A Nominalist's Dilemma and its Solution*

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Abstract

Current versions of nominalism in the philosophy of mathematics have the benefit of avoiding commitment to the existence of mathematical objects. But this comes with a cost: to avoid commitment to mathematical entities, nominalists cannot take mathematical theories literally, and so, they seem unable to accommodate mathematical practice. In a recent work, Jody Azzouni has challenged this conclusion, by formulating a nominalist view that doesn't have this cost (Azzouni 2004). In this paper, we argue that, as it stands, Azzouni's proposal doesn't yet succeed. It faces a dilemma to the effect that either the view isn't nominalist or it fails to take mathematics literally. So, in the end, it still doesn't do justice to mathematical practice. After presenting the dilemma, we suggest a solution which the nominalist should be able to accept.

1. Introduction

According to nominalism about mathematics, mathematical objects don't exist or, at least, they need not be taken to exist for us to make sense of mathematics. While nominalism has the *prima facie* benefit of not assuming the existence of mathematical objects, current nominalist interpretations face the problem of not being able to take mathematical theories literally. Typically, such theories have to be rewritten or, at least, reinterpreted in a nominalistically acceptable way, so as to avoid commitment to mathematical objects.

For example, in Geoffrey Hellman's modal-structural interpretation, a mathematical statement S has to be translated into a sentence of modal second-order language which (roughly) states that (i) if there were structures of the appropriate kind, S would hold in such structures, and (ii) it's possible that there are such structures (see Hellman 1989). In this way, the commitment to mathematical objects is avoided, and only the possibility of structures is asserted. But, clearly, mathematical language is not being taken literally.

In Hartry Field's version of nominalism, given that there are no mathematical objects, existential mathematical statements are false (Field 1980). In particular, according to Field, given that there are no numbers, the claim that "there are infinitely many prime numbers" is not true. But this simply flies in the face of the way mathematical discourse is used. To preserve verbal agreement with mathematicians, Field can, of course, introduce a fictional operator, "According to arithmetic", and, in this way, he can assert the true sentence "According to arithmetic, there are infinitely many prime numbers" without violating nominalism (Field 1989). But, once again, mathematical language wouldn't have been taken literally.

The trouble of not taking mathematical discourse literally is that it makes it hard for the nominalist to make sense of *mathematical practice*. After all, instead of understanding what mathematicians are doing when they are doing mathematics, the nominalist would have to be constantly rewriting mathematical discourse, and thus, changing a crucial part of the practice he or she is trying to understand. In its current forms, nominalism may provide a viable ontological alternative to those who worry about the existence of mathematical objects, but it's unclear whether it can yield an insightful picture of mathematical activity.

Jody Azzouni has recently developed a new way of conceptualizing nominalism in the philosophy of mathematics that is meant to overcome this problem (Azzouni 2004). On his view, mathematical theories can be taken literally, and so one can do justice to mathematical practice, but there is no commitment to the existence of mathematical objects. The key idea is that mathematical theories should be taken to be (deflationarily) true, but their truth doesn't entail the existence of mathematical entities. After all, on his view, an additional criterion of existence, which he calls 'ontological independence', has to be met.

In this paper, we argue that, as it stands, Azzouni's proposal doesn't yet succeed. It faces a dilemma to the effect that either the view isn't nominalist or it ultimately fails to take mathematics literally, and so it still doesn't do justice to mathematical practice. After presenting the dilemma, we suggest a possible way out for the nominalist.

2. A New Form Of Nominalism

According to Azzouni, two kinds of commitment should be distinguished: *quantifier* commitment and *ontological* commitment (see Azzouni 2004, 127; see also 49–122). We incur a quantifier commitment whenever our theories imply existentially quantified claims. But existential quantification, Azzouni insists, is not sufficient, in general, for ontological commitment. After all, we often quantify over objects we have no reason to believe exist, such as fictional entities. To incur an ontological

commitment—that is, to be committed to the existence of a given object—a criterion for what exists needs to be met. There are, of course, various possible criteria for what exists (such as causal efficacy, observability, and so on). But the criterion Azzouni favors, as the one that we have all collectively adopted, is ontological independence (2004, 99): what exist are the things that are ontologically independent of our linguistic practices and psychological processes. The idea here is that if we have just made something up through our linguistic practices or psychological processes, there's no need for us to be committed to the existence of the objects in question. And typically, we would resist any such commitment.

