Principia Metaphysica* (Draft)

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February 10, 1999

Contents

 This manuscript presents the *a priori* consequences of the theory of abstract objects. This theory has been developed and explained in print (Zalta [1983], [1988], [1993]) and in unpublished manuscripts (see the Bibilography). The latest version of this document can be obtained at the following URL: http://mally.stanford.edu/publications.html

§1: The Language

In the usual way, the language may be described by first listing the simple terms and then simultaneously defining the atomic formulas, complex formulas, and complex terms. To simplify the presentation, we divide this simultaneous definition into parts.

1) **Definitions:** Simple Terms. The *simple terms* of the language are:

.1) Individual Names: $a_1, a_2, \ldots \quad (a, b, c, \ldots)$

.2) Individual Variables: $x_1, x_2, \ldots \quad (x, y, z, \ldots)$

.3) n-place Relation Names ($n \geq 0$): $P_1^n, P_2^n, \ldots \quad (P^n, Q^n, R^n, \ldots)$

.4) n-place Relation Variables ($n \geq 0$): $F_1^n, F_2^n, \ldots \quad (F^n, G^n, H^n, \ldots)$

We use $p, q, r, \ldots$ as abbreviations for the 0-place relation variables. When $n=0$, we say that the relation names denote, and the relation variables range over, *propositions*. So the variables $p, q, r, \ldots$ range over propositions. When $n=1$, we say that the relation names denote, and the relation variables range over, *properties*. There is one distinguished relation name:

Distinguished 1-place relation name: $E!$

The predicate $E!$ denotes the property of having a location in spacetime (i.e., the property of being concrete).

2) **Definitions:** Atomic Formulas. To simplify the presentation, we use typical variables instead of Greek metavariables to describe the structure of atomic formulas. So we may describe the two kinds of atomic formulas as follows:

.1) Exemplification Formulas ($n \geq 0$): $F^n x_1 \ldots x_n$
(read: individuals $x_1, \ldots x_n$ *exemplify* relation F^n)

.2) Encoding Formulas: xF^1
(read: individual x *encodes* property F)

Hereafter, we drop the superscript on the predicate, since this can always be inferred from the number of individual terms in the formula. We may think of the formulas Fx and xF as expressing two distinct modes of predication. Note that for $n=0$, the simple variables $F^0, G^0, \ldots$ are defined to be atomic exemplification formulas. Since we are using '$p, q, \ldots$' as abbreviations '$F^0, G^0, \ldots$', the expressions $p, q, \ldots$ become both variables and formulas.

3) Definitions: Formulas and Terms. We use φ, ψ, χ to range over formulas, and α, β to range over all variables. We define the following kinds of complex formulas and terms:

.1) Formulas: $(\neg\varphi)$, $(\varphi \rightarrow \psi)$, $\forall\alpha(\varphi)$, $(\Box\varphi)$, $(\mathcal{A}\varphi)$

.2) Terms: $\imath x(\varphi)$ (for any formula φ) and $[\lambda x_1, \ldots x_n\ \varphi]$ (where $n \geq 0$ and φ contains no encoding subformulas)

We drop parentheses when there is little potential for ambiguity. We read $\neg\varphi$ as 'it is not the case that φ'; $\varphi \rightarrow \psi$ as 'if φ, then ψ'; $\forall\alpha\varphi$ as 'every α is such that φ'; $\Box\varphi$ as 'necessarily φ'; and '$\mathcal{A}\varphi$' as 'it is actually the case that φ'. The λ-*predicate* $[\lambda x_1, \ldots x_n\ \varphi]$ is a complex n-place relation term and is read 'being an $x_1, \ldots x_n$ such that φ'.[1,2] The

[1]Formulas φ appearing in λ-expressions may not contain encoding subformulas. The notion of 'subformula' is defined recursively as follows: (a) φ is a subformula of φ, (b) if φ is a formula of the form $\neg\psi$, $\psi \rightarrow \chi$, $\forall\alpha\psi$, or $\Box\psi$, then ψ, χ are subformulas of φ, and (c) if ψ is a subformula of φ, and χ is a subformula of ψ, then χ is a subformula of φ. In earlier work, we have called formulas φ with no encoding subformulas 'propositional formulas', for such a formula may be turned into the 0-place relation term $[\lambda\varphi]$, which denotes a proposition. Notice that definite descriptions with encoding formulas may appear in propositional formulas because those encoding formulas are not subformulas of the propositional formula. For example, the following atomic exemplification formula with the definite description $\imath x\, xQ$ is a propositional formula without encoding subformulas: $F\imath x\, xQ$.

[2]In previous work, we have also banished 'impredicative' formulas from λ-expressions; i.e., formulas containing quantifiers binding relation variables were not allowed in λ-expressions. The restriction was not added to avoid paradox, however, but rather to simplify the semantic interpretation of the language. In those previous works, we preferred to use 'algebraic semantics' to interpret λ-expressions in terms of a group of algebraic logical operations which harnessed simple and complex relations into more complex relations. These algebraic logical operations were the metaphysical counterpart of the logical functions Quine discusses in [1960]. Whereas Quine's logical functors operated on predicates, our logical operations operated on the relations denoted by those predicates. This algebraic semantics facilitates the view that necessarily equivalent relations may be distinct. However, the standard logical operations such as **PLUG**, **UNIV**, **CONV**, **REFL**, and **VAC** operate only on the *argument* places of relations (see Zalta [1983], p. 114-16, Zalta [1988a], pp. 236-7, Bealer [1982], and Menzel [1986]). So the *algebraic* interpretation of λ-expressions containing quantified relation variables would require additional logical operations, perhaps those formulated in Došen [1988]. Since the addition of further logical operations would complicate the simple semantic picture being developed, I thought it best, in those previous works, to

definite description $\imath x\varphi$ is a complex individual term and is read 'the x such that φ'. There are two important facts about definite descriptions: (1) If nothing (uniquely) satisfies φ, then $\imath x\varphi$ simply fails to denote, and any term containing $\imath x\varphi$ fails to denote. *Atomic* formulas containing such nondenoting terms are false. For example, if $\imath xPx$ fails to denote, then the atomic formulas $Q\imath xPx$ and $\imath xPxQ$ are false. So is the atomic formula $[\lambda y Ry\imath xPx]a$, which is constructed with a λ-predicate that contains $\imath xPx$. Molecular formulas, such as $\neg Q\imath xPx$ and $Q\imath xPx \rightarrow Q\imath xPx$, are true however. (2) Definite descriptions are *rigid designators*. For example, even if $\imath xPx$ appears under the scope of a modal operator, it denotes the object that *in fact* uniquely exemplifies the property P. Thus, $\Box Q\imath xPx$ asserts something about the unique individual that in fact exemplifies P, namely, that it is necessarily Q.

4) **Definitions:** Notational Conventions. We employ the usual definitions of $\varphi \,\&\, \psi$ ('φ and ψ'), $\varphi \vee \psi$ ('φ or ψ'), $\varphi \equiv \psi$ ('φ if and only if ψ'), $\exists\alpha\varphi$ ('there exists an α such that φ'), and $\Diamond\varphi$ ('possibly φ'). We let τ range over all *terms*: individual names and variables, relation names and variables, definite descriptions, and λ-predicates. We follow the usual definition of *free variable* and use the standard notion of term τ being *substitutable for* variable α. In terms of these notions, we let φ^{τ}_{α} stand for the result of substituting τ for α everywhere in φ (assuming τ is substitutable for α). In particular, $\varphi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$ stands for the results of substituting x_i for y_i everywhere in φ, for $1 \leq i \leq n$. We also use $\varphi(\alpha,\beta)$ to indicate the result of substituting β for one or more of the free occurrences of α in $\varphi(\alpha,\alpha)$. Finally, if τ is either a definite description or a complex term in which a definite description appears, we say that τ *contains* a definite description (otherwise τ is *free of* descriptions).

5) **Definition:** Ordinary and Abstract Objects. We define the property of *being ordinary* ('$O!$') as follows:

.1) $O! \ =_{df} \ [\lambda x \, \Diamond E!x]$

In other words, the property of being ordinary is the property of possibly

disallow quantifiers binding relation variables in λ-expressions.

However, in the present monograph, we shall not be concerned with the various ways of formulating a semantic interpretation of our language. The focus here is on the metaphysical system itself, for it is the conceptual framework embodied within the system itself that offers us metaphysical insights, if any. There is, therefore, no reason to retain the restriction 'no impredicative formulas in λ-expressions'.

being located in spacetime. We define the property of *being abstract* (‘$A!$’) as follows:

.2) $A! \;=_{df}\; [\lambda x\, \neg\Diamond E!x]$

The property of being abstract is the property of not possibly being located in spacetime. Abstract objects are not the kind of thing that could have a location in spacetime.

6) **Definitions:** The Identity$_E$ Relation ($=_E$) on Ordinary Objects. We say that objects x and y exemplify the *identity*$_{\mathrm{E}}$ relation iff x exemplifies being ordinary, y exemplifies being ordinary, and necessarily, x and y exemplify the same properties:

$$x =_E y \;=_{df}\; O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)$$

Note that $[\lambda xy\, x =_E y]$ is a well-formed λ-expression.

7) **Definitions:** Identity. Note that the normal symbol of identity ‘$=$’ is not among the primitives of the language. The defined symbol $=_E$ will provably designate a 2-place relation among ordinary objects. However, we may define a completely general notion of identity ‘$=$’ for individuals. Whereas ordinary objects x and y are identical iff they necessarily exemplify the same properties, abstract objects x and y are identical iff they necessarily encode the same properties. We may therefore formulate a single, completely general notion of identity as follows:

$$x = y \;=_{df}\; x =_E y \,\vee\, (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

We should emphasize that the defined symbols ‘$=$’ and ‘$=_E$’ are to be rigorously distinguished. The latter symbol may appear in λ-predicates, whereas the former may not ($[\lambda xy\, x = y]$ is ill-formed, due to the presence of the encoding subformulas).

8) **Definitions:** Relation Identity. We may also introduce the notion of identity appropriate to properties, relations, and propositions. Properties F and G are identical iff necessarily, F and G are encoded by the same objects:

.1) $F = G \;=_{df}\; \Box\forall x(xF \equiv xG)$

Relations F^n and G^n are identical iff for each way of plugging n–1 objects in the same order into F^n and G^n, the resulting 1-place properties are identical:

.2) $F^n = G^n =_{df}$ (where $n > 1$)

$$\forall x_1 \ldots \forall x_{n-1}([\lambda y\ F^n y x_1 \ldots x_{n-1}] = [\lambda y\ G^n y x_1 \ldots x_{n-1}]\ \&\ [\lambda y\ F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\ G^n x_1 y x_2 \ldots x_{n-1}]\ \&\ldots\&\ [\lambda y\ F^n x_1 \ldots x_{n-1} y] = [\lambda y\ G^n x_1 \ldots x_{n-1} y])$$

Finally, propositions p and q are identical iff the properties *being such that* p and *being such that* q are identical:

.3) $p = q =_{df} [\lambda y\ p] = [\lambda y\ q]$

§2: The Logic

We present the logic by describing the logical axioms and primitive rules. We use classical quantified modal logic modified only to admit the presence of definite descriptions. Definite descriptions have two important features: (1) they may fail to denote, and (2) they are rigid designators. We consider the consequences of these two facts in turn. (1) Since definite descriptions may fail to denote, any term τ containing a nondenoting definite description will also fail to denote. One may not validly instantiate such terms into universal claims, for this may lead from truth to falsity.[3] (2) Moreover, since descriptions are rigid designators, the logical axiom governing descriptions is contingently true, and so the Rule of Necessitation may not be applied to any line that depends on that axiom.[4] To accomodate these consequences of facts (1) and (2), we make two adjustments. (1) Quantification theory is formulated so that the logic of terms free of definite descriptions is classical, while the logic of terms containing

[3]Atomic formulas containing a nondenoting term are false. But in classical quantification theory, any term may be instantiated into a universal claim, and so classical quantification theory would allow us to derive the conclustion $Q\imath xPx$ from the premise $\forall xQx$. But the premise would be true and the conclusion false whenever $\forall xQx$ is true and $\imath xPx$ fails to denote. For then the conclusion would be false (given that atomic formulas with nondenoting descriptions are false). So the axiom that permits instantiation of terms into universal claims does not apply to terms containing definite descriptions.

[4]As an example, the following is derivable from the logical axiom governing descriptions: $G\imath xFx \rightarrow \exists xFx$. The Rule of Necessitation would then allow us to prove $\Box(G\imath xFx \rightarrow \exists xFx)$. But this is false when $\imath xFx$ rigidly desigates, for there are worlds in which: (a) the individual that is F at the actual world exemplifes G, and (b) there is nothing that exemplifies F. The law of descriptions and its consequences are *contingent* facts when descriptions are rigid, and so the Rule of Necessitation does not apply.

definite descriptions is 'free'. (2) We must explicitly formulate the Rule of Necessitation (RN) so that it does not apply to any line of a proof that depends on the logical axiom governing rigid definite descriptions. So instead of formulating RN as the primitive rule 'from φ, infer $\Box\varphi$', we take the modal closures of all axioms to be axioms of the systems, with the exception of the contingent axiom governing descriptions. The Rule of Necessitation may then be derived in a form which explicitly prevents its application to any formula that depends on a contingency such as the logical axiom for descriptions.

A. Logical Axioms, Rules of Inference, Theoremhood

In what follows, a *modal closure* of a formula φ is the result of prefacing any string of $\Box$s to φ. By convention, the modal closures of a schema are the result of applying any string of $\Box$s to any instance of the schema.

9) **Axioms and Rules:** Sentential Logic. The basis of our system is classical sentential logic. So the axioms are the modal closures of the following:

.1) $\varphi \to (\psi \to \varphi)$

.2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

.3) $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

The Rule of Inference is Modus Ponens (MP):

.4) MP: from φ and $\varphi \to \psi$, infer ψ

From this basis, all the tautologies of classical propositional logic are derivable. We did not simply cite 'the tautologies of propositional logic' as the axioms of our propositional logic, for that would require us to define this notion. While it is straightforward to do so, the definition is complicated by the fact that it must allow such sentences as $[\lambda x\, \neg Rx]\imath yQy \to (p \to [\lambda x\, Rx]\imath yQy)$ and $\Box p \to (q \to \Box p)$ to count as tautologies of propositional logic. To simplify the presentation, then, we have presented the traditional three axioms of classical propositional logic. The two formulas just presented as examples are instances of the first logical axiom.

10) **Axioms and Rules:** Logic of Quantification. We may describe the quantificational basis of our logic as a classical quantification theory that has been adjusted only for terms containing descriptions (which may fail to denote). So the logical axioms of our quantification theory are the modal closures of the following:

.1) $\forall\alpha\varphi \rightarrow (\exists\beta\, \beta=\tau \rightarrow \varphi^{\tau}_{\alpha})$, where τ is substitutable for α

.2) $\exists\beta\, \beta=\tau$, for any term τ free of descriptions

.3) $\psi^{\tau}_{\alpha} \rightarrow \exists\beta\, \beta=\tau$, where τ is any term containing a description and ψ^{τ}_{α} is any atomic formula in which τ has been substituted for the variable α

.4) $\forall\alpha(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall\alpha\psi)$, where α is not free in φ

The Rule of Generalization is the one rule of inference:

.5) GEN: from φ, infer $\forall\alpha\varphi$

Only the first three axioms require commentary. (10.1) asserts that if there is something which is τ (intuitively, if τ has a denotation), then τ may be substituted into universal claims. (10.2) asserts that for any term τ free of descriptions, there is something which is τ. (10.3) asserts that if a term τ containing a description appears in a true atomic formula, then there is something which is τ—atomic formulas are only if all of the terms in the formula have a denotation.

11) **Axioms:** Logic of Identity. The identity symbol '=' has been defined for both individual and relation variables. We associate with this notion the classical law of substitution, namely, that identical individuals and identical relations may be substituted for one another in any context. So where α, β are both individual variables, or both n-place relation variables ($n \geq 0$), the modal closures of the following are logical axioms of our system:

$$\alpha=\beta \rightarrow [\varphi(\alpha,\alpha) \equiv \varphi(\alpha,\beta)]$$

The other classical axiom of identity, namely that $\alpha=\alpha$, will be derived (see (30.1) below).

12) **Axioms:** Modal Logic. We take the modal closures of the three axioms of S5 as our propositional modal axioms:

.1) K axiom: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

.2) T axiom: $\Box\varphi \to \varphi$

.3) 5 axiom: $\Diamond\varphi \to \Box\Diamond\varphi$

The classical Rule of Necessitation (RN) will be derived (see (25) below), though it is modified only so that it does not apply to any line that depends on a nonmodal formula (the logical axiom governing definite descriptions is contingent and so we do not want to apply RN to any line of a proof that depends on that axiom). We also take the modal closures of the Barcan formulas as axioms:[5]

.4) $\forall\alpha\Box\varphi \to \Box\forall\alpha\varphi$, where α any variable

This means that the quantifier $\forall x$ ranges over a single, fixed domain of individuals and that that the quantifier $\forall F^n$ ($n \geq 0$) ranges over a single, fixed domain of relations.

13) **Axioms:** Logic of Actuality.[6] We shall take the ordinary, *non-modal* instances of the following axiom to be axioms of the system:

.1) $\mathcal{A}\varphi \equiv \varphi$

Such an axiom is a classic case of a logical truth that is not necessary, and so the modal closures of instances of this axiom will *not* be axioms of the system.[7] By contrast, the modal closures of the following axiom schema are taken as axioms:

.2) $\mathcal{A}\varphi \to \Box\mathcal{A}\varphi$

This asserts that if it is actually the case that φ, then necessarily, it is actually the case that φ. The modal closures of this claim are also axioms.

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[5]These formulas are usually derivable in an S5 quantified modal logic. But the derivation appeals to a primitive Rule of Necessitation (RN). In the present system, RN is derived and the Barcan Formulas play a crucial role in the derivation. So they are added as axioms.

[6]We interpret the actuality operator as rigid rather than as indexical. Intuitively, the formula $\mathcal{A}\varphi$ is true just in case φ is true at the distinguished actual world, i.e., the world such that all and only true propositions are true at that world. That is, even when $\mathcal{A}\varphi$ appears in a modal context, its truth depends only on the truth of φ at the actual world. That means that although $\mathcal{A}\varphi \equiv \varphi$ is logically true, it is not a necessary truth; $\Box(\mathcal{A}\varphi \to \varphi)$ will be false whenever both φ is true at the actual world and false at some possible world, and $\Box(\varphi \to \mathcal{A}\varphi)$ will be false whenever both *phi* is true at some possible world and false at the actual world.

[7]See Zalta [1988b] for a full discussion of logical and analytic truths that are not necessary.

14) **Axiom:** Logic of Encoding. If an object x possibly encodes a property F, it does so necessarily. That is, the modal closures of the following are logical axioms of the system:

$$\Diamond xF \rightarrow \Box xF$$

This axiom tells us that the properties an object encodes are not relative to any possible circumstance. Properties possibly encoded are encoded rigidly.

15) **Axioms:** Logic of Complex Predicates. If $[\lambda y_1 \ldots y_n\, \varphi]$ is any λ-predicate free of descriptions, the modal closures of the following are all assumed to be logical axioms of our system:

.1) $[\lambda y_1 \ldots y_n\, \varphi]x_1 \ldots x_n \equiv \varphi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$[8]

This is the classical λ-conversion principle.[9] There are also two other basic facts about the identity of relations:

.2) $[\lambda x_1 \ldots x_n\, F^n x_1 \ldots x_n] = F^n$

.3) $[\lambda x_1 \ldots x_n\, \varphi] = [\lambda y_1 \ldots y_n\, \varphi^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]$, where $y_1, \ldots, y_n$ are any variables distinct from, but substitutible for, $x_1, \ldots, x_n$

The second fact tells us that interchange of variables bound by the λ-operator makes no difference to the identity of the relations.

16) **Axioms:** Logic of Descriptions. If ψ is either an atomic formula containing the individual variable z or a defined identity formula of the form $\tau = z$ (for some individual term τ), then instances of the following are logical axioms of the system:

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[8] I am indebted to Alan McMichael, who helped me to formulate and interpret this axiom when the theory of objects was in an early stage of development.

[9] The restriction that φ be free of descriptions prevents non-denoting descriptions from playing havoc. For example, if $\imath xGx$ is a nondenoting description, the following sentence is false:

(a) $[\lambda y\, \neg Ry\imath xGx]z \equiv \neg Rz\imath xGx$

The reason is that the right side of the biconditional is true, because the atomic formula containing the nondenoting description is false. But the left side of the biconditional is false, since it is an atomic formula with a nondenoting λ-predicate. Note that if $\imath xGx$ has a denotation, it can be instantiated into universal claims. In particular, then we can substitute it into:

(b) $\forall u([\lambda y\, \neg Ryu]z \equiv \neg Rzu)$

Sentence (b) is an immediate consequence of our λ-conversion principle. In this way, we can produce only true sentences such as (a).

$$\psi_z^{\imath x\varphi} \equiv \exists x(\varphi \;\&\; \forall y(\varphi_x^y \rightarrow y\!=\!x) \;\&\; \psi_z^x)$$

This final group of logical axioms expresses Russell's analysis of definite descriptions. We emphasize that the modal closures of this axiom are *not* generally true and are not assumed to be proper axioms of the system.

17) **Definitions:** Derivability and Theoremhood. The following notions are defined in the usual way:

.1) A *proof* (or *deduction*) of ψ from some formulas Γ and the formulas $\varphi_1, \ldots, \varphi_n$ is a finite sequence of formulas ending with ψ in which every member of the sequence is either: (a) a logical axiom, (b) one of the formulas in Γ or one of the φ_i, or (c) inferred from one of the previous members of the sequence by the rule Modus Ponens (MP) or Generalization (GEN). A proof of ψ from no premises is any sequence of formulas ending with ψ in which every member either: (a) is a logical axiom or (b) follows from previous members of the sequence by the rule MP or GEN.

.2) We write $\Gamma, \varphi_1, \ldots, \varphi_n \vdash \psi$ to assert that there is a proof of ψ from the theory Γ and the assumptions $\{\varphi_1, \ldots, \varphi_n\}$. We write $\vdash \psi$ to assert that there is a proof of ψ from the empty set of premises.

.3) Whenever $\Gamma, \varphi_1, \ldots, \varphi_n \vdash \psi$, we say ψ *is a theorem of* the theory Γ and the assumptions $\{\varphi_1, \ldots, \varphi_n\}$. Whenever $\vdash \psi$, we say that ψ *is a theorem of logic* (or *is a logical theorem*).

B. Some Derived Rules

18) **Derived Rules:** The Natural Deduction Rules. Since we have both the standard axiomatization of propositional logic and employ the standard definitions of the connectives &, $\vee$, and $\equiv$, all of the usual natural deduction rules of inference apply to our system. These include Modus Tollens, Double Negation, Conjunction Introduction and Elimination rules, Disjunction Introduction and Elimination rules, etc.

19) **Definitions:** Dependence. Suppose that $\varphi_1, \ldots, \varphi_n$ is a proof of φ ($\varphi = \varphi_n$) from the theory Γ. Suppose further that the formula ψ is in Γ. Then we say that (the proof of) φ_i ($1 \leq i \leq n$) *depends upon* the formula ψ iff either (1) φ_i is ψ, and the justification for φ_i is that it is a logical axiom or in Γ, or (2) φ_i follows by MP or GEN from some previous members of the sequence at least one of which depends upon ψ.

20) **Derived Rules:** Conditional Proof and Reductio Ad Absurdum. The rules of Conditional Proof (CP) and Reductio Ad Absurdum (RAA) apply to our system:

.1) If $\Gamma, \varphi \vdash \psi$ in which application of GEN to a formula which depends upon φ has as its quantified variable a variable free in φ, then $\Gamma \vdash \varphi \rightarrow \psi$. (CP)

.2) If $\Gamma, \neg\varphi \vdash \neg\psi$ and $\Gamma, \neg\varphi \vdash \psi$, then $\Gamma \vdash \varphi$. (RAA)

These rules allow us to easily derive the Law of Noncontradiction, the Law of Excluded Middle, and the other laws of classical logic.

21) **Derived Rule:** Universal Instantiation. To show that our logic is a classical quantification theory for terms free of descriptions, we show that if τ is free of descriptions, it may be intantiated into universal claims:

If τ is free of descriptions and substitutable for α, then $\forall\alpha\varphi \vdash \varphi^{\tau}_{\alpha}$.

This rule, together with (10.4) and the rule GEN, establishes that we have a classical quantification theory for terms free of descriptions.

22) **Derived Rule:** Universal Generalization. The usual rule of Universal Generalization (UG) applies to our system:

If $\Gamma \vdash \varphi$ in which the constant symbol κ does not occur in Γ, then if α is a variable not occurring in φ, then $\Gamma \vdash \forall\alpha\varphi^{\alpha}_{\kappa}$ in which κ does not occur.

23) **Derived Rules:** Existential Generalization. The previous set of logical theorems allow us to derive the Rule of Existential Generalization (EG), in both classical and restricted forms, depending on whether the terms involved contain definite descriptions:

.1) $\varphi^{\tau}_{\alpha} \vdash \exists\alpha\varphi$, where τ is free of descriptions and substitutable for α

.2) $\varphi^{\tau}_{\alpha}, \exists\beta\, \beta\!=\!\tau \vdash \exists\alpha\varphi$, provided τ is any term substitutable for α

24) **Derived Rule:** Existential Instantiation. The usual rule of Existential Instantiation (EI) applies to our system:

If $\Gamma, \varphi^{\kappa}_{\alpha} \vdash \psi$ in which κ is a constant that does not occur in φ, ψ, or Γ, then $\Gamma, \exists\alpha\varphi \vdash \psi$ in which κ does not occur.

25) **Derived Rule:** Rule of Necessitation. Let us say that a *modal axiom* is any axiom other than an instance of (13.1) or (16). Let us also say that a formula ψ is *modal with respect to* the set Γ iff the formula $\Box\psi$ is in Γ. Then, the following Rule of Necessitation (RN) applies to our system:

> If $\Gamma \vdash \varphi$ such that φ depends only on modal axioms and formulas modal with respect to Γ, then $\Gamma \vdash \Box\varphi$.

This rule establishes that our propositional modal logic is indeed full-blooded S5 (KT5). Consequently, the modal operator $\Box$ can intuitively be conceived as a quantifier ranging over a single, fixed domain of all possible worlds (no notion of accessibility is required).[10]

C. Logical Theorems

26) **Logical Theorems:** The Converse Barcan Formulas (CBF).

$$\Box\forall\alpha\varphi \rightarrow \forall\alpha\Box\varphi$$

27) **Theorems:** The Domain of Objects is Partitioned. For any object x, either x is ordinary or x is abstract:

.1) $O!x \ \vee \ A!x$

Moreover, no object is both ordinary and abstract:

.2) $\neg\exists x(O!x \;\&\; A!x)$

28) **Theorem:** Identity$_{\mathrm{E}}$, Identity, and Necessity. (.1) If objects x and y are identical$_{\mathrm{E}}$, then they are identical; (.2) If objects x and y are identical$_{\mathrm{E}}$, then it is necessary that they are identical$_{\mathrm{E}}$:

.1) $x =_E y \rightarrow x = y$

.2) $x =_E y \rightarrow \Box x =_E y$

29) **Theorems:** Identity$_{\mathrm{E}}$ is Classical on Ordinary Objects. The relation identity$_{\mathrm{E}}$ relates ordinary objects in a classical manner:

.1) $=_E$ is an equivalence condition on ordinary objects, i.e.,

[10] But, of course, this is only an intuitive picture; our primitive variables x and F range only over primitive domains of individuals and relations. There is no primitive domain of possible worlds; rather worlds will be defined as objects of a certain sort.

$$O!x \rightarrow x =_E x$$
$$O!x \,\&\, O!y \rightarrow (x =_E y \rightarrow y =_E x)$$
$$O!x \,\&\, O!y \,\&\, O!z \rightarrow (x =_E y \,\&\, y =_E z \rightarrow x =_E z)$$

.2) Ordinary objects obey Leibniz's Law of the identity$_{\mathrm{E}}$ of indiscernables, i.e.,

$$O!x \,\&\, O!y \rightarrow [\forall F(Fx \equiv Fy) \rightarrow x =_E y]$$

.3) Distinct ordinary objects x, y have distinct haecceities $[\lambda z\, z =_E x]$ and $[\lambda z\, z =_E y]$, i.e.,

$$O!x \,\&\, O!y \,\&\, x \neq_E y \rightarrow [\lambda z\, z =_E x] \neq [\lambda z\, z =_E y]$$

30) **Theorems:** The Nature of Identity. General identity is an equivalence condition, both for individuals and for relations. Let α, β, γ be any three object variables or any three n-place relation variables:

.1) $\alpha = \alpha$

.2) $\alpha = \beta \rightarrow \beta = \alpha$

.3) $\alpha = \beta \,\&\, \beta = \gamma \rightarrow \alpha = \gamma$

31) **Theorems:** Every Individual and Relation Exists. It is an immediate consequence of the foregoing that every individual and every relation exists:

$$\forall \alpha \exists \beta (\beta = \alpha)$$

32) **Theorems:** The Necessity of Identity. If α and β are identical, then it is necessary that they are identical:

$$\alpha = \beta \rightarrow \Box \alpha = \beta$$

33) **Theorem:** Encoding and Necessary Encoding. An object x encodes property F iff necessarily, x encodes F.

$$xF \equiv \Box xF$$

34) **Theorem:** Objects that 'Have' the Same Properties are Identical. If abstract objects x and y encode the same properties, then they are identical:

.1) $[A!x \,\&\, A!y \,\&\, \forall F(xF \equiv yF)] \rightarrow x = y$

Similarly, if ordinary objects x and y exemplify the same properties, then they are identical:

.2) $[O!x \,\&\, O!y \,\&\, \forall F(Fx \equiv Fy)] \rightarrow x = y$

35) **Definitions:** Uniqueness. In the usual way, we say that there is a unique x such that φ ('$\exists!x\varphi$') just in case there is an x such that φ and anything y such that φ is identical to x:

$$\exists!x\varphi \;=_{df}\; \exists x(\varphi \,\&\, \forall y(\varphi^y_x \rightarrow y = x))$$

We should caution that the defined quantifier '$\exists!$' is distinct from the primitive predicate '$E!$'.

36) **Theorems:** Descriptions That Denote. There is a unique thing x such that φ iff there is something that is the x such that φ:

$$\exists!x\varphi \equiv \exists y(y = \imath x\varphi)$$

37) **Theorems:** Descriptions and Instantiation. If an individual y is identical with the x such that φ, then y is such that φ:

.1) $y = \imath x\varphi \rightarrow \varphi^y_x$

It now follows that if there is something that is the x such that φ, then the x such that φ is such that φ:

.2) $\exists y(y = \imath x\varphi) \rightarrow \varphi^{\imath x\varphi}_x$

D. Properties, Relations, Propositions, and Truth

In this subsection, we describe some important logical theorems that assert the existence of properties, relations, and propositions (PRPs).

38) **Theorems:** Comprehension Condition for Relations. The following is a logical theorem schema that is a direct consequence of (15.1) by RN and EG:[11]

.1) $\exists F^n \Box \forall x_1 \ldots \forall x_n(F^n x_1 \ldots x_n \equiv \varphi)$, where F^n is not free in φ and φ has no encoding subformulas and no descriptions.

[11] I am indebted to Alan McMichael, who first proposed the 'no encoding subformulas' restriction when this theorem had the status of an axiom in the first stage of the development of object theory.

This comprehension condition and the definition of $F^n = G^n$ in (8.2) constitute a precise theory of relations. As a special case, we have a comprehension condition on properties:

.2) $\exists F \Box \forall x(Fx \equiv \varphi)$, where F is not free in φ and φ has no encoding subformulas and no descriptions.

The theory of properties consists of this principle and the definition of property identity (8.1). Notice that there are two distinct senses of 'necessarily equivalent' for properties F and G: $\Box\forall x(Fx \equiv Gx)$ and $\Box\forall x(xF \equiv xG)$. Necessary equivalence in the former sense does not guarantee property identity, whereas necessary equivalence in the latter sense does (by stipulation). Intuitively, properties have two extensions: an exemplification extension that varies from world to world, and an encoding extension that does not vary from world to world. Therefore, we have an 'extensional' theory of 'intensional entities', since the identity conditions of properties are defined in terms of the identity of one of their extensions.

39) **Theorems:** Examples of Properties and Relations. The following typical instances of abstraction assert the existence of familiar complex properties.

$$\exists F \Box \forall x(Fx \equiv \neg Gx)$$
$$\exists F \Box \forall x(Fx \equiv Gx \,\&\, Hx)$$
$$\exists F \Box \forall x(Fx \equiv \Box(E!x \rightarrow Gx))$$
$$\exists F \Box \forall x(Fx \equiv \exists z Rzx)$$
$$\exists F \Box \forall x(Fx \equiv Gx \vee Hx)$$
$$\exists F \Box \forall x(Fx \equiv \Box Gx)$$
$$\exists F \Box \forall x(Fx \equiv p)$$

40) **Theorems:** Equivalence and Identity of Properties. If properties F and G are encoded by the same objects, (i.e., if they are materially equivalent with respect to the objects that encode them), then they are identical:

$$\forall x(xF \equiv xG) \rightarrow F = G$$

41) **Definition:** Truth. Recall that $p, q, r, \ldots$ are 0-place relation variables. They ranges over the domain of propositions. We may define the property of truth for a proposition by elimination as follows:

p *is true* $=_{df}$ p

42) **Theorems:** Comprehension Condition for Propositions. If we let $n = 0$ and suppose that φ meets the conditions in definition (3.2), then $[\lambda\, \varphi]$ is a well-formed complex 0-place relation term. We read $[\lambda\, \varphi]$ as 'that-φ'. Therefore, when φ is free of descriptions, (15.1) yields a group of axioms of the following form:

.1) $[\lambda\, \varphi] \equiv \varphi$

We may read this as follows: (the proposition) *that-*φ is true iff φ. Since 'p' is a 0-place relation variable, the following comprehension condition for propositions is an immediate consequence of (42.1) by RN and EG:

.2) $\exists p \Box(p \equiv \varphi)$, where φ has no encoding subformulas and no descriptions.

43) **Theorems:** Examples of Propositions. The following typical instances of abstraction assert the existence of familiar complex propositions:

$$\exists p \Box(p \equiv \neg q)$$
$$\exists p \Box(p \equiv q \;\&\; r)$$
$$\exists p \Box(p \equiv q \;\vee\; r)$$
$$\exists p \Box(p \equiv \Box q)$$

44) **Definition:** Propositional Properties. Let us call a property F *propositional* iff for some proposition p, F is the property *being such that* p:

$$Propositional(F) \;\; =_{df} \;\; \exists p(F = [\lambda y\, p])$$

45) **Theorems:** Propositional Properties. For any proposition (state of affairs) p, the propositional property $[\lambda y\, p]$ exists.

.1) $\forall p \exists F(F = [\lambda y\, p])$

In general, if F is the property *being such that* p, then necessarily, an object x exemplifies F iff p:

.2) $F = [\lambda y\, p] \rightarrow \Box \forall x(Fx \equiv p)$

This understanding of propositional properties sheds more light on the definition of proposition identity '$p = q$' in (8.3). The comprehension axiom (42.2) and definition (8.3) jointly offer a precise theory of propositions (states of affairs).

46) **Definition:** Objects Encode Propositions. We may introduce a sense in which abstract objects encode propositions. We say that x encodes proposition p ('$\Sigma_x p$') just in case x encodes the property of being such that p:

$$\Sigma_x p \ =_{df} \ x[\lambda y\, p]$$

§3: The Proper Axioms

47) **Axiom:** Ordinary Objects. Ordinary objects necessarily fail to encode properties:

$$O!x \rightarrow \Box\neg\exists F\, xF$$

48) **Axioms:** Abstract Objects. For every condition φ on properties, there is an abstract object that encodes just the properties satisfying φ:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)), \text{ where } \varphi \text{ has no free } x\text{s}$$

49) **Definition:** Theorems of Principia Metaphysica. The instances of the above proper axioms may be enumerated and collected into the set PM. In what follows, we develop interesting formulas φ such that PM$\vdash\varphi$. Proofs of selected theorems of PM are gathered in the Appendix.

§4: The Theory of Objects

50) **Theorems:** Uniqueness of Abstract Objects. From the definition of identity, we know that distinct abstract objects must differ with respect to at least one of their encoded properties. But, then, for each formula φ in (48), there couldn't be two distinct abstract objects that encode all and only the properties satisfying φ. So there is a unique object that encodes just the properties satisfying φ:

$$\exists!x(A!x \,\&\, \forall F(xF \equiv \varphi)), \text{ where } \varphi \text{ has no free } x\text{s}$$

51) **Theorems:** Well-Defined Descriptions. Consequently, for every formula φ with no free xs, there is such a thing as *the* abstract object that encodes just the properties satisfying φ:

$$\exists y\, y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

This means that descriptions of this form may always be validly instantiated into universal generalizations.

52) **Theorems:** Properties Encoded by Abstract Objects. It is now provable that the object that encodes just the properties satisfying φ encodes a property G iff G satisfies φ:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi^G_F$$

Notice that this theorem depends upon the contingent logical axiom (16) governing descriptions, and so we may *not* apply RN. So this theorem is contingent, for G need not necessarily satisfy φ even though it does in fact. It might be the case that both (a) the abstract object that encodes just the properties that *in fact* satisfy φ encodes G and (b) G fails to satisfy φ. For example, let G be the property of being snub-nosed, let Fs be the claim that Socrates exemplifies F, and assume that in fact Socrates is snub-nosed (i.e., φ^G_F). Then, at worlds where Socrates fails to exemplify being snub-nosed, the abstract object that encodes just the properties Socrates in fact exemplifies still encodes being snub-nosed, for properties encoded are rigidly encoded. Or, it might be the case that both (a) G satisfies φ, and (b) the abstract object that encodes just the properties that in fact satisfy φ doesn't encode G.

