterns", where these are taken as platonic entities in their own right).[12]

In footnote 12, Hellman goes on to question what 'positions in a structure' are.

[23]So, for example, Dedekind (1888 [1963], 68) tells us that to develop a structure such as a simply infinite system, "we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another." In Benacerraf (1965, 70), the elements of an abstract structure "have no properties other than those relating them to other 'elements' of the same structure." And in Shapiro (1997, 5–6) we find "There is no more to being the natural number 2 than being the successor of the successor of 0, the predecessor of 3, the first prime, and so on." Our question now is: what objects can 'have' only mathematical properties and no others?

alternative theory of such elements (or places) in a structure.

OT supplies a missing component of inferentialism by supplying an exact specification of the inferential role of the non-logical mathematical terms and predicates of a theory T. Since inferentialism assumes that there are deductive relationships among the truths of a mathematical theory, we may focus on axiomatic theories. To see what is needed, consider Warren's example of the *Peano rules* (2015, 1354; 2020, 200).[24] The Peano rules are just the Peano axioms reformulated as rules of inference.[25] He then draws a metasemantic conclusion (1355):

> The arithmetical inferentialist/conventionalist will want to say that the Peano Rules are meaning constituting rules for our arithmetical vocabulary (the number predicate [N], the zero constant [$\hat{0}$], and the successor function [$s(x)$]). ... This allows us to use these rules to explain the truth of any arithmetical sentence that follows from these rules, e.g., consider the truth of 'two is a number' or, in our formal toy model: '$Ns(s(\hat{0}))$' (two is a number).

Clearly an inferentialist can use these rules to explain the truth of any arithmetical sentence that follows from them. And the rules do indicate what role the non-logical expressions have in the transition from premises to conclusion. However, if Warren's metasemantic claim implies that in a semantics for the language of number theory, each of these distinct non-logical expressions could be assigned a distinct, theoretically-describable inferential role, then the Peano rules don't yet accomplish this. The rules don't provide distinct, meaning-constituting rules for each distinct non-logical symbol; for example, the rules don't specify, in theoretical terms, what *the* meaning is of the constant symbol '0' or of the predicate symbol 'N'. Of course, one might be able to use set theory

[24]As mentioned earlier, Warren's unrestricted inferentialism yields the second form of mathematical pluralism and so he has an account even of inconsistent mathematical theories. But his discussion of the inferential roles of the non-logical expressions in the case of Peano number theory serves as a good example.

[25]Specifically, Warren reformulates the axioms, stated in terms of the constant '0', the 1-place predicate 'N', and the unary function symbol 'S', as the following rules (2020, 200):

$$(P1)\ \frac{}{N0} \qquad (P2)\ \frac{Nx}{NS\alpha} \qquad (P3)\ \frac{N\alpha}{0\neq S\alpha} \qquad (P4)\ \frac{N\alpha \quad N\beta \quad S\alpha=S\beta}{\alpha=\beta}$$

$$(P5)\ \frac{\varphi(0) \quad \forall\xi(\varphi(\xi)\rightarrow\varphi(S\xi)) \quad N\beta}{\varphi(\beta)}$$

The final rule, P5, is a rule schema.

or other forms of mathematics to give a theoretical description of the total inferential pattern of usage for the symbols '0', 'N', etc., but OT gives a distinct, mathematics-free, description of the inferential role of each non-logical expression.

Since OT imports the theorems of T as analytic truths of the form of the form $T \models p$, then when τ is an individual symbol of T, say κ, its inferential pattern of usage is captured by (3) as κ_T, and when τ is a relation symbol of T, say Π, then its inferential pattern of usage is captured by (5). (3) and (5) abstract inferential roles from the body of theorems of T. Without some theoretical description of the inferential role on a per symbol basis, one can't give *compositional* truth conditions for mathematical sentences. Of course, inferentialism may simply abandon compositionality given its anti-realist approach to meaning, but OT preserves compositionality and the compositional truth conditions it offers, in (8) and (9) – (11) for example, yield a content for mathematical sentences that 'code up' proof-theoretic facts. We may then specify truth conditions in terms of predications and complex specifications thereof, making use of these objectified inferential roles.

Thus, we can regard principles (3) and (5) above as *objectifying the inferential roles* of any terms κ and Π, respectively, relative to T. κ and Π simply reify distinct subpatterns existing within the body of *theorems* of T. The example in (4) above objectifies the inferential role of $\emptyset$ in ZF, and the example in (6) above objectifies the inferential role of $\in$ in ZF.

These theoretical identifications of the inferential roles of the non-logical symbols of mathematical theories provide a heretofore missing component of the 'meaning as use' doctrine as applied to mathematics. The classical works on inferentialism in the philosophy of mathematics (Wittgenstein 1956; Sellars 1953 [1980], 1974; Dummett 1991) do not offer a theoretical account of the meaning of such symbols. And the recent developments of proof-theoretic semantics are limited to the inferential role of logical constants.[26]

Finally, we consider the multiverse view in Hamkins 2012. Hamkins appears to rely on an existence principle that asserts: every conception of sets is instantiated in a corresponding set theoretic universe (216). So, by comparison with the analysis offered in OT, it becomes an interesting, open question: how do we move from such a principle to producing truth conditions for the various axiom systems for set theory? A related concern about the view is Hamkins' identification of a set concept with "the model of set theory to which it gives rise" (2012, 417). Assuming this can be made precise, it raises the question: doesn't any attempt to specify a model of set theory presuppose some conception of set? In any case, the appeal to model theory seems to presuppose more mathematics, the language of which is precisely what is in question. Interestingly, OT's analysis seems to be consistent with Hamkins' claim (417) that "Often the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, ..." We've seen that in OT, any distinctive body of set-theoretic truths or theorems describes a set concept and, hence, a universe of sets. That understanding is incorporated into the OT analysis of mathematics generally and set theory in particular.