Quine, of course, identifies quantifier and ontological commitments, at least in a crucial case, namely, for the objects that are indispensable to our best theories of the world. Such objects are those that cannot be eliminated through paraphrase and over which we have to quantify when we regiment the theories in question (in first-order logic). For Quine, these are exactly the objects we are ontologically committed to. Azzouni insists that we should resist this identification. Even if the objects in our best theories are indispensable, even if we quantify over them, this is not sufficient for us to be ontologically committed to them. After all, the objects we quantify over might be ontologically dependent on us—on our linguistic practices or psychological processes—and thus we might have just made them up. But, in this case, clearly there is no reason to be committed to their existence. However, for those objects that are ontologically independent of us, we are committed to their existence.

As it turns out, on Azzouni's view, mathematical objects are ontologically dependent on our linguistic practices and psychological processes. And so, it's not surprising that, even though they might be indispensable to our best theories of the world, still we are not ontologically committed to them. Hence, Azzouni is a nominalist.

But in what sense do mathematical objects depend on our linguistic practices and psychological processes? In the sense that, in mathematical practice, the sheer postulation of certain principles is enough: "A mathematical subject with its accompanying posits can be created *ex nihilo* by simply writing down a set of axioms" (Azzouni 2004, 127). The only additional constraint that sheer postulation has to meet, in practice, is that mathematicians should find the resulting mathematics interesting. That is, briefly put, the consequences that follow from mathematical principles shouldn't be obvious, nor should they be intractable. Thus, given that sheer postulation is (basically) enough in mathematics, mathematical objects have no epistemic 'burdens' and so Azzouni calls such objects, or posits, as he uses term, 'ultrathin' (2004, 127).

The same move that Azzouni makes to distinguish ontological commitment from quantifier commitment is also used to distinguish ontological commitment to Fs from asserting the truth of "There are Fs". Although mathematical theories used in science are (taken to be) true, this is not sufficient to commit us to the existence of the objects these theories are supposed to be about. After all, on Azzouni's picture, it might be true that there are Fs, but to be ontologically committed to Fs, a criterion for what exists needs to be met. As Azzouni points out:

> I take true mathematical statements as literally true; I forgo attempts to show that such literally true mathematical statements are not indispensable to empirical science, and yet, nonetheless, I can describe mathematical terms as referring to nothing at all. Without Quine's criterion to corrupt them, existential statements are innocent of ontology. (Azzouni 2004, 4-5)

In Azzouni's picture, which is different from Quine's, ontological commitment is not signaled in any special way in natural (or even formal) language. We just don't read off the ontological commitment of scientific doctrines (even suitably regimented). After all, as noted, neither quantification over a given object (in a first-order language) nor formulation of true claims about such an object entails the existence of the latter.

Azzouni's proposal nicely expresses a view that should be taken seriously. And as opposed to current versions of nominalism, it has the significant benefit of aiming to take mathematical discourse literally. Does it succeed, though?

3. The Dilemma

Even if we grant Azzouni's move to distinguish ontological commitment to mathematical objects from quantifying over them and making true claims about them, we think Azzouni's view faces a dilemma when we consider the notion of reference. Briefly put, the dilemma goes as follows:

(a) Either '2' refers to 2, or it doesn't.

(b) If '2' refers to 2, then either Azzouni's view is not nominalist (according to his own definition of 'nominalism'), or it has a nonstandard notion of reference (in which case the language is not taken literally).

(c) If '2' doesn't refer to 2, then the proposal fails to take mathematical language literally.

Clearly, each option is problematic for Azzouni. But are the dilemma's premises true? Given that (a) is a logical truth, the crucial work is done by premises (b) and (c), and we will motivate them in what follows.