53) **Corollary:** Properties Encoded by Abstract Objects. It is a corollary to the previous theorem that the abstract object that encodes just the properties $G_1, \ldots, G_n$ encodes G_i $(1 \leq i \leq n)$:

$$\imath x(A!x \,\&\, \forall F(xF \equiv F{=}G_1 \vee F{=}G_n))G_i \quad (1 \leq i \leq n)$$

54) **Theorems:** The Null Object and the Universal Object. There is a unique object that encodes no properties, and a unique object that encodes every property:

.1) $\exists! x(A!x \,\&\, \forall F(xF \equiv F \neq F))$

.2) $\exists! x(A!x \,\&\, \forall F(xF \equiv F{=}F))$

55) **Theorems:** Some Non-Classical Abstract Objects. The following are facts concerning the existence of certain abstract objects: (.1) For any relation R, there are distinct abstract objects x and y for which the property of *bearing R to x* is identical to the property of *bearing R to y*, and (.2) For any relation R, there are distinct abstract objects x and y

for which the property of *being a z such that x bears R to z* is identical to the property of *being a z such that y bears R to z*:

.1) $\forall R \exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda z\, Rzx] = [\lambda z\, Rzy])$

.2) $\forall R \exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda z\, Rxz] = [\lambda z\, Ryz])$

Moreover, for any property F, there are distinct abstract objects x, y such that the proposition *that*-Fx is identical with the proposition *that*-Fy:

.3) $\forall F \exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda\, Fx] = [\lambda\, Fy])$

56) **Theorems:** Some Further Non-Classical Abstract Objects.[12] Since we allow the formation of impredicative relations (i.e., relations defined in terms of quantifiers that bind relation variables), we may prove the existence of distinct abstract objects a and b which exemplify exactly the same properties:

$$\exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, \forall F (Fx \equiv Fy))$$

57) **Remark:** Models of the Theory. Models of the theory reveal the structure that underlie the two previous theorems. For simplicity, we consider only the models of the monadic, non-modal portion of the theory.[13] In particular, we describe the models only in enough detail to see that the two comprehension principles $\exists x \forall F (xF \equiv \varphi)$ and $\exists F \forall x (Fx \equiv \varphi^*)$ are true (φ^* simply indicates that there are restrictions on permissible φs).

To appreciate the problem of modeling the theory, we first indicate how the theory avoids the paradoxes of encoding. Without the restrictions on λ-formation, the following two expressions would have been well-formed:

$$[\lambda x\, \exists F (xF \,\&\, \neg Fx)]$$

$$[\lambda x\, x = y]$$

Consider the first expression. This expression would describe the property that an object exemplifies iff it fails to exemplify a property that it encodes. Were there such a property, we could assert the existence of an abstract object that encoded such a property:

[12] I am indebted to Peter Aczel for pointing that this theorem results once we allow impredicative relations into the system.

[13] I have used Aczels' suggestion to develop more comprehensive models of the relational and modal portion of the theory. See my paper, 'The Modal Object Calculus and its Interpretation'. Also, interested readers should refer to the Appendix ('Modelling the Theory Itself') of Zalta [1983], where models of the theory proposed by Dana Scott are discussed in some detail.

$$\exists x(A!x \,\&\, \forall F(xF \equiv F = [\lambda x\, \exists F(xF \,\&\, \neg Fx)]))$$

Such an existence assertion would lead to a contradiction—it would be provable that any such abstract object exemplifies the property in question iff it does not. Consider, now, the second expression. This expression would describe the property of being identical (in the defined sense) to the object y. Were there such a property, we could assert the existence of an abstract object that encodes all the properties F such that for some object y, F is the property of being identical to y and y fails to encode F:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(F = [\lambda z\, z = y] \,\&\, \neg yF)))$$

Such an existence assertion would lead to a contradiction—just consider any such abstract object, say k, and ask whether it encodes $[\lambda z\, z = k]$.

Neither of the problematic existence assertions are axioms, however, for the λ-expressions involved are not well-formed. If we think model-theoretically for the moment, and assume some basic set theory, then the problem underlying these lurking paradoxes can be described as follows. The comprehension principle for abstract objects attempts to correlate abstract objects with sets of properties. Now if, for each distinct abstract object k, there were to exist a property (haecceity) of *being such that* k ($[\lambda z\, z = k]$), then there would be a one-one correlation between the power set of the set of properties and a subset of the set of properties, in violation of Cantor's Theorem. So abstract objects cannot have haecceities; i.e., there cannot be a distinct property of the form $[\lambda z\, z = y]$ for each abstract object y (this actually implied by (55)). However, ordinary objects remain perfectly well-behaved—they have haecceities of the form $[\lambda z\, z = y]$ which behave classically for ordinary y.

Notice that the comprehension principle for abstract objects attempts to populate a *subdomain* of the domain of individuals with as many abstract objects as there are sets of properties. Since the comprehension principle for properties intuitively correlates properties with the sets of individuals that serve as their exemplification extensions, it must turn out that for each property, there is some pair of distinct abstract objects that cannot be distinguished by exemplification predications. But the model that pictures this fact isn't quite as straightforward as one might imagine. One *cannot* use the following structure and definitions:

> Take the ordinary objects to be the urelements of ZF + Urelements.
> Take the properties to be the members of the power set of urelements

> (let the variable 'F' range over this set). Take the abstract objects to be the members of the power set of the set of properties. Put the ordinary objects and abstract objects together into one set (let the variable x range over this set) and then define: (a) 'xF' is true iff $F \in x$, and (b) 'Fx' is true iff $x \in F$.

This simple attempt at modeling the theory fails for two reasons: (1) the theory asserts that abstract objects exemplify properties (for example, every abstract object, by comprehension and definition, exemplifies the property of being necessarily nonspatiotemporal $[\lambda x \Box\neg E!x]$), yet the formula 'Fx' is always false, when x is an abstract object, in the proposed model, and (2) the theory asserts that there exists an object x such that xF and Fx (consider any object that encodes the property of being necessarily nonspatiotemporal), but in the proposed model, this would require $F \in x$ and $x \in F$, in violation of the foundation axiom of ZF.

Peter Aczel has proposed models that get around these obstacles.[14] An very simple, extensional model for the non-modal, monadic portion of the theory, based on Aczel's suggestion, can be described as follows:

> Suppose that the urelements of ZF + Urelements come divided into two mutually exclusive subdomains, the concrete objects and the special objects. Take the properties to be the members of the power set of urelements (let the variable 'F' range over this set). Take the abstract objects to be the members of the power set of the set of properties. Correlate each abstract object with one of the members of the urelement subdomain of special objects. Call the object correlated with abstract object k *the proxy of* k (so some distinct abstract objects will have the same proxy). Put the ordinary objects and abstract objects together into one set (let the variable x range over this set) and define a mapping $\|x\|$ from this set into the set of urelements as follows: $\|x\| = x$ if x is concrete and $\|x\| =$ the proxy of x if x is abstract. Finally, we may define: (a) 'xF' is true iff $F \in x$, and (b) 'Fx' is true iff $\|x\| \in F$.

In this model, both principles $\exists x \forall F(xF \equiv \varphi)$ and $\exists F \forall x(Fx \equiv \varphi^*)$ are true. It explains the earlier results about the absence of haecceities for abstract objects, and the collapse of $[\lambda x\, Rxk]$ and $[\lambda x\, Rxl]$ (and Fk and Fl) for some distinct abstract objects k and l. Exemplification, as un-

[14] Personal communication, January 10, 1991.

derstood in the model, cannot distinguish abstract objects that have the same proxy.

This simple model can be extended to a more general, intensional model structure for the relational, modal theory as follows.[15] The more general model assumes that the language has been interpreted in a structure containing several mutually exclusive domains of primitive entities:

1. a domain of *ordinary* objects $\mathbf{O}$ and a domain of *special* objects $\mathbf{S}$; the union of these domains is called the domain of ordinary* objects $\mathbf{O}^*$,

2. a domain $\mathbf{R}$ of relations, which is the general union of the sequence of domains $\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \ldots$, where each $\mathbf{R}_n$ is the domain of n-place relations,

3. a domain of possible worlds, which contains a distinguished actual world.

The domain of relations $\mathbf{R}$ is subject to two conditions: (a) it is closed under logical functions that harness the simple properties and relations into complex properties and relations (these logical functions are the counterparts of Quine's predicate functors, except that they operate on relations instead of predicates), and (b) there are at least as many relations in each $\mathbf{R}_n$ as there are elements of $\mathcal{P}([\mathbf{O}^*]^n)$ (= the power set of the n^{th}-Cartesian product of the domain of ordinary* objects). Relative to each possible world $\mathbf{w}$, each n-place relation $\mathbf{r}^n$ in $\mathbf{R}_n$ is assigned an element of $\mathcal{P}([\mathbf{O}^*]^n)$ as its exemplification extension at $\mathbf{w}$ (when $n = 0$, each proposition $\mathbf{r}^0$ is assigned a truth value at $\mathbf{w}$).[16] In what follows, we refer to the exemplification extension of a relation $\mathbf{r}^n$ at world $\mathbf{w}$ as $\mathbf{ext_w}(\mathbf{r}^n)$.

The model is completed by letting the domain of abstract objects $\mathbf{A}$ be the power set of the set of properties (i.e., $\mathbf{A} = \mathcal{P}(\mathbf{R}_1)$). Each abstract object in $\mathbf{A}$ is then mapped to one of the *special* objects in $\mathbf{S}$; the object correlated with abstract object a is called *the proxy of a*. Some distinct abstract objects will therefore get mapped to the same proxy. Finally, the ordinary and abstract objects are combined into one set $\mathbf{D}$ (= $\mathbf{O} \cup$

[15]See Zalta [1997], for a more thorough development.

[16]Constraints on the logical functions ensure that the exemplification extension of a complex relation $\mathbf{r}$ meshes in the proper way with the exemplification extensions of the simpler relations $\mathbf{r}$ may have as a part.

A).[17] Letting the variable $\mathbf{x}$ range over $\mathbf{D}$, we define a mapping $|\ |$ from $\mathbf{D}$ into the set of ordinary* objects as follows:

$$|\mathbf{x}| = \begin{cases} \mathbf{x}, \text{ if } \mathbf{x} \text{ is ordinary} \\ \text{the proxy of } \mathbf{x}, \text{ if } \mathbf{x} \text{ is abstract} \end{cases}$$

Now suppose that an assignment function g to the variables of the language has been extended to a denotation function $\mathbf{d}_g$ on all the terms of the language (so that in the case of the variables x and F, we know $\mathbf{d}_g(x) \in \mathbf{D}$ and $\mathbf{d}_g(F^n) \in \mathbf{R}^n$). We then define *true at* $\mathbf{w}$ (with respect to g) for the atomic formulas as follows:[18] (a) '$F^n x_1 \ldots x_n$' is true at $\mathbf{w}$ iff $\langle |\mathbf{d}_g(x_1)|, \ldots, |\mathbf{d}_g(x_n)| \rangle \in \mathbf{ext_w}(\mathbf{d}_g(F^n))$, and (b) '$xF$' is true at world $\mathbf{w}$ iff $\mathbf{d}_g(F) \in \mathbf{d}_g(x)$.[19] It is easy to constrain the models so that $\mathbf{ext_w}(\mathbf{d}_g(O!))$ is simply the subdomain of ordinary objects and that $\mathbf{ext_w}(\mathbf{d}_g(A!))$ is the subdomain of special objects.[20] Finally, the resulting structure can be constrained in ways to ensure that the logic of identity, logic of actuality, logic of descriptions, etc., are all preserved.[21]

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[17] This corrects the error in my [1997]. In that paper, the domain $\mathbf{D}$ was set to $\mathbf{O}^* \cup \mathbf{A}$ instead of $\mathbf{O} \cup \mathbf{A}$. But this won't yield a model, for the following reason. Consider distinct *special* objects a and b in $\mathbf{S}$. Then there will be some property, say P, such that Pa and $\neg Pb$. But given the definitions below, it will follow that both a and b are abstract, i.e., that $A!a$ and $A!b$ are both true. Moreover, since a and b are special objects, they necessarily fail to encode properties, so $\Box\forall F(aF \equiv bF)$. But, then, by the definition of identity for abstract objects, it follows that $a = b$, and thus $Pa \,\&\, \neg Pa$. By setting $\mathbf{D}$ to $\mathbf{O} \cup \mathbf{A}$, we avoid this result. I am indebted to Tony Roy for pointing this out to me.

[18] For simplicity, we are using representative atomic formulas containing only variables.

[19] Note that since the truth of encoding formulas at a world is defined independently of a world, an encoding formula will be true at all worlds if true at any. This validates the Logical Axiom of Encoding.

[20] We simply require that $\mathbf{ext_w}(\mathbf{d}_g(E!))$ be some subset of the domain of ordinary objects and that the domain of ordinary objects be the union, for every $\mathbf{w}$, of all the sets $\mathbf{ext_w}(\mathbf{d}_g(E!))$. Since $O!$ is defined as $[\lambda x \,\Diamond E!x]$ and $A!$ is defined as $[\lambda x \,\neg\Diamond E!x]$, this guarantees that all the ordinary objects are in the exemplification extension of $O!$ and all the special objects are in the exemplification extension of $A!$. It should now be straightforward to see that the proper axiom, $O!x \rightarrow \Box\neg\exists F\, xF$, is true in such a model.

[21] It shall not be our concern in the present work to demonstrate how to constrain the models so as to preserve the distinction between the logical axioms and proper axioms of the system.

These definitions have the following consequences: (1) the comprehension principle for abstract objects is true in this model;[22] (2) λ-Conversion and Relations are both true in this model;[23] (3) an abstract object x (i.e., set of properties) will exemplify (according to the model) a property F just in case the proxy of x exemplifies F in the traditional way; and (4) whenever distinct abstract objects x and y get mapped to the same proxy, x will exemplify a property F iff y exemplifies F.

This more general model structure helps us to picture the earlier results in (55) and (56). It would be a mistake, however, to construe abstract objects in what follows as sets of properties. With the consistency of the theory secure, we may go ahead and assume that the world is just the way that the theory says it. In particular, the theory does not presuppose any notions of set theory.

§5: The Theory of Platonic Forms:[24]

58) **Remark:** Are Platonic Forms Properties or Objects? It seems that in writing about the Forms, Plato had in mind entities that nowadays we would refer to as 1-place properties or universals. Furthermore, we would nowadays refer to Plato's notion of an object 'participating' in a Form in terms of the notion of an object exemplifying a property (or instantiating a universal). If this is all Plato intended, then such talk is systematized by our underlying logic. That is, the logic contains a theory of properties, a precise definition of the conditions under which properties F and G are identical, and implicitly axiomatizes the notion of exemplification. The conception of properties captured by this theory does seem to explicate some of the ideas Plato had in mind.

However, there is one major problem with this way of systematizing the Forms, namely, it turns Plato's most fundamental principle about the Forms into a logical truth as opposed to a proper thesis of metaphysics. We explain why this is so in the next paragraph. The logic of encoding

[22] For each condition φ on properties, there is a set of properties a such that: (a) the proxy of a is a special object and (b) a property F is an element of a iff F satisfies φ.

[23] The logical functions ensure that for any condition φ on objects $x_1, \ldots, x_n$ without encoding subformulas, there is a relation F^n whose exemplification extension, at any given possible world, contains all and only those n-tuples of objects satisfying φ. See Zalta [1983] and [1988a] for further details.

[24] The material in this section is a slightly revised version of material that first appeared in Zalta [1983].

gives us an alternative. We may treat Platonic Forms as abstract objects and analyze participation in terms of both exemplification and encoding. Under such an analysis, not only does Plato's main principle about the Forms become an interesting metaphysical thesis, but other problems concerning the nature of Forms have a simple solution.

Plato's foremost principle about the Forms is The One Over the Many Principle. It is stated in *Parmenides* (132a) as follows:

> If there are two distinct F-things, then there is a Form of F in which they both participate

The traditional view, according to which Forms are properties or universals, is captured by the following analytic hypotheses:

$$\textit{The Form of } F \;=_{df}\; F$$

$$x \textit{ participates in } F \;=_{df}\; Fx$$

But if we now translate The One Over the Many Principle using this analysis, we get the following:

$$x \neq y \;\&\; Fx \;\&\; Fy \rightarrow \exists G(G = F \;\&\; Gx \;\&\; Gy)$$

Clearly, this is just a logical truth. It can be derived from the logical axioms alone, without appeal to the proper theory of abstract objects. For assume the antecedent. Then, we may extract the second two conjuncts and conjoin them with the logical theorem that $F = F$ (30.1). So the consequent now follows by Existential Introduction.

So Plato's most important principle is a logical truth, under this analysis. The following series of definitions and theorems avoids this result, by offering a way of analyzing Platonic Forms so that The One Over the Many Principle becomes a proper thesis of metaphysics.

59) **Definitions:** Forms. We define: *x is a Form of G* iff x is an abstract object that encodes just the property G:

$$\textit{Form}(x, G) \;=_{df}\; A!x \;\&\; \forall F(xF \equiv F = G)$$

60) **Theorem:** Existence and Uniqueness of Forms. It follows that for every property G, there exists a unique Form of G:

$$\forall G \exists! x \textit{Form}(x, G)$$

61) **Definition:**. The Form of G. Since there is a unique Form of G, we may introduce a term ('Φ_G') to designate *the* Form of G:

$$\Phi_G \ =_{df} \ \imath x(\mathit{Form}(x, G))$$

62) **Theorem:** The Form of G Exists. The laws of description yield the following immediate result, namely, that for every property G, there exists something which is the Form of G:

$$\forall G \exists y(y = \Phi_G)$$

63) **Theorem:** The Form of G Encodes G.

$$\Phi_G G$$

If we think of xF as a way of predicating the property F of the object x, then this theorem is a kind of self-predication theorem. The property that constitutes the essence of The Form of G can be predicated of that form via encoding predication. The significance of this will be discussed below in (69).

64) **Definition:** Participation. An object y participates in x iff there is a property which x encodes and which y exemplifies:

$$\mathit{Participates}(y, x) \ =_{df} \ \exists F(xF \,\&\, Fy)$$

So a thing y participates in The Form of G iff there is property that Φ_G encodes and that y exemplifies.

65) **Theorem:** The Equivalence of Participation and Exemplification. It is an consequence of the definition of participation that an object x exemplifies a property F iff x participates in The Form of F:

$$Fx \equiv \mathit{Participates}(x, \Phi_F)$$

This theorem establishes that we can derive the traditional analysis of participation (as exemplification) from our analysis of participation.

66) **Theorem:** The One Over the Many Principle. The above definitions validate Plato's most important principle governing the Forms. The principle asserts that if there are two distinct F-things, then there is a Form of F in which they both participate:

$$x \neq y \,\&\, Fx \,\&\, Fy \rightarrow \exists z(z = \Phi_F \,\&\, \mathit{Participates}(x, z) \,\&\, \mathit{Participates}(y, z))$$

This is a proper theorem, requiring an appeal to one of the proper axioms of the theory of abstract objects.

67) Definition and Theorem: Platonic Being and Self-Participation. Consider the property being abstract $A!$; i.e., $[\lambda x\, \neg\Diamond E!x]$. We may define *Platonic Being* as The Form of $A!$:

.1) $\mathit{Platonic\ Being} \ =_{df} \ \Phi_{A!}$

It now follows that every Form participates in Platonic Being:

.2) $\forall x(\mathit{Form}(x) \rightarrow \mathit{Participates}(x, \Phi_{A!}))$

Of course, *Platonic Being* is itself a Form, and so by the theorem just proved, it participates in itself:

.3) $\mathit{Participates}(\Phi_{A!}, \Phi_{A!})$

So the question whether Forms can participate in themselves has a positive answer.

68) Remark: The Third Man Puzzle. We are now in a position to solve the third man puzzle. This is a puzzle discussed in the *Parmenides* (132a). It arises in connection with four of Plato's principles: The One Over the Many Principle (OM), The Self Predication Principle (SP), The Non-Identity Principle (NI), and the Uniqueness Principle (UP):

OM: If there are two distinct F-things, then there is a Form of F in which they both participate.

SP: The Form of F is F.

NI: If something participates in The Form of F, it is not identical with that Form.

UP: There is a unique Form of F.

It would seem that these four principles cannot be true together, given that there are two distinct F-things. For if there are x and y such that both are F-things, then by (OM) there is a Form of F in which they both participate. By (SP), The Form of F is an F-thing. But by two instances of (NI), it follows that The Form of F is distinct from both x and y. But then, (OM) requires that there is a second Form of F in which both x and the first Form participate. So, by (NI), the two Forms must be distinct, which contradicts the Uniqueness principle (UP).

Our work offers the following analysis of the third man puzzle. We have captured (OM) and (UP) as the theorems (66) and (60.2), respectively. (NI) is rejected because (67.3) is a counterexample. That is, if (NI) is analyzed as the claim:

$$Part(x, \Phi_F) \rightarrow x \neq \Phi_F,$$

then it is incompatible with (67.3) and the self-identity of every abstract object. Since a counterexample is derivable, the theory rejects (NI). The status of (SP) will be discussed in the next item, after we consider two important auxiliary hypotheses that have bearing on our understanding of (SP).

69) **Auxiliary Hypotheses:** Two Supporting Hypotheses.

.1) Abstract objects do *not* exemplify such ordinary *nuclear* properties as having a spatiotemporal location, having a shape, having a texture, having a size, having mass, undergoing change, having a color, having length, loving Hilary Rodham Clinton, meeting Yeltsin, dreaming about monsters, etc.[25]

.2) English predicative statements of the form 'x is F' are lexically ambiguous between Fx and xF, and English predication is structurally ambiguous between exemplification predication and encoding predication.

The first auxiliary hypothesis fills in the lacunae created by the fact that the theory does not say which which properties abstract objects exemplify (other than the property of being abstract). It captures our ordinary intuitions about the properties abstract objects exemplify. A variety of evidence for the second hypothesis was collected in Zalta [1983] and [1988a]. Given this hypothesis, the Self-Predication principle (SP) receives two readings: $G\Phi_G$ and $\Phi_G G$. Theorem (63) tells us that the latter reading is true, for it is derivable. The first auxiliary hypothesis tells us that, in the cases of ordinary nuclear properties, the former is false. Of course, $G\Phi_G$ is true whenever G is the negation of a nuclear property, and in such cases as the property $A!$.

[25] For our most systematic account of the notion of a nuclear property, see Parsons [1980].

§6: The Theory of Meinongian Objects

70) **Remark:** Meinongian Objects. Alexius Meinong believed that the realm of objects divides up into ordinary objects and ideal objects. Within each of these groups, there is a further division, namely, between those objects that have being and those that do not. Ordinary objects that have being are said to 'exist' and examples of such objects are existing humans, stones, trees, planets, atoms, etc. Examples of ordinary objects that have no being are Pegasus, Zeus, the fountain of youth, the Loch Ness monster, etc. Ideal objects that have being are said to 'subsist' and examples of such objects are the state of affairs *Clinton is president*, the state of affairs *the earth revolves about the sun*, the natural number 2, the real number π, etc. Examples of ideal objects that have no being are: the state of affairs *Clinton is not president*, the state of affairs *the earth is flat*, the even prime number greater than 2, the round square, the Russell set, etc.

In this section, we treat only those ordinary and ideal *individuals* that, according to Meinong, 'have no being'. We offer an analysis of the *descriptions* of those objects, not names of such objects. Names of nonexistent objects will be discussed in the later section on fiction. We shall say little about what Meinong regarded as 'existing ordinary objects', for we may think of these as objects x such that $O!x$. Nor shall we say much about the logically complex 'ideal objects' such as states of affairs, whether subsisting or not. For the present purposes, we may regard Meinong's states of affairs as propositions, i.e., entities systematized by the comprehension and identity principles for 0-place relations. So unlike Meinong, we assert, for every proposition p, that both $\exists q\, q = p$ and $\exists q\, q = \neg p$; a primitive notion of truth (indeed, an eliminable one) applies to propositions and if a proposition is false, we may still quantify over it. Finally, we should note that the ideal objects that are individuals, like the natural number 2, the real number π, etc., will be analyzed in the penultimate section and final section of this monograph.[26]

[26]We should also note that the typed theory of abstract objects developed in the last section of this monograph axiomatizes *higher-order* ordinary and abstract objects. Thus, we find ordinary and abstract properties, ordinary and abstract relations, and ordinary and abstract propositions. The abstract objects of a given type encode (and exemplify) properties appropriate to objects of that type. Though we may treat the usual sorts of propositions and their negations as ordinary propositions, it seems worthwhile to suggest that propositions like the liar proposition are not ordinary, but

71) **Remark:** Meinong's Unrestricted Satisfaction Principles. Meinong's theory of objects is a naive theory; it is based on principles that are not made explicit. He asserted such things as the following: the round square is round; the round square is square; the existent golden mountain is golden; the existent golden mountain is a mountain; the existent golden mountain is existent. If we try to make the principle Meinong is using here explicit, it would look something like the following informal principle of natural language:

The $F_1, \ldots, F_n$ is $F_i \quad (1 \leq i \leq n)$

The problem is that if the description 'the $F_1, \ldots, F_n$' is analyzed as denoting an object that *exemplifies* the enumerated properties and the copula 'is' is treated as an exemplification predication, then not only will the description frequently fail to denote but the resulting sentence will either be false (contradict contingent facts), contradict well-entrenched non-logical principles, or contradict logical principles. For example, if the entity denoted by 'the existent golden mountain' were to exemplify existence ($E!$), goldenness (G), and mountainhood (M), it would contradict the contingent fact that no golden mountains exist ($\neg\exists x(E!x \& Gx \& Mx)$). If the entity denoted by 'the round square' were to exemplify both roundness (R) and squareness (S), then it would contradict the non-logical law of geometry that everything exemplifying roundness fails to exemplify squareness ($\forall x(Rx \rightarrow Sx)$). If the entity denoted by 'the non-square square' were to exemplify both squareness and non-squareness ($\bar{S}$), then it would contradict the law of logic that anything exemplifying non-squareness fails to exemplify squareness ($\forall x(\bar{S}x \rightarrow \neg Sx)$). How then do we preserve the intuitive principle upon which Meinong seemed to rely?

One answer is to analyze the description 'the $F_1, \ldots, F_n$' as denoting the abstract object encoding just the enumerated properties. The following theorems justify this idea.

72) **Theorems:** Analysis of Meinongian Unrestricted Satisfaction. The following three claims are provable: (.1) for any properties $F_1, \ldots, F_n$, there is a unique abstract object that encodes just those properties; (.2) for any properties $F_1, \ldots, F_n$, the abstract object that encodes just $F_1, \ldots, F_n$

rather abstract propositions that encode the properties of propositions by which they are conceived. This does, in some sense, preserve Meinong's intuition that there are ideal objects like states of affairs that don't subsist.

exists; and (.3) the abstract object that encodes just $F_1, \ldots, F_n$ encodes F_i (where $1 \leq i \leq n$):

.1) $\forall F_1 \ldots \forall F_n \exists! x \forall G(xG \equiv G{=}F_1 \vee \ldots \vee G{=}F_n)$

.2) $\forall F_1 \ldots \forall F_n \exists y[y{=}\imath x \forall G(xG \equiv G{=}F_1 \vee \ldots \vee G{=}F_n)]$

Furthermore, we may appeal to our auxiliary hypothesis (69.2) to read the informal principle as an encoding predication. Putting these together, we get the following, which is a theorem:

.3) $\imath x \forall G(xG \equiv G{=}F_1 \vee \ldots \vee G{=}F_n)F_i \quad (1 \leq i \leq n)$

This last consequence asserts that the object that encodes just the properties $F_1, \ldots, F_n$ encodes the property F_i (where F_i is one of the enumerated properties). We therefore have an analysis not only on which the descriptions Meinong used in fact have a reading on which they denote individuals but also on which the predicative statements Meinong made to describe those individuals are true. This demonstrates that there is an explicit, formal, true principle which underlies Meinong's informal principle. The truth of Meinong's informal principle was obscured, however, by ambiguity in the natural language he used to express it.

§7: The Theory of Situations[27]

73) **Definition:** Situations. We say that a situation is any abstract object that encodes only propositional properties:

$$\mathit{Situation}(x) \ =_{df} \ A!x \ \& \ \forall F(xF \rightarrow \mathit{Propositional}(F))$$

74) **Theorems:** Comprehension Condition for Situations. Let us say that a formula φ is a condition on propositional properties iff every property F that satisfies φ is a propositional property. Where 's' is a variable ranging over situations, the following is a theorem schema comprehending the subdomain of situations:

.1) $\exists s \forall F(sF \equiv \varphi)$, where φ is any condition on propositional-properties having no free ss.

[27]The material in this section first appeared in Zalta [1993].

It is an immediate corollary that for each condition on propositional properties, there is a unique situation that encodes the propositional properties satisfying the condition;

.2) $\exists! s \forall F(sF \equiv \varphi)$, for any condition φ on propositional-properties having no free ss

75) **Definition:** Truth in a Situation. Appealing to the definition of 'x encodes p' in (46), we say that a proposition p is *true in* an situation s iff s encodes p. Whenever p is true in s, we write: $\models_s p$. Thus:

$$\models_s p \ =_{df} \ \Sigma_s p$$

We may also read '$\models_s p$' as: situation s *makes* proposition p *true*. Since we are not at present distinguishing propositions from states of affairs, we may also read '$\models_s p$' as follows: state of affairs p obtains in situation s. We always read '$\models$' with smallest possible scope. So $\models_s p \rightarrow p$ should henceforth be parsed as $(\models_s p) \rightarrow p$ rather than $\models_s (p \rightarrow p)$.

76) **Lemma:** Rigidity of Truth in a Situation. The truth of proposition p in a situation is a necessary fact about that situation:

.1) $\models_s p \equiv \Box \models_s p$

.2) $\Diamond \models_s p \equiv \Box \models_s p$

77) **Theorem:** Situation Identity. The fundamental fact concerning situation identity is now derivable, namely, that situations s and s' are identical iff they make the same propositions true:

$$s = s' \equiv \forall p(\models_s p \equiv \models_{s'} p)$$

78) **Definition:** Situation Parts and Wholes. We may say that x is a part-of y iff y encodes every property x encodes. To capture this definition formally, let us use the symbol '$\trianglelefteq$' to represent the notion *part-of*. We therefore have:

$$x \trianglelefteq y \ =_{df} \ \forall F(xF \rightarrow yF)$$

79) **Theorems:** Parts of Situations. It now follows that every part of a situation is a situation:

.1) $\forall x(x \trianglelefteq s \rightarrow \mathit{Situation}(x))$

It is also an immediate consequence that a situation s is a part of situation s' iff every proposition true in s is true in s'.

.2) $s \trianglelefteq s' \equiv \forall p(\models_s p \rightarrow \models_{s'} p)$

80) **Theorems:** Parts and Identity. The theory also makes two other simple predictions, namely, that two situations are identical iff each is part of the other, and that two situations are identical iff they have the same parts:

.1) $s = s' \equiv s \trianglelefteq s' \ \& \ s' \trianglelefteq s$

.2) $s = s' \equiv \forall s''(s'' \trianglelefteq s \equiv s'' \trianglelefteq s')$

In light of these results, we shall say that a situation s is a *proper* part of s' just in case s is a part of s' and $s \neq s'$.

81) **Theorems:** Part-of is a Partial Order. The notion of part-of is reflexive, anti-symmetric, and transitive on the situations.

.1) $s \trianglelefteq s$

.2) $s \trianglelefteq s' \ \& \ s \neq s' \rightarrow \neg(s' \trianglelefteq s)$

.3) $s \trianglelefteq s' \ \& \ s' \trianglelefteq s'' \rightarrow s \trianglelefteq s''$

82) **Definition:** Persistency. We say that a proposition p is persistent iff whenever p is true in a situation s, p is true in every situation s' of which s is a part:

$$Persistent(p) \quad =_{df} \quad \forall s[\models_s p \rightarrow \forall s'(s \trianglelefteq s' \rightarrow \models_{s'} p)]$$

83) **Theorem:** Propositions are Persistent.

$$\forall p \, Persistent(p)$$

84) **Definition:** Actual Situations. We say that a situation s is actual iff every proposition true in s is true:

$$Actual(s) \ =_{df} \ \forall p(\models_s p \rightarrow p)$$

85) **Theorems:** Actual and Nonactual Situations. It is an immediate consequence that there are both actual and nonactual situations:

.1) $\exists s \, Actual(s) \ \& \ \exists s \, \neg Actual(s)$

Also, if a situation is actual, then it doesn't make any proposition and its negation true:

.2) $\forall s[Actual(s) \rightarrow \neg\exists p(\models_s p \;\&\; \models_s \neg p)]$

Thus, some propositions are not true in any actual situation:

.3) $\exists p \forall s(Actual(s) \rightarrow \not\models_s p)$

86) **Theorems:** Properties of Actual Situations. Actual situations exemplify the properties they encode, and so, if an actual situation s makes p true, then s exemplifies being such that p:

.1) $\forall s[Actual(s) \rightarrow \forall p(\models_s p \rightarrow [\lambda y\ p]s)]$

.2) $\forall s[Actual(s) \rightarrow \forall F(sF \rightarrow Fs)]$

87) **Theorem:** Embedding Situations in Other Situations. For any two situations, there exists a third of which both are a part:

$$\forall s \forall s' \exists s''(s \trianglelefteq s'' \;\&\; s' \trianglelefteq s'')$$

88) **Definitions:** Maximality. Let us say that a situation s is maximal$_1$ iff every proposition or its negation is true in s. A situation s is partial$_1$ iff some proposition and its negation are not true in s. A situation s is maximal$_2$ iff every proposition is true in s. A situation s is partial$_2$ iff some proposition is not true in s. Formally:

.1) $Maximal_1(s) \;=_{df}\; \forall p(\models_s p \vee \models_s \neg p)$
.2) $Partial_1(s) \;=_{df}\; \exists p(\not\models_s p \;\&\; \not\models_s \neg p)$
.3) $Maximal_2(s) \;=_{df}\; \forall p(\models_s p)$
.4) $Partial_2(s) \;=_{df}\; \exists p(\not\models_s p)$

89) **Theorems:** Existence of Maximal and Partial Situations. There are maximal$_1$ and partial$_1$ situations, and there are maximal$_2$ and partial$_2$ situations:

.1) $\exists s Maximal_1(s)$
.2) $\exists s Partial_1(s)$
.3) $\exists s Maximal_2(s)$
.4) $\exists s Partial_2(s)$

90) **Definitions:** Possibility and Consistency of Situations. We say that a situation is possible iff it might have been actual.

.1) $Possible(s) \ =_{df} \ \Diamond Actual(s)$

A situation is consistent iff it doesn't make a proposition and its negation true.

.2) $Consistent(s) =_{df} \neg\exists p(\models_s p \ \& \ \models_s \neg p)$.

91) **Theorems:** Possible Situations are Consistent. It now follows that all possible situations are consistent:

$$Possible(s) \rightarrow Consistent(s)$$

§8: The Theory of Worlds

92) **Definition:** Worlds and Truth at a World. We say that an object x is a possible world iff it is possible that all and only the propositions encoded in x are true; i.e., iff x might have encoded all and only the true propositions:

$$World(x) \ =_{df} \ \Diamond\forall p(\models_s p \equiv p)$$

93) **Theorem:** Worlds Are Situations. It follows immediately that worlds are situations.

$$Worlds(x) \rightarrow Situation(x)$$

94) **Definition:** Truth at a World. We may therefore extend the definition of truth in a situation to that of truth in a world:

$$p \text{ is } \mathit{true\ at\ w} \ =_{df} \ \models_w p$$

95) **Theorem:** Maximal Worlds. Every world is maximal$_1$.

$$Worlds(x) \rightarrow Maximal_1(x)$$

96) **Theorems:** Worlds are Modally Closed. Necessary consequences of propositions true at a world are also true at that world:

$$\models_w p_1 \ \& \ \ldots \ \& \ \models_w p_n \ \& \ \Box(p_1 \ \& \ \ldots \ \& \ p_n \rightarrow q) \ \rightarrow \models_w q$$

97) **Theorems:** Possibility and Consistency of Worlds. It also follows that worlds are both possible and consistent:

.1) $World(s) \rightarrow Possible(s)$

.2) $World(s) \rightarrow Consistent(s)$

98) **Definition:** The Actual World. In what follows, we use the symbol 'w_α' to (rigidly) abbreviate the definite description 'the actual world':

$$w_\alpha \ =_{df} \ \imath x(World(x) \ \& \ Actual(x))$$

99) **Theorem:** The Actual World Exists.

$$\exists x(x = w_\alpha)$$

100) **Theorem:** Parts of the Actual World. Every actual situation is a part of the actual world:

$$\forall s(Actual(s) \equiv s \trianglelefteq w_\alpha)$$

101) **Theorem:** Truth and Truth at the Actual World. It now follows that a proposition p is true iff p is true at the actual world:

$$p \equiv \models_{w_\alpha} p$$

102) **Theorems:** Properties of the Actual World. A proposition p is true at the actual world iff the actual world exemplifes being such that p. Moreover, a proposition p is true iff the proposition that the actual world exemplifies being such that p is true at the actual world:

.1) $\models_{w_\alpha} p \equiv [\lambda y \ p]w_\alpha$

.2) $p \equiv \models_{w_\alpha} [\lambda y \ p]w_\alpha$

In situation theory, statements of the form $\models_s \varphi(s)$ constitute the defining characteristic of 'nonwellfounded' situations. Thus, for each true proposition p, there a fact about w_α, namely $[\lambda y \ p]w_\alpha$, which is true in w_α. Thus, w_α is non-well-founded.