Hamkins' view has engendered an interesting literature, including objections by Koellner 2009 and a defense by Freire (ms.). Barton (2016) proposes two ways to understand Hamkins, ontologically and structurally, and argues that though the structural interpretation doesn't address the Benacerraf (1973) problem of mathematical reference and knowledge, the ontological interpretation leads to a referential regress and so requires that one restrict one's pluralism (which Barton calls 'relativism'). But, from the present perspective, one can stop the regress Barton describes for the ontological interpretation by not identifying set concepts with models of set theory. OT's analysis doesn't make such an identification and so preserves a multiverse theory that is otherwise consistent with Hamkins' central view. Moreover, as we saw earlier, OT's analysis of mathematics implies that each body of theorems yields a structure, where a structure is defined in terms of the encoded *truths* that organize the individuals and relations of a theory; we'll return to the structural interpretation of OT in Section 4 below.

3 The Second Form of Mathematical Pluralism

To discuss the second form of mathematical pluralism, let's draw the following distinctions:

- An abstract object is *simply trivial* iff it encodes every property

[26] See, for example, the proof-theoretic semantics developed for certain fragments of language and logic in Prawitz 1973, 2006; Francez & Dyckhoff 2006; and Schroeder-Heister 2006.

whatsoever. There is exactly one such object, at each type t.[27]

- An abstract object is *mathematically trivial with respect to T* iff it encodes every property expressible in T. For each theory T, there is exactly one such object, at each type t.
- An abstract object is *impossible but not trivial* iff it encodes some incompatible properties but doesn't encode every property.
- An abstract object is *mathematically impossible but not trivial with respect to T* just in case it encodes some incompatible properties that are expressible in T but doesn't encode every property expressible in T.

With these distinctions, we can see how OT validates the second form of mathematical pluralism: the terms of an inconsistent mathematical theory T formulated in a paraconsistent logic denote mathematically impossible but not trivial objects with respect to T, whereas the terms of an inconsistent mathematical theory T formulated in a classical logic denote objects that are mathematically trivial with respect to T.

3.1 Inconsistent Theories in Paraconsistent Logic

Beall's (1999) RFBP gives rise to the same problem posed for FBP above, namely, the failure to assign unique denotations to the non-logical individual and relation terms of mathematical theories. But the version of RFBP available in OT is immune. Let L be some paraconsistent logic, and let T be one of the theories in Mortensen 1995 or 2009. Then we can apply OT as we did for non-classical mathematics, at the end of Section 2.2. We consider the deductive system T_L, i.e., T added to the logic L, and then add $T_L \models \varphi^*$ to OT whenever $T_L \vdash \varphi$. Then we identify the denotation of an individual term κ in T_L with the abstract object that encodes the properties F such that $T_L \models F\kappa$, and identify the denotation of a relation term Π in T_L with the abstract property that encodes the properties of relations attributed to Π in T_L. Given this analysis, the objects of such a theory T_L are mathematically impossible but not trivial with respect to T_L – they encode some incompatible properties that are expressible in T_L but they don't encode every property expressible in T_L. And truth conditions for the claims of T_L can be stated in exactly the way described above. Thus, OT overcomes the non-uniqueness problem in Beall 1999 and provides a precise and compositional semantic account of mathematical language that could supplement Bueno 2011 and Friend 2013 and 2014. But OT goes a step beyond mathematical relativism in that it *recovers* a sense of unrelativized truth; the claim "$\emptyset$ is a set", said in the context of ZF, has a reading on which it is a categorical (and thus, unprefixed and unrelativized) truth about the relativized objects $\emptyset_{ZF}$ and S_{ZF}, namely, that the former encodes the latter. Moreover, our analysis directly undermines the claim in Mortensen 2009, where he concludes (647):

> Certainly, the legitimacy of inconsistency ought to give pause to the Platonist. It poses the dilemma: either abandon Platonism, or admit inconsistent objects.

If we take the Quinean interpretation of the quantifiers of OT, then there is no dilemma; one can be Platonist *and* admit inconsistent objects.[28]

3.2 Inconsistent Theories in Classical Logic

This methodology can be taken one step further without triviality. The pluralism of OT can be applied in the analysis of the denotations and truth conditions for the terms of inconsistent mathematical theories expressed in *classical* logic. For example, an analysis of the language of Frege's theory (1893/1903) is called for, since the terms have a significance and we understand the language and the claims it makes. On the OT analysis, the terms denote entities that are mathematically trivial with respect to Frege's theory, but not simply trivial entities. To see this, take the Frege system $\mathfrak{G}$ to be second-order logic with λ-expressions,

[27] Let x be a variable of type t, F be a variable of type $\langle t\rangle$, and $A!$ denote the property *being abstract* with type $\langle t\rangle$. Then as an instance of (1), we know that there is an abstract individual that encodes every property:

$$\exists x(A!x \,\&\, \forall F(xF \equiv F{=}F))$$

And such an object is unique, given the identity conditions described for abstract objects in Section 2.