Premise (b): Suppose that '2' refers to 2. Now either it follows from this that there is something to which '2' refers or that there is nothing to which '2' refers. Suppose Azzouni thinks the former. In this case, '2' refers to something, namely, the number 2 and so there is something which is the number 2. If so, then the resulting view is not nominalist by Azzouni's own definition, since he says "my nominalism ... describe[s] mathematical terms as referring to *nothing at all*" (2004, 4–5) [his emphasis].

Perhaps Azzouni could respond that when he said that mathematical terms don't refer to anything at all, he meant they don't refer to any *existing* object. Thus, he could say that although '2' refers to something, what it refers to doesn't exist and that is what preserves nominalism. For Azzouni, quantifier commitment doesn't entail ontological commitment; the number 2 doesn't exist, given that it depends only on our linguistic and psychological practices and so fails the criterion for what exists.[1]

But this response doesn't seem open to Azzouni, for when he wants to use a notion of 'reference' in this manner, he introduces the notion reference* (2004, 61–62).[2] Our dilemma, however, concerns reference and

[1]If Azzouni were to respond in the way just described, it looks like he would be placing himself squarely into a neo-Meinongian camp, since a neo-Meinongian would both deny that quantifier commitment entails ontological commitment (Parsons 1980) and would agree that there is something which is the number 2 though it doesn't exist. Azzouni explicitly denies he is a Meinongian (2004, 72), but it is worth noting that this form of Meinongianism seems to be consistent with his form of nominalism, modulo the problem we discuss next in the text.

[2]Azzouni writes:

> When terms that refer to nothing at all occur in a language with identity, and with consistent identity conditions that allow statements like $A = B$ to be true for *distinct* terms A and B, I'll describe such terms as *co-referring** and will also, in general, speak of a term A as referring* in this sense even should it (like "Mickey Mouse" or "1", as I eventually

not reference*. Morever, even if we grant Azzouni that '2' refers to something that doesn't exist, there is still a problem. Azzouni also claims (2004, 57) that "things that don't exist ... don't *have* properties" [his emphasis]. Yet it follows, on this horn of the dilemma, that the number 2 does have properties, or at least we can formulate well-defined predications about the number 2. From the claim that '2' refers to 2, it follows both that '2' exemplifies (the property of) *referring to* 2 (i.e., that '2' is such that it refers to 2) and that 2 exemplifies (the property of) *being referred to by* '2' (i.e., that 2 is such that '2' refers to it). Thus, the resulting theory is not consistent with the extended conception of nominalism that Azzouni outlines.

Alternatively, suppose Azzouni thinks it doesn't follow from the fact that '2' refers to 2 that '2' refers to something. If so, then on his view, '2' refers to 2 but '2' refers to nothing. But then he clearly has a non-standard notion of reference. Note that we are not saying that the notion of reference requires the existence of an object to which we refer to. But it does require that there be something to which we refer. In any case, a non-standard notion of reference is indicative that mathematical language is not being taken literally. Thus, either Azzouni's proposal is not nominalist or the language is not being taken literally.

Premise (c): Suppose that '2' doesn't refer to 2. In this case, '2' would either refer to nothing or else refer to some object other than 2. In the former case, '$\exists y(y = 2)$' would be false. But this result is inconsistent with Azzouni's claim to be taking mathematical statements to be true. (After all, a mathematician could correctly infer that $\exists y(y = 2)$ from the true premises that $\exists! y(1 < y < 3)$, $1 < 2$, and $2 < 3$.) In the latter case, we wouldn't be referring to the object that '2' is normally taken to refer to. In either case, mathematical language isn't being taken literally.[3]

claim) actually refer to nothing at all (62).

In light of this very last remark, since Premise (b) talks about reference and not reference*, it would seem that Azzouni would accept Premise (b) as true, on the grounds that it has a false antecedent.

[3]Perhaps Azzouni could respond in the following way:

> "Refer," as it's often used, is ontologically committing—if a term refers, then it refers to something; so too, if two terms co-refer, then they refer to the same something. I'll acquiesce in this usage even though, as its well known, Tarski's theory of truth allows us to define a "theory of denotation." For given what I've said about the ontological neutrality of objectual quantifiers, it follows that such a theory of denotation can easily be produced for terms that refer to nothing at all. (2004, 61–2)

Both horns of the dilemma lead to the result that Azzouni's view doesn't do justice to mathematical practice, because it doesn't take mathematical language literally. Is there a way out?