103) **Theorem:** Fundamental Theorem of World Theory. The foremost principle of world theory is that a proposition is necessary iff it is true in all worlds. An alternative, but equivalent statement is: a proposition is possible iff there is some world in which it is true:

.1) $\Box p \equiv \forall w(\models_w p)$

.2) $\Diamond p \equiv \exists w(\models_w p)$

From the latter, and the assumption that $\neg q \,\&\, \Diamond q$, for some proposition q, we may prove the existence of nonactual possible worlds. For if $\neg q$, then q is not true at the actual world. Yet if $\Diamond q$, there is a world w where q is true. But such a world must be distinct from the actual world, and since there is only one of the latter, the former must be a nonactual possible world.

104) **Theorems:** A Useful Equivalence Concerning Worlds. It is a consequence of the foregoing that p is true at world w iff x exemplifies being such that p at w:

$$\models_w p \;\equiv\; \models_w [\lambda y\, p]x$$

§9: The Theory of Times

Suppose we add the tense operators $\mathcal{G}$ ('it will always be the case that') and $\mathcal{H}$ ('it was always the case that') to our language, defining their duals $\mathcal{F}$ ('it will once be the case') and $\mathcal{P}$ ('it was once the case') in the usual way. Then, we may define $\blacksquare\varphi$ ('it is always the case that φ') as: $\mathcal{H}\varphi \,\&\, \varphi \,\&\, \mathcal{G}\varphi$. Let a tense closure of φ be the result of prefacing any string of $\blacksquare$s to φ. Then add as axioms: (1) all the tense closures of all the modal axioms, (2) the tense-theoretic Barcan Formulas, and (3) the axioms of minimal tense logic K_t. We can then derive the omnitemporalization rules of inference for $\mathcal{G}$ and $\mathcal{H}$ (these are the counterparts to the Rule of Necessitation). And we can define $\blacklozenge\varphi$ ('Sometime, φ') as: $\mathcal{P}\varphi \vee \varphi \vee \mathcal{F}\varphi$. Now suppose that the property *being ordinary* is $[\lambda x \Diamond\blacklozenge E!x]$ (so ordinary objects are those that might *at some time* be located in spacetime). And the property *being abstract* is $[\lambda x \neg\Diamond\blacklozenge E!x]$ (so abstract objects are those that could not *at any time* be located in spacetime). Then, we may define a certain class of abstract objects, the times, as follows:

$$s \textit{ is a time} =_{df} \blacklozenge\forall p(s \models p \equiv p)$$

The tense-theoretic counterparts of the theorems pertaining to worlds described in the previous subsection are now derivable as theorems. So times are maximal, tense-theoretically consistent, and tense theoretically closed. In addition, there is a unique present moment, and a proposition is always true iff it is true at all times.

§10: The Theory of Leibnizian Concepts

A. Concepts

105) **Definition:** Concepts. The principal idea underlying the analysis of Leibniz's calculus of concepts is the identification of Leibnizian concepts as abstract objects. Let us say that *x is a Leibnizian concept* iff *x* is an abstract object:

$$Concept(x) \ =_{df} \ A!x$$

Instances of the comprehension principle for abstract objects (48) therefore assert the existence of Leibnizian concepts, and so we may think of (48) as the comprehension schema that comprehends the domain of concepts. Moreover, definition (7) presents the notion of identity that governs concepts.

106) **Definitions:** Concepts G. We shall define a concept G to be any abstract object that encodes just the properties necessarily implied by G. This requires us to first define: *G necessarily implies F* ('$G \Rightarrow F$') iff necessarily, for any object y, if y exemplifies G then y exemplifies F:

.1) $G \Rightarrow F \ =_{df} \ \Box\forall y(Gy \rightarrow Fy)$

We then define *x is a concept G* as: x is a concept which encodes just the properties necessarily implied by G:

.2) $Concept(x, G) \ =_{df} \ Concept(x) \ \& \ \forall F(xF \equiv G \Rightarrow F)$

107) **Theorem:** Existence of a Unique Concept G. It now follows that for any property G, there is a unique concept G:

$$\forall G\exists!x Concept(x, G)$$

108) **Theorem:** The Concept G. Given the previous theorem, we know that there is something which is *the* concept G exists and so we may introduce a name ('c_G') for it:

$$c_G \ =_{df} \ \imath x Concept(x, G)$$

It follows from these definitions that the concept rational ('c_R') is the abstract object that encodes just the properties necessarily implied by the property of being rational. Clearly, then, we are distinguishing the property of being rational from the concept rational.[28]

[28]Reasons for doing this are detailed in my paper 'A (Leibnizian) Theory and Calculus of Concepts'. See the Bibliography.

B. Concept Identity

Throughout the rest of this section, let $x, y, z, \ldots$ range over Leibnizian concepts.

109) **Theorems:** Concept Identity. Leibniz proves the (reflexivity,) symmetricality and transitivity of concept identity in *LLP* (131). We can derive these same facts about our Leibnizian concepts from (30.1) – (30.3):

.1) $x = x$

.2) $x = y \to y = x$

.3) $x = y \,\&\, y = z \to x = z$

Note that Leibniz states his symmetry claim as follows:

If $A = B$, then $B = A$

But we would represent Leibniz's claim as the claim:

$$c_G = c_H \to c_H = c_G$$

C. Concept Addition

110) **Definitions:** A Sum of Concepts x and y. If given concepts x, y, we say that z is *a sum of x and y* iff z is a concept that encodes the properties encoded by either x or y:

$$Sum(z, x, y) \ =_{df} \ Concept(z) \,\&\, \forall F(zF \equiv xF \ \vee \ yF)$$

111) **Theorems:** Existence of Unique Sums. It now follows that for any concepts x, y, there is a unique sum of x and y:

$$\exists! z Sum(z, x, y)$$

112) **Definition:** The Sum of x and y. Since the sum of x and y exists, we may introduce a name ('$x \oplus y$') for this object:

$$x \oplus y \ =_{df} \ \imath z Sum(z, x, y)$$

For example, the sum of the concept rational and the concept animal ('$c_R \oplus c_A$') is the (abstract) object that encodes just the properties encoded by either concept.

113) **Theorem:** Identity of Sums. It is a simple consequence of these definitions that the sum of the concept G and the concept H is identical to the (abstract) object that encodes just the properties implied by G or implied by H:

$$c_G \oplus c_H = \imath x(\mathit{Concept}(x) \,\&\, \forall F(xF \equiv G \Rightarrow F \;\vee\; H \Rightarrow F))$$

114) **Theorems:** $\oplus$ is Algebraic. Now to confirm that $\oplus$ behaves in the manner that Leibniz prescribed, note that it follows immediately from these definitions that the sum operation is idempotent and commutative:

.1) $x \oplus x = x$

.2) $x \oplus y = y \oplus x$

Leibniz takes these two principles as *axioms* of his calculus in *LLP* (132), whereas we derive them as theorems. It is almost as immediate to see that $\oplus$ is associative:

.3) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

In virtue of this fact, we may leave off the parentheses in '$(x \oplus y) \oplus z$' and '$x \oplus (y \oplus z)$'. Leibniz omits associativity from his list of axioms for $\oplus$, but needs associativity for his proofs to go through.

115) **Definition:** General Sums. The previous theorem suggests that we may generalize the sum operation. Let us say that y is a general sum of concepts $x_1, \ldots, x_n$ iff y is a concept that encodes all of the properties encoded by the x_i:

$$\mathit{GSum}(y, x_1, \ldots, x_n) \;=_{df}\; \mathit{Concept}(y) \,\&\, \forall F(yF \equiv x_1F \vee \ldots \vee x_nF)$$

116) **Theorem:** Existence of Unique General Sums. It now follows that for any concepts $x_1, \ldots, x_n$, there is a unique general sum of $x_1, \ldots, x_n$:

$$\forall x_1 \ldots \forall x_n \exists ! y \, \mathit{GSum}(y, x_1, \ldots, x_n)$$

117) **Definition:** The General Sum of $x_1, \ldots, x_n$. Since the general sum of $x_1, \ldots, x_n$ exists, we may introduce a name ('$x_1 \oplus \ldots \oplus x_n$') for this object:

$$x_1 \oplus \ldots \oplus x_n \;=_{df}\; \imath y \, \mathit{GSum}(y, x_1, \ldots, x_n)$$

118) **Theorem:** Identity of General Sums. We may now generalize an earlier theorem:

$$c_{G_1}\oplus\ldots\oplus c_{G_n} = \imath x(Concept(x) \& \forall F(xF \equiv G_1 \Rightarrow F \vee \ldots \vee G_n \Rightarrow F))$$

119) **Theorems:** Concept Addition and Identity. Leibniz proves two other theorems pertaining solely to concept addition and identity in *LLP* (133):

.1) $x = y \ \rightarrow \ x\oplus z = y\oplus z$

.2) $x = y \,\&\, z = u \ \rightarrow \ x\oplus z = y\oplus u$

These are straightforward consequences of our definitions of concept identity and concept addition. (Leibniz notes that) counterexamples to the converses of these theorems can be produced. As a counterexample to the converse of (119.1), let x be $c_F\oplus c_G\oplus c_H$, y be $c_F\oplus c_G$, and z be c_H. Then $x\oplus z = y\oplus z$, but $x \neq y$. Similarly, as a counterexample to the converse of (119.2), let x be $c_F\oplus c_G$, z be $c_H\oplus c_I$, y be c_F, and u be $c_G\oplus c_H\oplus c_I$. Then, $x\oplus z = y\oplus u$, but neither $x = y$ nor $z = u$.

D. Concept Inclusion and Containment

Leibniz defined the notions of concept inclusion and concept containment in terms of 'coincidence', and in *LLP* (132) he took them to be converses of each other.

120) **Definitions:** Inclusion and Containment. We may define these notions of concept inclusion ('$x \preceq y$') and concept containment ('$x \succeq y$') in our system as follows:

.1) $x \preceq y \ =_{df} \ \forall F(xF \rightarrow yF)$

.2) $x \succeq y \ =_{df} \ y \preceq x$

To show that these definitions are good ones, we establish that the notions defined behave the way Leibniz says they are supposed to behave. In what follows, we identify the relevant theorems in pairs, a theorem governing concept inclusion and the counterpart theorem governing concept containment. We prove the theorem only as it pertains to concept inclusion.

121) **Theorems:** First, we note that concept inclusion and containment are reflexive, anti-symmetric, and transitive:

.1) $x \preceq x$
$x \succeq x$

.2) $x \preceq y \ \rightarrow \ (x \neq y \rightarrow y \npreceq x)$
$x \succeq y \ \rightarrow \ (x \neq y \rightarrow y \nsucceq x)$

.3) $x \preceq y \,\&\, y \preceq z \ \rightarrow \ x \preceq z$
$x \succeq y \,\&\, y \succeq z \ \rightarrow \ x \succeq z$

Leibniz asserts the reflexivity of concept containment in *LLP* (33), and proves the transitivity of concept inclusion in *LLP* (135).

122) **Theorems:** Inclusion, Containment, and Identity. Leibniz proves, in *LLP* (136), that when concepts x and y are included or contained in each other, they are identical:

.1) $x \preceq y \,\&\, y \preceq x \rightarrow x = y$
$x \succeq y \,\&\, y \succeq x \rightarrow x = y$

Two interesting consequences of concept inclusion and identity are that x and y are identical concepts if either of the following biconditionals hold: (i) concept z is included in x iff z is included in y, and (ii) x is included in z iff y is included in z:

.2) $\forall z(z \preceq x \equiv z \preceq y) \rightarrow x = y$
$\forall z(x \succeq z \equiv y \succeq z) \rightarrow x = y$

.3) $\forall z(x \preceq z \equiv y \preceq z) \rightarrow x = y$
$\forall z(z \succeq x \equiv z \succeq y) \rightarrow x = y$

123) **Theorems:** General Inclusion. Two further consequences of the foregoing are: (.1) For any properties $G_1, \ldots, G_n$, the concept G_1 is included in the sum of the concept G_1 and the concept G_2, which in turn is included in the sum of the concepts $G_1\, G_2\, G_3$, etc., and (.2), no matter how one scrambles the properties in the sum of the concepts $G_1 \ldots G_n$, the resulting concepts are included in it:

.1) $c_{G_1} \preceq c_{G_1} \oplus c_{G_2} \preceq \ldots \preceq c_{G_1} \oplus \ldots \oplus c_{G_n}$
$c_{G_1} \oplus \ldots \oplus c_{G_n} \succeq \ldots \succeq c_{G_1} \oplus c_{G_2} \succeq c_{G_1}$

.2) $c_{G_{i_1}} \oplus \ldots \oplus c_{G_{i_j}} \preceq c_{G_1} \oplus \ldots \oplus c_{G_n}$,
where $1 \leq i_1 \leq \ldots \leq i_j \leq n$
$c_{G_1} \oplus \ldots \oplus c_{G_n} \succeq c_{G_{i_1}} \oplus \ldots \oplus c_{G_{i_j}}$,
where $1 \leq i_1 \leq \ldots \leq i_j \leq n$

124) **Theorems:** Inclusion and Addition. In *LLP* (33), we find that every concept x is included in the sum $x \oplus y$, and moreover, so is the concept y:

.1) $x \preceq x \oplus y$
$x \oplus y \succeq x$

.2) $y \preceq x \oplus y$
$x \oplus y \succeq y$

Moreover, Leibniz notes in *LLP* (41) that if y is included in z, then $x \oplus y$ in included in $x \oplus z$:

.3) $y \preceq z \;\rightarrow\; x \oplus y \preceq x \oplus z$
$y \succeq z \;\rightarrow\; x \oplus y \succeq x \oplus z$

It also follows, not just according to *LLP* (136), that if $x \oplus y$ is included in z, then both x and y are included in z:

.4) $x \oplus y \preceq z \;\rightarrow\; x \preceq z \,\&\, y \preceq z$
$z \succeq x \oplus y \;\rightarrow\; z \succeq x \,\&\, z \succeq y$

And it follows that if both x and y are included in z, then $x \oplus y$ is included in z (noted in *LLP* (137)):

.5) $x \preceq z \,\&\, y \preceq z \;\rightarrow x \oplus y \preceq z$
$z \succeq x \,\&\, z \succeq y \;\rightarrow z \succeq x \oplus y$

Finally, we may prove that if x is included in y and z is included in u, then $x \oplus z$ is included in $y \oplus u$ (*LLP* (137)):

.6) $x \preceq y \,\&\, z \preceq u \;\rightarrow x \oplus z \preceq y \oplus u$
$x \succeq y \,\&\, z \succeq u \;\rightarrow x \oplus z \succeq y \oplus u$

E. Concept Inclusion, Addition, and Identity

Now we show that our definitions of concept inclusion (containment), addition, and identity are all related in the appropriate way.

125) **Theorem:** Leibnizian Definition of Inclusion. Whereas Leibniz defines $x \preceq y$ by saying there is a concept z such that $x \oplus z = y$ (*LLP* (132)), we prove this definition as a theorem:

$$x \preceq y \equiv \exists z(x \oplus z = y)$$
$$x \succeq y \equiv \exists z(x = y \oplus z)$$

126) **Theorem:** Leibniz's Equivalence. Our definition of $\preceq$ also validates the principal theorem governing Leibniz's calculus, namely, that x is included in y iff the sum of x and y is identical with y (*LLP* (135)):

$$x \preceq y \equiv x \oplus y = y$$
$$x \succeq y \equiv x = x \oplus y$$

Though Leibniz apparently proves this theorem using our (125) as a definition, on our theory, no appeal to (125) needs to be made. So Leibniz's main principle governing the relationship between concept inclusion ($\preceq$), concept identity, and concept addition ($\oplus$) is derivable.

127) **Theorem:** Existence of the Missing Concept. Since Leibniz's Equivalence establishes the equivalence of $x \oplus y = y$ and $x \preceq y$, and (125) establishes the equivalence of $x \preceq y$ and $\exists z(x \oplus z = y)$, it follows that:

$$x \oplus y = y \equiv \exists z(x \oplus z = y)$$
$$x = x \oplus y \equiv \exists z(x = y \oplus z)$$

128) **Remark:** Identity of the Concept Person. We conclude this discussion of the basic theory of Leibnizian concepts with the following observation. Since we have derived (126), the following instance is therefore provable: the concept person contains the concept rational iff the concept person is identical with the sum of the concept person and the concept rational. In formal terms:

$$c_P \succeq c_R \equiv c_P = c_P \oplus c_R$$

Clearly, if every property encoded in the concept rational is encoded in the concept person (i.e., if every property implied by the property of being rational is implied by the property of being a person), then the concept person is identical with (i.e., encodes the same properties as) the concept that encodes all the properties encoded in 'both' the concept person and the concept rational (since c_R contributes to $c_P \oplus c_R$ no properties not already encoded in c_P). However, recall that Leibniz believed that the concept person is itself the sum of the concept rational and the concept animal. In formal terms, Leibniz's suggestion amounts to:

$$c_P = c_R \oplus c_A$$

If this is an accurate representation of Leibniz's view, then it is clear that the left side of the above instance of (126) is also derivable, i.e.,

$$c_P \succeq c_R$$

Given the identity of c_P with $c_R \oplus c_A$, then $c_P \succeq c_R$ is a simple consequence of (124.1): $x \oplus y \succeq x$.

However, it seems reasonable to argue that, strictly speaking, c_P is not the same concept as $c_R \oplus c_A$. The latter concept, by definition, encodes all and only the properties either implied by the property of *being rational* or implied by the property of *being an animal*. But such a concept does not contain the properties implied by the conjunctive property *being a rational animal*; i.e., such a concept does not encode the properties implied by the property $[\lambda x Rx \& Ax]$. The concept $c_R \oplus c_A$, for example, fails to encode the property $[\lambda x\, Rx \,\&\, Ax]$, since this conjunctive property is neither implied by the property of being rational nor implied by the property of being an animal. This suggests that it might be preferable to analyze the concept person as the concept that encodes just the properties implied by the conjunctive property of being a rational animal:[29]

$$c_P = c_{[\lambda x\, Rx \,\&\, Ax]}$$

This makes good sense. For one might argue that, strictly speaking, it is the property of being a person that is identical with the property of being a rational animal (just as the property of being a brother is identical with the property of being a male sibling, and the property of being a circle is identical with the property of being a closed, plane figure every point of which lies equidistant from some given point, etc.). So simply by adding the hypothesis that $P = [\lambda x\, Rx \,\&\, Ax]$ to our system, we could derive that c_P is identical with the concept $c_{[\lambda x\, Rx \,\&\, Ax]}$ (since abstract objects that encode the same properties are identical).

Notice that even if we were to represent the concept person in this way, it still follows that the concept person contains the concept rational, since the conjunctive property of being a rational animal implies the property of being rational. But, strictly speaking, this is a departure from the letter of the Leibnizian corpus, in which a complex concept such as c_P is analyzed in terms of the sum of its simpler concepts. Our theory of concepts suggests that there is subtle and important difference between

[29]Recall that $c_{[\lambda x\, Rx \,\&\, Ax]}$ is defined as the abstract object that encodes all the properties necessarily implied by the conjunctive property $[\lambda x\, Rx \,\&\, Ax]$.

$c_R \oplus c_A$ and $c_{[\lambda x\ Rx\ \&\ Ax]}$. This difference may not have been observed in Leibniz's own theory.

§11: The Theory of Leibnizian Monads

In this section, we use the variables u and v to range over ordinary objects.

129) **Definitions:** Realization and Appearance. A concept x is realized by an ordinary object u at world w just in case u exemplifies at w all and only the properties x encodes:

.1) $\mathit{RealizedBy}(x, u, w) \ =_{df} \ \forall F(\models_w Fu \equiv xF)$

Now the notion of appearance at can be defined in terms of realization:

.2) $\mathit{Appears}(x, w) \ =_{df} \ \exists u \mathit{RealizedBy}(x, u, w)$

130) **Definition:** Monads and Mirroring. A monad is any object that appears at some world:

.1) $\mathit{Monad}(x) \ =_{df} \ \exists w \mathit{Appears}(x, w)$

One of the more interesting notions of monadology is that of mirroring. An object x will mirror a world w just in case x encodes all and only the propositions true in w:

.2) $\mathit{Mirrors}(x, w) \ =_{df} \ \forall p(\Sigma_x p \equiv \models_w p)$, i.e.,
$\forall p(x[\lambda y\, p] \equiv w[\lambda y\, p])$

So an object x mirrors a world w just in case x encodes just the propositional properties that w encodes.

131) **Theorem:** Appearance and Mirroring. It now follows that an object x mirrors any world where it appears:

$$\mathit{Appears}(x, w) \rightarrow \mathit{Mirrors}(x, w)$$

132) **Definition:** The World of a Monad. We may introduce the notation 'w_m' to abbreviate the description 'the world where monad m appears':

$$w_m =_{df} \imath w(\mathit{Appears}(m, w))$$

133) **Theorem:** The World of a Monad Exists. We now derive one of the fundamental theorems of monadology, namely, that for every monad, there is something that is the world where it appears:

$\forall m \exists x (x = w_m)$

In other words, every monad appears at a unique world.

134) **Corollary:** A Monad Mirrors *Its* World. The previous two theorems, that anything appearing at a world mirrors that world, and that every monad appears at a unique world, allow us to assert that every monad m mirrors *its* world:

$Mirrors(m, w_m)$

135) **Definition:** Compossibility. Two monads are compossible just in case they appear at the same world:

$Compossible(m_1, m_2) =_{df} \exists w(Appears(m_1, w) \ \& \ Appears(m_2, w))$

136) **Lemma:** The World of Compossible Monads. It is useful to know that if monads m_1 and m_2 are compossible, then the worlds where they appear are identical:

$Compossible(m_1, m_2) \equiv w_{m_1} = w_{m_2}$

137) **Theorems:** Compossibility Partitions the Monads. It is an immediate consequence of the lemma that compossibility is reflexive, symmetric, and transitive:

.1) $Compossible(m_1, m_1)$

.2) $Compossible(m_1, m_2) \rightarrow Compossible(m_2, m_1)$

.3) $Compossible(m_1, m_2) \ \& \ Compossible(m_2, m_3) \rightarrow$
$Compossible(m_1, m_3)$

Since compossibility is reflexive, symmetrical, and transitive, we know the monads are partitioned. For each cell of the partition, there is a unique world where all of the monads in that cell appear.

138) **Definition:** Complete Concepts. Monads are suppose to be *complete, individual concepts*. We may say that a complete concept is any concept x such that, for every property F, either x encodes F or x encodes $\bar{F}$ (where $\bar{F}$ abbreviates $[\lambda y \, \neg Fy]$):

$Complete(x) =_{df} \forall F(xF \ \vee \ x\bar{F})$

139) **Theorem:** Monads and Completeness. It now follows that every monad is complete:

$$Monad(x) \rightarrow Complete(x)$$

140) **Definition:** Individual Concepts. A concept x is an individual concept just in case, for any world w, if x is realized by ordinary objects u and v at w, then $u = v$.

$$IndividualConcept(x) =_{df} \forall w(RealizedBy(x,u,w) \ \& \ RealizedBy(x,v,w) \rightarrow u = v)$$

141) **Theorem:** Monads and Individual Concepts. It now follows that every monad is an individual concept:

$$Monad(x) \rightarrow IndividualConcept(x)$$

So monads are complete, individual concepts.

§12: The Containment Theory of Truth

In this section, we continue to use u, v as variables ranging over ordinary objects.

142) **Definition:** Concepts of Ordinary Objects. If given an ordinary object u, we say that x is a concept of u iff x is a concept that encodes just the properties that u exemplifies:

$$ConceptOf(x,u) \ =_{df} \ Concept(x) \ \& \ \forall F(xF \equiv Fu)$$

143) **Theorem:** The Existence of a Unique Concept of u. For any ordinary object u, there is a unique concept of u:

$$\exists! x Concept(x,u)$$

144) **Definition:** The Concept u. Given that the concept u exists, we may introduce a name ('c_u') for it:

$$c_u =_{df} \imath x Concept(x,u)$$

Given the premise that Alexander ('a') is an ordinary object, the concept Alexander ('c_a') is well-defined.

145) **Lemmas:** Facts About Concepts of Ordinary Objects. It is a immediate consequence of the foregoing that: (.1) the concept u encodes F iff u exemplifies F, and (.2) the concept u encodes F iff the concept u contains the concept F:

.1) $c_u F \equiv Fu$

.2) $c_u F \equiv c_u \succeq c_F$

146) **Definition:** The Containment Theory of Truth. We can now develop Leibniz's containment theory of truth as follows. Let κ be a name of an ordinary object. A statement of the form 'κ is (a, an) F' in ordinary language is true iff c_κ contains c_F:

$$\kappa \text{ is (a, an) } F \ =_{df} \ c_\kappa \succeq c_F$$

147) **Theorem:** Exemplification and Containment. It is a consequence of the immediately preceding lemmas and definition that the Leibnizian analysis of predication 'κ is (a, an) F' is equivalent to the contemporary analysis of predication in terms of exemplification, for it follows that an ordinary individual u exemplifies F iff the concept u contains the concept F:

$$Fu \equiv c_u \succeq c_F$$

148) **Remark:** Example of Concept Containment. Now let us assume that Alexander is an ordinary object. Then the concept Alexander is well-defined, by (143). Now Leibniz analyzes the contingent fact:

Alexander is a king

in the following terms:

The concept Alexander contains the concept king

On our analysis, this becomes:

$$c_a \succeq c_K$$

So given the contingent facts that Alexander is an ordinary object and that Alexander is a king (as assumptions), we may use (147) and either use the contemporary analysis of the second fact to derive Leibniz's analysis or use Leibniz's analysis to derive the contemporary analysis. Whatever one decides to do, we have established, given our assumptions, that the sentence 'Alexander is a king' is true just in case the subject concept contains the predicate concept.

149) **Remark:** Hypothetical Necessity. These results present us with a defense of Leibniz's discussion in Article 13 of the *Discourse on Metaphysics*.[30] Arnauld charged that Leibniz's analyses turn contingent truths into necessary truths. Arnauld worried that the contingent statement that 'Alexander is a king' is analyzed in terms of the necessary truth 'the concept Alexander contains the concept king'. Leibniz defends himself against Arnauld's charges by appealing to the notion of a 'hypothetical necessity'. On our analysis, there are two related ways to defend this idea.

For one thing, we can point to the fact that the *proof* of $c_a \succeq c_K$ rests on the contingent premise that Alexander is a king (i.e., it rests on Ka). Just consider (145.1) and (145.2). The contingent premise Ka is required if we are to *prove* that c_a encodes the property of being a king and so is required to demonstrate that $c_a \succeq c_K$. Now we observe that it is a theorem that if $x \succeq y$, then $\Box(x \succeq y))$, for the logic of encoding (14) guarantees that encoded properties are rigidly encoded. So, as an instance, if $c_a \succeq c_K$, then $\Box(c_a \succeq c_K)$. But since the former is conditional on a contingent fact, it follows that the latter is as well. So we have a case where a necessary truth is conditional upon a contingent fact, or rather, a case where the proof of necessity depends on a contingent hypothesis. This is one way to understand 'hypothetical necessities'.

A second way to understand this idea is to note that (145.1) is a an example of a logical truth that is not necessary. It is a logical truth because it is derivable from our logical axioms and rules alone. But it is not metaphysically necessary because its negation is metaphysically possible. Just suppose that Ka is true (at the actual world) and consider a world w where Alexander is (was) not a king. At such a world, Ka is false. But note that at such a world, c_aK is true. That's because the object that at the actual world encodes all and only the properties that Alexander actually exemplifies encodes K even at w. (The description that serves to define c_a is a rigid designator and so denotes at w the object x that satisfies '$\forall F(xF \equiv Fa)$' at the actual world.) So w is a world where the left condition (145.1) is true but the right condition is false. Hence it is not metaphysically necessary. Similarly, (147) is a logical truth that is not necessary. That means that it is logically required (provable) that Ka is equivalent to $c_a \succeq c_K$, but this equivalence is not necessary in the metaphysical sense. Maybe this is another way of understanding Leibniz's idea of 'hypothetical necessities'. (For a fuller discussion of such logical

[30] *LPW 23*.

truths that are not necessary, see Zalta [1988b].)

150) **Theorem:** The Existence of Monads. For any ordinary object u, it follows that the concept u is a monad:

.1) $Monad(c_u)$

It is a simple corollary to this theorem that for any ordinary object u, the concept u is a complete individual concept.

.2) $Complete(c_u)$ & $Individual(c_u)$

So, for example, if Alexander is an ordinary object, the concept Alexander exists (by (143), is a monad (by (150.1)), and is a complete, individual concept (by (150.2)). Now, of course, our theory doesn't assert that Alexander is an ordinary object, nor does it assert the existence of concrete objects in general or the existence of any particular concrete object such as Alexander; that is, $O!a$, $\exists xE!x$, and $E!a$ are not axioms. However, it seems reasonable to extend our theory by adding the *a priori* assumption that it is *possible* that there are concrete objects; i.e., $\Diamond\exists xE!x$. If this is true, then by the Barcan formulas, it follows that $\exists x\Diamond E!x$. In other words, it follows from *a priori* assumptions alone that there are ordinary objects. In which case, it follows *simpliciter*:

.3) $\exists x Monad(x)$

Thus we have a proof of the existence of monads from *a priori* assumptions alone.

151) **Definition:** Containment and Quantification. Leibniz may have anticipated not only Montague's [1974] subject-predicate analysis of basic sentences of natural language, but also the whole idea of a generalized quantifier. Recall that Montague was able to give a uniform subject-predicate analysis of a fundamental class of English sentences by treating such noun phrases as 'John' and 'every person' as sets of properties. He supposed that the proper name 'John' denotes the set of properties F such that the individual John exemplifies F and supposed that the noun phrase 'every person' denoted the set of properties F such that every person exemplifies F. Then, English sentences such as 'John is happy' and 'Every person is happy' could be given a subject-predicate analysis: such sentences are true iff the property denoted by the predicate 'is happy' is a member of the set of properties denoted by the subject term.

Leibniz's conceptual containment theory of truth may also be used to analyze such quantified claims and give a unified analysis of the basic sentences of natural language. Claims such as 'Every person is happy' and 'Some person is happy' can be given a Leibnizian, subject-predicate analysis. We may define 'the concept every G' ('$c_{\forall G}$') as the concept that encodes just the properties F such that every G exemplifies F. And we define 'the concept some G' ('$c_{\exists G}$') as the concept that encodes just the properties F such that some G exemplifies F:

.1) $c_{\forall G} \ =_{df} \ \iota x(Concept(x) \,\&\, \forall F(xF \equiv \forall z(Gz \rightarrow Fz)))$

.2) $c_{\exists G} \ =_{df} \ \iota x(Concept(x) \,\&\, \forall F(xF \equiv \exists z(Gz \,\&\, Fz)))$

As an example of the latter, the concept *some person* is the concept that encodes just the properties F such that some person exemplifes F. Now we may analyze 'Every G is F' as: $c_{\forall G} \succeq c_F$. And we may analyze 'Some G is F' as: $c_{\exists G} \succeq c_F$.

152) **Theorem:** The Concepts $c_{\forall G}$ and $c_{\exists G}$ are not Monads. If there are distinct ordinary things that exemplify G, then the concepts $c_{\forall G}$ and $c_{\exists G}$ are not monads. Letting u, v be variables ranging over ordinary objects:

.1) $\exists u, v(u \neq v \,\&\, Gu \,\&\, Gv) \rightarrow \neg Monad(c_{\forall G})$

.2) $\exists u, v(u \neq u \,\&\, Gu \,\&\, Gv) \rightarrow \neg Monad(c_{\exists G})$

153) **Remark:** The analyses described in the previous section are somewhat different from the ones developed in Zalta [1983]. In that earlier work, a simpler analysis was used, in which the Leibnizian concept G was treated as the property G and the Leibnizian claim that "The individual concept u contains the concept G" was analyzed more simply as '$c_u G$'. But under that scheme, concept containment became a connection between two different kinds of things (namely, A-objects and properties). Clearly, though, Leibnizian concept containment and concept inclusion are suppose to connect things of a single kind, namely, concepts. That idea is preserved in the way we have analyzed containment and inclusion in the present work—the notions of containment and inclusion that we use both in the calculus of concepts and in the monadology relate concepts (i.e., A-objects).

Of course there is a simple thesis that connects the present analysis with the earlier analysis, namely, (145.2). This thesis shows that the former analysis is equivalent to the present one. This lays the groundwork for connecting the work in [1983], in which containment is analyzed as encoding, with the definitions developed in the present work. It demonstrates that our Leibnizian analysis of 'Alexander is a king' in [1983] (p. 90) as $c_a K$ is equivalent to our present analysis of this claim as: $c_a \succeq c_K$.

§13: The Theory of Fiction

In this section, the variables x, y, z are all unrestricted. The variable s, however, ranges over stories, i.e., a special group of abstract objects that will presently be defined.

154) **Remark:** For the general theory of fiction, we shall need two primitive notions which are required for the treatment of fiction: authorship and temporal precedence. Let 'Axy' represent the claim that x is the author of y. Let let '$p < q$' represent the claim that p occurred before q. In particular, $Azs' < Ays$ asserts: z authored s' before y authored s. Whereas we may assume that a standard logic governs the notion of temporal precedence, the exact analysis of the authorship relation is still somewhat mysterious. Intuitively, if x authors y, x is an ordinary object, y has a content defined by certain propositions, and certain cognitive activities activities on the part of x result in a storytelling which establishes the propositional properties that define y. As the reader shall soon discover, we assume that a story s is a situation closed under 'relevant' entailment. That is, we also suppose that some system of relevance logic has been found adequate to the data. We use $p \vdash_R q$ indicate that q is derivable from p by the axioms and rules of relevance logic.

155) **Definition:** Stories and Truth in a Story.

.1) $\mathit{Story}(x) \ =_{df} \ \mathit{Situation}(x) \ \& \ \exists y(E!y \ \& \ Ayx)$

We assert that *Crime and Punishment* is a story in this sense. Since stories are situations, the defined notion of 'truth in a situation' may be used as the analysis of 'truth according to a story':

.2) *According to the story* s, $p \ =_{df} \ \models_s p$

So where the abbreviations are obvious, we may analyze:

> According to *Crime and Punishment*, Raskolnikov is a student who kills an old moneylender.

as follows:

$$\models_{CP} Sr \,\&\, \exists y(Oy \,\&\, My \,\&\, Kry))$$

156) **Theorem:** Identity of Stories. From our definitions, it follows that a story s is that abstract object that encodes just the propositional properties $[\lambda y\, p]$ such that p is true in s:

$$s = \imath x(A!x \,\&\, \forall F(xF \equiv \exists p(\models_s p \,\&\, F = [\lambda y\ p])))$$

Thus, in so far as we assume, for each story s, a fixed group of truths of the form 'According to the story s, p', we have a rule for identifying the story. People may disagree about which propositions are in fact true according to a given story, but they use this same understanding of what the story is.

157) **Rule of Inference:** Stories and Relevant Entailment. Any proposition relevantly entailed by the propositions true in a story is also true in that story:

$$\text{If } p_1, \ldots, p_n \vdash_R q, \text{ then from } \models_s p_1, \text{ and } \ldots \text{ and } \models_s p_n, \text{ infer } \models_s q$$

158) **Rules of Inference:** Some Rules of Relevant Entailment. We shall assume that the following are rules of relevant entailment: from the fact that objects $x_1, \ldots, x_n$ exemplify the relation $[\lambda y_1 \ldots y_n\, \varphi]$, we may infer$_{\mathrm{R}}$ $\varphi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$, and vice versa:

$$[\lambda y_1 \ldots y_n\ \varphi]x_1 \ldots x_n \quad {}_R\dashv\vdash_R\ \varphi^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$$

As an example: from the proposition that Achilles fought Hector (Fah), we may conclude both that Achilles had the property of fighting Hector ($[\lambda x\, Fxh]a$), and that Hector had the property of having fought Achilles ($[\lambda x\, Fax]h$). So, by principle (157), we may conclude from the fact that according to *The Iliad*, Achilles fought Hector ($\models_I Fah$), that according to *The Iliad*, Achilles had the property of fighting Hector ($\models_I [\lambda y\, Fyh]a$), and that according to *The Iliad* Hector had the property of being fought by Achilles ($\models_I [\lambda y\, Fay]h$).

159) **Definition:** Characters of Stories. We say that x is a character of s just in case there is some property x exemplifies according to s:

$$\mathit{Character}(x, s) \ =_{df}\ \exists F(\models_s Fx)$$

160) **Theorem:** Raskolnikov is a Character. From the above theoretical definitions, we may prove that if Raskolnikov is that thing which, according to *Crime and Punishment*, killed an old moneylender, then Raskolnikov is a character of *Crime and Punishment*.

$$r = \imath x(\models_{CP} Sx \;\&\; \exists y(Oy \;\&\; My \;\&\; Kxy)) \rightarrow \mathit{Character}(r, \mathrm{CP})$$

161) **Definition:** Native Characters. A native character of story s is an abstract object that is a character of s and which is not a character of any earlier story s':

$$\mathit{Native}(x, s) \;=_{\mathit{df}}\; A!x \;\&\; \mathit{Character}(x, s) \;\& \\ \forall y \forall z \forall s'((Azs' < Ays) \rightarrow \neg \mathit{Character}(x, s'))$$

We assert that Raskolnikov is a native character of *Crime and Punishment* in this sense.