[28] Colyvan (2008, 122, footnote 13) cites this passage in Mortensen 2009 when discussing a possible objection to his [Colyvan's] argument for inconsistent objects, namely, that they would constitute a reductio of Quine's naturalized metaphysics or even of metaphysics generally. The OT analysis reconciles Platonism and inconsistent objects and forestalls such an objection. In Section 4 below, we'll see that the existence of such objects doesn't undermine a Quinean naturalized metaphysics or metaphysics generally.

extended with the primitive, non-logical function term ϵG and the non-logical axiom Basic Law V. Then, to keep our discussion simple, let's apply the OT analysis just to the individual terms of $\mathfrak{G}$. Since $\mathfrak{G} \vdash \psi$ holds for every closed formula ψ expressible in the language of $\mathfrak{G}$, we import *every* sentence ψ of $\mathfrak{G}$ into object theory as an analytic claim of the form: $\mathfrak{G} \models \psi$. This yields analytic truths of the form $\mathfrak{G} \models F\kappa$, for each individual term κ. So the analysis:

$$\kappa_{\mathfrak{G}} = \imath x(A!x \,\&\, \forall F(xF \equiv \mathfrak{G} \models F\kappa))$$

identifies the denotation of every individual term κ of $\mathfrak{G}$ as the same abstract object, namely, the one that encodes every property F of individuals expressible in the language of $\mathfrak{G}$. So the non-logical terms of $\mathfrak{G}$ denote an object that is mathematically trivial with respect to $\mathfrak{G}$, but one that isn't simply trivial ($\kappa_{\mathfrak{G}}$ doesn't encode every property whatsoever). Note that OT itself doesn't become inconsistent in virtue of representing the terms of Frege's theory this way, nor does it require paraconsistent logic to make sense of the significance of those terms.

Though OT's analysis implies, for example, that all the individual terms of $\mathfrak{G}$ denote the same individual, it doesn't imply that they all have the same sense. The senses of expressions are also representable in OT – as abstract objects that encode properties. OT assumes that these senses vary from person to person and even from time to time (Zalta 1988a, ch. 9–12; 1988b). The reason Frege didn't see the contradiction is that his sense (representation) of '0' and his sense (representation) of '1' were *distinct* – his senses of the terms encoded different properties, at least prior to being informed about the paradox. Had he cognitively associated the same representations with '0' and '1' and concluded that 0 and 1 were characterized by the same properties, he would have judged that $0 = 1$ and that his system led to an absurdity.

We've now seen how OT offers a version of the first two forms of mathematical pluralism; it preserves the idea that each consistent and inconsistent mathematical theory is about distinctive objects and relations. And in the case where T assumes classical logic and is inconsistent, the analysis tells us (a) that all the constants of T denote the unique mathematically trivial object with respect to T, and all the n-place relation symbols of T denote the unique mathematically trivial n-place relation with respect to T; and (b) the non-logical terms of any similarly trivial theory T' that is expressed in the same mathematical language as

T denote the same mathematically trivial objects and relations with respect to T. Indeed, given our earlier identification of inferential roles, OT predicts that the non-logical terms of such theories have the same inferential role, as to be expected.

4 The Third Form of Mathematical Pluralism

The third kind of mathematical pluralism that OT exhibits is metaphilosophical in character – its formalism can be interpreted in ways that preserve the central theses of various philosophies of mathematics. In Section 4.1, we focus on those philosophies of mathematics that take mathematical language at face value and attempt to give an account of that language. In Section 4.2, we examine why the most important elements of deductivism, as embodied by modal structuralism, can't and shouldn't be preserved in OT.

4.1 Metaphilosophy of Mathematical Language

Linsky & Zalta (1995) explain in some detail how the main principles of platonism and naturalism are preserved.[29] We can extend this reconciliation a bit here, by first remembering that the quantifier '$\exists\alpha$' in OT, on this interpretation, is the one Quine prefers, as implying the existence of the entities over which α ranges. Then OT becomes committed to the existence of abstract individuals and abstract relations. While this preserves the main thesis of Platonism, note that we can *apply* OT to assert the existence of mathematical entities *only once* a mathematical theory T is identified and analyzed. This requires the behavior of mathematicians, who set up patterns in the natural world that give rise to truths of the form "In mathematical theory T, ...". Without such patterns, OT

[29]We can summarize the argument by starting with the idea that the mind-independence and objectivity of abstract objects is *not* to be understood along the model of the mind-independence and objectivity of objects in the natural world. Abstract objects are not subject to an appearance-reality distinction, but rather 'have' (in the encoding sense) only the properties attributed to them in their respective theories. Nor are they 'out there' in a sparse way waiting to be discovered – they constitute a plenitude and so one need not account for our knowledge of abstracta based on the same epistemological principles used in the account of our knowledge of natural scientific theories. Finally, in the case of mathematical objects (and abstract objects generally), the distinction between knowledge by acquaintance and knowledge by description just *collapses* – acquaintance is by way of description; see the discussion of nominalism below.

can't be applied to theoretical mathematics.[30] Thus, the existence of the theoretical mathematical objects to which OT is committed depends on *mathematical practice*. The theory 'kicks in' only when the natural world exhibits the relevant patterns; OT then gives rise to instances of (1) that objectify those patterns. This is a form of naturalism – the patterns being objectified ontologically depend on contingencies.

Some of the main ideas underlying fictionalism and nominalism can be sustained as well. Note how OT is consistent with Field's claim (1989, 3) that "the fictionalist can say that the sense in which '$2+2=4$' is pretty much the same as the sense in which 'Oliver Twist lived in London' is true". Field does go on to say that "Oliver Twist lived in London" and '$2+2=4$' are true *only* in the story- or theory-relative sense, but we need not follow him in this regard. Furthermore, Colyvan & Zalta 1995 (347–348) develop an interpretation of OT under which one can *derive* the claim that mathematical objects *don't exist*. To see how, take the 'Meinongian' interpretation of the quantifier '$\exists\alpha$' as 'there are' (not 'there exist'), and interpret the predicate '$E!$' as 'exists'. Under such an interpretation, abstract objects become entities that couldn't possibly exist, since the definition is $A!x \equiv_{df} \neg\Diamond E!x$. And since mathematical objects, as identified above, are abstract, it follows *a fortiori* that they don't exist.