4. A Nominalist Solution

We offer a friendly amendment to Azzouni's view; we think he can reject the first horn of the dilemma, i.e., reject Premise (b). We think Azzouni can say that '2' refers to 2 yet still argue that his view is nominalist. Since he can say that 2 doesn't exist, Azzouni should be able to say that he hasn't increased the ontology. However, we also think he can accept that $\exists y(y = 2)$. This latter claim employs the ontologically neutral quantifier, as distinct from the existence predicate employed in the previous claim. (This distinction between an ontologically neutral quantifier and an existence predicate is something Azzouni accepts; see 2004, 82.) Thus, given that '2' refers to something (though not something that exists), we can take mathematical discourse literally. In just a moment, we shall explain our suggestion in precise detail and show how it is consistent with Azzouni's views.

But there is complicating factor. Azzouni is reluctant to accept that $\exists y(y = 2)$ because he thinks this implies that 2 is an object and that once he accepts that, he would be required to say that 2 exemplifies properties. This, he assumes, leads to Platonism (or worse, Meinongianism). So, we are also going to show how one can agree that 2 is an object but in a deflationary sense of "object" which doesn't entail Platonism (or Meinongianism). We think this is a position consistent with Azzouni's intentions.

So how can one say that 2 doesn't exist, but is nevertheless an object in a deflated sense of "object" that shouldn't offend a nominalist like Azzouni? For the answer, we turn to object theory, as developed in Zalta 1983 and 1988, and as applied to mathematics in Linsky & Zalta 1995 and in Zalta 2000. We will offer a new, deflationary interpretation of object

Without seeing the theory, however, it is hard to determine whether this response addresses this horn of the dilemma. Moreover, by saying that '2' refers to something, we are not claiming that '2' refers to something that exists. We could agree that both quantification and reference are ontologically neutral. But, at a minimum, reference must involve reference to something. One can't simultaneously hold that '2' refers to nothing but agree with the mathematicians that '$\exists y(y = 2)$' is true.

theory that should preserve Azzouni's intuitions and still do the work that is needed to reject the first horn of the dilemma. The reader should note that we will not be adopting the interpretation of object theory used in Linsky & Zalta 1995 and Zalta 2000, though we will nevertheless use the formalism employed on those papers. Our new interpretation will not assume the Quinean reading of the quantifiers of object theory, but rather take an ontologically neutral reading, such as the one developed in Zalta 1983, when object theory was first formulated. Our new interpretation takes this reading of the quantifiers one step further, by formulating a deflationary account of abstract objects they quantify over.

Object theory is developed within a 'syntactically' second-order logic that includes a new mode of predication.[4] In addition to the traditional mode of predication '$F^n x_1 \ldots x_n$', object theory includes the "encoding" mode of predication, 'xF', which asserts that object x encodes the property F. The idea is that whereas ordinary objects exemplify (or instantiate) their properties in the ordinary sense of predication, abstract objects encode the properties by which we conceive and identify them. So it is an axiom that ordinary objects ('$O!x$') don't encode properties; i.e., $\forall x(O!x \rightarrow \neg\exists F xF)$. By contrast, abstract objects both encode and exemplify properties. The main comprehension principle for abstract objects ('$A!x$') asserts the conditions under which there are abstract objects encoding properties:

(A) $\exists x(A!x \,\&\, \forall F(xF \equiv \phi))$, where ϕ has no free xs.

Intuitively, given any expressible condition on properties ϕ, (A) asserts that there is an abstract object that encodes just the properties satisfying the condition. We will see some instances of (A) below, and there are numerous examples of instances of (A) in the works on object theory cited previously. Abstract objects are governed by the following principle of identity:

(B) $A!x \,\&\, A!y \rightarrow (x = y \equiv \forall F(xF \equiv yF))$.