162) **Axiom:** Identity of Native Characters. If an object x is native to story s, then x encodes just those properties that x exemplifies according to s:

$$\mathit{Native}(x, s) \rightarrow x = \imath y(A!y \;\&\; \forall F(yF \equiv \models_s Fy))$$

So if we assume that Raskolnikov is a native character of *Crime and Punishment*, then we may identify Raskolnikov as the abstract object that encodes just the properties F such that Raskolnikov exemplifies F according to *Crime and Punishment*.

163) **Definition:** Fictional Characters. We say that an object is a fictional character just in case it is native to some story:

.1) $\mathit{Fictional}(x) \;=_{\mathit{df}}\; \exists s \mathit{Native}(x, s)$

If we assume that Achilles is native to *The Iliad*, then we can conclude that Achilles is a fictional character. We say that x is a fictional G just in case x is native to some story according to which x exemplifies G.

.2) $\mathit{FictionalG}(x) \;=_{\mathit{df}}\; \exists s(\mathit{Native}(x, s) \;\&\; \models_s Gx)$

Since Achilles is native to *The Iliad* and according to *The Iliad*, Achilles is Greek warrior ($\models_{\mathrm{I}} [\lambda y\, Gy \;\&\; Wy]a$), we may conclude that Achilles is a fictional Greek warrior.

164) **Definition:** Possibility and Fictional Characters. As abstract objects, fictional characters couldn't possibly be located in spacetime. However, certain fictional characters are such that it is possible that there is some spatiotemporal object that exemplifies every property the character encodes. We say such characters might have existed:

$$x \textit{ might have existed } =_{df} \Diamond\exists y(E!y \,\&\, \forall F(xF \rightarrow Fy))$$

So Superman doesn't 'exist' in the popular sense of having a location in spacetime, but he might have existed in the sense that, possibly, something exemplifies everything Superman encodes.

165) **Remark:** Analysing the Data: Here are some other data which we may analyze in terms of our logic and metaphysical framework:

(1) Sherlock Holmes inspires real detectives: $\exists x(Dx \,\&\, E!x \,\&\, Ihx)$

(2) Something inspires real detectives: $\exists y\exists x(Dx \,\&\, E!x \,\&\, Iyx)$

(3) Schliemann didn't search for Atlantis: $\neg Ssa$

(4) Fictional characters don't exist: $\forall x(\mathit{Fictional}(x) \rightarrow \neg E!x)$
Jupiter is a fictional character: $\mathit{Fictional}(j)$
Augustus Caesar worshipped Jupiter: Wcj
Therefore, Caesar worshipped something that doesn't exist:
$\exists x(\neg E!x \,\&\, Wcx)$

§14: The Theory of Mathematical Objects and Relations[31]

In this section, we introduce the philosophical theory of types to facilitate our analysis of mathematical theories and objects. We shall not only identify mathematical objects such as the real numbers and the Zermelo Fraenkel sets as abstract individuals, but also identify mathematical relations, such as the *successor* relation in Peano Number Theory, the *greater than* relation in real number theory, and the *membership* relation of ZF, as *abstract relations* (i.e., relations that encode properties of relations).

[31] I am indebted to Bernard Linsky for his help in the development of a philosophy and epistemology of mathematics that best supplements the ideas in this section. See our paper, Linsky and Zalta [1995].

To do this, we need a theory of abstract relations that can encode properties of relations. Such a theory may be formulated simply by typing the language, logic, and proper axioms of the preceding.

We shall then analyze a mathematical theory as a certain kind of situation. As such, mathematical theories are identified abstract objects that encodes only propositional properties. The propositional properties they encode will be constructed out of mathematical propositions, i.e., propositions expressible in terms of notions pretheoretically accepted as mathematical. We shall then define *truth in a mathematical theory T* in the same way that we defined truth in a situation, truth at a world, and truth in a story. Instead of being governed by the constraint that propositions *relevantly entailed* (using relevance logic) by truths of the story are also true in the story, mathematical theories are governed by the constraint that propositions logically implied (using ordinary logic) by propositions true in the theory are also true in the theory. Given this analysis, we shall then go on to analyze mathematical objects and relations with respect to their theories.

So, in what follows, we work our way to the formulation of the following typed version of our comprehension principle for abstract objects of every type:

$$\exists x^t (A!^{\langle t \rangle} x \;\&\; \forall F^{\langle t \rangle}(xF \equiv \varphi)), \text{ where } \varphi \text{ has no free } x\text{s}$$

Then we shall be able not only to identify ordinary theoretical mathematical objects as individuals of type i, but also identify theoretical mathematical properties and relations as abstract objects of type $\langle i \rangle$ and type $\langle i, i \rangle$, respectively. For example, the number π of real number theory will be identified as that abstract individual that encodes just the properties exemplified by π in real number theory. Simiarly, the relation $\in$ of Zermelo-Fraenkel set theory will be identified as that abstract relation that encodes just the properties of relations exemplified by the relation $\in$ in ZF. Those readers who with an intuitive grasp of how the type-theoretic version of our theory will be formulated may just skip ahead to (170), where we begin the exposition of our analysis of mathematical theories and objects.

166) **Definition:** Types. Let the symbol 'i' be the type of *individuals*. Then we define *type* as follows:

.1) (a) i is a type, and

.2) (b) where $t_1, \ldots, t_n$ are any types, $\langle t_1, \ldots, t_n \rangle$ is a type ($n \geq 0$)

We say that $\langle t_1, \ldots, t_n \rangle$ is the type of n-place *relations* that have as arguments objects of type $t_1, \ldots, t_n$, respectively. So $\langle i \rangle$ is the type of properties of individuals, and $\langle i, i \rangle$ is the type of 2-place relations between individuals. We define:

.3) $p \ =_{df} \ \langle\ \rangle$

We say p is the type of *propositions*.

167) **Definition:** Typing the Language. In the usual and straightforward manner, we type the language. We suppose that there are constants $a^t, b^t, \ldots$, and variables $x^t, y^t \ldots$, for every type t. To make the language easier to read, we use appropriate abbreviations: capital letters $P, Q, \ldots$ and $F, G, \ldots$ will abbreviate constants and variables for properties and relations of various types; untyped letters $p, q, \ldots$ will be variables for propositions (i.e., they abbreviate variables of type F^p). The two basic atomic formulas of our language shall be:

.1) $F^{\langle t_1, \ldots t_n \rangle} x^{t_1} \ldots x^{t_n}$ ($n \geq 0$) (atomic exemplification)

.2) $x^t F^{\langle t \rangle}$ (atomic encoding)

We use the following distinguished symbol, for each type t: $E!^{\langle t \rangle}$. So the formula $E!^{\langle t \rangle} x^t$ asserts that the object x (of type t) exemplifies the property $E!$ (of type $\langle t \rangle$). We next describe the complex formulas and complex terms:

.3) Complex Formulas: $\neg\varphi$, $\varphi \rightarrow \psi$, $\forall x^t \varphi$, and $\Box\varphi$

.4) Complex Terms: $\imath x^t \psi$ (ψ any formula), and $[\lambda x^{t_1} \ldots x^{t_n}\, \varphi]$ ($n \geq 0$), provided φ contains no encoding subformulas

The restriction on λ-expressions that φ contain no encoding formulas avoids the paradoxes. Now to complete the typing of our language, we introduce the two defined predicates, $O!$ and $A!$, and the definitions of identity. The predicates are defined as follows:

.7) $O!^{\langle t \rangle} \ =_{df} \ [\lambda x^t \, \Diamond E!^{\langle t \rangle} x]$

.8) $A!^{\langle t \rangle} \ =_{df} \ [\lambda x^t \, \neg\Diamond E!^{\langle t \rangle} x]$

The definitions for identity are:

.9) $x^t =_{E^{\langle t\rangle}} y^t \ =_{df}\ O!^{\langle t\rangle}x \,\&\, O!y \,\&\, \Box\forall F^{\langle t\rangle}(Fx \equiv Fy)$

.10) $x^t = y^t \ =_{df}\ x^t =_{E^{\langle t\rangle}} y^t \vee$
$A!^{\langle t\rangle}x \,\&\, A!y \,\&\, \Box\forall F^{\langle t\rangle}(xF \equiv yF))$

.11) $F^{\langle t\rangle} = G^{\langle t\rangle} \ =_{df}\ \Box\forall x^t(xF \equiv xG)$

.12) $F^{\langle t_1,\ldots t_n\rangle} = G^{\langle t_1,\ldots t_n\rangle} \ =_{df}$ (where $n > 1$)

$$\forall x^{t_2}\ldots\forall x^{t_n}([\lambda y^{t_1}\, Fyx^{t_2}\ldots x^{t_n}] = [\lambda y^{t_1}\, Gyx^{t_2}\ldots x^{t_n}])\,\&$$
$$\forall x^{t_1}\forall x^{t_3}\ldots\forall x^{t_n}([\lambda y^{t_2}\, Fx^{t_1}yx^{t_3}\ldots x^{t_n}] = [\lambda y^{t_2}\, Gx^{t_1}yx^{t_3}\ldots x^{t_n}])\,\&$$
$$\ldots \& \,\forall x^{t_1}\ldots\forall x^{t_{n-1}}([\lambda y^{t_n}\, Fx^{t_1}\ldots x^{t_{n-1}}y] = [\lambda y^{t_n}\, Gx^{t_1}\ldots x^{t_{n-1}}y])$$

.13) $p = q \ =_{df}\ [\lambda y^i\, p] = [\lambda y^i\, q]$

168) **Logical Axioms and Rules:** Typing the Logic. We type all of the logical axioms and rules in the obvious way. Since this is straightforward, we do not formulate the type-theoretic versions of the logical axioms and rules.

169) **Proper Axioms:** Typing the Proper Axioms. The proper axioms of our theory may now all be typed:

.1) Ordinary objects of every type necessarily fail to encode properties:

$$O!^{\langle t\rangle}x^t \rightarrow \Box\neg\exists F^{\langle t\rangle}\, xF$$

.2) For every condition φ on properties (having type $\langle t\rangle$), there is an abstract object of type t that encodes just the properties satisfying φ:

$$\exists x^t(A!^{\langle t\rangle}x \,\&\, \forall F^{\langle t\rangle}(xF \equiv \varphi)),\ \text{where } \varphi \text{ has no free } x\text{s}$$

170) **Definition:** Mathematical Theories. In what follows, we take the notion of a *mathematical proposition* as basic. We do not propose here to say which notions are mathematical and which are not; rather, we judge, if only pretheoretically, that some notions are mathematical while others are not, and therefore, that certain propositions are mathematical. Now using the notion of a mathematical proposition, we shall identify mathematical theories as a situations that (a) are constructed out of mathematical propositions, and (b) are authored. We therefore repeat the definition of a situation, this time in our type-theoretic context:

.1) $Situation(x^i) \;\; =_{df} \;\; A!^{\langle i \rangle} \;\&\; \forall F^{\langle i \rangle}(xF \rightarrow \exists p(F = [\lambda y^i\, p]))$

Now a mathematical situation may be defined:

.2) $MathSituation(x^i) \;\; =_{df}$

$$A!^{\langle i \rangle} \;\&\; \forall F^{\langle i \rangle}(xF \rightarrow \exists p(Mathematical(p) \;\&\; F = [\lambda y^i\, p]))$$

Now a mathematical theory is a mathematical situation that some existing individual authored:

.3) $MathTheory(x^i) \;\; =_{df}$

$$MathSituation(x) \;\&\; \exists y^i(E!^{\langle i \rangle} y \;\&\; A^{\langle i,i \rangle} yx)$$

Note that a possible mathematical theory is just a mathematical theory with the exception that it is possible that some existing object authored it.

171) **Definition:** Truth in a Theory. Let T be a mathematical theory. We analyse statements of the form 'p is true in T' as $T[\lambda y\, p]$ (i.e., as: T encodes being such that p):

$$p \textit{ is true in } T \;\; =_{df} \;\; \models_T p$$

And specifically, we analyze statements of the form 'x^t exemplifies $F^{\langle t \rangle}$ in theory T' as: $\models_T Fx$.

172) **Theorem:** Identity of Theories. It now follows that a theory T just is that unique abstract object that encodes any property $F^{\langle i \rangle}$ of the form $[\lambda y^i\, p]$ such that p is true in T:

$$T \;=\; \imath x^i \forall F^{\langle i \rangle}(xF \equiv \exists p(\models_T p \;\&\; F = [\lambda y^i\, p]))$$

Notice that this view allows us to distinguish a theory from its axiomatization. The same theory can be axiomatized in different ways—an axiomatization is just an initial group of propositions from which the theorems of the theory can be derived. But we identify the theory with the (abstract object that encodes all the) theorems, rather than with some particular axiomatization of it.

173) **Rule of Inference:** Mathematical Theories Closed Under Implication. Any mathematical proposition logically implied by propositions true in a theory is also true in that theory:

> Suppose $p_1, \ldots p_n$ and q are mathematical propositions and that $p_1, \ldots, p_n \vdash q$. Then from $\models_T p_1$, and … and $\models_T p_n$, infer $\models_T q$

Consider the following extended example, in which our rule of inference is applied. First, by (15), the following constitute facts about our logic, whatever the real numbers 3 and π turn out to be:[32]

.1) $\pi > 3 \vdash [\lambda x\, x > 3]\pi$

.2) $\pi > 3 \vdash [\lambda x\, \pi > x]3$

It is also a fact that in Real Number Theory, π is greater than 3, i.e.,

.3) $\models_{RNT} \pi > 3$

So, by principle (173), we may conclude both that according to Real Number Theory, π has the property of being greater than 3 and 3 has the property of being such that π is greater than it; i.e., we may conclude both that:

.4) $\models_{RNT} [\lambda x\, x > 3]\pi$

.5) $\models_{RNT} [\lambda x\, \pi > x]3$

These facts prove useful as we turn next to the identification of mathematical objects.

174) **Axiom:** Mathematical Individuals and Relations. The mathematical individuals and relations of a given theory may now be described theoretically as follows. Let κ^t be a constant of type t or complex term of type t formed from primitive function symbols and constants in the language of theory T. Then we say that object κ^t of mathematical theory T ('κ^t_T') is that abstract object that encodes just the properties $F^{\langle t \rangle}$ such that, in theory T, κ^t_T exemplifies $F^{\langle t \rangle}$:

$$\kappa^t_T = \imath y^t \forall F^{\langle t \rangle}(yF \equiv \models_T F\kappa^t_T)$$

For example, the number π of Real Number Theory is the abstract individual that encodes exactly those properties $F^{\langle i \rangle}$ such that π_{RNT} exemplifies F in Real Number Theory:

.1) $\pi_{RNT} = \imath y^i \forall F^{\langle i \rangle}(yF \equiv \models_{RNT} F\pi_{RNT})$

[32] Strictly speaking, the names in these examples should have subscripts that identify them as objects of Real Number Theory. See the next item.

Similarly, the set ω of Zermelo-Fraenkel set theory is that abstract individual that encodes just the properties $F^{\langle i\rangle}$ such that ω_{ZF} exemplifies F in ZF.[33] It is important to recognize that these are *not* definitions of the objects in question but rather theoretical descriptions! The descriptions are well-defined because we've established that for each condition on properties, there is a unique abstract object that encodes just the properties satisfying the condition. So the identity of the mathematical object in each case is ultimately secured by our ordinary mathematical judgements of the form: in theory T, x is F.

Our type theory may now be employed to extend the analysis to mathematical relations. The relation $\in$ of ZF is that abstract relation of type $\langle i,i\rangle$ that encodes just the properties (having type $\langle\langle i,i\rangle\rangle$) that $\in_{ZF}$ exemplifies according to ZF:

.2) $\in_{ZF} = \imath y^{\langle i,i\rangle}\forall F^{\langle\langle i,i\rangle\rangle}(yF \equiv\, \models_{ZF} F\!\in_{ZF})$

Here are some examples of facts that can be derived concerning $\in_{ZF}$. In what follows, we sometimes (for readability) drop the subscripts on terms, it being understood that we are talking about the sets $\emptyset$, $\{\emptyset\}$, and relation $\in$ of ZF. From the fact that:

.3) $\models_{ZF} \emptyset \in \{\emptyset\}$

and the fact that:

.4) $\emptyset \in \{\emptyset\} \vdash [\lambda F^{\langle i,i\rangle}\ F\emptyset\{\emptyset\}]\!\in$

we may use (173) to infer:

.5) $\models_{ZF} [\lambda F^{\langle i,i\rangle}\ F\emptyset\{\emptyset\}]\!\in$

Since $[\lambda F^{\langle i,i\rangle}\ F\emptyset\{\emptyset\}]$ is a property of type $\langle\langle i,i\rangle\rangle$ that the relation $\in$ of ZF exemplifies according to ZF, we may infer from our identification principle that $\in_{ZF}$ encodes $[\lambda F^{\langle i,i\rangle}\ F\emptyset\{\emptyset\}]$:

[33] In cases where theories employ definite descriptions instead of function symbols, we theoretically describe the objects as follows, where φ is the propositional formula which formally expresses the matrix of the ordinary description 'the ... of theory T':

$$\text{The} \ldots \text{of theory } T \;=\; \imath y\forall F(yF \equiv\, \models_T F\imath x(\models_T \varphi))$$

For example, to identify 'the empty set of ZF', let the formula φ be: $\neg\exists z(z \in x)\ \&\ \forall y[\neg\exists z(z \in y) \rightarrow y = x]$. Notice that the definite description $\imath x(\models_{ZF} \varphi)$ has a denotation, since the principles of ZF imply that there is a unique set which has no members. Thus, it makes sense to ask what properties does the object denoted by the description exemplify in ZF.

.6) $\in_{ZF}[\lambda F^{\langle i,i\rangle}\ F\emptyset\{\emptyset\}]$.

175) **Remark**: Significance of the Foregoing. The philosophical significance of the foregoing ideas are discussed in Linsky and Zalta [1995]. However, it would serve well to just sketch the basic claims of that paper here. According to a widespread philosophical view (i.e., naturalism), our belief in abstract objects is justified only if the objects are required by our best natural scientific theories. So we should believe in the existence of mathematical objects, on this view, only if our best scientific theories quantify over those objects. This view further suggests that such mathematical theories are, in an important sense, *continuous* with natural science, and that the existence claims are therefore contingent, and so subject to revision and refutation along with the rest of the scientific theory of which they are a part (though, because they stand at the heart of the scientific theory, they have greater immunity to revision and refutation than less central aspects of the scientific theory).

From the present perspective, this method of justifying mathematical existence assertions is fundamentally flawed: (a) It offers no understanding of how beings in the natural world, such as ourselves, acquire knowledge of individual abstract mathematical objects (the null set has no locus in spacetime). (b) It disenfranchises all of the mathematics that is not applied in natural science (and so it fails to explain the meaningfulness of mathematics that might one day be used in a scientific theory). (c) Mathematical theories are not *revised* in the face of recalcitrant data; instead scientists just *switch* to different mathematical theory. (d) Scientific theories do not tell us anything about the non-mathematical properties of mathematical objects (and so there is no guarantee that the names and descriptions of mathematical objects are well-defined within the context of the wider scientific theory). Indeed, this entire philosophical view is based on the mistaken analogy that abstract mathematical objects are like physical objects: it is presupposed that abstract objects, like physical objects, are subject to an appearance/reality distinction, exist in a sparse way, are complete in all their details, and are subject to the principle of parsimony.

We suggest, however, that the proper conception of abstract objects is not derived by analogy with physical objects. Once the proper conception is developed, a justification of existence assertions that is consistent with naturalism can be developed. On our conception, abstract objects: (a) are not subject to an appearance/reality distinction; rather they have (i.e., encode) exactly the properties that satisfy their defining conditions, (b) do not exist in a sparse way but rather constitute a plenitude; (c) are not necessarily complete objects (i.e., they do not encode, for every property F, either F or its negation). As a species of abstract object, mathematical objects fit this pattern: (a) they encode exactly the properties attributed to them in their respective theories, (b) every consistent mathematical theory with existence assumptions is about a special domain of objects, and (c) mathematical objects are essentially incomplete objects. Once we rectify our conception of abstract and mathematical objects in this way, it becomes clear that the proper way to justify the assertion that mathematical and other abstract objects exist is to: (1) show that they are comprehended by a principle which asserts that there exists a plenitude of such objects, and (2) show that this principle is required, not just for our present scientific theories, but rather for our very understanding of any possible scientific theory. We believe that the comprehension principle for abstract objects is such a principle, and we develop a precise argument for this in the paper mentioned above.

Assuming that the argument in the paper is correct, and that belief in abstract objects is justified in the way we suggest, then a reconception of the search for mathematical foundations is in order. We do not question the fact that the search for a mathematical theory to which the rest of mathematics can be reduced is an inherently interesting logical enterprise. Nor do we question the inherent interest of the search for the weakest mathematical theory to which all of the mathematics presently required by the natural sciences can be reduced. However, we should not justify such searches on the grounds that they satisfy the demands of naturalism. The discovery of such theories does not imply that the mathematical objects that can be reduced *really are* just species of the 'fundamental' mathematical objects. Such reductions imply nothing about the identity of the reduced objects, for if we are correct, each different mathematical theory is about a different domain of mathematical objects. Moreover, such reductions of one variety of mathematical object to another are usually arbitrary (such reductions can usually be accomplished in an infinite number of ways), and such arbitrariness offers another reason for *not* thinking that the real identity of a mathematical object is revealed when it is reduced to some other mathematical object. On our view, reductions are not philosophically necessary, though they do establish the logical and mathematical power inherent in certain primitive notions and mathematical axioms.

§15: The Theory of Fregean Logical Objects

In this section[34] we prove the existence of the following Fregean logical objects: the (natural) set of Fs (‘$\{u|Fu\}$’), the cardinal number of (ordinary) Fs (‘$\#_F$’), the natural numbers $0, 1, 2, \ldots$, the direction of (ordinary) line a (‘$\vec{a}$’), the shape of (ordinary) figure b (‘$\breve{b}$’), the truth-value of proposition p (‘p°’), etc. These will be identified in the following subsections and subsubsections:

A. Natural Sets
B. Natural Cardinals
C. Natural Numbers
  1. Predecessor
  2. The Ancestrals of a Relation R
  3. The Dedekind/Peano Axioms
  4. Natural Arithmetic
D. Directions and Shapes
E. Truth Values
F. Higher-Order Logical Objects

Though we continue to use the variables x, y, z as variables for any kind of object, the variables u, v, w will serve as *restricted* variables ranging over ordinary objects.

176) **MetaDefinition:** Logical Objects. In the subsections that follow, the definitions of the natural sets, natural cardinals, directions, shapes, truth-values, and higher order ‘logical’ objects all follow a certain pattern. This pattern is easy to recognize: first a condition on objects and properties, say *Condition*(x, G), is defined in terms of an equivalence condition of the form $\psi(F, G)$:

$$\mathit{Condition}(x, G) \ =_{df} \ A!x \,\&\, \forall F(xF \equiv \psi(F, G))$$

[34]The material in this section was first sketched and explained in the paper with the same title (unpublished). To obtain a draft, see the Bibliography. Note however that sections on natural cardinals and natural numbers in the unpublished paper was revised and improved in the present monograph. The improved work appearing here also appears in my paper ‘Natural Cardinals and Natural Numbers as Abstract Objects’ (also unpublished; see the Bibliography).

To say that $\psi(F, G)$ is an equivalence condition is simply to say that ψ is a formula such that the following all hold $\psi(F, F)$, $\psi(F, G) \rightarrow \psi(G, F)$, and $\psi(F, G) \;\&\; \psi(G, H) \rightarrow \psi(F, H)$. Intuitively, such equivalence conditions partition the domain of properties into equivalence classes. Then, if given a property G, we may prove the existence of a unique abstract object that encodes all the properties in the same cell of the partition as G with respect to the given condition; that is, we may prove that the definite description $\imath x Condition(x, G)$ is well-defined. Whenever such a series of definitions and proofs takes place, we refer to the objects defined as 'logical objects'. It will be seen that the objects defined over the course of the remaining subsections are all defined in this manner (with the exception of the natural numbers—they are a species of natural cardinal and so will acquire the status of 'logical object' in virtue of being natural cardinals).

A. Natural Sets[35]

177) **Definition:** Properties Equivalent on the Ordinary Objects. Let us say that properties F and G are materially equivalent with respect to the ordinary objects ('$F \equiv_E G$') just in case all and only the ordinary objects that exemplify F exemplify G:

$$F \equiv_E G \;\; =_{df} \;\; \forall u(Fu \equiv Gu)$$

178) **Definitions:** Natural (Exemplification) Extensions of Properties. In virtue of theorems in (55) and (56), we shall restrict our attention to the natural extension of a property F, i.e., to the *ordinary* objects that exemplify F. Let us say that x is a natural extension of the property G iff x is an abstract object that encodes just the properties materially equivalent to G with respect to the ordinary objects:

$$NaturalExtension(x, G) \;\; =_{df} \;\; A!x \;\&\; \forall F(xF \equiv F \equiv_E G)$$

179) **Theorem:** The Existence of Unique Natural Extensions. It follows that for every property G, there is a unique natural extension of G:

$$\forall G \exists! x NaturalExtension(x, G)$$

[35] Bernard Linsky assisted in the development of parts of this subsection. In a joint working session on the paper Linsky and Zalta [1995], Linsky suggested the definition 'the set of Gs' and formulated the law of extensionality. Together we formulated Frege's Basic Law V. I later added Pairs, Unions, Null Set, and Separation, and reformulated all the definitions in terms of the notion of *x is an extension of G*.

180) **Definition:** Set Abstracts. Since the natural extension of the property G exists, we may introduce the notation $\{u|Gu\}$ to denote this object:

$$\{u|Gu\} \ =_{df} \ \imath x NaturalExtension(x, G)$$

181) **Definition:** Natural Sets. We say that x is a *natural set* iff x is the natural extension of some property G:

$$NaturalSet(x) \ =_{df} \ \exists G(x = \{u|Gu\})$$

Hereafter, we sometimes refer to $\{u|Gu\}$ as the (natural) set of Gs.

182) **Definition:** Membership. In the present context, the notion of membership for natural sets is definable: *y is a member of x* iff y is an ordinary object and x is the extension a property exemplified by y:[36]

$$y \in x \ =_{df} \ O!y \ \& \ \exists G(x = \{u|Gu\} \ \& \ Gy)$$

183) **Lemmas:** Features of Natural Sets. The previous definition immediately yields: (.1) the natural set of Gs encodes G, and (.2) the natural set of Gs encodes a property F iff F is materially equivalent to G with respect to the ordinary objects.

.1) $\{u|Gu\}G$

.2) $\{u|Gu\}F \equiv \forall v(Fv \equiv Gv)$

184) **Lemmas:** Membership in a Natural Set. The usual condition on set abstraction now applies: (.1) an ordinary object v is an element of the natural set of Gs iff v exemplifies G. Moreover, (.2) an arbitrary object y is an element of the natural set of Gs iff y is an ordinary object that exemplifies G:

.1) $v \in \{u|Gu\} \equiv Gv$[37]

.2) $y \in \{u|Gu\} \equiv O!y \ \& Gy$

185) **Theorem:** Frege's Basic Law V. A consistent version of Frege's Basic Law V is now derivable, namely, the extension of the property F is identical with the extension of the property G iff F and G are materially equivalent (on the ordinary objects):[38]

[36] cf. Frege, *Grundgesetze I*, §34.
[37] cf. Frege, *Grundgesetze I*, §55, Theorem 1.
[38] [cf. Frege, *Grundgesetze I*, §20.

$$\{u|Fu\} = \{u|Gu\} \equiv \forall v(Fv \equiv Gv)$$

186) **Theorem:** The Law of Extensionality. If we temporarily use the variables x, y, z to range over natural sets, we may conveniently formulate the five most basic axioms of set theory. These become theorems that apply to natural sets. First, the basic law of extensionality asserts that if natural sets x and y have the same members, they are identical:

$$\forall u(u \in x \equiv u \in y) \rightarrow x = y$$

187) **Theorem:** The Pair Set Axiom. The pair set axiom is now a theorem asserting that for any two ordinary things u and v, there is a natural set x that has just u and v as members:

$$\forall u \forall v \exists x \forall w(w \in x \equiv w = u \ \vee \ w = v)$$

188) **Theorem:** The Unions Axiom. The unions axiom is now a theorem asserting that for any natural sets x and y, there is another natural set z which has as elements all and only the members of x and y:

$$\forall x \forall y \exists z \forall u(u \in z \equiv u \in x \ \vee \ u \in y))$$

189) **Theorem:** Null Set. The existence of a null set is derivable:

$$\exists x \neg \exists u(u \in x)$$

190) **Theorem:** A Separation Schema. Let $\varphi(z)$ be any formula without encoding subformulas in which the variable z may or may not be free. Then there is a natural set that contains as members all and only those ordinary objects that are such that φ:

$$\exists x \forall y(y \in x \equiv O!y \;\&\; \varphi^y_z)$$

191) **Remark**: Given the Law of Extensionality and the preceding theorems we may introduce the usual definite descriptions for sets and prove that they are well-defined: $\{u, v\}$, $x \cup y$, $x \cap y$, $\emptyset$, and $\{u|\varphi\}$, where u, v are any ordinary objects, x, y are any natural sets, and φ is any formula without encoding formulas in which u may or may not be free.

192) **Remark**: Natural, Logical, and Mathematical Conceptions of 'Set'. The logical conception of a set is an extension of the natural conception of a set. The natural conception is based on two ideas: (1) that sets are defined by predicating properties of objects; i.e., that the objects which a property is 'true of' constitute a set, and (2) that the primary examples of sets are those constituted by ordinary objects, such as *the set of red things*, *the set of tables*, *the set of planets*, etc. This conception is captured by the foregoing theorems: whenever we have a *bona fide* property F, we may formulate a set of ordinary (i.e., possibly concrete) objects that exemplify F.

The logical conception of a set arises by generalizing on this idea. This conception simply *assumes* that there are abstract objects and that they exemplify properties as well, then extends the natural conception with the idea that for any property F, there is a set of objects (whether ordinary or abstract) that exemplify F. It is further assumed that properties can be defined by arbitrary conditions expressible in the language. The existence of Russell's paradox reveals that the logical conception of set is incoherent. The present framework reveals that when abstract objects are axiomatized in a completely general way and property comprehension is restricted to avoid paradox, there are too many abstract objects for properties to distinguish in terms of the usual predicative mode of *exemplification*. If we are to postulate abstract objects corresponding to every condition of properties, then there must be, given any property F, distinct abstract objects that F cannot distinguish. And, moreover, there must be abstract objects a and b that exemplify exactly the same properties. This is the lesson to be learned from theorems in (55) and (56). Of course, if abstract objects are postulated by weaker and less general means, there are ways to rehabilitate the logical conception.

The mathematical conception of a set is based on the idea of iteration by stages. Once we are given some objects, we can take any combination of those objects and form a set. Then, once we have those sets, we can take any combination of sets and objects, and form new sets. And so on. The iterative conception of a set is made precise by Zermelo-Fraenkel set theory, but since ZF constitutes a mathematical theory, it should be analyzed according to the ideas developed in Section 14.

B. Natural Cardinals

193) **Definition:** Equinumerosity with Respect to the Ordinary Objects. Consideration of the theorems in (55) and (56) also suggest an unproblematic notion of equinumerosity that relates properties F and G whenever the ordinary objects that exemplify F and G can be put into one-to-one correspondence. We say that properties F and G are *equinumerous with* *respect to the ordinary objects* ('$F \approx_E G$') just in case there is a relation R that constitutes a one-to-one and onto function from the ordinary objects in the exemplification extension of F to the ordinary objects in the exemplification extension of G:

$$F \approx_E G =_{df} \exists R[\forall u(Fu \rightarrow \exists!v(Gv \,\&\, Ruv)) \,\&\, \forall u(Gu \rightarrow \exists!v(Fv \,\&\, Rvu))]$$

[cf. *Grundlagen*, §71.] So F and G are equinumerous$_E$ just in case there is a relation R such that: (a) every ordinary object that exemplifies F bears R to a unique ordinary object exemplifying G (i.e., R is a function from the ordinary objects of F to the ordinary objects of G), and (b) every ordinary object that exemplifies G is such that a unique ordinary object exemplifying F bears R to it (i.e., R is a one-to-one function from the ordinary objects of F onto the ordinary objects of G). In the proofs of what follows, we say that such a relation R is a *witness* to the equinumerosity$_E$ of F and G.

194) **Theorems:** Equinumerosity$_E$ Partitions the Domain of Properties.

.1) $F \approx_E F$

.2) $F \approx_E G \rightarrow G \approx_E F$

.3) $F \approx_E G \,\&\, G \approx_E H \rightarrow F \approx_E H$

195) **Remark:** Why Not Plain Equinumerosity. It is important to give the reason for employing, in what follows, the condition '$F \approx_E G$' instead of the more traditional notion of one-to-one correspondence between properties, '$F \approx G$', which is defined without the restriction to ordinary objects. It is a consequence of the theorems concerning the existence of non-classical abstract objects (items (55) and (56)) that '$F \approx G$' does not define an equivalence condition on properties.[39] Although $F \approx G$ is not an equivalence relation, $F \approx_E G$ proves to be a useful substitute .

[39] Here is the argument that establishes this. By (56), we know that there are at least two distinct abstract objects, say a and b, which are 'indiscernible' (i.e., which exemplify the same properties). Now consider any property that both a and b exemplify, say, P. Then there won't be any property to which P is equinumerous. For suppose, for some property, say Q, that $P \approx Q$. Then there would be a relation R which is a one-one and onto function from the Ps to the Qs. Since R maps each object exemplifying P to some unique object exemplifying Q, R maps a to some object, say c, that exemplifies Q. So a exemplifies the property $[\lambda z\, Rzc]$. But, since a and b are indiscernible, b exemplifies the property $[\lambda z\, Rzc]$, i.e., Rbc. But this contradicts the

196) **Definitions:** Numbering a Property. We may appeal to our definition of equinumerosity$_E$ to say when an object numbers a property: x *numbers* (the ordinary objects exemplifying) G iff x is an abstract object that encodes just the properties equinumerous$_E$ with G:

$$Numbers(x,G) =_{df} A!x \mathbin{\&} \forall F(xF \equiv F \approx_E G)$$

197) **Theorem:** Existence of Unique Numberers. It now follows that for every property G, there is a unique object which numbers G:

$$\forall G \exists ! x Numbers(x,G)$$

198) **Definition:** The Number of Gs. Since there is a unique number of Gs, we may may introduce the notation '$\#_G$' to refer to *the* number of Gs:

$$\#_G =_{df} \imath x Numbers(x,G)$$

199) **Theorem**: The Number of Gs Exists. It follows that for every property G, there is something which is the number of Gs:

$$\forall G \exists y(y = \#_G)$$

200) **Lemmas:** Equinumerosity$_E$ and The Number of Gs. It now follows that: (.1) the number of Gs encodes F iff F is equinumerous$_E$ with G, and (.2) the number of Gs encodes G. In the following, formal renditions, note that '$\#_G F$' asserts that the number of Gs encodes F.

.1) $\#_G F \equiv F \approx_E G$

.2) $\#_G G$

201) **Theorem:** Hume's Principle. The following claim has now become known as 'Hume's Principle': The number of Fs is identical to the number of Gs if and only if F and G are equinumerous$_E$.[40]

$$\#_F = \#_G \equiv F \approx_E G$$

202) **Definition:** Natural Cardinals. We may now define: x is a *natural cardinal* iff there is some property F such that x is the number of Fs.[41]

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one-one character of R, for both Rac and Rbc and yet a and b are distinct. Thus, P can't be equinumerous to any property, including itself! Since $F \approx G$ is not a reflexive condition on properties, it is not an equivalence condition.

[40] cf. Frege, *Grundlagen*, §72.

[41] cf. Frege, *Grundlagen*, §72; and *Grundgesetze I*, §42.

$$NaturalCardinal(x) \;=_{df}\; \exists F(x = \#_F)$$

203) **Theorem:** Encoding and Numbering F. A natural cardinal encodes a property F just in case it is the number of Fs:

$$NaturalCardinal(x) \rightarrow (xF \equiv x = \#_F)$$

204) **Definition:** Zero.[42]

$$0 \;=_{df}\; \#_{[\lambda z\, z \neq_E z]}$$

205) **Theorem:** 0 is a Natural Cardinal.

$$NaturalCardinal(0)$$

206) **Theorem**: 0 Encodes the Properties Unexemplified by Ordinary Objects. 0 encodes all and only the properties which no ordinary object exemplifies:

$$0F \equiv \neg\exists u Fu$$

207) **Corollary**: Empty Properties Numbered 0. It is a simple consequence of the previous theorems and definitions that F fails to be exemplified by ordinary objects iff the number of Fs is zero:[43]

$$\neg\exists u Fu \equiv \#_F = 0$$

208) **Lemmas:** Equinumerous$_E$ and Equivalent$_E$ Properties. The following consequences concerning equinumerous$_E$ and materially equivalent$_E$ properties are easily provable: (.1) if F and G are materially equivalent$_E$, then they are equinumerous$_E$; (.2) if F and G are materially equivalent$_E$, then the number of Fs is identical to the number of Gs; and (.3) if F is equinumerous$_E$ to G and G is materially equivalent$_E$ to H, then F is equinumerous$_E$ to H:

.1) $F \equiv_E G \rightarrow F \approx_E G$

.2) $F \equiv_E G \rightarrow \#_F = \#_G$

.3) $F \approx_E G \;\&\; G \equiv_E H \rightarrow F \approx_E H$

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[42] cf. Frege, *Grundlagen*, §74; and *Grundgesetze I*, §41.
[43] cf. Frege, *Grundlagen*, §75; and *Grundgesetze I*, §99, Theorem 97.