Recall that in addition to offering true readings, OT offers a false reading for unprefixed mathematical claims such as '2 is prime'. It therefore validates the intuition that such claims of mathematics are false (Field 1980 [2016], Leng 2010). So under this interpretation, OT preserves the fictionalist claims (a) that "In PA, $2+2=4$" is true, (b) that "$2+2=4$" is false [at least on one reading], and (c) that none of 2, 4, $\emptyset$, ω, π, etc., exist. Moreover, if one's fictionalism about mathematics takes on board the views in Walton 1990 (as does Leng 2010), then it is important to point out that significant parts of Walton's language of make-believe can be systematized in OT (Zalta 2000b).

A variant of the interpretation just described preserves the main idea of nominalism. Bueno & Zalta 2005 interpret the quantifier '$\exists\alpha$' of OT not by appealing to the 'nominalist platonism' described in Boolos 1985 for understanding the 2nd-order quantifiers, but by applying the dis-

[30] As mentioned before,though OT can be applied to *natural* mathematical objects, such as the natural numbers (Zalta 1999), and natural set theory, directions, shapes, etc. (Anderson & Zalta 2004). But such applications are not the part of OT that is relevant for mathematical pluralism, since they aren't part of the analysis of theoretical mathematics.

tinctions in Azzouni 2004. The latter uses an 'existentially-unloaded' understanding of $\exists$ as 'some'. On this reading, a quantified claim doesn't even imply the *being* of anything. Azzouni distinguishes mere *quantifier commitment* from *ontological commitment*, and if we interpret object theory's quantifier in terms of mere *quantifier commitment*, the theory becomes nominalistic, at least according to some philosophers. And Priest, building on ideas in Routley [Sylvan] (1980), says we can regard quantifiers as existentially unloaded (2005 [2016], vii):

> But the main technical trick is just thinking of one's quantifiers as existentially neutral. '$\forall$' is understood as 'for every'; '$\exists$' is understood as 'for some'. Existential commitment, when required, has to be provided explicitly, by way of an existence predicate.

A bit further on, he suggests that we should not read '$\exists\alpha\varphi$' as *there is* something such that φ, but rather as *some* α is such that φ:[31] Since this position has been ably defended, apply it to OT's formalism and the result is Azzouni-Priest-Routley nominalism.

Indeed, OT helps us to make sense of ideas that, at present, are somewhat metaphorical, namely, that mathematical objects are 'ultrathin' (Azzouni 2004, 127; Rayo forthcoming) and are objects whose "existence does not make a substantial demand upon the world" (Linnebo 2018, 4). Azzouni suggests that mathematicians just need to write down axioms and the resulting 'posits' have no epistemic 'burdens'. And Rayo (forthcoming) develops a conception of 'ultrathin' objects on which they arise in virtue of language-based networks. This notion of thinness is evident in OT, in several ways. The mere statement of a mathematical theory T triggers OT to articulate a distinctive group of mathematical objects and relations for T. These objects and relations are *thin* along the dimension of what they encode, for they have only a partial (i.e., not complete) com-

[31] Priest writes (2005 [2016], 11, 13):

> For reasons that I will explain in due course, I will write the quantifiers, not as $\forall$ and $\exists$, as usual, but as $\mathfrak{A}$ (for all) and $\mathfrak{S}$ (for some). . . . (11)
>
> . . . one must precisely not read $\mathfrak{S}xA(x)$ as 'There exists something, x, such that $A(x)$'. Assuming that existence and being are the same thing, one should not even read it as 'There is something, x, such that $A(x)$'. The reading 'Something, x, is such that $A(x)$' will do nicely. (13)

plement of encoded properties (namely, only the properties attributed to them in their respective theories). With OT in the background, no additional 'demands upon the world' are needed for the terms of T to acquire significance. Indeed, as Linsky & Zalta 1995 (547) point out, all one has to do to become acquainted with a mathematical entity such as 0_{PA}, $\emptyset_{\mathrm{ZF}}$, $\in_{\mathrm{ZF}}$, or $\in_{\mathrm{ZFC}}$, etc., is to *understand* its defining description, as given by such identity claims as (4), (6), (12), and (13). We don't need a special faculty, or an 'information pathway' for acquiring knowledge of abstract objects; we just need the faculty of the understanding. This should satisfy whatever epistemological concerns are driving the demands of nominalism.

We've already seen, in Section 2.3, how OT supplies components missing from structuralism and inferentialism. Given this discussion, we can then interpret the OT formalism in a way that preserves the central insights of both philosophies of mathematics, starting with structuralism. The OT analysis is that mathematical theories *are* structures, where these are identified without any mathematical assumptions other than analytic truths about mathematical theories. Let me reiterate that by identifying mathematical individuals and relations as abstracta that encode only their mathematical properties and no others, OT neglects their 'special character' (i.e., neglects their exemplified properties). And, as previously noted, this analysis complements the standard (non-modal) structuralist views, which only discuss 'places in structures', but rarely talk about 'relational places', i.e., the places that 'partial' or 'indeterminate' relations occupy in a structure.

Once we interpret OT as a form of structuralism, a variety of puzzles about structuralism become soluble, as explained in Nodelman & Zalta 2014. To take an example, consider the puzzle Shapiro described for his view (2006, 115): previously he had claimed (1997) that individual natural numbers do not have non-structural *essential* properties, but now he admits that numbers in fact do seem to have some such properties:

> For example, the number 2 has the property of being an abstract object, the property of being non-spatio-temporal, and the property of not entering into causal relations with physical objects. ... Abstractness is certainly not an *accidental* property of a number—or is it? (2006, 116)

He then develops an extended discussion of the issues (2006, 117–20)

and concludes not only that *abstractness* is not a mathematical property but that it isn't therefore an essential property of natural numbers. From the point of view of OT, this conclusion is a *consequence* of the analysis in Section 2.1. The essential properties of numbers are just the mathematical properties they encode (Zalta 2006), and these are distinct from the properties they necessarily exemplify (such as being abstract, not being a building, having no causal powers, etc.).[32]

Turning now to inferentialism, we again restrict our attention to axiomatic theories, since inferentialism presupposes some sort of deductive relationships among the truths of T. But with this restriction, we can be brief, since the discussion in Section 2.3 already provides the essentials. Given any axiomatic theory T, we can interpret the schematic and specific principles (3) – (6) as picking out the inferential roles of the non-logical, mathematical terms of T. These principles identify a specific role for each non-logical term of T. They are not axioms reinterpreted as rules of inference; as we saw earlier, the rules reconstructed from axioms are not fine-grained enough to theoretically describe the inferential roles of the individual terms and predicates of T.

While this interpretation preserves the basic insight of inferentialism, the more interesting fact is how it reconciles the referential and use-theoretic approaches to the meaning of mathematical language and renders them consistent (cf. Murzi & Steinberger 2017). The idea is straightforward: the objectified inferential roles can serve as the denotations of mathematical terms in a compositional semantics. Thus, the formalism of OT suggests that the traditional distinction between inferentialism, on the one hand, and referential theories such as Platonism and struc-

[32] To take another example, Shapiro says (2006, 133) "Presumably, a structuralist cannot accept haecceities for places, since a haecceity seems to be a non-structural property." But on one reading, this conclusion is predicted by OT – the theory implies that not every abstract object has a haecceity, for cardinality reasons. For if we temporarily assume set theory to construct models of OT, then abstract objects can be modeled as sets of properties, and if every abstract object had a haecceity, there would be a 1-1 mapping from the power set of the set of properties into a subset of the set of properties, in violation of Cantor's theorem. For a full discussion of this issue see Section 4.3 ("No Haecceities") in Nodelman & Zalta 2014, 64–66.

One the other hand, OT does allow one to introduce, for each theory T, a restricted identity relation, $=_T$, on the individuals of T. Then, OT does assert the existence of haecceities$_T$, i.e., properties of the form $[\lambda x\, x =_T y]$, where y is an object of T. For a full discussion of this issue, see Section 3.2 ("Elements and Relations of Structures") in Nodelman & Zalta 2014, 52–53.

turalism, is partly a matter of focus. We have one formalism that has multiple realist interpretation and multiple anti-realist interpretations.

To see how the basic insight of formalism is preserved, we can ignore many of the differences between Hilbertian formalism,[33] term formalism,[34] and game formalism.[35] That's because in each case, the essential idea is that mathematics is about (formula and symbol) *types* and not *tokens*. That is, on any kind of formalism, mathematics is not about any particular marks on the page or about any particular sound waves emanating from the mouths of mathematicians, but rather about the types that the marks or sound waves are tokens of. The 'formal rules' that the principles of T represent apply to types, not to tokens.

To preserve this insight, we use OT to identify *types* as abstract objects that encode properties. A type encodes the properties that the tokens of that type exemplify. For example, a pure symbol type encodes the shape and/or sound properties that the tokens of the types exemplify. In the case of a mathematical theory T, the formal objects denoted by the terms and predicates of T are not pure symbol types, but symbol types as abstracted from the role they play in the formulas true in T. Under this interpretation, the individual terms of mathematics denote individual-symbol types that encode the abstract property-symbol types denoted by the predicates.[36] For example, $\emptyset_{\mathrm{ZF}}$, as identified in (6),

[33] I'm focusing here on what Hilbert regarded as the *ideal* part of mathematics, which deals with infinity. Thus, the formulas of ideal mathematics are uninterpreted and though they have the syntactic form of sentences, and thereby allow us to apply formal, inferential rules of thought, they have no semantics (Hilbert 1927 [1967, 475]; Weir 2021, §1). See Detlefsen 1993 for a careful review of Hilbert's evolving formalist views. It seems that earlier, thought that consistent theories defined *forms* of existence. Detlefsen (1993, 288) criticizes this view, on the grounds that definitions aren't creative, but I think Hilbert was relying on the principle that if a theory is consistent, then it is not only a definition but a creative one!

[34] This is the view that the expressions of mathematics, e.g., the singular terms, are referring expressions, but refer to symbols rather than to mathematical entities distinct from symbols. See Shapiro 2000 (141); Weir 2021, §2.

[35] This is the view that the terms in mathematical formulas do not pick out objects and properties, but instead the formulas are simply elements of a game in which symbol strings are transformed according to fixed rules. See Shapiro 2000 (144); Weir 2021, §2.

[36] It is important to remember that OT *doesn't use model theory* to define what the individuals and relations of a theory T are, i.e., it doesn't say that to be an individual or relation of T is to be the value of a variable x or F used in T. Rather, OT defines the individuals and relations of T to be entities that are distinguishable in the formalism of T. For a full discussion of this issue, see Nodelman & Zalta 2014, §3.2, 52–53 (a definition of the elements and relations of T), and §4.4, 66–73 (indiscernibles are not elements of a theory).

becomes a symbol-type that encodes just those property-symbol types F such that the formula type "In ZF, $F\emptyset$" constitutes part of the data. Since ZF is given axiomatically, this data comes from sentence types of the form "$\mathrm{ZF} \vdash F\emptyset$".