C. Natural Numbers[44]

1. Predecessor

209) **Definition:** Predecessor. We say that x *precedes* y iff there is a property F and ordinary(!) object u such that (a) u exemplifies F, (b) y is the number of Fs, and (c) x is the number of (the property) *exemplifying-F-but-not-identical$_E$-to-u*:[45]

$$Precedes(x, y) \ =_{df} \ \exists F \exists u (Fu \ \& \ y = \#_F \ \& \ x = \#_{[\lambda z \ Fz \ \& \ z \neq_E u]})$$

Note that the definition of $Precedes(x, y)$ contains the identity sign '=', which is defined in terms of encoding subformulas. As such, there is no guarantee as yet that $Precedes(x, y)$ is a relation,[46] though objects x and y may satisfy the condition nonetheless.

210) **Theorem**: Nothing is a Precedessor of Zero.[47]

$$\neg \exists x \ Precedes(x, 0)$$

Since nothing precedes zero, it follows that no cardinal number precedes zero.

211) **Lemma**: Let F^{-u} designate $[\lambda z \ Fz \ \& \ z \neq_E u]$ and G^{-v} designate $[\lambda z \ Gz \ \& \ z \neq_E v]$. Then if F is equinumerous$_E$ with G, u exemplifies F, and v exemplifies G, F^{-u} is equinumerous$_E$ with G^{-v}:[48]

$$F \approx_E G \ \& \ Fu \ \& \ Gv \rightarrow F^{-u} \approx_E G^{-v}$$

212) **Theorem:** Predecessor is One-to-One. If x and y precede z, then $x = y$:[49]

[44] I am greatly indebted to Bernard Linsky, for suggesting that I try to prove the Dedekind/Peano Axioms using just the machinery of object theory (he pointed out that the definition of predecessor was formulable in the language of the theory) and for reading and correctly criticizing numerous earlier drafts of the technical material in the following subsections. I am also indebted to Godehard Link, who critiqued this same material in conversation. I have also benefited from reading Heck [1993].

[45] See Frege, *Grundlagen*, §76; and *Grundgesetze I*, §43.

[46] That is, the Comprehension Principle for Relations does not ensure that it is a relation.

[47] cf. *Grundgesetze I*, Theorem 108.

[48] cf. *Grundgesetze I*, Theorem 87ϑ. This is the line on p. 126 of *Grundgesetze I* which occurs *during* the proof of Theorem 87. Notice that Frege proves the contrapositive. Notice also that this theorem differs from Frege's theorem only by two applications of Hume's Principle: in Frege's Theorem, $\#_F = \#_G$ is substituted for $F \approx_E G$ in the antecedent and $\#_{F^{-u}} = \#_{G^{-v}}$ is substituted for $F^{-u} \approx_E G^{-v}$ in the consequent.

[49] cf. Frege, *Grundlagen*, §78; and *Grundgesetze I*, Theorem 89.

$$Precedes(x,z) \;\&\; Precedes(y,z) \rightarrow x\!=\!y$$

213) **Lemma**: Let F^{-u} designate $[\lambda z\, Fz \,\&\, z \neq_E u]$ and G^{-v} designate $[\lambda z\, Gz \,\&\, z \neq_E v]$. Then if F^{-u} is equinumerous$_E$ with G^{-v}, u exemplifies F, and v exemplifies G, then F is equinumerous$_E$ with G:[50]

$$F^{-u} \approx_E G^{-v} \;\&\; Fu \;\&\; Gv \rightarrow F \approx_E G$$

214) **Theorem**: Predecessor is Functional. If z precedes both x and y, then x is y:[51]

$$Precedes(z,x) \;\&\; Precedes(z,y) \rightarrow x\!=\!y$$

2. The Ancestrals of a Relation R

215) **Definition**. Properties Hereditary with Respect to Relation R. We say that a property F *is hereditary with respect to* R iff every pair of R-related objects are such that if the first exemplifies F then so does the second:

$$Hereditary(F,R) \;\; =_{df} \;\; \forall x,y(Rxy \rightarrow (Fx \rightarrow Fy))$$

Hereafter, whenever *Hereditary*(F,R), we sometimes say that F is R-hereditary.

216) **Definition:** The Ancestral of a Relation R. We define: x comes before y in the R-series iff y exemplifies every R-hereditary property F which is exemplified by every object to which x is R-related:[52]

$$R^*(x,y) \;\; =_{df} \;\; \forall F[\forall z(Rxz \rightarrow Fz) \,\&\, Hereditary(F,R) \rightarrow Fy]$$

So if we are given a genuine relation R, it follows by comprehension for relations that $R^*(x,y)$ is a genuine relation as well (the quantifier over relations in the definition of $R^*(x,y)$ is permitted by the comprehension principle for relations).

217) **Lemmas**: The following are immediate consequences of the two previous definitions: (.1) if x bears R to y, then x comes before y in the

[50]cf. Frege, *Grundgesetze I*, Theorem 66. This theorem differs from Frege's Theorem 66 only by two applications of Hume's Principle: in Frege's Theorem, $\#_{F^{-u}} = \#_{G^{-v}}$ is substituted for $F^{-u} \approx_E G^{-v}$ in the antecedent, and $\#_F = \#_G$ is substituted for $F \approx_E G$ in the consequent.

[51]cf. Frege, *Grundgesetze I*, Theorem 71.

[52]cf. Frege, *Grundlagen*, §79; and *Grundgesetze I*, §45.

R-series; (.2) if x comes before y in the R-series, F is exemplified by every object to which x bears R, and F is R-hereditary, then y exemplifies F; (.3) if x exemplifies F, x comes before y in the R-series, and F is R-hereditary, then y exemplifies F; (.4) if x bears R to y and y comes before z in the R series, then x comes before z in the R series; and (.5) if x comes before y in the R series, then something bears R to y:

.1) $Rxy \rightarrow R^*(x,y)$

.2) $R^*(x,y) \,\&\, \forall z(Rxz \rightarrow Fz) \,\&\, \mathit{Hereditary}(F,R) \rightarrow Fy$[53]

.3) $Fx \,\&\, R^*(x,y) \,\&\, \mathit{Hereditary}(F,R) \rightarrow Fy$[54]

.4) $Rxy \,\&\, R^*(y,z) \rightarrow R^*(x,z)$[55]

.5) $R^*(x,y) \rightarrow \exists z Rzy$[56]

218) **Definition:** Weak Ancestral. We say that y *is a member of the R-series beginning with* x iff either x comes before y in the R-series or $x = y$.[57]

$$R^+(x,y) \;=_{df}\; R^*(x,y) \vee x = y$$

The definition of $R^+(x,y)$ involves the identity sign, which is defined in terms of encoding subformulas. So though x and y may satisfy the condition $R^+(x,y)$, there is no guarantee as yet that they stand in a relation in virtue of doing so.

219) **Lemmas**: The following are immediate consequences of the previous three definitions: (.1) if x bears R to y, then y is a member of the R-series beginning with x; (.2) if x exemplifies F, y is a member of the R-series beginning with x, and F is R-hereditary, then y exemplifies F; (.3) if y is a member of the R series beginning with x, and y bears R to z, then x comes before z in the R-series; (.4) if x comes before y in the R-series and y bears R to z, then z is a member of the R-series beginning with x; (.5) if x bears R to y, and z is a member of the R series beginning with y, then x comes before z in the R series; and (.6) if x comes before y in the R series, then some member of the R-series beginning with x bears R to y:

[53] cf. Frege, *Grundgesetze I*, Theorem 123.
[54] cf. Frege, *Grundgesetze I*, Theorem 128.
[55] cf. Frege, *Grundgesetze I*, Theorem 129.
[56] cf. Frege, *Grundgesetze I*, Theorem 124.
[57] cf. Frege, *Grundlagen*, §81; and *Grundgesetze I*, §46.

.1) $Rxy \rightarrow R^{+}(x, y)$

.2) $Fx \,\&\, R^{+}(x, y) \,\&\, \mathit{Hereditary}(F, R) \rightarrow Fy$[58]

.3) $R^{+}(x, y) \,\&\, Ryz \rightarrow R^{*}(x, z)$[59]

.4) $R^{*}(x, y) \,\&\, Ryz \rightarrow R^{+}(x, z)$

.5) $Rxy \,\&\, R^{+}(y, z) \rightarrow R^{*}(x, z)$[60]

.6) $R^{*}(x, y) \rightarrow \exists z(R^{+}(x, z) \,\&\, Rzy)$[61]

3. The Dedekind/Peano Axioms

220) **Definition:** Natural Numbers. We may now define:[62]

$$\mathit{NaturalNumber}(x) \ =_{\mathit{df}} \ \mathit{Precedes}^{+}(0, x)$$

We sometimes use 'm, n, o' as restricted variables ranging over natural numbers.

221) **Theorem**: Natural Numbers are Natural Cardinals. It is a consequence of a previous lemma that natural numbers are natural cardinals:

$$\mathit{NaturalNumber}(x) \rightarrow \mathit{NaturalCardinal}(x)$$

222) **Theorem:** 0 is a Natural Number.[63]

$$\mathit{NaturalNumber}(0)$$

With this theorem, we have derived the 'first' Dedekind/Peano axiom.

223) **Theorems:** 0 Is Not the Successor of Any Natural Number. It now follows that: (.1) 0 does not ancestrally precede itself, and (.2) no natural number precedes 0.[64]

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[58] cf. Frege, *Grundgesetze I*, Theorem 144.

[59] cf. Frege, *Grundgesetze I*, Theorem 134.

[60] cf. Frege, *Grundgesetze I*, Theorem 132.

[61] cf. Frege, *Grundgesetze I*, Theorem 141.

[62] cf. Frege, *Grundlagen*, §83; and *Grundgesetze I*, §108. In this section, Frege informally reads the formula $\mathit{Precedes}^{+}(0, x)$ as 'x is a finite number', though he doesn't officially introduce new notation for this notion.

[63] cf. Frege, *Grundgesetze I*, Theorem 140. Frege here proves only the general theorem that $\forall x \mathit{Precedes}^{+}(x, x)$, but doesn't seem to label the the result of instantiating the universal quantifier to the number zero as a separate theorem.

[64] cf. Frege, *Grundgesetze I*, Theorem 126.

.1) $\neg Precedes^*(0, 0)$

.2) $\neg\exists n(Precedes(n, 0))$

With (223.2), we have derived the 'second' Dedekind/Peano axiom.

224) **Theorems:** No Two Natural Numbers Have the Same Successor. From (212), it follows that no two natural numbers have the same successor:

$$\forall n, m, o(Precedes(n, o) \;\&\; Precedes(m, o) \rightarrow m = n)$$

With (224), we have derived the 'third' Dedekind/Peano axiom. We now work our way towards a proof that for every natural number, there is a unique natural number which is its successor.

225) **Lemma**: Successors of Natural Numbers are Natural Numbers. If a natural number n precedes an object y, then y is itself a natural number:

$$Precedes(n, y) \rightarrow NaturalNumber(y)$$

226) **Modal Axiom**. Richness of Possible Objects.[65] The following modal claim is true *a priori*: if x is a natural number and x is the number of Gs, then there might have been a concrete object y which is distinct$_E$ from every ordinary object u that *actually* exemplifies G. We may formalize this *a priori* truth as the following modal axiom:

$$NaturalNumber(x) \;\&\; x = \#_G \rightarrow \\ \quad \Diamond\exists y(E!y \;\&\; \forall u(\mathcal{A}Gu \rightarrow u \neq_E y))$$

227) **Modal Lemma**. Distinctness of Possible Objects. It is a consequence of the logic of actuality and the logic of the identity$_E$ relation that

[65] I am indebted to Karl-Georg Niebergall for conversations about this axiom. In a previous draft of this monograph, I used, instead of this axiom, a modal axiom schema of the form 'There might have been at least n concrete objects' (where each 'n' is eliminated in terms of numerical quantifiers). The derivation that every number has a successor required an appeal to the Omega Rule. In the attempt to find a proof that did not appeal to the Omega Rule, I sought to replace the modal axiom schema with a simpler modal axiom. However, my first sketch of the axiom failed to include the conjunctive antecedent and failed to use the actuality operator. Discussions with Niebergall about general number systems led me to see that by including this antecedent and strategically employing the actuality operator, the following modal axiom would imply that every number has a successor without an appeal to the Omega Rule.

if it is possible that every ordinary object which actually exemplifies G is distinct$_E$ from ordinary object v, then in fact every ordinary object which exemplifies G is distinct$_E$ from v:

$$\Diamond\forall u(\mathcal{A}Gu \rightarrow u \neq_E v) \rightarrow \forall u(\mathcal{A}Gu \rightarrow u \neq_E v)$$

228) **Theorem**: Every Natural Number Has a Unique Successor. It now follows from (226) and (227) that for every natural number n, there exists a unique natural number m which is the successor of n:

$$\forall n \exists! m Precedes(n, m)$$

With this theorem, we have derived the 'fourth' Dedekind-Peano Axiom.

229) **Remark**: Justification of the Modal Axiom Schema. Axiom (226) is *not* contingent. It does not assert the existence of concrete objects. Rather, it merely asserts the possible existence of concrete objects whenever a certain condition holds. The difference is vast. The claim that concrete objects exist is an empirical claim, but the claim that it is possible that concrete objects exist is not. By the rules of S5, (226) is a necessary truth. Moreover, this is the kind of fact that logicians appeal to when defending the view that logic should have no existence assumptions and make no claims about the size of the domain of objects.[66]

The fact that every number has a successor does not imply that there are an infinite number of concrete objects. Rather, it implies only that there are an infinite number of possibly concrete objects.[67] These possible

[66] For example, in Boolos [1987], we find (p. 18):

> In logic, we ban the empty domain as a concession to technical convenience but draw the line there: We firmly believe that the existence of even two objects, let alone infinitely many, cannot be guaranteed by logic alone. ... Since there might be fewer than two items that we happen to be talking about, we cannot take even $\exists x \exists y(x \neq y)$ to be valid.

From the antecedent of his last sentence, it would seem that Boolos takes claims of the form 'there might have been fewer than n objects' to be true *a priori*. It seems clear that he would equally accept the claim 'there might have been more than n objects' to be true *a priori*. We are formalizing this assumption and in the process, make it clear that (possibly) concrete objects are the ones in question.

[67] Indeed, there is a natural cardinal that numbers the ordinary objects, namely, $\#[\lambda z\, z =_E z]$. It is easy to see that that this natural cardinal is not a natural number. For suppose it is a natural number. Then by our modal axiom (226), it is possible for there to be a concrete object distinct from all the objects actually exemplifying $[\lambda z\, z =_E z]$. But this is provably not possible, for such an object would be an ordinary objects distinct$_E$ from itself.

concrete objects are all 'ordinary' objects (by definition) and so can be counted by our natural numbers. So the fourth Dedekind/Peano axiom has no contingent consequences. Moreover, we have employed no axiom of infinity such as the one asserted in Russell and Whitehead [1910] or in Zermelo-Fraenkel set theory.

230) **Remark**. Frege's Proof That Every Number Has a Successor. Frege proved that every number has a successor from Hume's Principle without any additional axioms or rules. If we try to follow Frege's procedure, we run into the following obstacle when attempting to prove that every number has a successor. We may formulate the claim as follows:

a) $\forall n \exists m (\mathit{Precedes}(n, m))$

To prove this claim, it suffices to prove the following in virtue of Theorem (225):

b) $\forall n \exists x (\mathit{Precedes}(n, x))$

To prove this, Frege's strategy was to prove that every number n immediately precedes the number of members in the *Predecessor* series ending with n (intuitively, that n immediately precedes the number of natural numbers less than or equal to n), i.e.,

c) $\forall n\, \mathit{Precedes}(n, \#_{[\lambda x\, \mathit{Precedes}^{+}(x,n)]})$

[cf. *Grundgesetze I*, Theorem 155.] However, given the definitions we have introduced so far, (c) fails to be true; indeed, no number n is the number of members of the predecessor series ending with n. For suppose, for some n, that:

d) $\mathit{Precedes}(n, \#_{[\lambda x\, \mathit{Precedes}^{+}(x,n)]})$

Then, by the definition of *Predecessor*, (d) implies that there is a property P and ordinary object a such that:

e) $Pa \,\&\, \#_{[\lambda x\, \mathit{Precedes}^{+}(x,n)]} = \#_P \,\&\, n = \#_{[\lambda x\, Px \,\&\, x \neq_E a]}$

From the second conjunct of (e), it follows by Hume's Principle that:

f) $[\lambda x\, \mathit{Precedes}^{+}(x, n)] \approx_E P$

Since Pa, it follows from the definition of $\approx_E$ that there is an ordinary object, say b, such that $\mathit{Precedes}(b, n)$. But this contradicts the fact that any predecessor of a number n is an abstract object, by definition of *Predecessor*.

Clearly, the simple reason why (c) is false is that our natural numbers and natural cardinals only count ordinary (possibly natural) objects. They do not count abstract objects. It is provable that if an object is in the exemplification extension of the property $[\lambda x\ \mathit{Precedes}^{+}(x,n)]$, it is not an ordinary object. So our natural numbers and natural cardinals will number such a property as having zero ordinary objects. Moreover, our definitions can't be extended so that natural cardinals and numbers count both the ordinary and the abstract objects in the exemplification extension of a property. There are just too many abstract objects to count by means of the classical notions of exemplification and one-to-one correspondence (the latter which involves exemplification).

231) **Axioms:** Predecessor and Its (Weak) Ancestral Are Relations. The definitions of *Predecessor* and its weak ancestral involve encoding subformulas, and so they are not automatically guaranteed to be relations. In what follows, we assume that these conditions do in fact define relations: objects:

.1) $\exists F \forall x \forall y (Fxy \equiv \mathit{Precedes}(x,y))$

It follows from this by the comprehension principle for relations that *Precedes*$^{*}(x,y))$ is a relation:

.2) $\exists F \forall x \forall y (Fxy \equiv \mathit{Precedes}^{*}(x,y))$

However, since the definition of $\mathit{Precedes}^{+}(x,y)$ involves an encoding formula, we explicitly assume the following:

.3) $\exists F \forall x \forall y (Fxy \equiv \mathit{Precedes}^{+}(x,y))$

Notice that (231.2) need not be assumed, for the existence of the ancestral of a relation R is guaranteed by comprehension (the definition of $\mathit{Precedes}^{*}(x,y)$ involves only quantifiers binding relation variables, and so given (231.1), it follows that there is such a relation as $\mathit{Precedes}^{*}(x,y)$).

232) **Remark:** In light of the theorems in (55) and (56), we may reflect on the consequences of assuming that *Predecessor* (or its (weak) ancestral) is a relation. (55) tells us that there are distinct abstract objects, say a and b, such that $[\lambda z\ \mathit{Precedes}(z,a)]$ just is (identical with) $[\lambda z\ \mathit{Precedes}(z,b)]$ and such that $[\lambda z\ \mathit{Precedes}(a,z)]$ is identical with $[\lambda z\ \mathit{Precedes}(b,z)]$. So if one could prove the claim that something exemplifies either member of the first (or second) pair of properties, one could derive a contradiction from the fact that *Predecessor* is functional (or one-to-one). That is, the assumption that *Predecessor* is a relation should be safe as long as we don't assert that a or b, or any other such objects, are members of the *Predecessor* series. So, in what follows, we must be very careful that we do not assert anything that would imply such a claim.

In (235), we describe a model that shows that our assumptions in (231) are consistent.

233) **Theorem:** Generalized Induction. If R^+ is a relation, then if an object a exemplifies F and F is hereditary with respect to R when R is restricted to the members of the R-series beginning with a, then every member of the R-series beginning with a exemplifies F:[68]

$$\exists G \forall x,y(Gxy \equiv R^+(x,y)) \rightarrow \\ \quad \forall F[Fa \;\&\; \forall x,y(R^+(a,x) \;\&\; R^+(a,y) \;\&\; Rxy \rightarrow (Fx \rightarrow Fy)) \rightarrow \\ \quad\quad \forall x(R^+(a,x) \rightarrow Fx)]$$

234) **Corollary**: Principle of Induction. The Principle of Induction falls out as a corollary to the previous theorem and the assumption that *Predecessor*$^+$ is a relation:

$$F0 \;\& \\ \forall x,y[\mathit{NaturalNumber}(x) \;\&\; \mathit{NaturalNumber}(y) \;\&\; \mathit{Precedes}(x,y) \rightarrow \\ \quad (Fx \rightarrow Fy)] \rightarrow \\ \forall x(\mathit{NaturalNumber}(x) \rightarrow Fx)$$

We may put this even more simply by using our restricted variables n, m which range over numbers:

$$F0 \;\&\; \forall n,m(\mathit{Precedes}(n,m) \rightarrow (Fn \rightarrow Fm)) \rightarrow \forall n Fn$$

With the Principle of Induction, we have derived the 'fifth' and final DedekindPeano axiom.

235) **Remark:** Consistency of the Extended Theory. To prove the fourth and fifth Dedekind/Peano axioms, we added (226) and (231), respectively, to the theory of abstract objects. To show that one may consistently add these assumptions, we develop a suggestion by Peter Aczel for extending the model construction in (57):[69] To follow Aczel's suggestions, we

[68] cf. Frege, *Grundgesetze I*, Theorem 152.
[69] Aczel suggests that we:

proceed as follows. We start with a denumerably infinite domain **O** and we include a copy of the natural numbers $0^*, 1^*, 2^*, \ldots$ in **S**. Then we identify the natural numbers $0, 1, 2, \ldots$ as those abstract objects (i.e., sets of properties) which are sets of equinumerous properties. Next, we set the proxy function so that $|n| = n^*$. We then stipulate that the domain of relations $\mathbf{R}_2$ contains the *Predecessor* relation, and that its extension at the actual world is the distinguished set of ordered pairs of proxies of predecessors: $\{\langle 0^*, 1^*\rangle, \langle 1^*, 2^*\rangle, \langle 2^*, 3^*\rangle, \ldots\}$. To constrain the model so that the modal axiom (226) is true, we simply require that the domain of possible worlds includes an ω-sequence of possible worlds, $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$ and stipulate that at $\mathbf{w}_n$, there are n ordinary objects in $\mathbf{ext}_{\mathbf{w}_n}(\mathbf{d}_g(E!))$. Thus, no matter what the characteristics of the distinguished actual world and no matter which property G is chosen, whenever natural number n is the number of Gs at the actual world, there is a world $\mathbf{w}_{n+1}$ where there is an ordinary object y that is distinct$_E$ from all the objects that are actually G.

Note that our reconstruction of the natural numbers is weaker than Frege's logicism not only because we include the comprehension principle for abstract objects and the modal axiom in our logic and metaphysics, but also because we can not reconstruct the rationals and the reals (natural sets, recall, only have ordinary objects as members, and so we cannot reconstruct natural pairs of natural numbers). These latter mathematical objects are creatures of mathematical theories, and so are to be analyzed in the manner described in Section 14.

4. Natural Arithmetic

236) **Definition:** Notation for Successors. We introduce the functional notation n' to abbreviate the definite description 'the successor of n' as follows:

> ... use my model construction with an infinite set of urelements and have among the special objects a copy of the natural numbers. For each natural number n let α_n be the set of those ordinary properties that are exemplified by exactly n ordinary objects. Now in choosing proxies just make sure that the special object that is the copy of the natural number n is chosen as the proxy of α_n for each n. The copy of the Predecessor relation will be an ordinary relation, as are its (weak) ancestral.

This is quoted from a personal communication of November 11, 1996, with the symbol 'α_n' replacing his symbol 'G_n' (since I have been using 'G' to range over properties) and with 'Predecessor' replacing 'proceeds'.

$$n' =_{df} \imath y(Precedes(n,y))$$

By (228), we know that every natural number has a unique successor. So n' is always well-defined.

237) **Definitions:** Introduction of the Integer Numerals. We introduce the integer numerals '1', '2', '3', ... , as abbreviations, respectively, for the descriptions 'the successor of 0', 'the successor of 1', 'the successor of 2', etc.:

$$1 =_{df} 0'$$

$$2 =_{df} 1'$$

$$3 =_{df} 2'$$

$$\vdots$$

Note that the definite descriptions being abbreviated are well-defined terms of our formal language. So it is provable that the numerals have denotations.

238) **Definitions:** The Numerical Quantifiers. We inductively define the *exact* numerical quantifiers 'there are exactly n ordinary F-things' ('$\exists!_n uFu$') as follows:

$$\exists!_0 uFu =_{df} \neg\exists uFu$$

$$\exists!_n uFu =_{df} \exists u(Fu \,\&\, \exists!_n v(Fv \,\&\, v \neq_E u))$$

Note that from the point of view of third-order logic, such a definition induces properties of the property F, one for each cardinality. Our natural numbers, in effect, encode the properties satisfying these third-order properties. This is revealed by the following metatheorem:

239) **Metatheorem:** Numbers 'Encode' Numerical Quantifiers. For each numeral n, it is provable that n is the abstract object that encodes just the properties F such that there are exactly n ordinary objects which exemplify F, i.e.,

$$\vdash n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n uFu))$$

240) **Definitions**: Addition, Less Than. We may now define, in the usual way:

$$n + 0 =_{df} n$$

$$n + m' =_{df} (n + m)'$$

Moreover, we also define:

$$n < m =_{df} \mathit{Precedes}^*(n, m)$$

$$n \leq m =_{df} \mathit{Precedes}^+(n, m)$$

241) **Definition:** One-to-One Relations. We say that a relation R is *one-to-one* ('$\mathit{One\text{-}One}(R)$') iff for any objects x, y, z, if Rxz and Ryz, then $x = y$:

$$\mathit{One\text{-}One}(R) =_{df} \forall x, y, z(Rxz \,\&\, Ryz \rightarrow x = y)$$

242) **Lemmas:** One-One Relations and Their Ancestrals. The following facts about one-one relations and their ancestrals are used in the proof that no natural number (ancestrally) precedes itself.

.1) $\mathit{One\text{-}One}(R) \,\&\, Rxy \,\&\, R^*(z, y) \rightarrow R^+(z, x)$

.2) $\mathit{One\text{-}One}(R) \rightarrow \forall x, y[Rxy \rightarrow (\neg R^*(x, x) \rightarrow \neg R^*(y, y))]$

.3) $\mathit{One\text{-}One}(R) \,\&\, \neg R^*(x, x) \rightarrow [R^+(x, y) \rightarrow \neg R^*(y, y)]$

243) **Corollaries:** No Natural Number (Ancestrally) Precedes Itself. It is a consequence of the previous lemmas that no member of the *Predecessor* series beginning with 0 repeats itself anywhere in this series.

.1) $\mathit{NaturalNumber}(x) \rightarrow \neg \mathit{Precedes}^*(x, x)$

.2) $\mathit{NaturalNumber}(x) \rightarrow \neg \mathit{Precedes}(x, x)$

244) **Definition:** The Predecessor of an Object. We introduce the notation '$x - 1$' to abbreviate the definite description 'the predecessor of x':

$$x - 1 =_{df} \imath y(\mathit{Precedes}(y, x))$$

245) **Definition:** (Positive) Integers. A (positive) integer is any natural number not equal to zero:

$$\mathit{Integer}(x) =_{df} \mathit{NaturalNumber}(x) \,\&\, x \neq 0$$

246) **Theorem:** The Predecessors of Integers Exist and Are Numbers. If x is an integer, then the predecessor of x exists and is a natural number:

$$Integer(x) \rightarrow \exists n(n = x - 1)$$

This theorem and the previous theorem assure us that if given an arbitrary integer n', there is a unique way to count backwards from n' reaching 0 without encountering any number twice. In other words, if given any integer n', we may enumerate the integers between n' and 0 as follows: n', $n' - 1$, $(n' - 1) - 1$, ...,0.

D. Directions and Shapes

247) **Definitions:** The Directions of Lines. Let $u \| v$ assert that (ordinary) line u *is parallel to* (ordinary) line v, where this constitutes an equivalence relation on lines. For the present purposes, we may take this relation as primitive. Further, let us call the property $[\lambda z\ z =_E u]$ the *haecceity* of u. We may define: *x is a direction of v* iff x is an abstract object that encodes all and only the haecceities of lines parallel to v:

$$Direction(x, v) \ =_{df} \ A!x \ \& \ \forall F(xF \equiv \exists u(u \| v \ \& \ F = [\lambda z\ z =_E u]))$$

248) **Theorem:** Existence of Unique Directions. It now follows that there is a unique direction of line v:

$$\exists! x Direction(x, v)$$

249) **Definition:** The Direction of Line v. Since the direction of line v exists, we may introduce a term ('$\vec{v}$') to name it:

$$\vec{v} \ =_{df} \ \imath x Direction(x, v)$$

250) **Definition:** Directions. A *direction* is any object that is the direction of some ordinary line is:[70]

$$Direction(x) \ =_{df} \ \exists v(x = \vec{v})$$

It follows that if there is at least one existing line, then there are (abstract) directions.

251) **Theorem:** The Direction of Two Lines is The Same iff They Are Parallel. It now follows that the direction of line u is identical to the direction of line v iff u is parallel to v then :

[70] cf. Frege, *Grundlagen*, §66.

$$\vec{u} = \vec{v} \equiv u \| v$$

[cf. *Grundlagen*, §65.] We have therefore established the Fregean biconditional principle for directions.

252) **Results:** Shapes. Similar definitions can be constructed with respect to shapes. Let $u \cong v$ assert that ordinary figure u is congruent or similar to ordinary figure v, where this is an equivalence relation on figures. Then we define:

.1) $\mathit{Shape}(x, v) \ =_{df}\ A!x \,\&\, \forall F(xF \equiv \exists u(u \cong v \,\&\, F = [\lambda z\, z =_E u]))$

It now follows that there is a unique shape of figure v:

.2) $\exists! x \mathit{Shape}(x, v)$

Since the shape of v exists, we use '$\breve{v}$' to denote it:

.3) $\breve{v} \ =_{df}\ \imath x \mathit{Shape}(x, v)$

We now say that a shape is any abstract object that is the shape of some ordinary figure:

.4) $\mathit{Shape}(x) \ =_{df}\ \exists v(x = \breve{v})$

It now follows that the shape of figure u is identical to the shape of figure v iff u and v are congruent:

.5) $\breve{u} = \breve{v} \equiv u \cong v$

E. Truth Values

In the following subsection, we use $p, q, r, \ldots$ as variables ranging over propositions.

253) **Definition:** The Extension of a Proposition. Recall item (46) in which we define: x encodes a proposition p ('$\Sigma_x p$') iff x encodes the property *being such that* p (i.e., iff $x[\lambda y\, p]$). We may then define: *x is an extension of* proposition p iff x is an abstract object that encodes just the propositions materially equivalent to p:

.1) $\mathit{Extension}(x, p) \ =_{df}\ A!x \,\&\, \forall F(xF \equiv \exists q(q \equiv p \,\&\, F = [\lambda y\, q]))$

254) **Theorem**: Existence of Unique Extensions. It follows that every proposition p has a unique extension.

$\forall p \exists ! x Extension(x, p)$

255) **Definition:** The Extension of p. Since the extension of p exists, we introduce the notation 'p°' to refer to it:

$$p^\circ \;=_{df}\; \imath x Extension(x, p)$$

256) **Definition:** Truth Values. We define: *x is a truth value* iff x is the extension of some proposition:

$$T\text{-}value(x) \;=_{df}\; \exists p(x = p^\circ)$$

In light of this definition, we often refer to p° in what follows as the truth value of p.

257) **Lemmas**: Two Lemmas Concerning Truth Values. The following two lemmas are simple consequences of our definitions: (.1) the truth-value of p encodes the proposition q iff q is materially equivalent to p, and (.2) the truth-value of p encodes p.

.1) $\Sigma_{p^\circ} q \;\equiv\; q \equiv p$

.2) $\Sigma_{p^\circ} p$

258) **Theorem**: The Fregean Biconditional Principle for Truth Values.

$$p^\circ = q^\circ \;\equiv\; p \equiv q$$

259) **Definition**. The True and The False. We can define *The True* ('$\top$') as the object that encodes all and only true propositions, and define *The False* ('$\bot$') as the object that encodes all and only false propositions:

$$\top \;=_{df}\; \imath x(A!x \,\&\, \forall F(xF \equiv \exists p(p \,\&\, F = [\lambda y\, p])))$$

$$\bot \;=_{df}\; \imath x(A!x \,\&\, \forall F(xF \equiv \exists p(\neg p \,\&\, F = [\lambda y\, p])))$$

260) **Theorem**: The True and The False Are Truth Values:

.1) $T\text{-}value(\top)$

.2) $T\text{-}value(\bot)$

261) **Theorem**: The True is The (Actual) World. It also follows that *The True* is identical with the actual world:

$$\top = w_\alpha$$

This is a simple consequence of the definition of the actual world w_α, which can be found in Zalta [1983], [1988a], or [1993].

262) **Lemma**: The Truth Values of Truths and Falsehoods. To help prove our final theorem, we establish: (.1) the truth value of a true proposition p is The True, and (.2) the truth value of a false proposition p is The False:

.1) $p \rightarrow (p^\circ = \top)$

.2) $\neg p \rightarrow (p^\circ = \bot)$

263) **Theorem**: There are Exactly Two Truth Values.

$$\exists x, y[\textit{T-value}(x) \ \& \ \textit{T-value}(y) \ \& \ x \neq y \ \& \\ \forall z(\textit{T-value}(z) \rightarrow z = x \ \vee \ z = y)]$$

F. Higher Order Fregean Logical Objects

264) **Results**: Generalized Abstraction. So far, we have seen that whenever we can: (a) formulate an equivalence condition $\varphi(F, G)$ on properties (with respect to the ordinary objects that exemplify F and G), (b) assert the existence of an equivalence relation R on ordinary objects, or (c) formulate an equivalence condition on propositions $\varphi(p, q)$, we may define abstract objects that correspond to the cells of the partition. Examples of the abstract objects corresponding to methods (a), (b), and (c) are extensions of properties, directions of lines, and truth values, respectively. We may generalize this procedure in a couple of ways. First, we may formulate equivalence conditions $\varphi(F, G)$ without considering whether there are any ordinary objects that exemplify F and G. For example, consider the condition $F = G$, i.e., $\Box\forall x(xF \equiv xG)$. This is clearly an equivalence condition on properties, for it is reflexive, symmetric and transitive. Thus, one may define an abstract object that corresponds to each cell in the partition. Indeed, we have defined such objects elsewhere: the Platonic Form of G (Φ_G) and the Leibnizian concept of G (c_G) are both defined as the abstract object that encodes just the properties identical with G.[71] Notice that our conditions $\varphi(F, G)$ will either be conditions that partition properties F with respect to the ordinary objects that exemplify F or it

[71] See Zalta [1983] for the treatment of Forms, and Zalta [1995] for the treatment of Leibnizian concepts.

will partition F with respect to the abstract objects that encode F. There are no (encoding) conditions $\varphi(F,G)$ that partition the properties F with respect to both the abstract and ordinary objects that encode F (since it is axiomatic that ordinary objects do not encode properties), nor are there exemplification conditions $\varphi(F,G)$ that partition properties F with respect to the abstract and ordinary objects that exemplify F (since such conditions constitute relations and it is provable that for each relation R, there is a pair of abstract objects the R-relational properties to which cannot be distinguished).

Second, we may formulate equivalence conditions and assert the existence of equivalence relations at higher types to yield higher order Fregean logical objects. Recall the classifications of the type theory developed in §14. Consider the type $\langle i\rangle$. This is the type associated with properties of individuals. Let $x^{\langle i\rangle}$ be a variable ranging over objects with this type. Now suppose that $F^{\langle\langle i\rangle\rangle}$ and $G^{\langle\langle i\rangle\rangle}$ are variables that range over properties of properties of individuals. Then consider a condition $\varphi(F^{\langle\langle i\rangle\rangle},G^{\langle\langle i\rangle\rangle})$ that is provably an equivalence condition on properties of properties of individuals. φ need not be restricted to just the ordinary properties with type $\langle i\rangle$ exemplifying F; however, for reasons that we have already encountered, φ will partition the domain of properties of properties either on the basis of the ordinary properties that exemplify such properties of properties or on the basis of the abstract properties that encode such properties of properties. Our typed theory of abstract objects then asserts that, for any property of properties $G^{\langle\langle i\rangle\rangle}$, there is a unique abstract property with type $\langle i\rangle$ that encodes just the properties $F^{\langle\langle i\rangle\rangle}$ such that $\varphi(F^{\langle\langle i\rangle\rangle},G^{\langle\langle i\rangle\rangle})$:

$$\exists!x^{\langle i\rangle}(A!^{\langle\langle i\rangle\rangle} \;\&\; \forall F^{\langle\langle i\rangle\rangle}(xF \equiv \varphi(F^{\langle\langle i\rangle\rangle},G^{\langle\langle i\rangle\rangle}))$$

Clearly, then, we may define a special abstract property $\hat{G}$ with type $\langle i\rangle$ and prove that the Fregean biconditional principle holds for such objects:

$$\hat{F}=\hat{G} \equiv \varphi(F^{\langle\langle i\rangle\rangle},G^{\langle\langle i\rangle\rangle})$$

Except for the fact that we have moved from type i to type $\langle i\rangle$, this procedure is analogous to the definitions of $\{u|Gu\}$, $\#_G$, Φ_G, and c_G.