We've already discussed how OT's comprehension principle (1) and principles such (3) and (5) allow a Carnapian to explain how the constants and predicates of each framework come to denote the right objects and relations, so that the internal question "Do Xs exist?" is always true, or provably true, within the framework. This fact about OT suffices to show how it preserves the basic insight of Carnapianism.

No metaphilosophy of mathematics would be complete without some discussion of logicism. But my discussion here will be only a sketch, since this is a topic of ongoing research. Many philosophers now believe that logicism is a non-starter, since mathematics has strong existence claims and logic has very weak ones, making any reduction of mathematics to logic impossible. Indeed, logicism *is* a non-starter *if* one's conception of logic makes it impossible for existence claims to be logical truths and *relative interpretability* is the standard of reduction. But if one (a) develops a conception of logic that allows certain kinds of existence claims (such as 2nd-order comprehension and (1) above) to be logically true, and (b) uses an alternative, but equally precise, standard of reduction (on which each well-defined term is assigned a unique denotation and the theorems are assigned readings on which they are true), then OT can be viewed as part of logic and mathematics becomes reducible to logic plus analytic truths. The key to this conception of logic is the idea that the principles of 2nd-order logic and OT are *required* for a correct understanding of predication, logically complex thought (including complex predications and abstract mathematical thought), and the validity of consequences inferred from such thoughts. We'll leave the matter here, however, since this is the subject of ongoing research (see Leitgeb, Nodelman, & Zalta, ms).

4.2 Paraphrasing Mathematical Language

OT shares with deductivism the idea that the fundamental truths of a mathematical theory T are statements under the scope of an operator: "In T, …" in the case of OT, and "If the conjunction of the axioms of

T hold, then ..." in the case of deductivism.[37] But the similarity ends there, especially when we consider the more sophisticated version of deductivism embodied by modal structuralism (MS). OT doesn't preserve the ideas underlying MS because the two theories are attempts to address different problems. OT takes mathematical language at face value as containing constants and predicates that have a semantic value (at our world), and attempts to preserve the long-standing tradition in logic in which (axiomatic) mathematical theories are formally represented in a classical, non-modal predicate calculus extended with (a) the non-logical constants and non-logical predicates, and (b) non-logical axioms that are *categorically stated*. OT's analysis, which is based on an ambiguity in predication, assigns the categorical predication "0 is a number" of PA a true reading ('$0N$') and a false reading ('$N0$'), in which the terms '0' and 'N' are indexed to PA. And the categorical axiom "0 is not the successor of any number" of PA is analyzed in a manner analogous to (8) – (11) above. And so on.

But MS doesn't adopt this methodology; instead it denies that the constants and predicates of mathematical theories have a semantic significance at our world, and denies that categorical predications and categorical quantified claims serve as the proper analysis of mathematical axioms. Instead, it replaces each distinguished constant and predicate in the language of a mathematical theory T by a distinct variable of the appropriate type, so that the categorical claims φ of T become open

[37]This connection makes OT and deductivism subject to the same objection: how to distinguish mathematics from fiction, since both approaches relativize the basic truths with respect to these operators. Quine 1936 [1976, 83] argues that deductivism w.r.t. geometry:

> ... reduces merely to an exclusion of geometry from mathematics, a relegation of geometry to the status of sociology or Greek mythology; the labeling of the 'theory of deduction of non-mathematical geometry' as 'mathematical geometry' is a verbal *tour de force* which is equally applicable to the case of sociology or Greek mythology.

It's true that OT analyzes names in fiction ('Zeus', 'Sherlock Holmes', etc.) and predicates in fiction ('being a hobbit', 'being an orc', etc.) in a manner similar to constants and predicates in mathematical theories. Intuitively speaking, both fictions and mathematical objects are good examples of 'partial' abstract objects. But the objection, that a common treatment somehow disrespects mathematics or collapses mathematics and fiction, loses force in the context of mathematical pluralism. Even with the liberal attitude of mathematical pluralism, there are still a number of differences between the rigors of mathematical practice and the freedoms of fictional practice, and these result in somewhat different methodologies for analyzing mathematics and fiction in OT. But I won't pursue the matter here.

formulas of the form $\varphi(\vec{x},\vec{F})$, where $\vec{x}$ and $\vec{F}$ represent the sequence of individual and relation variables introduced to replace the non-logical primitives. Then, since the conjunction of the axioms, $\wedge T$, becomes an open formula, $\wedge T(\vec{x},\vec{F})$, MS paraphrases the categorical theorems φ of T as logical theorems of the form:

$$\Box\forall\vec{x}\forall\vec{F}(\wedge T(\vec{x},\vec{F}) \rightarrow \varphi(\vec{x},\vec{F}))$$

I.e., necessarily, for any objects $\vec{x}$ and relations $\vec{F}$, if the conjunction of the axioms of T holds w.r.t. $\vec{x}$ and $\vec{F}$, then $\varphi(\vec{x},\vec{F})$ holds. To complete its analysis of mathematics, MS then requires an additional group of assertions; for every theory T, MS asserts or implies:

$$\Diamond\exists\vec{x}\exists\vec{F}(\wedge T(\vec{x},\vec{F}))$$

I.e., it is possible that there are objects $\vec{x}$ and relations $\vec{F}$ such that the conjunction of the axioms of T hold w.r.t. $\vec{x}$ and $\vec{F}$.[38] Finally, MS encourages the nominalistic interpretation of the second-order quantifiers of the background formalism.

This methodology doesn't attempt to analyze the axioms of T as categorical predications or universal claims. Indeed, one may consistently suppose that none of the constants or predicates of T have denotations, much less denote specifically mathematical objects or relations. Moreover, since the Barcan formulas are invalid in the S5 modal logic assumed in MS (Hellman 1989, 17), one cannot infer $\exists\vec{x}\,\exists\vec{F}\Diamond(\wedge T(\vec{x},\vec{F}))$ from $\Diamond\exists\vec{x}\,\exists\vec{F}(\wedge T(\vec{x},\vec{F}))$. So mathematical theories are not *about* structures or indeed about anything (there are, in fact, no objects and relations in the right structural relationships), though it is *possible* that they are about something.[39] In many ways, MS is a form of *mathematical eliminativism* rather than a form of *mathematical pluralism*, since the distinc-

[38]I've tried to describe the ideas in Hellman 1989 and 1996 as simply as possible; I won't be criticizing any aspect of MS arising from this simplification. I think the simplification helps one to see how MS offers a general analysis, for as it is presented in Hellman 1989 and 1996, the 'possible existence' axioms required by MS seem to be *tailor-made* for each mathematical theory T and constructed to be as 'weak' as possible. This leads one to wonder whether there is a general algorithm for giving a modal structuralist account for an *arbitrary* theory T. I've side-stepped this question for now by formulating MS as above, but we discuss this issue further below

[39]I not sure that Hellman & Bell (2006, 75–76) are justified when they say, near the end of their paper:

> It turns out, however, that there is a way out of this impasse, but at a price.
> If we introduce modality and tolerate talk of the possibility of large domains

tive primitive notions employed by mathematicians are all eliminated in favor of variables and modally quantified conditionals.

In any case, even if OT and MS aren't attempts to solve different problems, it is still difficult to compare them. Here are some questions that can be raised. One concerns the status of the possibility claims that MS has to assert to complete the analysis of mathematical theories. MS has to add at least one special axiom of the form $\Diamond\exists\vec{x}\,\exists\vec{F}(\wedge T(\vec{x},\vec{F}))$, for each mathematical theory T that it analyzes. So, the question is, can one actually state MS generally? Should we suppose that MS really includes the universal claim: $\forall T\Diamond\exists\vec{x}\,\exists\vec{F}(\wedge T(\vec{x},\vec{F}))$? If not, then instead of analyzing T, MS seems to be analyzing $T + \Diamond\exists\vec{x}\,\exists\vec{F}(\wedge T(\vec{x},\vec{F}))$. If the latter, then one might expect to see explicit modal claims in mathematical practice. Since we don't, there is a question about how MS achieves generality as a theory. And since the MS methodology described above was a simplification, the problem may prove to be a difficult one, especially if the possibility claims added to MS have to be customized so as to be the weakest claims that can do the job.[40]

By contrast, OT doesn't have to add modal claims for each new mathematical theory it analyzes – it just takes the theory-prefaced statements

> of discourse—essentially just large numbers of objects—then we have a natural way of recognizing a plurality of models of set theory, and toposes, living side-by-side within these domains, of which there also can be many, but without ever allowing for any totality of all such domains. ... In fact, surprisingly, second-order logical machinery is available to describe not only large domains, in the sense of having inaccessible cardinality, but also structures for set theory and category theory, but without ever officially quantifying over classes or relations as objects. ... Similar methods yield characterizations of other key mathematical structures such as the natural numbers, full models of set theory, and various topoi, etc., again, without ever countenancing classes or relations as objects.

This passage seems clearly to be about MS, but as far as I can tell, MS doesn't 'recognize' or 'describe' structures or domains for set theory, category theory, natural numbers, etc., for *there are none*. MS doesn't even allow talk about entities that are possible structures or possible domains, given the invalidity of the Barcan formulas.

[40]See Hellman 1989, pp. 27–30, for the claim needed for PA (concerning the possible existence of ω-sequences); p. 45, for the claim needed for 2nd-order real analysis (RA) (concerning the possible existence of complete, ordered, separable continua); and p. 71, for the claim needed for 2nd-order ZF (concerning the possibility of natural set-theoretic models). And see Hellman 1996 for the possibility claims needed for other mathematical theories. Given these discussions, it may that that the simplified methodology for MS presented above obscures the fact that customized, special axioms are needed on a case-by-case basis.

as the data and the rest falls out from OT comprehension (1), the identification principles (3) and (5), and the various readings of the unprefixed mathematical claims that this methodology makes possible.[41]

A second question concerns the analysis of mathematical constants and predicates that appear outside purely mathematical contexts. Presumably, MS can't accept the following claims at face value:

- π is more well-known than Euler's number e.
- At one time, mathematicians didn't believe that $\sqrt{-1}$ exists.
- Fraenkel wondered whether the existence of $\omega+\omega$ could be proved in Zermelo set theory.
- The number Zero wasn't always used for counting.

These claims can be analyzed in OT without any special heroics. But I suspect the same can't be said for MS – there is no *de re* knowledge of mathematical entities of any kind.

But the final, and most important question for MS, arises in connection with our understanding of possibility claims in terms of truth at some possible world. If the modal claim for T, namely $\Diamond\exists\vec{x}\,\exists\vec{F}(\wedge T(\vec{x},\vec{F}))$, is to be true, then speaking semantically, there are possible worlds where $\exists\vec{x}\,\exists\vec{F}(\wedge T(\vec{x},\vec{F}))$ is true. Even if MS interprets the quantifier(s) $\exists\vec{F}$ non-objectually, it still treats the quantifier(s) $\exists\vec{x}$ as objectual. So, again speaking in the metalanguage, mathematical objects exist at other possible worlds. Then what exactly is the conception of these objects that explains why they exist only at other possible worlds but not our own?