Moreover, let us assume that $R^{\langle\langle i\rangle,\langle i\rangle\rangle}$ is an equivalence relation that holds between *ordinary* properties $u^{\langle i\rangle}$ and $v^{\langle i\rangle}$. Recall that when u is an ordinary property with type $\langle i\rangle$, it will have a haecceity $[\lambda z^{\langle i\rangle}\, z\!=_E\!u]$ (where the identity sign $=_E$ is a relation with type $\langle\langle i\rangle,\langle i\rangle\rangle$). This haecceity exists by typed comprehension and it is logically well-behaved. Then for any ordinary property $v^{\langle i\rangle}$, there is a unique abstract property with type $\langle i\rangle$ that encodes just the haecceities of ordinary properties u that bear R to v:

$$\exists!x^{\langle i\rangle}(A!^{\langle\langle i\rangle\rangle} \,\&\, \forall F^{\langle\langle i\rangle\rangle}(xF \equiv \exists u^{\langle i\rangle}(Ruv \,\&\, F = [\lambda z\; z =_E u])))$$

So we may define a special abstract property $\hat{v}$ with type $\langle i\rangle$ and prove the Fregean biconditional principle for $\hat{v}$. In other words, this procedure is entirely analogous to the definition of $\vec{a}$ and $\breve{b}$, except that we have moved from type i to type $\langle i\rangle$.

The foregoing constructions then generalize to any type t. This establishes a type-theoretic Fregean Platonism of higher-order logical objects.

Appendix: Proofs of Selected Theorems

(25): Suppose there is a proof of φ from the set Γ in which φ depends only on modal axioms and formulas modal with respect to Γ. We show by induction on the length of a proof that there is a proof of $\Box\varphi$ from the set Γ. Then if the proof of φ is one line, φ must be either a logical axiom or an element of Γ. If φ is a logical axiom, it must be a modal axiom and so its modal closure $\Box\varphi$ is an axiom. So $\Gamma \vdash \Box\varphi$. If φ is an element of Γ, then φ must be a formula modal with respect to Γ (otherwise, since every formula depends on itself, the proof of φ would depend on a formula not modal with respect to Γ, contrary to hypothesis). But if φ is modal with respect to Γ, then $\Box\varphi$ is in Γ. So $\Gamma \vdash \Box\varphi$. If the proof of φ is more than one line, then either φ was derived from previous lines ψ and $\psi \rightarrow \varphi$ by MP or φ $(= \forall\alpha\psi)$ was derived from a previous line ψ by GEN. If the former, then by induction hypothesis, $\Gamma \vdash \Box\psi$ and $\Gamma \vdash \Box(\psi \rightarrow \varphi)$. But then, by the K axiom (12.1), $\Gamma \vdash \Box\varphi$. If the latter, then by inductive hypothesis, $\Gamma \vdash \Box\psi$, and by GEN, $\Gamma \vdash \forall\alpha\Box\psi$. So by the Barcan Formula (12.4), $\Gamma \vdash \Box\forall\alpha\psi$, i.e., $\Gamma \vdash \Box\varphi$. $\bowtie$

(26): Assume $\forall\alpha\varphi$. By (21), it follows that φ, and so by conditional proof: $\forall\alpha\varphi \rightarrow \varphi$. By RN, $\Box(\forall\alpha\varphi \rightarrow \varphi)$. By the K axiom, $\Box\forall\alpha\varphi \rightarrow \Box\varphi$. By GEN, $\forall\alpha(\Box\forall\alpha\varphi \rightarrow \Box\varphi)$. But by (10.4), $\Box\forall\alpha\varphi \rightarrow \forall\alpha\Box\varphi$. $\bowtie$

(27): By propositional logic, either $\Diamond E!x \vee \neg\Diamond E!x$. If $\Diamond E!x$, then by (15.1), $[\lambda y\, \Diamond E!y]x$, i.e., $O!x$, by (5.1). If $\neg\Diamond E!x$, then $[\lambda y\, \neg\Diamond E!y]x$, i.e., $A!x$, by (5.2). So, either $O!x \vee A!x$. $\bowtie$

(29.2): Suppose $O!x$, $O!y$, and $\forall F(Fx \equiv Fy)$. To show $x =_E y$, we simply have to show that $\Box\forall F(Fx \equiv Fy)$. But, for reductio, suppose not, i.e., suppose $\Diamond\neg\forall F(Fx \equiv Fy)$. Without loss of generality, suppose $\Diamond\exists F(Fx \,\&\, \neg Fy)$. Then, by the Barcan formula, $\exists F\Diamond(Fx \,\&\, \neg Fy)$. Say P, for example, is our property is such that $\Diamond(Px \,\&\, \neg Py)$. Now, consider the property: $[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]$. We know by λ-Conversion that:

$$[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]x \equiv \Diamond(Px \,\&\, \neg Py)]$$

But we know the right hand side of this biconditional, and so it follows that: $[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]x$. But it is also a consequence of λ-Conversion that:

$$[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]y \equiv \Diamond(Py \,\&\, \neg Py)]$$

But clearly, by propositional modal logic, $\neg\Diamond(Py \,\&\, \neg Py)$. So we may conclude: $\neg[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]y$. So we have established:

$$[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]x \,\&\, \neg[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]y$$

So, by EG, $\exists F(Fx \& \neg F)$, which contradicts our hypothesis $\forall F(Fx \equiv Fy)$. $\bowtie$

(30.1): (a) Suppose α is an object variable, say x. To show $x = x$, we have to prove the definens of (7) with x substituted for the variable y. Well, by (27), either $O!x \vee A!x$. If $O!x$, then by propositional logic, $Fx \equiv Fx$, and so by GEN, $\forall F(Fx \equiv Fx)$, and so by RN, $\Box\forall F(Fx \equiv Fx)$. So we have proved the left disjunct of (7). If $A!x$, then by similar reasoning, we have $\Box\forall F(xF \equiv xF)$, and thus, the right hand disjunct of (7). So $x = x$. (b) Suppose α is a 1-place property variable, say F. We have to prove the definiens of (8.1), with the variable F substituted for the variable G. But, by quantified modal logic, $\Box\forall x(xF \equiv xF)$. So $F = F$. (c) Suppose α is an n-place relation variable, $n \geq 2$, say F^n. (Exercise) (d) Suppose α is a 0-place relation variable, say p. (Exercise) $\bowtie$

(30.2): Assume $\alpha = \beta$. Let $\varphi(\alpha, \alpha)$ be $\alpha = \alpha$ and $\varphi(\alpha, \beta)$ be $\beta = \alpha$. Then by (11), we know:

$$\alpha = \beta \rightarrow (\alpha = \alpha \rightarrow \beta = \alpha)$$

But our hypothesis is that $\alpha = \beta$. And by (30.1), $\alpha = \alpha$. So $\beta = \alpha$. $\bowtie$

(30.3): Assume $\alpha = \beta$ and $\beta = \gamma$. Let $\varphi(\beta, \beta)$ be $\beta = \gamma$ and let $\varphi(\beta, \alpha)$ be $\alpha = \gamma$. Then by (11) (with α and β interchanged), we know:

$$\beta = \alpha \rightarrow (\beta = \gamma \rightarrow \alpha = \gamma)$$

But by (30.2), $\alpha = \beta \rightarrow \beta = \alpha$. So, given our two hypotheses, we may conclude $\alpha = \gamma$. $\bowtie$

(32):

(a) Suppose α, β are object variables, say x and y. We take $x = y$ as hypothesis and prove $\Box x = y$. If $x = y$, then by (7):

$$(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \;\vee\; (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

Suppose the left disjunct is true. Then $O!x$, i.e., $[\lambda z \Diamond E!z]x$. But by (15.1), we know $\Diamond E!x$. By (12.3), it follows that $\Box\Diamond E!x$. But by applying RN to (15.1), we also know $\Box([\lambda z \Diamond E!z]x \equiv \Diamond E!x)$. Since we have both $\Box(\varphi \equiv \psi)$

and $\Box\psi$, we may conclude $\Box\varphi$, i.e., $\Box[\lambda z\, \Diamond E!z]x$, by propositional modal logic. So, by definition, $\Box O!x$. And by similar reasoning, $\Box O!y$. By the S5 law that $\Box\varphi \rightarrow \Box\Box\varphi$, we may conclude $\Box\Box\forall F(Fx \equiv Fy)$ from our hypothesis that $\Box\forall F(Fx \equiv Fy)$ (this is true in the left disjunct). So we have assembled the conjunction:

$$\Box O!x \,\&\, \Box O!y \,\&\, \Box\Box\forall F(Fx \equiv Fy)$$

So by propositional modal logic, it follows that:

$$\Box(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$$

On the other hand, suppose the right disjunct of our hypothesis is true. Then $A!x$, i.e., $[\lambda z\, \neg\Diamond E!z]x$, by (5.2). So by (15), $\neg\Diamond E!x$, i.e., $\Box\neg E!x$, by modal logic. So by S5, $\Box\Box\neg E!x$, i.e., $\Box\neg\Diamond E!x$. Since we know that $\Box([\lambda z\, \neg\Diamond E!z]x \equiv \neg\Diamond E!x)$, it follows that $\Box[\lambda z\, \neg\Diamond E!z]x$, i.e., $\Box A!x$. And by S5, we may infer $\Box\Box\forall F(xF \equiv yF)$ from $\Box\forall F(xF \equiv yF)$ (the latter being true in the right disjunct). So we may assemble the following conjunction:

$$\Box A!x \,\&\, \Box A!y \,\&\, \Box\Box\forall F(xF \equiv yF)$$

By modal logic, it follows that:

$$\Box(A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

So both of our original disjuncts are necessarily true. But given the modal law that $\Box\varphi \vee \Box\psi \rightarrow \Box(\varphi \vee \psi)$, we have established $\Box x = y$, by (7).

(b) Suppose α, β are property variables, say F and G. Our hypothesis is that $F = G$, i.e., $\Box\forall x(xF \equiv xG)$. But then, by S5, $\Box\Box\forall x(xF \equiv xG)$. So $\Box F = G$.

(c) Suppose α, β are n-place relation variables, $n \geq 2$, say F^n and G^n. (Exercise)

(d) Suppose α, β are proposition variables, say p and q. (Exercise) $\bowtie$

(33): ($\rightarrow$) If xF, then $\Diamond xF$, by propositional modal logic. But by (14), $\Box xF$. ($\leftarrow$) If $\Box xF$, then xF, by (12.2). $\bowtie$

(34.1): Suppose $A!x$, $A!y$, and $\forall F(xF \equiv yF)$. Then pick an arbitrary property, say P. So $xP \equiv yP$. But, by (33), $xP \equiv \Box xP$ and $yP \equiv \Box yP$. So $\Box xP \equiv \Box yP$. So, by propositional logic, either $\Box xP \,\&\, \Box yP$ or $\neg\Box xP \,\&\, \neg\Box yP$. If the former, then by modal logic, $\Box(xP \,\&\, yP)$. If the latter, by (14), $\neg\Diamond xP \,\&\, \neg\Diamond yP$, i.e., $\Box\neg xP \,\&\, \Box\neg yP$, which by modal logic, implies $\Box(\neg xP \,\&\, \neg yP)$. So either $\Box(xP \,\&\, yP)$ or $\Box(\neg xP \,\&\, \neg yP)$, which means that necessarily, xP and yP have the same truth value, i.e., $\Box(xP \equiv yP)$. But P was arbitrary, so $\forall F\Box(xF \equiv yF)$. Thus, by the Barcan formula, $\Box\forall F(xF \equiv yF)$. Since $A!x$ and $A!y$, it follows by (7) that $x = y$. ⋈

(34.2): Suppose $O!x$, $O!y$, and $\forall F(Fx \equiv Fy)$. To show $x = y$, we have to show that $\Box\forall F(Fx \equiv Fy)$. But, for reductio, suppose not, i.e., suppose $\Diamond\neg\forall F(Fx \equiv Fy)$. Without loss of generality, suppose $\Diamond\exists F(Fx \,\&\, \neg Fy)$. Then, by the Barcan formula, $\exists F\Diamond(Fx \,\&\, \neg Fy)$. Say P, for example, is our property such that $\Diamond(Px \,\&\, \neg Py)$. Now, consider the property: $[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]$. We know by (15.1) that:

$$[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]x \equiv \Diamond(Px \,\&\, \neg Py)]$$

But we know the right hand side of this biconditional, and so it follows that: $[\lambda z\, \Diamond(Pz \,\&\, \neg Py)]x$. But it is also a consequence of (15.1) that:

$$[\lambda z\, \Diamond Pz \,\&\, \neg Py]y \equiv \Diamond(Py \,\&\, \neg Py]$$

But clearly, by propositional modal logic, $\neg\Diamond(Py \,\&\, \neg Py)$. So we may conclude: $\neg[\lambda z\, \Diamond Pz \,\&\, \neg Py]y$. So we have established:

$$[\lambda z\, \Diamond Pz \,\&\, \neg Py]x \,\&\, \neg[\lambda z\, \Diamond Pz \,\&\, \neg Py]y$$

So, by EG, $\exists F(Fx \,\&\, \neg Fy)$, which contradicts our hypothesis $\forall F(Fx \equiv Fy)$. ⋈

(36): ($\rightarrow$) Assume $\exists!x\varphi$. Then, by definition, $\exists x(\varphi \,\&\, \forall y(\varphi^y_x \rightarrow y = x))$. So there is some object, say a. such that $\varphi^a_x \,\&\, \forall y(\varphi^y_x \rightarrow y = a)$. We want to show: $a = \imath x\varphi$. If we let ψ be the formula $a = z$, then $a = \imath x\varphi$ has the form $\psi^{\imath x\varphi}_z$, and so we may appeal to (16). We therefore have to show: $\exists x(\varphi \,\&\, \forall y(\varphi^y_x \rightarrow y = x) \,\&\, a = x)$. But, by (30.1), we know $a = a$. So we have:

$$\varphi^a_x \,\&\, \forall y(\varphi^y_a \rightarrow y = a) \,\&\, a = a$$

From which it follows that:

$$\exists x(\varphi \,\&\, \forall y(\varphi^y_x \rightarrow y = x) \,\&\, a = x)$$

So by (16), $a = \imath x\varphi$, and since a contains no descriptions, $\exists y(y = \imath x\varphi)$. So by EI, we are done. ($\leftarrow$) Reverse the reasoning. ⋈

(37.1): Assume $y = \imath x\varphi$. Then since this defined identity formula results from substituting $\imath x\varphi$ for z in $y = z$, we may apply (16) to conclude: $\exists x(\varphi \,\&\, \forall u(\varphi^u_x \rightarrow u = x) \,\&\, y = x)$. So there must be some object, say a, such that $\varphi^a_x \,\&\, \forall u(\varphi^u_x \rightarrow u = a) \,\&\, y = a$. If so, then since φ^a_x and $y = a$, we may apply (11) to conclude φ^y_x. $\bowtie$

(37.2): Assume $\exists y(y = \imath x\varphi)$. Then by (10.1), $\imath x\varphi$ may be substituted for y into the universal generalization of (37.1). So:

$$\imath x\varphi = \imath x\varphi \rightarrow \varphi^{\imath x\varphi}_x$$

But, by (10.1), $\imath x\varphi$ can also be instantiated into (30.1) to yield $\imath x\varphi = \imath x\varphi$. So $\varphi^{\imath x\varphi}_x$. $\bowtie$

(37.2): (Alternative) Assume $\exists y(y = \imath x\varphi)$. Then, there is some object, say a, such that $a = \imath x\varphi$. Then by (37.1) and (10.1), we know φ^a_x. But by (10.1), $\imath x\varphi$ is substitutable into the law of substitution (11) as follows:

$$a = \imath x\varphi \rightarrow (\varphi^a_x \equiv \varphi^{\imath x\varphi}_x)$$

But since $a = \imath x\varphi$ and φ^a_x, it follows that $\varphi^{\imath x\varphi}_x$. $\bowtie$

(40): Suppose $\forall x(xF \equiv yG)$. If we can establish that $\forall x\Box(xF \equiv xG)$, then an application of the Barcan Formula and an appeal to the definition of identity for properties completes the proof. So pick an arbitrary object, say a (to show: $\Box(aF \equiv aG)$). So given our initial assumption, we know $aF \equiv aG$. But, by (33), $aF \equiv \Box aF$ and $aG \equiv \Box aG$. So $\Box aF \equiv \Box aG$. So, by propositional logic, either $\Box aF \,\&\, \Box aG$ or $\neg\Box aF \,\&\, \neg\Box aG$. If the former, then by modal logic, $\Box(aF \,\&\, aG)$ and so $\Box(aF \equiv aG)$ (given that $\Box[(\varphi \,\&\, \psi) \rightarrow (\varphi \equiv \psi)]$). If the latter, by (14), $\neg\Diamond aF \,\&\, \neg\Diamond aG$, i.e., $\Box\neg aF \,\&\, \Box\neg aG$, which by modal logic, implies $\Box(\neg aF \,\&\, \neg aG)$. So again, $\Box(aF \equiv aG)$ (since $\Box[(\neg\varphi \,\&\, \neg\psi) \rightarrow (\varphi \equiv \psi)]$). But a was arbitrary, so $\forall x\Box(xF \equiv xG)$. Thus, by the Barcan formula, $\Box\forall x(xF \equiv yG)$. So by (8.1), it follows that $F = G$. $\bowtie$

(45.1): Consider an arbitrary proposition, say p_1. Then, since $[\lambda y\, p]$ contains no descriptions, we know that $[\lambda y\, p_1] = [\lambda y\, p_1]$. But then, we may also apply EG to conclude: $\exists F(F = [\lambda y\, p_1])$. And since p_1 was arbitrary, $\forall p\exists F(F = [\lambda y\, p])$. $\bowtie$

(50): Pick φ. Then by (48), there is an object, say a, such that $A!a \& \forall F(aF \equiv \varphi)$. Now we have to show that $\forall y[(A!y \,\&\, \forall F(yF \equiv \varphi) \rightarrow y = a]$. Pick an arbitrary object, say b and assume $A!b \,\&\, \forall F(bF \equiv \varphi)$. We want to show that $b\!=\!a$. But, clearly, $\forall F(aF \equiv bF)$, and so by (34.1), $b\!=\!a$. ⋈

(51): In the statements of (50) and (36), replace φ by ψ. Then let ψ be $A!x \,\&\, \forall F(xF \equiv \varphi))$ and apply the revised versions of (50) and (36). ⋈

(52): ($\rightarrow$) Assume $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$. Note that this is an atomic encoding formula of the form zG in which z has been replaced by the description $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$. So we may apply (16) and conclude that some object, say a, is such that:

$$A!a \,\&\, \forall F(aF \equiv \varphi) \,\&\, \forall y(A!y \,\&\, \forall F(yF \equiv \varphi) \rightarrow y\!=\!a) \,\&\, aG$$

But from this fact about a, we both know: (1) $\forall F(aF \equiv \varphi)$, and so in particular, $aG \equiv \varphi^G_F$ and (2) aG. So φ^G_F. ($\leftarrow$) Assume φ^G_F. We know, by (50) that there is an object, say a, such that

$$A!a \,\&\, \forall F(aF \equiv \varphi) \,\&\, \forall y(A!y \,\&\, \forall F(yF \equiv \varphi) \rightarrow y\!=\!a)$$

But since $\forall F(aF \equiv \varphi)$, we know that $aG \equiv \varphi^G_F$. Since φ^G_F by hypothesis, aG. So we now know:

$$A!a \,\&\, \forall F(aF \equiv \varphi) \,\&\, \forall y(A!y \,\&\, \forall F(yF \equiv \varphi) \rightarrow y\!=\!a) \,\&\, aG$$

So by (16), $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$. ⋈

(55.1): Pick an arbitrary relation R. Consider the following instance of abstraction:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(F\!=\![\lambda z\, Rzy] \,\&\, \neg yF)))$$

Call such an object a. So we know the following about a:

$$\forall F(aF \equiv \exists y(F\!=\![\lambda z Rzy] \,\&\, \neg yF))$$

Now consider the property $[\lambda z\, Rza]$ and ask the question whether a encodes this property. Assume, for reductio, $\neg a[\lambda z Rza]$. Then, by definition of a, for any object y, if the property $[\lambda z\, Rza]$ is identical with the property $[\lambda z\, Rzy]$, then y encodes $[\lambda z\, Rza]$. Since the property $[\lambda z\, Rza]$ is self-identical, it follows that a encodes $[\lambda z\, Rza]$, contrary to assumption. So $a[\lambda z\, Rza]$. So by the definition of a, there is an object, say b, such that the property $[\lambda z\, Rza]$ is identical to the property $[\lambda z\, Rzb]$ and such that b doesn't encode $[\lambda z\, Rza]$. But since a encodes, and b does not encode,

$[\lambda z\, Rza]$, $a \neq b$. So there are objects x and y such that $x \neq y$, yet such that $[\lambda z\, Rzx] = [\lambda z\, Rzy]$. $\bowtie$

(55.2): By analogous reasoning. $\bowtie$

(55.3): Pick an arbitrary property P. Consider the following instance of abstraction:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(F = [\lambda z\, Py] \,\&\, \neg yF)))$$

By reasoning analogous to the above, it is straightforward to establish that there are distinct abstract objects k, l such that $[\lambda z\, Pk]$ is identical to $[\lambda z\, Pl]$. But, then by the definition of proposition identity ($p = q \quad =_{df}$ $[\lambda z\, p] = [\lambda z\, q]$), it follows that the proposition $[\lambda\, Pk]$ is identical to the proposition $[\lambda\, Pl]$. $\bowtie$

(56): Let R_0 be the relation $[\lambda xy\, \forall F(Fx \equiv Fy)]$. We know from the previous theorems that, for any relation R, there exist distinct abstract objects a, b such that $[\lambda z\, Rza] = [\lambda z\, Rzb]$. So, in particular, there are distinct abstract objects a, b such that $[\lambda z\, R_0za] = [\lambda z\, R_0zb]$. But, by the definition of R_0, it is easily provable that R_0aa, from which it follows that $[\lambda z\, R_0za]a$. But, then, $[\lambda z\, R_0zb]a$, from which it follows that R_0ab. Thus, by definition of R_0, $\forall F(Fa \equiv Fb)$.

(74.1): The comprehension principle for abstract objects is:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)),$$

where φ has no free xs. But the conditions φ on propositional properties constitute a subset of the conditions that may be used in this comprehension schema for abstract objects. Moreover, any object 'generated' by such a condition on propositional properties encodes only propositional properties, and will therefore be a situation. $\bowtie$

(74.2): Use the same reasoning in the proof of (50). $\bowtie$

(77): ($\leftarrow$) Our hypothesis is that the same propositions are true in both s and s', and we want to show that these two situations are identical. Since both s and s' are situations, and hence abstract objects, to show that they are identical, we must show that necessarily, they encode the same properties. We reason by showing, for an arbitrary property Q, that $sQ \equiv s'Q$, for then by universal generalization and (34.1), we are done.

($\rightarrow$) Assume sQ. Then since s is a situation, Q must be a propositional property, say $[\lambda y\, q]$ (for some state of affairs q). So s encodes $[\lambda y\, q]$, and by the definition of 'true in', q is true in s ($\models_s q$). But our initial hypothesis is that the same propositions are true in s and s', and so $\models_{s'} q$, i.e., $s'[\lambda y\, q]$. So s' encodes the property Q. ($\leftarrow$) Reverse reasoning. $\bowtie$

(79.1): Suppose x is a part of situation s and that x encodes G (to show G is a propositional property). Then, since s encodes every property x encodes, s encodes G. But since s is a situation, every property it encodes is a propositional property. So G is a propositional property. $\bowtie$

(79.2): ($\rightarrow$) Assume s is a part of s' and that $\models_s q$ (to show $\models_{s'} q$). By definition of $\models_s q$, we have $s[\lambda y\ q]$. Since every property encoded by s is encoded by s', $s'[\lambda y\ q]$, i.e., $\models_{s'} q$. ($\leftarrow$) Assume $\forall p(\models_s p \rightarrow \models_{s'} p)$ and that s encodes G (to show s' encodes G). Since s is a situation, $G = [\lambda y\ q]$, for some proposition q, and so $\models_s q$. By hypothesis, then, $\models_{s'} q$, and so s' encodes G. $\bowtie$

(80.1): ($\leftarrow$) If s and s' are both parts of each other, then they encode exactly the same properties. So by the definition of identity for abstract objects, they are identical. $\bowtie$

(80.2): ($\leftarrow$) Suppose that s and s' have the same parts. To show that s and s' are identical, we must show that they encode the same properties. So, without loss of generality, assume s encodes Q (to show: s' encodes Q). From the definition of part-of, we know that s is a part of itself, and so it follows from our first assumption that s is a part of s'. So s' encodes Q.

(81.2): To see that *part-of* is anti-symmetric, assume $s \trianglelefteq s'$ and $s \neq s'$. Then, there is a property s' that is not encoded in s. So, $\neg(s' \trianglelefteq s)$. $\bowtie$

(81.3): To see that *part-of* is transitive, assume $s \trianglelefteq s'$ and $s' \trianglelefteq s''$ and that s encodes property G (to show that s'' encodes G). Since s is a part of s', s' encodes G. Since s' is a part of s'', s'' encodes G. $\bowtie$

(83): Assume $\models_s p$ and that $s \trianglelefteq s'$. Then by (79.2), $\models_{s'} p$. $\bowtie$

(85.1): Consider the following two instances of (74.1):

$$\exists s \forall F(sF \equiv F = [\lambda y\ q])$$
$$\exists s \forall F(sF \equiv F = [\lambda y\ \neg q])$$

Now if q is true, then $\neg q$ isn't. So the first instance gives us an actual situation (in which q and no other proposition is true), while the second instance gives us a non-actual situation (in which $\neg q$ and no other proposition is true). However, if $\neg q$ is true, then q isn't. Then, the first instance gives us a non-actual situation whereas the second gives us an actual one. But either q or $\neg q$ is true. $\bowtie$

(85.2): Assume s is actual. Then $\forall p(\models_s p \rightarrow p)$. For *reductio*, assume that there is a proposition q such that both $\models_s q$ and $\models_s \neg q$. Then, since s is actual, both q and $\neg q$ are true, which is impossible. $\bowtie$

(85.3): By (42.2), for an arbitrary proposition q, there is a complex proposition $q \,\&\, \neg q$. Assume for an arbitrary situation s that s is actual and that $\models_s (q \,\&\, \neg q)$. Then by the actuality of s, $q \,\&\, \neg q$, which is impossible. So, for any actual situation s, if s is actual, $s \not\models (q \,\&\, \neg q)$. So there is a proposition p that is not true in any actual situation. $\bowtie$

(86.1): Assume s is actual and s makes p true. Since s is actual, p obtains. But, by λ-abstraction, necessarily, an object x exemplifies $[\lambda y\ p]$ iff p obtains (i.e., $[\lambda y\ p]x \equiv p$). So, in particular, s exemplifies $[\lambda y\ p]$. $\bowtie$

(86.2): From (86.1), the definition of $\models$, and the fact that situations encode only propositional properties. $\bowtie$

(87): By (74.1), there is a situation that encodes all and only the propositional properties F constructed out of propositions true in either s or s'; i.e.,

$$\exists s''\forall F[s''F \equiv \exists p((\models_s p \vee \models_{s'} p) \,\&\, F = [\lambda y\ p])]$$

Note that if s and s' are both actual, so is s''. $\bowtie$

(89): Consider the following two instances of (74.):

$$\exists s\forall F[sF \equiv \exists p(F = [\lambda y\ p])]$$
$$\exists s\forall F(sF \equiv F = [\lambda y\ q])$$

The first instance yields a situation ('s_1') in which every proposition is true. *A fortiori*, s_1 is maximal$_1$. The second instance yields a situation ('s_2') in which q is true. Then, for any proposition r such that $q \neq r$ and $q \neq \neg r$, neither r nor $\neg r$ is true in s_2. So s_2 is partial$_1$. Now consider the same two instances of (74.1) utilized in the previous proof. Situation s_1 is maximal$_2$, and s_2 is partial$_2$. $\bowtie$

(91): Assume s is possible. Then, $\Diamond\forall p(\models_s p \rightarrow p)$. For *reductio*, assume s is not consistent. Then, there is a proposition q such that both $\models_s q$ and $\models_s \neg q$. Note that by (76), these last two facts are necessary. Moreover, they are all that is needed to establish: $\forall p(\models_s p \rightarrow p) \rightarrow (q \,\&\, \neg q)$. Since this conditional is provable using only necessary truths, the rule of necessitation applies and yields: $\Box[\forall p(\models_s p \rightarrow p) \rightarrow (q \,\&\, \neg q)]$. But, by hypothesis, $\Diamond\forall p(\models_s p \rightarrow p)$. So it follows by a previously mentioned principle of modal logic that $\Diamond(q \;\&\; \neg q)$, which contradicts the law of modal logic that $\neg\Diamond(q \;\&\; \neg q)$. $\bowtie$

(95): Suppose s is a world. Then $\Diamond\forall p(\models_s p \equiv p)$. We first try to establish, for an arbitrary proposition q, that $\Diamond(\models_s q \vee \models_s \neg q)$, for then it will follow by (76) that $\models_s q \vee \models_s \neg q$, and hence that $Maximal_1(s)$. Now if we momentarily assume $\forall p(\models_s p \equiv p)$, we can use the fact that $\Box(q \vee \neg q)$ to establish that $\models_s q \vee \models_s \neg q$. So by conditional proof: $\forall p(\models_s p \equiv p) \rightarrow (\models_s q \vee \models_s \neg q)$. Since this conditional was proved without appealing to any contingencies, the rule of necessitation applies and we get: $\Box[\forall p(\models_s p \equiv p) \rightarrow (\models_s q \vee \models_s \neg q)]$. From this fact, and the original fact that $\Diamond\forall p(\models_s p \equiv p)$, we may apply the following well known theorem of modal logic: $\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$. Applying this theorem yields: $\Diamond(\models_s q \vee \models_s \neg q)$, which is our first objective.

From this fact, it follows that $\Diamond \models_s q \vee \Diamond \models_s \neg q$. But by (76.1) and (76.2), each disjunct gives us a nonmodal truth about s, and so it follows that $\models_s q \vee \models_s \neg q$. Since q was arbitrary, we have shown: $Maximal_1(s)$. $\bowtie$

(97.1): By the definition of a world, *a fortiori*, from the definition of a possible situation (90.1).

(97.2): Suppose s is a world. So by (97.1), it follows that s is possible. And by (91), it follows that s is consistent. $\bowtie$

(99): Consider the situation that encodes all and only those properties F constructed out of propositions that are true; i.e,

$$\exists s\forall F[sF \equiv \exists p(p \;\&\; F = [\lambda y\, p])]$$

Call such a situation 's_0.' It is straightforward to show that s_0 has the following feature for an arbitrary proposition q: $\models_{s_0} q \equiv q$. So, *a fortiori*, s_0 is both a world and actual. Now to see that there couldn't be two distinct actual worlds, suppose for *reductio* that s' is a distinct actual world. Since s' and s_0 are distinct, there must be a proposition q true in one but not in the other (by (77)). Suppose, without loss of generality, that $\models_{s_0} q$ and $\not\models_{s'} q$. Then since s' is a world, it is maximal$_1$. So $\models_{s'} \neg q$. But since both s_0 and s' are actual, both q and $\neg q$ must obtain, which is a contradiction. $\bowtie$

(100): ($\rightarrow$) Suppose s is actual and that q is a state of affairs true in s. Then q must be true. But since all and only the true propositions are true in w_α (by definition of w_α), q is true in w_α. So by (79.2), $s \trianglelefteq w_\alpha$. (←) By reverse reasoning. ⋈

(101): By definition of w_α. ⋈

(102.1): (→) By (86.2) and the fact that w_α is actual. (←) Suppose $[\lambda y\, p]w_\alpha$. Then, by λ-abstraction, p is true. So, by (101), $\models_{w_\alpha} p$. ⋈

(102.2): (→) Suppose p. Then by (101) and (102.1), $[\lambda y\, p]w_\alpha$. But let $q = [\lambda y\, p]w_\alpha$. Then, by (101), $\models_{w_\alpha} q$, i.e., $\models_{w_\alpha} [\lambda y\, p]w_\alpha$. (←) By reverse reasoning. ⋈

(103.1): (→) Assume $\Box q$. We want to show, for an arbitrarily chosen world w, that $\models_w q$. Since w is a world, $\Diamond\forall p(\models_w p \equiv p)$. Moreover, by appealing to $\Box q$, it is easy to establish: $\Box[\forall p(\models_w p \equiv p) \rightarrow \models_w q]$. Since we know $\Diamond\forall p(\models_w p \equiv p)$, it follows by now familiar reasoning that $\Diamond \models_w q$, and by (76), $\models_w q$. (←) Assume that $\forall w(\models_w q)$. Our strategy is to appeal to the K axiom (12.1), namely, $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, in which $\varphi = \forall w(\models_w q)$ and let $\psi = q$. So we simply need to show: (a) $\Box[\forall w(\models_w q) \rightarrow q]$ and (b) $\Box\forall w(\models_w q)$. (a) We prove this by applying the Rule of Necessitation to a theorem that we establish by Conditional Proof, namely, $\forall w(\models_w q) \rightarrow q$. So assume $\forall w(\models_w q)$. Suppose for reductio, however, that $\neg q$. We know by comprehension that:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists p(p \,\&\, F = [\lambda y\, p])))$$

Call such an object 'x_0'. It should be clear that x_0 is a world (it encodes all and only those propositional properties constructed out of true propositions). But, since by assumption $\neg q$, it follows that $x_0[\lambda y\, \neg q]$. Since x_0 is a world and worlds are maximal, it follows that $\neg x_0[\lambda y\, q]$. From this it follows that $\exists w(\not\models_w q)$, contrary to our initial assumption that $\forall w(\models_w q)$. So we have established:

$$\forall w(\models_w q) \rightarrow q$$

and since the proof does not depend on any instance of (16), we may apply the Rule of Necessitation to conclude:

$$\Box[\forall w(\models_w q) \rightarrow q].$$

(b) Pick an arbitrary world, say w_1 and assume that $\models_{w_1} q$. By the rigidity of encoding, it follows that $\Box(\models_{w_1} q)$. So by Conditional Proof:

$$\models_{w_1} q \rightarrow \Box(\models_{w_1} q)$$

and by Universal Generalization:

$$\forall w[\models_w q \rightarrow \Box(\models_w q)]$$

But from this conclusion and our initial assumption that $\forall w(\models_w q)$, it follows that $\forall w\Box(\models_w q)$. And by the Barcan Formula, it follows that $\Box\forall w(\models_w q)$. $\bowtie$

(103.2): By contraposition and modal negation of (103.1), applying the fact that if a $\neg p$ is true at w, then p fails to be true at w. $\bowtie$

(104): Consider proposition r, world w_1, and object a. Then, by (15.1), $\Box([\lambda y\, r]a \equiv r)$. ($\rightarrow$) Suppose $\models_w r$. Then, since $\Box(r \rightarrow [\lambda y\, r]a)$, it follows by (96) that $\models_w [\lambda y\, r]a$. ($\leftarrow$) By analogous reasoning. $\bowtie$

(113):[72] We prove the concepts in question encode the same properties. ($\leftarrow$) Assume $c_G{\oplus}c_H P$. We need to show:

$$\imath x\forall F(xF \equiv G{\Rightarrow}F \;\vee\; H{\Rightarrow}F)P$$

So by (52), we must show: $G{\Rightarrow}P \;\vee\; H{\Rightarrow}P$. By hypothesis, we know:

$$\imath x\forall F(xF \equiv c_G F \;\vee\; c_H F)P$$

But by (52), it follows that $c_G P \vee c_H P$. Again, for disjunctive syllogism, suppose $c_G P$. Then by definition of of c_G, it follows that:

$$\imath x\forall F(xF \equiv G{\Rightarrow}F)P$$

So by (52), we know $G \Rightarrow P$. And by similar reasoning, if $c_H P$, then $H \Rightarrow P$. So by disjunctive syllogism, it follows that $G \Rightarrow P \;\vee\; H \Rightarrow P$, which is what we had to show.

($\rightarrow$) Assume:

$$\imath x\forall F(xF \equiv G{\Rightarrow}F \;\vee\; H{\Rightarrow}F)P$$

. We want to show: $c_G{\oplus}c_H P$. By the definition of real sum, we have to show:

$$\imath x\forall F(xF \equiv c_G F \;\vee\; c_H F)P$$

[72] In the following proof, we shall assert that the sum concept $c_G \oplus c_H$ encodes property P by writing: $c_G{\oplus}c_H P$. This is an atomic encoding formula of the form xG with a complex object term substituted for the variable x.

By (52), we therefore have to show that $c_G P \vee c_H P$. By applying (52) to our hypothesis, though, we know:

$$G \Rightarrow P \vee H \Rightarrow P$$

So, for disjunctive syllogism, suppose $G \Rightarrow P$. Then, by (52):

$$\imath x \forall F(xF \equiv G \Rightarrow F)P$$

That is, by definition of the concept G, we know: $c_G P$. By similar reasoning, if $H \Rightarrow P$, then $c_H P$. So by our disjunctive syllogism, it follows that $c_G P \vee c_H P$, which is what we had to show. ⋈

(114.3):[73] We show that $(x \oplus y) \oplus z$ and $x \oplus (y \oplus z)$ encode the same properties. ($\rightarrow$) We first assume that $(x \oplus y) \oplus z$ encodes an arbitrary property, say P, and show that $x \oplus (y \oplus z)$ encodes P. Assume $(x \oplus y) \oplus zP$. Then, by definition of $\oplus$, we have:

$$\imath w \forall F(wF \equiv x \oplus yF \vee zF)P$$

This, by (52), entails:

$$x \oplus yP \vee zP$$

Expanding the left disjunct by the definition of $\oplus$, we have:

$$\imath u \forall F(uF \equiv xF \vee yF)P \vee zP$$

And reducing the left disjunct by applying (52), we have:

$$(xP \vee yP) \vee zP$$

This, of course, is equivalent to:

$$xP \vee (yP \vee zP)$$

Applying (52) in the reverse direction to this line, we obtain:

$$xP \vee \imath u \forall F(uF \equiv yF \vee zF)P$$

[73] In the proof, we express the facts that $(x \oplus y) \oplus z$ encodes P and that $x \oplus (y \oplus z)$ encodes P in our formal notation, respectively, as follows:

$$(x \oplus y) \oplus zP$$
$$x \oplus (y \oplus z)P$$

These are atomic encoding formulas of the form xP, where x is replaced by a complex term.

By definition of $\oplus$, this is equal to:

$$xP \vee y{\oplus}zP$$

And by another application of (52), this becomes:

$$\imath w\forall F(wF \equiv xF \vee y{\oplus}zF)P$$

At last, by definition of $\oplus$, we reach:

$$x{\oplus}(y{\oplus}z)P$$

So if $(x{\oplus}y){\oplus}z$ encodes P, so does $x{\oplus}(y{\oplus}z)$. ($\leftarrow$) To show that $(x{\oplus}y){\oplus}z$ encodes P given that $x{\oplus}(y{\oplus}z)$ encodes P, reverse the reasoning. $\bowtie$

(121.2): Suppose $x \preceq y$ and $x \neq y$. To show that $y \not\preceq x$, we need to find a property F such that $yF \,\&\, \neg xF$. But, if $x \neq y$, either there is a property x encodes y doesn't, or there is a property y encodes that x doesn't. But, since $x \preceq y$, it must be the latter. $\bowtie$

(125): ($\rightarrow$) Assume $x \preceq y$.

a) Suppose $x = y$. By the idempotency of $\oplus$, $x{\oplus}x = x$, in which case, $x{\oplus}x = y$. So, we automatically have $\exists z(x{\oplus}z = y)$.

b) Suppose $x \neq y$. Then since $x \preceq y$, we know there must be some properties encoded by y which are not encoded by x. Consider, then, the object that encodes just such properties; i.e., consider:

$$\imath u\forall F(uF \equiv yF \,\&\, \neg xF)$$

Call this object 'u' for short (we know such an object exists by the abstraction schema for abstract objects). We need only establish that $x{\oplus}u = y$, i.e., that $x{\oplus}u$ encodes the same properties as y. ($\rightarrow$) Assume $x{\oplus}uP$ (to show: yP). By definition of $\oplus$ and (52), it follows that $xP \vee uP$. If xP, then by the fact that $x \preceq y$, it follows that yP. On the other hand, if uP, then by definition of u, it follows that $yP \,\&\, \neg xP$. So in either case, we have yP. ($\leftarrow$) Assume yP (to show $x{\oplus}uP$). The alternatives are xP or $\neg xP$. If xP, then $xP \vee uP$, so by (52):

$$\imath z\forall F(zF \equiv xF \vee uF)P$$

So, by the definition of $\oplus$, we have $x{\oplus}uP$. Alternatively, if $\neg xP$, then we have $yP \,\&\, \neg xP$. So by definition of u, uP, and by familiar reasoning, it follows that $x{\oplus}uP$. Combining both directions of our biconditional, we have established that $x \oplus u P$ iff yP, for an arbitrary P. So $x \oplus u = y$, and we therefore have $\exists z(x \oplus z = y)$.

($\leftarrow$) Assume $\exists z(x \oplus z = y)$. Call such an object '$u$'. To show $x \preceq y$, assume xP (to show yP). Then, $xP \vee uP$, which by (52) and the definition of $\oplus$, entails that $x \oplus uP$. But by hypothesis, $x \oplus u = y$. So yP. $\bowtie$

(126): ($\rightarrow$) Assume $x \preceq y$. So $\forall F(xF \rightarrow yF)$. To show that $x \oplus y = y$, we need to show that $x \oplus y$ encodes a property P iff y does. ($\rightarrow$) So assume $x \oplus y$P. Then, by definition,

$$\imath z \forall F(zF \equiv xF \ \vee \ yF)P$$

By (52), it then follows that $xP \ \vee \ yP$. But if xP, then by the fact that $x \preceq y$, it follows that yP. So both disjuncts lead us to conclude yP. ($\leftarrow$) Assume yP. Then $xP \ \vee \ yP$. So by (52),

$$\imath z \forall F(zF \equiv xF \ \vee \ yF)P$$

In other words, $x \oplus yP$.

($\leftarrow$) Assume that $x \oplus y = y$.[74] To show that $x \preceq y$, assume, for an arbitrary property P, that xP (to show: yP). Then $xP \ \vee \ yP$. By (52), it follows that:

$$\imath z \forall F(zF \equiv xF \ \vee \ yF)P$$

By definition of $\oplus$, it follows that $x \oplus yP$. But given that $x \oplus y = y$, it follows that yP. $\bowtie$

(131): Suppose x appears at w. So x is realized by some ordinary object, say b, at w; i.e., $\forall F(\models_w Fb \equiv xF)$. We want to show, for an arbitrary proposition q, that $x[\lambda y q]$ iff $w[\lambda y q]$. ($\rightarrow$) Assume $x[\lambda y q]$. So, by definition of b, $\models_w [\lambda y \, q]b$. And by (104), it follows that $\models_w q$, i.e., $w[\lambda y \, q]$. ($\leftarrow$) Reverse the reasoning. $\bowtie$

(133): Assume m is a monad. We want to show $\exists! w(\mathit{Appears}(m, w))$, for then w_m will be well-defined. Since m is a monad, then by definition, it appears at some world, say, w_1. For reductio, assume that m also appears at w_2, $w_2 \neq w_1$. Since the worlds are distinct, there must be some proposition true at one but not the other (by the definition of abstract object identity and the fact that worlds only encode properties of the

[74] If we allow ourselves an appeal to (125), we are done. For it follows from this that $\exists z(x \oplus z = y)$, which by (125), yields immediately that $x \preceq y$.

form $[\lambda y\, p]$, for some proposition p). So without loss of generality, assume that $\models_{w_1} p$ and $\not\models_{w_2} p$. Since worlds are maximal (by a theorem of world-theory), it follows that $\models_{w_2} \neg p$. But, by the previous theorem, m mirrors w_1, since it appears there. So since $\models_{w_1} p$, we know $m[\lambda y\, p]$. But m also mirrors w_2, since it appears there as well. So, from our last fact, it follows that $\models_{w_2} p$. This contradicts the consistency of w_2. ⋈

(139): Suppose m is a monad. Then m is realized by some ordinary object b at some world w_1. Consider an arbitrary property P. It is necessary that $Pb \vee \neg Pb$. So by (103.1), $\models_{w_1} Pb \vee \neg Pb$. But note that since $\Box(\bar{P}b \equiv \neg Pb)$ (by (15.1), we also have $\models_{w_1} \bar{P}b \equiv \neg Pb$. But by the closure of worlds (96), we know $\models_{w_1} Pb \vee \bar{P}b$. But since m is realized by b at w_1, $xP \vee x\bar{P}$. And since P was arbitrary, m is complete. ⋈

(141): Suppose x is a monad. To show that x is an individual concept, pick an arbitrary world, say w_1, and assume that there are ordinary objects, say b and c, by which x is realized at w_1. For reductio, suppose that $c \neq b$. Then since b and c are distinct ordinary objects, we know that both $b =_E b$ and $c =_E c$, by (29.1). From (28.2), it is clear that the following is a fact:

$$x =_E x \rightarrow \Box(x =_E x)$$

It therefore follows that $\Box(b =_E b)$ and $\Box(c =_E c)$. We also know, by (15.1), that:[75]

$$\Box([\lambda y\, y =_E x]x \equiv x =_E x)$$

So, in particular, $\Box([\lambda y\, y =_E b]b \equiv b =_E b)$, and a similar claim holds for c. So by a simple principle of modal logic, we know both $\Box[\lambda y\, y =_E b]b$ and $\Box[\lambda y\, y =_E c]c$. Now by (103.1), it follows that $\models_{w_1} [\lambda y\, y =_E b]b$ and $\models_{w_1} [\lambda y\, y =_E c]c$. But, if the former, then since x is realized by b at w_1, we know that $x[\lambda y\, y =_E b]$. But since x is realized by c at w_1, $[\lambda y\, y =_E b]c$. Now by (103.1) and the above consequence of (15.1), we also know that $\models_{w_1} [\lambda y\, y =_E b]c \equiv c =_E b$. So since worlds are closed under necessary implication, by (96), it follows that $c =_E b$. But, by the definition of identity, if $c =_E b$, then $c = b$, which contradicts our hypothesis. ⋈

(145.2): Let a be any ordinary object and Q be an arbitrary property. $(\rightarrow)$ Assume $c_a Q$. To show $c_a \succeq c_Q$, we need to show that $\forall F(c_Q F \rightarrow$

[75]Since $=_E$ is a primitive relation symbol, we may use it to construct complex relations. So $[\lambda y\, y =_E x]$ is an acceptable λ-expression.

c_aF). So assume c_QP, for an arbitrary property P. Then by definition of c_Q and (51), we know $Q \Rightarrow P$. Note that we have assumed c_aQ, and that it follows by definition of c_a and (51) that Qa. So from Qa and $Q \Rightarrow P$ it follows that Pa. So, again by the definition of c_a and (51), it follows that c_aP. ($\leftarrow$) Assume $c_a \succeq c_Q$ (to show: c_aQ). But by *A-descriptions* and the fact that $Q \Rightarrow Q$, it follows that:

$$\imath z\forall F(zF \equiv Q \Rightarrow F)Q$$

So, by definition of c_Q, c_QQ. But, by hypothesis, $c_a \succeq c_Q$, and so it follows that c_aQ. $\bowtie$

(150.1): Assume a is an ordinary object. Then c_a exists. Pick an arbitrary property, say P. By definition of the concept a, $c_aP \equiv Pa$. But, it is a consequence of the definition of the actual world w_α that $\models_{w_\alpha} Pa \equiv Pa$. So by properties of the biconditional, $\models_{w_\alpha} Pa \equiv c_aP$. Since P was arbitrary, it follows that $\mathit{RealizedBy}(c_a, a, w_\alpha)$. So c_a appears at w_α, and hence, is a monad. $\bowtie$

(152.1): From the fact that u fails to exemplify $[\lambda z\, z =_E v]$. $\bowtie$

(152.2): From the fact that no ordinary object can exemplify both $[\lambda z\, z =_E u]$ and $[\lambda z\, z =_E v]$. $\bowtie$

(160): Assume:

$$r = \imath x(\models_{CP} Sx \,\&\, \exists y(Oy \,\&\, My \,\&\, Kxy))$$

Then by (16):

$$\models_{CP} Sr \,\&\, \exists y(Oy \,\&\, My \,\&\, Kry)$$

Note also that from our rule of inference for relevance logic (158), we know:

$$Sr \,\&\, \exists y(Oy \,\&\, My \,\&\, Kry) \vdash_{\mathrm{R}} [\lambda x\, Sx \,\&\, \exists y(Oy \,\&\, My \,\&\, Kxy)]r$$

So from our last two results, we may conclude, by (157), that:

$$\models_{CP} [\lambda x\, Sx \,\&\, \exists y(Oy \,\&\, My \,\&\, Kxy)]r$$

So, $\exists F(\models_{CP} Fr)$, which by (159), implies $\mathit{Character}(r, \mathit{CP})$. $\bowtie$

(184.1): Eliminating the restricted variable, our theorem says that if y is an ordinary object, then y is a member of $\{u|Gu\}$ iff Gy. So assume

$O!a$, where a is an arbitrary ordinary object. ($\rightarrow$) Assume $a \in \{u|Gu\}$. We want to show Ga. By the definition of membership (182), we know that there is a property, say H, such that $\{u|Gu\} = \{u|Hu\}\ \&\ Ha$. But, by the previous lemma (183.1), we know $\{u|Gu\}G$. So $\{u|Hu\}G$. So, by (183.2), G is materially equivalent to H with respect to the ordinary objects. But, then, since a is an ordinary object that exemplifies H, it also exemplifies G. ($\leftarrow$) Assume Ga. We want to show that $a \in \{u|Gu\}$. Since $\{u|Gu\}$ is self-identical, we know:

$$\{u|Gu\} = \{u|Gu\}\ \&\ Ga$$

So by generalization, it follows that:

$$\exists H(\{u|Gu\} = \{u|Hu\}\ \&\ Ha)$$

But given our original hypothesis that $O!a$, we have:

$$O!a\ \&\ \exists H(\{u|Gu\} = \{u|Hu\}\ \&\ Ha)$$

And by (182), it follows that $a \in \{u|Gu\}$. $\bowtie$

(184.2): By (184.1), we know that, for any property G, $O!y \rightarrow (y \in \{u|Gu\} \equiv Gy)$. We want to show, for an arbitrary property, say P, that $y \in \{u|Pu\}$ iff $O!y\ \&\ Py$. ($\rightarrow$) Assume $y \in \{u|Pu\}$ (to show $O!y$ and Py). Then by (182), $O!y$ and there is a property, say F, such that $\{u|Pu\} = \{u|Fu\}\ \&\ Fy$. But since y is ordinary, we may apply (184.1) to yield: $y \in \{u|Pu\} \equiv Py$. And since $y \in \{u|Pu\}$ by assumption, Py. ($\leftarrow$) Assume $O!y$ and Py (to show $y \in \{u|Pu\}$). Again by (184.1), we know that $y \in \{u|Pu\} \equiv Py$. So $y \in \{u|Pu\}$. $\bowtie$

(185): ($\rightarrow$) Assume $\{u|Fu\} = \{u|Gu\}$. To show $\forall v(Fv \equiv Gv)$, we pick an arbitrary ordinary object, say a, and show $Fa \equiv Ga$. ($\rightarrow$) Assume Fa. Then, by (184.1), $a \in \{u|Fu\}$. But then $a \in \{u|Gu\}$, and so Ga, again by (184.1). ($\leftarrow$) By analogous reasoning. ($\leftarrow$) Assume $\forall v(Fv \equiv Gv)$. To show $\{u|Fu\} = \{u|Gu\}$, we pick an arbitrary property, say P, and show that $\{u|Fu\}P$ iff $\{u|Gu\}P$. ($\rightarrow$) So assume $\{u|Fu\}P$. Then by the definition of $\{u|Fu\}$ (180), and its supporting definition (178), it follows that $\forall v(Fv \equiv Pv)$. So $\forall v(Gv \equiv Pv)$, and hence, by the definition of $\{u|Gu\}$, $\{u|Gu\}P$. ($\leftarrow$) By analogous reasoning. $\bowtie$

(186): Assume $\forall v(v \in a \equiv v \in b)$, where a, b are natural sets. Since a, b are both natural sets, there are properties Q, R such that $a = \{u|Qu\}$ and

$b = \{u|Ru\}$. So we know $\forall v(v \in \{u|Qu\} \equiv v \in \{u|Ru\})$. So, by a previous Lemma, $\forall v(Qv \equiv Rv)$. So by Basic Law V, $\{u|Qu\} = \{u|Ru\}$, i.e., $a = b$. $\bowtie$

(187): Consider arbitrarily chosen ordinary objects a and b. Consider the property $[\lambda y\ y =_E a \ \vee\ y =_E b]$. This property exists by property abstraction. Call this property H. Then there is a set $\{w|Hw\}$, by a previous theorem. We can prove that something w is in this set iff $w = a$ or $w = b$. $\bowtie$

(188): Consider arbitrarily chosen sets a and b. Then there are properties P and Q such that $a = \{v|Pv\}$ and $b = \{w|Qw\}$. Consider the property $[\lambda z\ Pz \ \vee\ Qz]$, which exists by property abstraction. Call this property H. We show that for any ordinary object u, $u \in \{w|Hw\}$ iff $u \in a$ or $u \in b$. This follows by the λ-conversion principle and the definition of membership. $\bowtie$

(189.1): Consider the property $[\lambda z\ u \neq_E u]$. This property exists by property abstraction. Call it H. Then, by λ-conversion and the definition of membership, $u \in \{v|Hv\}$ iff $u \neq_E u$. But no ordinary objects are non-self-identical. $\bowtie$

(190): Let $\varphi(z)$ be any formula without encoding subformulas or definite descriptions and in which the variable z may or may not be free. Then $[\lambda z\ \varphi]$ denotes a property. So $\{u \,|\, [\lambda z\ \varphi]u\}$ exists, by (179), (180), and (36). So by (184.2):

$$y \in \{u \,|\, [\lambda z\ \varphi]u\} \equiv O!y \ \&\ [\lambda z\ \varphi]y,$$

where y is any object. By λ-conversion, it then follows that:

$$y \in \{u \,|\, [\lambda z\ \varphi]u\} \equiv O!y \ \&\ \varphi^y_z$$

So by universally generalizing on y and existentially generalizing on our set abstract, it follows that there is a natural set that contains as members all and only those ordinary objects that are such that φ:

$$\exists x \forall y (y \in x \equiv O!y \ \&\ \varphi^y_z)$$

$\bowtie$

(194.1): Pick an arbitrary property P. To show that equinumerosity$_E$ is reflexive, we must find a relation that is a one-to-one function from the ordinary objects of P onto the ordinary objects of P. However, we need look no further than the relation $=_E$. We have to show: (a) that $=_E$ is a function from the ordinary objects of P to the ordinary objects of P, and (b) that $=_E$ is a one-to-one function from the ordinary objects of P onto the ordinary objects of P. To show (a), pick an arbitrary ordinary object, say b, such that Pb. We need to show that there is an ordinary object v which is such that $\forall w(Pw \,\&\, b =_E w \equiv w =_E v)$. But b is such a v, for pick an arbitrary ordinary object, say c. ($\rightarrow$) If $Pc \,\&\, b =_E c$, then $c =_E b$. ($\leftarrow$) If $c =_E b$, then since Pb by assumption, we know Pc. So $Pc \,\&\, b =_E c$. Therefore, since c was arbitrary, we know $\forall w(Pw \,\&\, b =_E w \equiv w =_E b)$, and so there is an object v such that $\forall w(Pw \,\&\, b =_E w \equiv w =_E v)$. This demonstrates (a). To demonstrate (b), we need only show that $=_E$ is a one-to-one function from the ordinary objects of P to the ordinary objects of P, for by previous reasoning, we know that $=_E$ is a function from the ordinary objects of P onto the ordinary objects of P (i.e., we already know that every ordinary object exemplifying P bears $=_E$ to an ordinary object exemplifying P). For reductio, suppose that $=_E$ is not one-to-one, i.e., that there are distinct ordinary objects P which bear $=_E$ to some third P-object. But this is impossible, given that $=_E$ is a classical equivalence relation. $\bowtie$

(194.2): To show that that equinumerosity$_E$ is symmetric, assume that $P \approx_E Q$ and call the relation that is witness to this fact R. We want to show that there is a relation R' from Q to P such that (a) $\forall u(Qu \rightarrow \exists!v(Pv \,\&\, R'uv))$, and (b) $\forall u(Pu \rightarrow \exists!v(Qv \,\&\, R'vu))$. Consider the converse of R: $[\lambda xy\, Ryx]$, which we may call R^{-1}. We need to show that (a) and (b) hold for R^{-1}. To show (a) holds for R^{-1}, pick an arbitrary ordinary object, say b, such that Qb. We want to show that there is a unique ordinary object exemplifying P to which b bears R^{-1}. By the definition of R and the fact that Qb, we know that there is a unique ordinary object that exemplifies P and bears R to b. But such an object bears R to b iff b bears R^{-1} to it. So there is a unique ordinary object exemplifying P to which b bears R^{-1}. To prove that (b) holds for R^{-1}, the reasoning is analogous: consider an arbitrary object, say a, that exemplifies P. We want to show that there is a unique object exemplifying Q that bears R^{-1} to a. But by the definition of R and the fact that a exemplifies P, we know that there is a unique object that exemplifies Q and to which a bears R. But then, by the definition of R^{-1}, there is a unique object exemplifying Q that bears R^{-1} to a. $\bowtie$

(194.3): To show that equinumerosity$_E$ is transitive, assume both that $P \approx_E Q$ and $Q \approx_E S$. Call the relations that bear witness to these facts R_1 and R_2, respectively. Consider the relation: $[\lambda xy \exists z(Qz \& R_1xz \& R_2zy)]$. Call this relation R. To show that R bears witness to the equinumerosity$_E$ of P and S, we must show: (a) R is a function from the ordinary objects of P to the ordinary objects of S, and (b) R is a one-to-one function from the ordinary objects of P onto the ordinary objects of S. To show (a), consider an arbitrary ordinary object, say a, such that Pa. We want to find a unique ordinary object exemplifying S to which a bears R. To find such an object, note that given the equinumerosity$_E$ of P and Q, it is a fact about R_1 that there is a unique ordinary object exemplifying Q, say b, to which a bears R_1. And from the equinumerosity$_E$ of Q and S, it is a fact about R_2 that there is a unique ordinary object exemplifying S, say c, to which b bears R_2. So if we can show that c is a unique ordinary object exemplifying S to which a bears R, we are done. Well, by definition, c exemplifies S. By the definition of R, we can establish Rac if we can show $\exists z(Qz \, \& \, R_1az \, \& \, R_2zc)$. But since b is such a z, it follows that Rac. So it remains to prove that any object exemplifying S to which a bears R just is (identical$_E$ to) c. So pick an arbitrary ordinary object, say d, such that both d exemplifies S and Rad. We argue that $d =_E c$ as follows. Since Rad, we know by the definition of R that there is an object, say e, such that $Qe \, \& \, R_1ae \, \& \, R_2ed$. But recall that a bears R to a unique object exemplifying Q, namely b. So $b =_E e$. But since R_2ed, it then follows that R_2bd. So we know $Sd \, \& \, R_2bd$. But recall that b bears R to a unique object exemplifying S, namely c. So $c =_E d$.

To show (b), pick an arbitrary ordinary object b such that b exemplifies S. We want to show that there is a unique ordinary object exemplifying P that bears R to b. Since R_2 is, by hypothesis, a one-to-one function from the ordinary objects of Q onto the ordinary objects of S, there is a unique object, say c, such that c exemplifies Q and R_2cb. And since R_1 is, by hypothesis, a one-to-one function from the ordinary objects of P onto the ordinary objects of Q, there is a unique object, say d, such that d exemplifies P and R_1dc. We now establish that d is a unique object exemplifying P that bears R to b. Clearly, d is an object that exemplifies P. Moreover, d bears R to b, for there is an object, namely c, that exemplifies Q and is such that both R_1dc and R_2cb. To show that d is unique, suppose, for reductio, that there is an object e, $e \neq_E d$, such that Pe and Reb. Then by the definition of R, there is an object, say f such that Qf and R_1ef and R_2fb. Since $e \neq_E d$, we know by the functionality of R_1, that $f \neq_E c$. But we now have that Qc, R_2cb, Qf, R_2fb, and $f \neq_E c$, and this contradicts the fact that c is the unique object exemplifying Q that bears R_2 to b. ⋈

(200.1): This is immediate from the definition of $\#_G$ and the definition of $Numbers(x, G)$. ⋈

(200.2): This follows from (200.1) and the fact that equinumerosity$_E$ is reflexive. ⋈

(201): ($\rightarrow$) Assume that the number of Ps is identical to the number of Qs. Then, by the definition of identity for abstract objects, we know that $\#_P$ and $\#_Q$ encode the same properties. By (200.2), we know that $\#_P$ encodes P. So $\#_Q$ encodes P. But, by (200.1), it follows that $P \approx_E Q$. ($\leftarrow$) Assume $P \approx_E Q$. We want to show that $\#_P = \#_Q$, i.e., that they encode the same properties. ($\rightarrow$) Assume $\#_P$ encodes S (to show: $\#_Q$ encodes S). Then by (200.1), $S \approx_E P$. So by the transitivity of equinumerosity$_E$, $S \approx_E Q$. But, then, by (200.1), it follows that $\#_Q$ encodes S. ($\leftarrow$) Assume $\#_Q$ encodes S (to show: $\#_P$ encodes S). Then by (200.1), $S \approx_E Q$. By the symmetry of equinumerosity$_E$, it follows that $Q \approx_E S$. So, given our hypothesis that $P \approx_E Q$, it follows by the transitivity of equinumerosity$_E$ that $P \approx_E S$. Again, by symmetry, we have: $S \approx_E P$. And, thus, by (200.1), it follows that $\#_P$ encodes S. ⋈

(203): Assume k is a natural cardinal. Then, by definition, there is a property, say P such that $k = \#_P$. ($\rightarrow$) Assume k encodes Q. Then $\#_P$ encodes Q. So by (200.1), it follows that $Q \approx_E P$. And by Hume's Principle, it follows that $\#_Q = \#_P$. So, $k = \#_Q$. ($\leftarrow$) Assume $k = \#_Q$. By (200.2), we know that $\#_Q$ encodes Q. So, k encodes Q. ⋈

(206): ($\rightarrow$) Assume 0 encodes P. Then P is equinumerous$_E$ to $[\lambda z\, z \neq_E z]$, by (200.1). So there is an R that is a one-to-one function from the ordinary objects of P onto the ordinary objects of $[\lambda z\, z \neq_E z]$. So, for every ordinary object x such that Px, there is an (unique) ordinary object y such that $[\lambda z\, z \neq_E z]y$ and Rxy. Suppose, for reductio, that $\exists u Pu$, say Pa. Then there is an ordinary object, say b, such that Rab and $[\lambda z\, z \neq_E z]b$. But this contradicts the fact that no ordinary object exemplifies this property. ($\leftarrow$) Suppose $\neg\exists u Pu$. It is also a fact about $[\lambda z\, z \neq_E z]$ that no ordinary object exemplifies it. But then P is equinumerous$_E$ with $[\lambda z\, z \neq_E z]$, for any relation R you pick bears witness to this fact: (a) every ordinary object exemplifying P bears R to a unique ordinary object exemplifying $[\lambda z\ z \neq_E z]$ (since there are no ordinary objects exemplifying P), and (b) every ordinary object exemplifying $[\lambda z\ z \neq_E z]$ is such that there is a unique ordinary object exemplifying P that bears R to it (since there are no ordinary objects exemplifying $[\lambda z\ z \neq_E z]$). Since $P \approx_E [\lambda z\ z \neq_E z]$, it follows by (200.1), that $\#_{[\lambda z\ z \neq_E z]}$ encodes P. So 0 encodes P. ⋈

(207): By (205), 0 is a natural cardinal, and so by (203), $0P$ iff $0 = \#_P$. But by (206), $0P$ iff $\neg\exists uPu$. So $\neg\exists uPu$ iff $0 = \#_P$. ⋈

(210): Suppose, for reductio, that something, say a, is a precedessor of 0. Then, by the definition of predecessor, it follows that there is an property, say Q, and an ordinary object, say b, such that Qb, $0 = \#_Q$, and $a = \#_{[\lambda z\ Qz\ \&\ z \neq_E b]}$. But if $0 = \#_Q$, then by (207), $\neg\exists uQu$, which contradicts the fact that Qb. ⋈

(211): Assume that $P \approx_E Q$, Pa, and Qb. So there is a relation, say R, that is a one-to-one function from the ordinary objects of P onto the ordinary objects of Q. Now we use P^{-a} to designate $[\lambda z\ Pz\ \&\ z \neq_E a]$, and we use Q^{-b} to designate $[\lambda z\ Qz\ \&\ z \neq_E b]$. We want to show that $P^{-a} \approx_E Q^{-b}$. By the definition of equinumerosity$_E$, we have to show that there is a relation R' which is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}. We prove this by cases.

Case 1: Suppose Rab. Then we choose R' to be R itself. Clearly, then, R' is a one-to-one function from the ordinary objects of P^{-a} to the ordinary objects of Q^{-b}. But the proof can be given as follows. We show: (A) that R is a function from the ordinary objects of P^{-a} to the ordinary objects of Q^{-b}, and then (B) that R is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}.

(A) Pick an arbitrary ordinary object, say c, such that $P^{-a}c$. We want to show that there is a unique ordinary object which exemplifies Q^{-b} and to which c bears R. Since $P^{-a}c$, we know that $Pc\ \&\ c \neq_E a$, by the definition of P^{-a}. But if Pc, then by our hypothesis that R is a witness to the equinumerosity$_E$ of P and Q, it follows that there is a unique ordinary object, say d, such that Qd and Rcd. But we are considering the case in which Rab and so from the established facts that Rcd and $c \neq_E a$, it follows by the one-to-one character of R that $b \neq_E d$. So we have that Qd and $d \neq_E b$, which establishes that $Q^{-b}d$. And we have also established that Rcd. So it remains to show that every other ordinary object that exemplifies Q^{-b} to which c bears R just is identical$_E$ to d. So suppose $Q^{-b}e$ and Rce. Then by definition of Q^{-b}, it follows that Qe. But now $e =_E d$, for d is the unique ordinary object exemplifying Q to which c bears R. So there is a unique ordinary object which exemplifies Q^{-b} and to which c bears R.

(B) Pick an arbitrary ordinary object, say d, such that $Q^{-b}d$. We want to show that there is a unique ordinary object exemplifying P^{-a} that bears R to d. Since $Q^{-b}d$, we know Qd and $d \neq_E b$. From Qd and the fact that R witnesses the equinumerosity$_E$ of P and Q, we know that there is a unique ordinary object, say c, that exemplifies P and which bears R to d. Since we are considering the case in which Rab, and we've established Rcd and $d \neq_E b$, it follows that $a \neq_E c$, by the functionality of R. Since we now have Pc and $c \neq_E a$, we have established that c exemplifies P^{-a}, and moreover, that Rcd. So it remains to prove that any other ordinary object that exemplifies P^{-a} and which bears R to d just is (identical$_E$ to) c. But if f, say, exemplifies P^{-a} and bears R to d, then Pf, by definition of P^{-a}. But recall that c is the unique ordinary object exemplifying P which bears R to d. So $f =_E c$.

Case 2: Suppose $\neg Rab$. Then we choose R' to be the relation:

$$[\lambda xy\ (x \neq_E a \,\&\, y \neq_E b \,\&\, Rxy) \vee (x =_E \imath u(Pu \,\&\, Rub) \,\&\, y =_E \imath u(Qu \,\&\, Rau))]$$

To see that there is such a relation, note that the following is an instance of the comprehension principle for Relations, where u, w are *free* variables:

$$\exists F \forall x \forall y (Fxy \equiv (x \neq_E a \ \&\ y \neq_E b \ \&\ Rxy) \ \vee\ (x =_E u \ \&\ y =_E w))$$

By two applications of the Rule of Generalization, we know:

$$\forall u \forall w \exists F \forall x \forall y (Fxy \equiv (x \neq_E a \ \&\ y \neq_E b \ \&\ Rxy) \ \vee\ (x =_E u \ \&\ y =_E w))$$

Now by the assumptions of the lemma, we know that the descriptions $\imath u(Pu \ \&\ Rub)$ and $\imath u(Qu \ \&\ Rau)$ are well-defined and have denotations (if R is a one-to-one function from the ordinary objects of P onto the ordinary objects of Q, and Pa and Qb, then there is a unique ordinary object that exemplifies P that bears R to b and there is a unique ordinary object that exemplifies Q to which a bears R). So we may instantiate these descriptions for universally quantified variables u and w, respectively, to establish that our relation R' exists.

Now we want to show that R' is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}. We show (A) that

R' is a function from the ordinary objects of P^{-a} to the ordinary objects of Q^{-b}, and then (B) that R' is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}.

(A) To show that R' is a function from the ordinary objects of P^{-a} to the ordinary objects of Q^{-b}, pick an arbitrary ordinary object, say c, such that $P^{-a}c$. Then by definition of P^{-a}, we know that Pc and $c \neq_E a$. We need to find an ordinary object, say d for which the following three things hold: (i) $Q^{-b}d$, (ii) $R'cd$, and (iii) $\forall w(Q^{-b}w \,\&\, R'cw \rightarrow w =_E d)$. We find such a d in each of the following, mutually exclusive cases:

Case 1: Rcb. So $c =_E \imath u(Pu \& Rub)$. Then choose $d =_E \imath u(Qu \& Rau)$ (we already know that there is such an ordinary object). So Qd, Rad, and $\forall w(Qw \,\&\, Raw \rightarrow w =_E d)$. We now show that (i), (ii) and (iii) hold for d:

i) Since we know Qd, all we have to do to establish $Q^{-b}d$ is to show $d \neq_E b$. But we know Rad and we are considering the case where $\neg Rab$. So, by the laws of identity$_E$, $d \neq_E b$.

ii) To show $R'cd$, we need to establish:

$$(c \neq_E a \& d \neq_E b \& Rcd) \vee (c =_E \imath u(Pu \& Rub) \& d =_E \imath u(Qu \& Rau))$$

But the conjunctions of the right disjunct are true (by assumption and by choice, respectively). So $R'cd$.

iii) Suppose $Q^{-b}e$ (i.e., Qe and $e \neq_E b$) and $R'ce$. We want to show: $e =_E d$. Since $R'ce$, then:

$$(c \neq_E a \& e \neq_E b \& Rce) \vee (c =_E \imath u(Pu \& Rub) \& e =_E \imath u(Qu \& Rau))$$

But the left disjunct is impossible (we're considering the case where Rcb, yet the left disjunct asserts Rce and $e \neq_E b$, which together contradict the functionality of R). So the right disjunct must be true, in which case it follows from the fact that $e =_E \imath u(Qu \,\&\, Rau)$ that $e =_E d$, by the definition of d.

Case 2: $\neg Rcb$. We are under the assumption $P^{-a}c$ (i.e., Pc and $c \neq_E a$), and so we know by the definition of R and the fact that Pc that there is a unique ordinary object which exemplifies Q and to which c bears R. Choose d to be this object. So Qd, Rcd, and $\forall w(Qw \,\&\, Rcw \rightarrow w =_E d)$. We can now show that (i), (ii) and (iii) hold for d:

i) Since we know Qd, all we have to do to establish that $Q^{-b}d$ is to show $d \neq_E b$. We know that Rcd and we are considering the case where $\neg Rcb$. So it follows that $d \neq_E b$, by the laws of identity$_E$. So $Q^{-b}d$.

ii) To show $R'cd$, we need to establish:

$$(c \neq_E a \& d \neq_E b \& Rcd) \vee (c =_E \imath u(Pu \& Rub) \& d =_E \imath u(Qu \& Rau))$$

But the conjuncts of the left disjunct are true, for $c \neq_E a$ (by assumption), $d \neq_E b$ (we just proved this), and Rcd (by the definition of d). So $R'cd$.

iii) Suppose $Q^{-b}e$ (i.e., Qe and $e \neq_E b$) and $R'ce$. We want to show: $e =_E d$. Since $R'ce$, then:

$$(c \neq_E a \& e \neq_E b \& Rce) \vee (c =_E \imath u(Pu \& Rub) \& e =_E \imath u(Qu \& Rau))$$

But the right disjunct is impossible (we're considering the case where $\neg Rcb$, yet the right disjunct asserts $c =_E \imath u(Pu \,\&\, Rub)$, which implies Rcb, a contradiction). So $c \neq_E a \,\&\, e \neq_E b \,\&\, Rce$. Since we now know that Qe and Rce, we know that $e =_E d$, since d is, by definition, the unique ordinary object exemplifying Q to which c bears R.

(B) To show that R' is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}, pick an arbitrary ordinary object, say d, such that $Q^{-b}d$. Then by definition of Q^{-b}, we know that Qd and $d \neq_E b$. We need to find an ordinary object, say c, for which the following three things hold: (i) $P^{-a}c$, (ii) $R'cd$, and (iii) $\forall w(P^{-a}w \,\&\, R'wd \rightarrow w =_E c)$. We find such a c in each of the following, mutually exclusive cases:

Case 1: Rad. So $d =_E \imath u(Qu \& Rau)$. Then choose $c =_E \imath u(Pu \& Rub)$ (we know there is such an object). So Pc, Rcb, and $\forall w(Pw \& Rwb \rightarrow w =_E c)$. We now show that (i), (ii) and (iii) hold for c:

i) Since we know Pc, all we have to do to establish $P^{-a}c$ is to show $c \neq_E a$. But we know Rcb, and we are considering the case where $\neg Rab$. So, by the laws of identity$_E$, it follows that $c \neq_E a$.

ii) To show $R'cd$, we need to establish:

$$(c \neq_E a \& d \neq_E b \& Rcd) \vee (c =_E \imath u(Pu \& Rub) \& d =_E \imath u(Qu \& Rau))$$

But the conjuncts of the right disjunct are true (by choice and by assumption, respectively). So $R'cd$.

iii) Suppose $P^{-a}f$ (i.e., Pf and $f \neq_E a$) and $R'fd$. We want to show: $f =_E c$. Since $R'fd$, then:

$$(f \neq_E a \;\&\; d \neq_E b \;\&\; Rfd) \;\vee\; (f =_E \imath u(Pu \;\&\; Rub) \;\&\; d =_E \imath u(Qu \;\&\; Rau))$$

But the left disjunct is impossible (we're considering the case where Rad, yet the left disjunct asserts Rfd and $f \neq_E a$, which together contradict the fact that R is one-to-one). So the right disjunct must be true, in which case it follows from the fact that $f =_E \imath u(Pu \;\&\; Rub)$ that $f =_E c$, by the definition of c.