5 Final Observations

It is important to mention what *hasn't* been attempted in the foregoing. I've said little about the epistemology of mathematics, though I've occasionally referenced the ideas in Linsky & Zalta 1995, which describes the

[41]The fact that the analysis of mathematics in MS requires the addition of possibility claims also raises a question of whether it can provide an analysis of inconsistent (but not trivial) mathematical theories. It seems doubtful, unless MS either explicitly adopts a modal logic in which $\Diamond$ is interpreted as quantifier over both possible and impossible worlds or adopts other measures to analyze such theories.

epistemology of 'principled platonism'. I've not discussed how OT analyzes natural mathematics (i.e., the mathematical statements from ordinary language, outside the context of theoretical mathematics). Though some of the relevant work was cited above in footnotes, this is the subject of ongoing new research.[42] I've not tried to give an account of the use of mathematical language during the process of theory formation or theory comparison. Nor have I elaborated on the facts that (a) the modal logic of encoding is captured by the principle $xF \to \Box xF$ (i.e., encoding claims are necessary if true), (b) the true readings of unprefixed mathematical statements are encoding claims, but (c) the necessity of the mathematical truths that can be derived from (a) and (b) rests on analytic truths of the form "In theory T, p".[43] These issues are all worthy of being discussed, but haven't been pursued in any detail here.

Let me instead close with two thoughts. The first concerns a real obstacle to theory acceptance about the nature of mathematics, namely, the fact that many philosophers of mathematics don't even agree on the data to be explained. Some (platonists, structuralists, logicists, etc.) think that the unprefixed theorems of our most well-entrenched mathematical theories are true, others (fictionalists, nominalists, modal structuralists, etc.), take these claims to be false, and still others suggest that the claims are just not truth-apt or always relative. This lack of agreement about the data should, and can, be explained. OT does so via the distinction between exemplification and encoding predications, which attempts to resolve a subtle ambiguity in predication and thus an ambiguity in the data. This ambiguity is resolved by formulating both true and false readings that disambiguate unprefixed mathematical claims. One would expect disagreement about the data if (a) some philosophers, on the basis of certain background assumptions, focus on the true readings, (b) other philosophers, on the basis of different background assumptions, focus on the false readings, and (c) still other philosophers, in the presence of arguments by (a) and (b) philosophers, conclude that the data is neither strictly true nor strictly false (i.e., as not truth-apt or as always relative). If none of these groups admit to an ambiguity, the various sides are bound to disagree and talk past each other concerning solutions and explanations of the data.

We can see how this explanation ties in with OT's metaphilosophical pluralism by examining the conclusion in Balaguer 1998a. He lists eight points on which platonism (as embodied by FBP) and anti-platonism (as embodied by fictionalism) agree (152–155), and notes that they disagree only on one point (155), namely, that "FBP-ists think that mathematical objects exist and, hence, that our mathematical theories are true, whereas fictionalists think there are no such things as mathematical objects and, hence our mathematical theories are fictional." He then draws a strong *epistemic* conclusion (namely, that we could never have a cogent argument that settles the dispute over mathematical objects), and a strong *metaphysical conclusion* (namely, that there is no fact of the matter as to whether platonism or anti-platonism is true). But, as first suggested in Colyvan & Zalta (1999, 347), these conclusions could be explained by the following hypotheses: (a) platonism focuses on the sense in which unprefixed mathematical claims are true, while fictionalism focuses on the sense in which they are false, (b) both platonism and fictionalism are different, *incompatible interpretations* of the *same formalism* (these interpretations were stated above in the 2nd paragraph of Section 4.1), and (c) natural language can be equally well regimented in two ways: one consistent with platonism and one consistent with fictionalism. These hypotheses would *predict* Balaguer's conclusion that platonism and fictionalism are on a dialectical par and would *explain* why Balaguer comes to the conclusion that there may be no fact of the matter as to which is true.

[42] Nodelman and I have extended the work in Zalta 1999; we derive second-order Peano Arithmetic (and not just the Dedekind-Peano axioms) in an extension of OT, without adding any mathematical primitives or analytic claims about what is true in PA. A draft of this work is available online: see Chapter 14 of http://mally.stanford.edu/principia.pdf.

[43] This is important because such analytic truths are not asserted as *necessary* truths. When a mathematical theory is specified in terms of distinguished constants and predicates, the expressive power of the language is thereby changed and so claims of the form "In theory T, p", such as (7), rest on this contingent change. When these claims are represented in OT and asserted as axiomatic, we do *not* also assert their necessitations, but instead mark them as 'modally fragile' (i.e., as resting on a contingency). OT is built so as to allow for axioms that are contingent or rest on a contingency; such axioms are *not* subject to the Rule of Necessitation (RN), nor can RN be applied to any theorem derived from such axioms. When such an axiom is formalized as an encoding claim, then we derive its necessity, and the necessity of any encoding theorems that depend on it, via the modal logic of encoding, not RN. So the resulting necessary theorems are flagged as derived from an axiom marked as modally fragile. For example, the analytic truth (7) becomes represented as the modally fragile axiom (8). Consequently, when we derive the encoding claims (9) – (11), we can apply the modal logic of encoding and consequently derive their necessitations. But these are necessary truths derived from an axiom marked as resting on a contingency. See Zalta 1988c for a discussion of logical and analytic truths that are not necessary.

The concluding thought is to consider that OT wasn't developed specifically for the analysis of mathematics. Rather, it was formulated for systematically analyzing abstract objects generally. It therefore has additional explanatory power, in so far as it provides us with a theory of possible worlds, concepts, fictions, Fregean numbers, senses, etc. The present effort is informative only with respect to OT's application to mathematics. And I would argue that it gives one a better overall perspective on the subject. If no other theory provides a better understanding of both the language and objects of mathematics, or better unifies apparently incompatible philosophical accounts of mathematics, then OT is a conceptual framework to consider seriously until a better overall theory comes along.

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