Case 2: $\neg Rad$. We are under the assumption $Q^{-b}d$ (i.e., Qd and $d \neq_E b$), and so we know by the definition of R and the fact that Qd that there is a unique ordinary object which exemplifies P and which bears R to d. Choose c to be this object. So Pc, Rcd, and $\forall w(Pw \;\&\; Rwd \rightarrow w =_E c)$. We can now show that (i), (ii), and (iii) hold for c:

i) Since we know Pc, all we have to do to establish that $P^{-a}c$ is to show $c \neq_E a$. But we know that Rcd, and we are considering the case in which $\neg Rad$. So it follows that $c \neq_E a$, by the laws of identity$_E$. So $P^{-a}c$.

ii) To show $R'cd$, we need to establish:

$$(c \neq_E a \& d \neq_E b \& Rcd) \vee (c =_E \imath u(Pu \& Rub) \& d =_E \imath u(Qu \& Rau))$$

But the conjuncts of the left disjunct are true, for $c \neq_E a$ (we just proved this), $d \neq_E b$ (by assumption), and Rcd (by the definition of c). So $R'cd$.

iii) Suppose $P^{-a}f$ (i.e., Pf and $f \neq_E a$) and $R'fd$. We want to show: $f =_E c$. Since $R'fd$, then:

$$(f \neq_E a \,\&\, d \neq_E b \,\&\, Rfd) \vee (f =_E \imath u(Pu \,\&\, Rub) \,\&\, d =_E \imath u(Qu \,\&\, Rau))$$

But the right disjunct is impossible (we're considering the case where $\neg Rad$, yet the right disjunct asserts $d =_E \imath u(Qu \,\&\, Rau)$, which implies Rad, a contradiction). So $f \neq_E a \,\&\, d \neq_E b \,\&\, Rfd$. Since we now know that Pf and Rfd, we know that $f =_E c$, since c is, by definition, the unique ordinary object exemplifying P which bears R to d. $\bowtie$

$\bowtie$

(212): Assume that both a and b are precedessors of c. By the definition of predecessor, we know that there are properties and ordinary objects P, Q, d, e such that:

$$Pd \,\&\, c = \#_P \,\&\, a = \#_{P^{-d}}$$

$$Qe \,\&\, c = \#_Q \,\&\, b = \#_{Q^{-e}}$$

But if both $c = \#_P$ and $c = \#_Q$, then $\#_P = \#_Q$. So, by Hume's Principle, $P \approx_E Q$. And by (211), it follows that $P^{-d} \approx_E Q^{-e}$. If so, then by Hume's Principle, $\#_{P^{-d}} = \#_{Q^{-e}}$. But then, $a = b$. $\bowtie$

(213): Assume that $P^{-a} \approx_E Q^{-b}$, Pa, and Qb. So there is a relation, say R, that is a one-to-one function from the ordinary objects of P^{-a} onto the ordinary objects of Q^{-b}. We want to show that there is a function R' which is a one-to-one function from the ordinary objects of P onto the ordinary objects of Q. Let us choose R' to be the following relation:

$$[\lambda xy \, (P^{-a}x \,\&\, Q^{-b}y \,\&\, Rxy) \vee (x =_E a \,\&\, y =_E b)]$$

We know such a relation exists, by the comprehension principle for relations. We show: (A) that R' is a function from the ordinary objects of P to the ordinary objects of Q, and (B) that R' is a one-to-one function from the ordinary objects of P onto the ordinary objects of Q.

(A) Pick an arbitrary ordinary object, say c, such that Pc. We want to show that there is a unique object that exemplifies Q to which R' relates c. We show that there is such an object in each of the following two mutually exclusive cases:

Case 1. $c =_E a$. Then we show that b is a unique object that exemplifies Q to which R' bears c:

i) Qb, by hypothesis.

ii) $R'cb$, since $c =_E a$ and $b =_E b$ and so the right disjunct for $R'cb$ obtains

iii) Suppose Qe and $R'ce$ (to show $e =_E b$). Then since $R'ce$, we know $(P^{-a}c \;\&\; Q^{-b}e \;\&\; Rce)$ or $(c =_E a \;\&\; e =_E b)$. But the first disjunct is impossible, since $P^{-a}c$ implies Pc and $c \neq_E a$, contradicting the present case, in which $c =_E a$. So $e =_E b$.

Case 2. $c \neq_E a$. Then we consider the object $d =_E \imath u(Q^{-b}u \;\&\; Rcu)$ (we know there is such an object by the assumptions of the lemma). So $Q^{-b}d$, Rcd, and $\forall w(Q^{-b}w \;\&\; Rcw \rightarrow w =_E d)$. We show that d is a unique object exemplifying Q to which R' relates c:

i) Qd, since this is implied by $Q^{-b}d$.

ii) To see that $R'cd$, note first that Pc (by assumption) and $c \neq_E a$ (present case). So $P^{-a}c$. Since we also know $Q^{-b}d$ and Rcd (by definition of d), we have that c and d stand in the relation R'.

iii) Assume Qe and $R'ce$ (to show $e =_E d$). Since $R'ce$, we know that either $(P^{-a}c \;\&\; Q^{-b}e \;\&\; Rce)$ or $(c =_E a \;\&\; e =_E b)$. But in the present case, $c \neq_E a$, and so the right disjunct is false. But from the facts that $Q^{-b}e$ and Rce, it follows that $e =_E d$, since d is, by definition, the unique object exemplifying Q^{-b} to which c bears R.

(B) Pick an arbitrary object, say d, such that Qd. We want to show that there is a unique object exemplifying P which bears R to d. We find such an object in each of the following two mutually exclusive cases:

Case 1. $d =_E b$. Then we show that a is a unique object exemplifying P which bears R to d:

i) Pa, by hypothesis.

ii) $R'ad$, from the facts that $R'ab$ (by definition of R') and $d =_E b$ (true in the present case).

iii) Assume Pf and $R'fd$ (to show: $f =_E a$). Since $R'fd$, then either $(P^{-a}f \;\&\; Q^{-b}d \;\&\; Rfd)$ or $(f =_E a \;\&\; d =_E b)$. But the left disjunct is impossible, since $Q^{-b}d$ implies Qd and $d \neq_E b$, contradicting the assumption of the present case. So $f =_E a$.

Case 2. $d \neq_E b$. Then choose $c =_E \imath u(P^{-a}u \;\&\; Rud)$ (we know there is such an object by the assumptions of the lemma). So $P^{-a}c$, Rcd, and $\forall w(P^{-a}w \,\&\, Rwd \rightarrow w =_E c)$. We show that c is a unique object exemplifying P such that $R'cd$:

i) Pc, from the fact that $P^{-a}c$.

ii) To see that $R'cd$, note first that $P^{-a}c$ and Rcd, by the definition of c. Note also that from our assumption that Qd and the present case in which $d \neq_E b$, it follows that $Q^{-b}d$. So $R'cd$, by the definition of R'.

iii Suppose that Pf and $R'fd$ (to show that $f =_E c$). Then since $R'fd$, we know that either ($P^{-a}f$ & $Q^{-b}d$ & Rfd) or ($f =_E a$ & $d =_E b$). But the right disjunct is inconsistent with the present case. So from the facts that $P^{-a}f$ and Rfd, it follows that $f =_E c$, since c is, by definition, the unique object that exemplifies P^{-a} and which bears R to d. ⋈

⋈

(214): Assume that a is a precedessor of both b and c. By the definition of predecessor, we know that there are properties and ordinary objects P, Q, d, e such that:

$$Pd \,\&\, b = \#_P \,\&\, a = \#_{P^{-d}}$$

$$Qe \,\&\, c = \#_Q \,\&\, a = \#_{Q^{-e}}$$

But if both $a = \#_{P^{-d}}$ and $a = \#_{Q^{-e}}$, then $\#_{P^{-d}} = \#_{Q^{-e}}$. So, by Hume's Principle, $P^{-d} \approx_E Q^{-e}$. And by (213), it follows that $P \approx_E Q$. Now, by Hume's Principle, $\#_P = \#_Q$. But then, $b = c$. ⋈

(217.1): Assume Rab. Pick an arbitrary property, say P and assume $\forall z(Raz \rightarrow Pz)$ and *Hereditary*(P, R). Then Pb, by the first two of our three assumptions. ⋈

(217.2): This follows immediately from the definition of R^*. ⋈

(217.3): Assume Pa, $R^*(a, b)$, and that *Hereditary*(P, R). Then by the lemma we just proved (217.2), all we have to do to show that b exemplifies P is show that P is exemplified by every object to which R relates a. So suppose R relates a to some arbitrarily chosen object c (to show Pc). Then by the fact that P is hereditary with respect to R and our assumption that Pa, it follows that Pc. ⋈

(217.4): Assume Rab and $R^*(b,c)$. To prove $R^*(a,c)$, further assume $\forall z(Raz \rightarrow Pz)$ and $Hereditary(P,R)$ (to show Pc). So Pb. But from Pb, $R^*(b,c)$, and $Hereditary(P,R)$, it follows that Pc, by (217.3). $\bowtie$

(217.5): Assume $R^*(a,b)$ to show $\exists zRzb$. If we instantiate the variables x,y in (217.2) to the relevant objects, and instantiate the variable F to $[\lambda w\, \exists zRzw]$, the we know the following fact (after λ-conversion):

$$[R^*(a,b) \& \forall x(Rax \rightarrow \exists zRzx) \& \forall x,y(Rxy \rightarrow (\exists zRzx \rightarrow \exists zRzy))] \rightarrow \exists zRzb$$

So we simply have to prove the second and third conjuncts of the antecedent. But these are immediate. For an arbitrarily chosen object c, $Rac \rightarrow \exists zRzc$. So $\forall x(Rax \rightarrow \exists zRzx)$. Similarly, for arbitrarily chosen c,d, the assumptions that Rcd and $\exists zRzc$ immediately imply $\exists zRzd$. So $\forall x,y(Rxy \rightarrow (\exists zRzx \rightarrow \exists zRzy))$. $\bowtie$

(219.1): This is immediate from (217.1). $\bowtie$

(219.2): Assume Pa and $R^+(a,b)$. Then by the definition of weak ancestral, either $R^*(a,b)$ or $a=b$. If the former, then Pb, by (217.3). If the latter, then Pb, from the assumption that Pa. $\bowtie$

(219.3): Assume $R^+(a,b)$ and Rbc. Then either (I) $R^*(a,b)$ and Rbc or (II) $a=b$ and Rbc. We want to show, in both cases, $R^*(a,c)$:

> Case I: $R^*(a,b)$ and Rbc. Pick an arbitrary property P. To show R^*ac, we assume that $\forall z(Raz \rightarrow Pz)$ and $Hereditary(P,R)$. We now try to show: Pc. But from the fact that $R^*(a,b)$, it then follows that Pb, by the definition of R^*. But from the facts that $Hereditary(P,R)$, Rbc, and Pb, it follows that Pc.
>
> Case II: $a=b$ and Rbc. Then Rac, and so by (217.1), it follows that $R^*(a,c)$.

$\bowtie$

(219.4): Assume $R^*(a,b)$ and Rbc (to show $R^+(a,c)$). Then by the first assumption and the definition of R^+, it follows that $R^+(a,b)$. So by (219.3), it follows that $R^*(a,c)$. So $R^+(a,c)$, by the definition of R^+. $\bowtie$

(219.5): Assume Rab and $R^+(b,c)$ (to show: $R^*(a,c)$). By definition of the weak ancestral, either $R^*(b,c)$ or $b=c$. If $R^*(b,c)$, then $R^*(a,c)$, by (217.4). If $b=c$, then Rac, in which case, $R^*(a,c)$, by (217.1). $\bowtie$

(219.6): Assume $R^*(a,b)$ (to show: $\exists z(R^+(a,z) \,\&\, Rzb)$). The following is an instance of (217.2):

$$R^*(a,b) \,\&\, \forall x(Rax \rightarrow Fx) \,\&\, \mathit{Hereditary}(F,R) \rightarrow Fb$$

Now let F be the property $[\lambda w\, \exists z(R^+(a,z) \,\&\, Rzw)]$. So, by expanding definitions and using λ-conversion, we know:

$$\begin{array}{l} R^*(a,b) \,\&\, \forall x(Rax \rightarrow \exists z(R^+(a,z) \,\&\, Rzx)) \,\&\, \\ \quad \forall x,y[Rxy \rightarrow (\exists z(R^+(a,z) \,\&\, Rzx) \rightarrow \exists z(R^+(a,z) \,\&\, Rzy))] \rightarrow \\ \qquad \exists z(R^+(a,z) \,\&\, Rzb) \end{array}$$

Since the consequent is what we have to show, we need only establish the three conjuncts of the antecedent. The first is true by assumption. For the second, assume Rac, where c is an arbitrarily chosen object (to show: $\exists z(R^+(a,z) \& Rzc)$). But, by definition of R^+, we know that $R^+(a,a)$. So, from $R^+(a,a) \& Rac$, it follows that $\exists z(R^+(a,z) \& Rzc)$. For the third conjunct, assume Rcd and $\exists z(R^+(a,z) \& Rzc)$ (to show: $\exists z(R^+(a,z) \& Rzd)$). Since we know Rcd, we simply have to show $R^+(a,c)$ and we're done. But we know that for some object, say e, $R^+(a,e) \,\&\, Rec$. So by (219.3), it follows that $R^*(a,c)$. But, then $R^+(a,c)$, by definition of R^+. ⋈

(221): Let n be a natural number. Then, by definition, $\mathit{Precedes}^+(0,n)$. By definition of R^+, it follows that either $\mathit{Precedes}^*(0,n)$ or $0=n$. (I) If the former, then by (217.5), there is an object, say a, such that *Precedes*(a,n). So by the definition of *Predecessor* it follows that there is a property, say P, and an ordinary object, say b, such that:

$$Pb \,\&\, n = \#_P \,\&\, a = \#_{P^{-b}},$$

(where P^{-b} is defined as in the proof of Lemma (211)). Since n is the number of some property, n is a natural cardinal number. (II) If the latter, then since 0 is a natural cardinal, by (205), it follows that n is a natural cardinal. ⋈

(223.1): By (217.5), we know that if x ancestrally precedes y, then there is something that precedes y. But, by (210), we know that nothing precedes zero. So nothing ancestrally precedes zero, and in particular, zero doesn't ancestrally precede itself. ⋈

(223.2): By (210), nothing precedes zero. So no natural number precedes zero. ⋈

(224): By (212), *Predecessor* is one-one. *A fortiori*, it is one-one when restricted to the members of the *Predecessor* series beginning with 0. $\bowtie$

(225): Assume $Precedes(n, a)$. Since n is a number, $Precedes^+(0, n)$. So by (219.3), it follows that $Precedes^*(0, a)$, and so by the definition of weak ancestral, it follows that $Precedes^+(0, a)$; i.e., $NaturalNumber(a)$. $\bowtie$

(227): Suppose, for an arbitrary ordinary object b, that $\Diamond\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$. We want to show that $\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$. So assume, for an arbitrary ordinary object c, that $\mathcal{A}Gc$ (to show: $c \neq_E b$). Since, $\mathcal{A}Gc$, it follows that $\Box\mathcal{A}Gc$, by $\Box$ Actuality. Since we know that $\Diamond\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$, we know there is a world where $\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$ is true. Let us, for the moment, reason with respect to that world. Since $\Box\mathcal{A}Gc$ is true at our world, we know that $\mathcal{A}Gc$ is true at the world where $\forall u(\mathcal{A}Gu \rightarrow u \neq_E b)$ is true. So $c \neq_E b$ is true at that world. So, from the point of view of our world, we know that $\Diamond c \neq_E b$ (since $c \neq_E b$ is true at some world). But, by the logic of $=_E$, we know that $x =_E y \rightarrow \Box x =_E y$. That is, by modal duality, we know that $\Diamond x \neq_E y \rightarrow x \neq_E y$. So since $\Diamond c \neq_E b$, it follows that $c \neq_E b$, which is what we had to show. [NOTE: This proof involved the natural deduction version of the modal axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$.] $\bowtie$

Alternative Proof of (227):

By (13.2), we know that $\mathcal{A}Gu \rightarrow \Box\mathcal{A}Gu$. So by contraposition,

$$\Diamond\neg\mathcal{A}Gu \rightarrow \neg\mathcal{A}Gu \tag{1}$$

Note also that the following claim is true

$$\forall v(\mathcal{A}Gv \rightarrow v \neq_E u) \equiv \neg\mathcal{A}Gu \tag{2}$$

(Here is the proof: $(\rightarrow)$ Assume $\forall v(\mathcal{A}Gv \rightarrow v \neq_E u)$. So $\mathcal{A}Gu \rightarrow u \neq_E u$. But since u is ordinary, $u =_E u$. So $\neg\mathcal{A}Gu$. $(\leftarrow)$ Assume $\neg\mathcal{A}Gu$. To show $\forall v(\mathcal{A}Gv \rightarrow v \neq_E u)$, assume $\mathcal{A}Gb$, where b is an arbitrary ordinary object. But if $b =_E u$, then $\mathcal{A}Gu$, contrary to hypothesis. So $b \neq_E u$.) Given (2), however, we know the equivalence of $\forall v(\mathcal{A}Gv \rightarrow v \neq_E u)$ and $\neg\mathcal{A}Gu$. So we may substitute the former for the latter in (1). The result is what we had to prove, namely,

$$\Diamond\forall v(\mathcal{A}Gv \rightarrow v \neq_E u) \rightarrow \forall v(\mathcal{A}Gv \rightarrow v \neq_E u)$$

$\bowtie$

(228): Suppose $NaturalNumber(a)$. We want to show $\exists! m Precedes(a, m)$. But in virtue of (214), we simply have to show that $\exists m Precedes(a, m)$, and in virtue of (225), it suffices to show that $\exists y Precedes(a, y)$. Since $NaturalNumber(a)$, it follows that $Precedes^{+}(0, a)$, and so we know that either $Precedes^{*}(0, a)$ or $0 = a$. We show that in either case, a precedes something.

Suppose $0 = a$. We know both that $NaturalNumber(0)$ and $0 = \#_{[\lambda z\, z \neq_E z]}$. Let us use '$P$' to designate the property $[\lambda z\, z \neq_E z]$. Then the antecedent of our modal axiom (226) applies, and we may conclude:

$$\Diamond \exists y (E!y \,\&\, \forall z (\mathcal{A}Pz \rightarrow z \neq_E y))$$

By the Barcan formula, this implies:

$$\exists y \Diamond (E!y \,\&\, \forall z (\mathcal{A}Pz \rightarrow z \neq_E y))$$

Let b be an arbitrary such object. So we know:

$$\Diamond (E!b \,\&\, \forall z (\mathcal{A}Pz \rightarrow z \neq_E b))$$

It then follows by the laws of possibility that:

$$\Diamond E!b \,\&\, \Diamond \forall z (\mathcal{A}Pz \rightarrow z \neq_E b))$$

Thus, we know that b is an ordinary object. Moreover, by comprehension for properties, we know that $[\lambda z\, z =_E b]$ exists. Call this property 'K'. By (199), we know that $\#_K$ exists. We now show that $Precedes(0, \#_K)$. This requires that we show:

$$\exists F \exists u (Fu \,\&\, \#_K = \#_F \,\&\, 0 = \#_{[\lambda z\, Fz \,\&\, z \neq_E u]})$$

So we now show that K and b are such a property and object. b exemplifies K, by the definition of K and the fact that b is an ordinary object. $\#_K = \#_K$ is true by the laws of identity. So it simply remains to show:

$$0 = \#_{[\lambda z\, Kz \,\&\, z \neq_E b]}$$

But by (207), we simply have to show that no ordinary object exemplifies $[\lambda z\, Kz \,\&\, z \neq_E b]$. So, for reductio, assume that some ordinary object d exemplifies this property. Then, $Kd \,\&\, d \neq_E b$. But, by definition of K, $d =_E b$, which is a contradiction.

Now suppose $Precedes^{*}(0, a)$. From (221) and the fact that $NaturalNumber(a)$, it follows that $NaturalCardinal(a)$. So $\exists F(a = \#_F)$. Suppose $a = \#_Q$. Then we may apply our modal axiom (226) to conclude the following:

$$\Diamond\exists y(E!y \,\&\, \forall z(\mathcal{A}Qz \rightarrow z \neq_E y))$$

By the Barcan formula, this implies:

$$\exists y\Diamond(E!y \,\&\, \forall z(\mathcal{A}Qz \rightarrow z \neq_E y))$$

Let c be an arbitrary such object. So we know:

$$\Diamond(E!c \,\&\, \forall z(\mathcal{A}Qz \rightarrow z \neq_E c))$$

By the laws of possibility, it follows that:

$$\Diamond E!c \,\&\, \Diamond\forall z(\mathcal{A}Qz \rightarrow z \neq_E c)$$

From the first conjunct, it follows that $O!c$, and by (227), the second conjunct implies:

$$\forall z(\mathcal{A}Qz \rightarrow z \neq_E c) \tag{I}$$

Now the property $[\lambda z\, Qz \ \vee\ z =_E c]$ exists by comprehension. Call this property Q^{+c}. By (199), it follows that $\#_{Q^{+c}}$ exists. Now if we can show $\mathit{Precedes}(a, \#_{Q^{+c}})$, we are done. So we have to show:

$$\exists F \exists u(Fu \,\&\, \#_{Q^{+c}} = \#_F \,\&\, a = \#_{[\lambda z\, Fz \,\&\, z \neq_E u]})$$

So we now show that Q^{+c} and c are such a property and object. c exemplifies Q^{+c}, by the definition of Q^{+c} and the fact that c is an ordinary object. $\#_{Q^{+c}} = \#_{Q^{+c}}$ is true by the laws of identity. So it simply remains to show:

$$a = \#_{[\lambda z\, Q^{+c}z \,\&\, z \neq_E c]}$$

Given that, by definition of Q, $a = \#_Q$, we have to show:

$$\#_Q = \#_{[\lambda z\, Q^{+c}z \,\&\, z \neq_E c]}$$

By Hume's Principle, it suffices to show:

$$Q \approx_E [\lambda z\, Q^{+c}z \,\&\, z \neq_E c]$$

But, given (208.3), we need only establish the following two facts:

(a) $Q \approx_E [\lambda z\, Qz \,\&\, z \neq_E c]$

(b) $[\lambda z\, Qz \,\&\, z \neq_E c] \equiv_E [\lambda z\, Q^{+c}z \,\&\, z \neq_E c]$

Now to show (a), we simply need to prove that Q and $[\lambda z\, Qz \;\&\; z \neq_E c]$ are materially equivalent$_E$, in virtue of (208.1). ($\rightarrow$) Assume, for some arbitrary ordinary object d, that Qd. Then by the logical axiom (13.1), it follows that $\mathcal{A}Qd$. But then by fact (I) proved above, it follows that $d \neq_E c$. Since $Qd \;\&\; d \neq_E c$, it follows that $[\lambda z\, Qz \;\&\; z \neq_E c]d$. ($\leftarrow$) Trivial. Finally, we leave (b) as an exercise. $\bowtie$

(233): Assume that R^+ is a relation and:

$$Pa \;\&\; \forall x, y(R^+(a,x) \;\&\; R^+(a,y) \;\&\; Rxy \rightarrow (Px \rightarrow Py)).$$

We want to show, for an arbitrary object b, that if $R^+(a,b)$ then Pb. So assume $R^+(a,b)$. To show Pb, we appeal to Lemma (219.2):

$$Fx \;\&\; R^+(x,y) \;\&\; \mathit{Hereditary}(F,R) \rightarrow Fy$$

Instantiate the variable F in this lemma to the property $[\lambda z\, Pz \& R^+(a,z)]$ (that there is such a property is guaranteed by the comprehension principle for relations and the assumption that R^+ is a relation), and instantiate the variables x and y to the objects a and b, respectively. The result is, therefore, something that we know to be true (after λ-conversion):

$$Pa \;\&\; R^+(a,a) \;\&\; R^+(a,b) \;\&\; \mathit{Hereditary}([\lambda z\, Pz \;\&\; R^+(a,z)], R) \rightarrow Pb \;\&\; R^+(a,b)$$

So if we can establish the antecedent of this fact, we establish Pb. But we know the first conjunct is true, by assumption. We know that the second conjunct is true, by the definition of R^+. We know that the third conjunct is true, by further assumption. So if we can establish:

$$\mathit{Hereditary}([\lambda z\, Pz \;\&\; R^+(a,z)], R),$$

we are done. But, by the definition of heredity, this just means:

$$\forall x, y[Rxy \rightarrow ((Px \;\&\; R^+(a,x)) \rightarrow (Py \;\&\; R^+(a,y))].$$

To prove this claim, we assume Rxy, Px, and $R^+(a,x)$ (to show: $Py \& R^+(a,y)$). But from the facts that $R^+(a,x)$ and Rxy, it follows from (219.3) that $R^*(a,y)$, and this implies $R^+(a,y)$, by the definition of R^+. But since we now have $R^+(a,x)$, $R^+(a,y)$, Rxy, and Px, it follows from the first assumption in the proof that Py. $\bowtie$

(234): By assumption (231), $\mathit{Predecessor}^+$ is a relation. So by (233), it follows that:

$$Fa \,\&$$
$$\forall x, y[\mathit{Precedes}^{+}(a,x) \,\&\, \mathit{Precedes}^{+}(a,y) \,\&\, \mathit{Precedes}(x,y) \rightarrow$$
$$(Fx \rightarrow Fy)] \rightarrow$$
$$\forall x(\mathit{Precedes}^{+}(a,x) \rightarrow Fx)$$

Now substitute 0 for a, *NaturalNumber*(x) for *Precedes*$^{+}(0,x)$, and *NaturalNumber*(y) for *Precedes*$^{+}(0,y)$. ⋈

(239): We prove this for $n = 0$ and then we give a proof schema for any numeral n' which assumes that a proof for n has been given. This proof schema has an instance which constitutes a proof of:

$$\vdash n' = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))$$

from the assumption:

$$\vdash n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n} uFu))$$

$n = 0$. We want to show:

$$\vdash 0 = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{0} uFu))$$

That is, we want to show:

$$\vdash 0 = \imath x(A!x \,\&\, \forall F(xF \equiv \neg\exists uFu))$$

But this is an immediate corollary of (206).

Proof schema for any numeral n'. Our assumption is:

$$\vdash n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n} uFu))$$

We want to show that this holds for the numeral n':

$$\vdash n' = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))$$

To do this, we have to show that there is a proof that the objects flanking the identity sign encode the same properties; i.e.,

$$\vdash \forall G[n'G \leftrightarrow \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))G]$$

($\rightarrow$) Assume that $n'P$, where P is an arbitrary property. We want to show that:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))P$$

By the laws of description, we have to show

$$\exists!_{n'}uPu,$$

i.e.,

$$\exists u(Pu \,\&\, \exists!_n v(Pv \,\&\, v \neq_E u))$$

Note that by λ-conversion, it suffices to show:

$$\exists u(Pu \,\&\, \exists!_n v([\lambda z\, Pz \,\&\, z \neq_E u]v))$$

In other words, we have to show:

$$\exists u(Pu \,\&\, \exists!_n vP^{-u}v),$$

where P^{-u} stands for $[\lambda z\, Pz \,\&\, z \neq_E u]$.

Since $Precedes(n, n')$, there is some property, say Q and some ordinary object, say a, such that:

$$Qa \,\&\, n' = \#_Q \,\&\, n = \#_{Q^{-a}},$$

where Q^{-a} denotes $[\lambda z\, Qz \,\&\, z \neq_E a]$. Note that since $n'P$ (our initial assumption) and $n' = \#_Q$, we know that $\#_Q P$, and thus that P is equinumerous$_E$ to Q and vice versa. So there is a relation R which is a one-to-one and onto function from Q to P. Since Qa, we know that $\imath u(Pu \,\&\, Rau)$ exists. Call this object b. If we can show:

$$Pb \,\&\, \exists!_n vP^{-b}v$$

then we are done. But Pb follows by definition of b and the laws of description. To see that $\exists!_n vP^{-b}v$, note that $n = \#_{Q^{-a}}$. And by (211), we may appeal to the facts that $Q \approx_E P$, Qa, and Pb to conclude that $Q^{-a} \approx_E P^{-b}$. So by Hume's Principle, $\#_{Q^{-a}} = \#_{P^{-b}}$. So $n = \#_{P^{-b}}$ and by (203) and the fact that n is a natural cardinal, it follows that nP^{-b}. But we are assuming that the theorem holds for the numeral n:

$$n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n uFu))$$

This entails, by the laws of descriptions, that:

$$nF \equiv \exists!_n vFv$$

So since nP^{-b}, it follows that $\exists!_n vP^{-b}v$, which is what we had to show.

($\leftarrow$) Exercise.

$\bowtie$

(242.1): Suppose R is one-one, Rab, and $R^*(c,b)$ (to show: $R^+(c,a)$). By (219.6) and the fact that $R^*(c,b)$, it follows that there is some object, say z, such that $R^+(c,z)$ and Rzb. By facts that R is one-one, Rab and Rzb, it follows that $a\!=\!z$. So $R^+(c,a)$. ⋈

(242.2): Assume R is one-one and Rab (to show: $\neg R^*(a,a) \rightarrow \neg R^*(b,b)$). We shall prove the contrapositive, so assume $R^*(b,b)$ (to show: $R^*(a,a)$). Apply (242.1), letting $x\!=\!a$, $y\!=\!b$, and $z\!=\!b$:

$$\textit{One-One}(R)\ \&\ Rab\ \&\ R*(b,b) \rightarrow R^+(b,a)$$

So, $R^+(b,a)$. Now apply (219.5), letting $x\!=\!a$, $y\!=\!b$, and $z\!=\!a$:

$$Rab\ \&\ R^+(b,a) \rightarrow R^*(a,a)$$

Thus, $R^*(a,a)$. ⋈

(242.3): Assume that R is one-one, that $\neg R^*(a,a)$, and that $R^+(a,b)$ (to show: $\neg R^*(b,b)$). Apply (219.2), letting $F = [\lambda z\ \neg R^*(z,z)]$, $x\!=\!a$, and $y\!=\!b$:

$$R^+(a,b)\ \&\ \neg R^*(a,a)\ \&\ \forall x,y[Rxy \rightarrow (\neg R^*(x,x) \rightarrow \neg R^*(y,y))] \rightarrow \neg R^*(b,b)$$

Since the consequent is what we want to show, we establish the antecedent. The first two conjuncts of the antecedent are true by assumption. But since R is one-one, the third conjunct of the antecedent is also true, by (242.2). ⋈

(243.1): Assume a is a natural number; i.e., $\textit{Precedes}^+(0,a)$ (to show: a doesn't ancestrally precede itself). By (242.3), we know the following about the *Predecessor* relation and its ancestrals:

$$\textit{One-One}(\textit{Predecessor})\ \&\ \neg\textit{Precedes}^*(0,0) \rightarrow [\textit{Precedes}^+(0,a) \rightarrow \neg\textit{Precedes}^*(a,a)]$$

But by (224), *Predecessor* is one-one. And by (223.1), 0 doesn't ancestrally precede itself. And by hypothesis, a is a member of the *Predecessor* series beginning with 0. So $\neg\textit{Precedes}^*(a,a)$. ⋈

(243.2): By (243.1) and (217.1). ⋈

(246): Assume a is an integer, i.e., that a is natural number and $a \neq 0$ (to show: there is a unique natural number that precedes a). By the definitions of natural number and weak ancestral, we know that either $Precedes^*(0,a)$ or $a = 0$. So $Precedes^*(0,a)$. But, then, by (219.6), it follows that $\exists y(Precedes^+(0,y)\ \&\ Precedes(y,a))$; in other words, some natural number precedes a. Now to show that some natural number uniquely precedes a, apply (224). $\bowtie$

(251): $(\rightarrow)$ We prove the contrapositive: assume that $\neg a\|b$ (to show: $\vec{a} \neq \vec{b}$). So we want to show that $\vec{a}$ encodes a property that $\vec{b}$ fails to encode, or vice versa. Call $[\lambda z\ z =_E a]$ *the haecceity of a*. Since $\neg a\|b$, then the the haecceity of a is not the haecceity of a line parallel to b. So, by definition of $\vec{b}$, $\vec{b}$ doesn't encode the haecceity of a (by definition, $\vec{b}$ encodes only all and only the haecceities of lines parallel to b). But, clearly, $\vec{a}$ does encode the haecceity of a. So, $\vec{a} \neq \vec{b}$. $(\leftarrow)$ Assume $a\|b$. To show that $\vec{a} = \vec{b}$, we prove they encode the same properties. $(\rightarrow)$ Assume $\vec{a}P$. Then, by definition of $\vec{a}$,

$$\exists u(u\|a\ \&\ P = [\lambda z\ z =_E u])$$

So call an arbitrary such object c. Since $c\|a$ and $a\|b$, it follows by the transitivity of $\|$ that $c\|b$. So:

$$\exists u(u\|b\ \&\ P = [\lambda z\ z =_E u])$$

By the definition of $\vec{b}$, it follows that $\vec{b}P$. $(\leftarrow)$ By analogous reasoning. $\bowtie$

(257.1): Consider arbitrary propositions p_1 and q_1. $(\rightarrow)$ Assume that the truth value of p_1 encodes q_1 (to show that q_1 is materially equivalent to p_1). Then, by definition of the truth value of p_1, there is a proposition, say r_1, such that r_1 is materially equivalent to p_1 and such that the property *being such that* q_1 just is the property *being such that* r_1. So by the definition of proposition identity (namely, that $p = q$ whenever $[\lambda y\ p] = [\lambda y\ q]$), it follows that q_1 just is the proposition r_1. So, q_1 is materially equivalent to p_1. $(\leftarrow)$ Suppose that q_1 is materially equivalent to p_1. So:

$$(q_1 \equiv p_1)\ \&\ ([\lambda y\ q_1] = [\lambda y\ q_1])$$

Therefore, there is a proposition r (namely, q_1) such that r is materially equivalent to p_1 and such that *being such that* q_1 is identical to *being such that* r. So, by the definition of the truth value of p_1, it follows that the truth value of p_1 encodes *being such that* q_1. $\bowtie$

(258): Pick arbitrary propositions p_1 and q_1. ($\rightarrow$) Assume the truth value of p_1 is identical to the truth value of q_1. Now by (257.2), we know that the truth value of p_1 encodes p_1. So the truth value of q_1 encodes p_1. So by (257.2), p_1 is materially equivalent to q_1. ($\leftarrow$) Assume p_1 is materially equivalent to q_1. To show that the truth value of p_1 is identical to the truth value of q_1, we show they encode the same properties. ($\rightarrow$) Assume p_1 encodes P. Then there is a proposition, say r_1, such that r_1 is materially equivalent to p_1 and such that P is identical to *being such that* r_1. So there is a proposition r (namely r_1) such that r is materially equivalent to q_1 and such that P is identical to $[\lambda y\, r]$. So, by the definition of the truth value of q_1, it follows that the truth value of q_1 encodes P. ($\leftarrow$) By analogous reasoning. $\bowtie$

(260.1): We want to show that $\top$ is identical with the truth-value of some proposition. So pick any arbitrary proposition you please, say p_1, and consider the proposition $p_1 \rightarrow p_1$. Call this proposition p_2. Since p_2 is a logical truth, it is true. We want to show that $\top$ encodes a property Q iff the truth value of p_2 encodes Q. ($\rightarrow$) Assume $\top$ encodes Q. Then there is a proposition, say r_1, such that r_1 is true and Q is the property *being such that* r_1. Since p_2 and r_1 are both true, they are materially equivalent. So there is a proposition r (namely, r_1) such that r is materially equivalent to p_2 and such that Q is the property *being such that* r. So, by the definition of the truth value of p_2, it follows that the truth value of p_2 encodes Q. ($\leftarrow$). Assume that the truth value of p_2 encodes Q. So, there is a proposition, say r_1, such that r_1 is materially equivalent to p_2 and such that Q is *being identical with* r_1. But since p_2 is true, r_1 is true. So there is a proposition r (namely, r_1) which is true and such that Q is *being such that* r. So $\top$ encodes Q. $\bowtie$

(262.1): Pick an arbitrary proposition, say p_1, and assume p_1 is true. We want to show that p_1° is identical to $\top$. ($\rightarrow$) Assume that $p_1^\circ Q$ (to show that $\top Q$). Then, by the definition of extension, there is a proposition, say r_1, such that r_1 is materially equivalent to p_1 and Q is *being such that* r_1. So r_1 is true, since it is materially equivalent to a true proposition. So there is a proposition r (namely, r_1) such that r is true and Q is *being such that* r. So by the definition of $\top$, it follows that $\top$ encodes Q. ($\leftarrow$). Assume that $\top$ encodes Q. Then there is a true proposition, say r_1, such that Q is *being such that* r_1. But since r_1 is true, it is materially equivalent to p_1. So there is a proposition r materially equivalent to p_1 such that Q

is *being such that* r. So p_1° encodes Q. $\bowtie$

(263): Since by (260) we know that $\top$ and $\bot$ are both truth values, it suffices to show that they are distinct and that any other truth value is identical to either $\top$ or $\bot$. Since it is obvious that they are distinct, we simply prove that every truth value is identical to either $\top$ or $\bot$. So assume that z is a truth value. So there is some proposition, say p_1, such that $z = p_1^\circ$. Now either p_1 or $\neg p_1$. So by disjunctive syllogism, (262.1), and (262.2), either $p_1^\circ = \top$ or $p_1^\circ = \bot$. So either $z = \top$ or $z = \bot$. $\bowtie$

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