Given (B), one can easily prove that for any condition ϕ, there is a unique abstract object that encodes just the properties satisfying ϕ. This theorem implies that canonical descriptions of the form $\imath x(A!x \,\&\, \forall F(xF \equiv \phi))$ are always well-defined.

[4]We say 'syntactically' second-order because object theory doesn't require full second-order logic. Its models don't require that the domain of properties be as large as the power set of the set of individuals.

Notice that if one takes the Quinean interpretation of the quantifier $\exists$, then the instances of (A) assert the existence of abstract objects. However, we agree with Azzouni that one can interpret formalisms such as the above without assuming the quantifier $\exists$ has existential import. Indeed, this was the way influential philosophers, such as Meinong 1904 and Russell 1903, used the quantifier in the very early part of the 20th century. Using this interpretation of the quantifier, one could then introduce an existence predicate ('$E!$'),[5] define ordinary objects as ones that possibly exist ('$\Diamond E!x$') and abstract objects as ones that couldn't possibly exist ('$\neg\Diamond E!x$'). The point here is that once a mathematical object such as the number 2 is identified as an abstract object, we can say that there is such a thing as the number 2 ('$\exists y(y = 2)$') but 2 doesn't exist ('$\neg E!2$'). So far, then, we are preserving Azzouni's distinction between quantifier commitment and ontological commitment when we claim that mathematical objects don't exist. Moreover, we've just defined a notion on which the number 2 is both abstract and doesn't exist.

Now to develop an account on which 2 is an object only in a deflationary sense of "object", we have to remind the reader how object theory has been applied to mathematical objects and theories. In Linsky & Zalta 1995, and in Zalta 2000, mathematical theories were themselves treated as abstract objects that encode propositions. The claim "In theory T, p" was analyzed in object theory as the atomic encoding claim "T encodes the propositional property being such that p" (i.e., $T[\lambda y\, p]$). (In what follows, we call "In theory T" the theory operator, and we abbreviate the encoding claim that $T[\lambda y\, p]$ as $T \models p$. We can read the latter more simply as: p is true in T.) Then, the axioms of mathematical theory T are added to object theory prefaced by their corresponding theory operator and with the well-defined terms and predicates of T indexed to T. Thus, the following axioms of Peano Number Theory (PNT) and ZF, respectively:

$$\text{Number}(0)$$

$$\exists x \neg y(y \in x)$$

would get added to object theory as follows:

[5]Of course, one must not suppose '$\exists y(y = x)$' and '$E!x$' are equivalent. The existence predicate has a proper subset of the domain as its extension. The idea is that the ontology to which one is committed consists only of the objects in the extension of the existence predicate.

$$\text{PNT} \models Number_{\text{PNT}}(0_{\text{PNT}})$$

$$\text{ZF} \models \exists x \neg \exists y (y \in_{\text{ZF}} x)$$

Finally, we ensure that the theory operator is closed under proof-theoretic consequence, by adding the following rule to object theory:

If $T \models p_1$, and ... and $T \models p_n$ and $p_1, \ldots, p_n \vdash q$, then $T \models q$.

Now we are in a position to identify the objects denoted by arbitrary well-defined terms of theory T. Where κ is any well-defined term of T, the following principle identifies the denotation of κ:

(C) $\kappa_T = \iota x(A!x \;\&\; \forall F(xF \equiv T \models F\kappa))$.

In other words, the object κ of theory T is the abstract object that encodes exactly the properties that κ exemplifies in theory T. (Recall that object theory guarantees that the description used in the right hand side of the identity in (C) is always well-defined.) (C) is not a definition (schema) but rather a principle that tells us that the identity of κ_T is tied to the role it has in theory T. One instance of (C) asserts, for example: 0_{PNT} is the abstract object that encodes all and only those properties F such that in PNT, 0_{PNT} exemplifies F.[6] This is a principled way of identifying the number 0 in Peano Number Theory in terms of the truths of that theory.

So far, this all still sounds Platonistic or Meinongian. But we are now in a position to see how to give a deflationary interpretation of object theory so that mathematical objects are "objects" only in a sense that should be acceptable to Azzouni. Note that Azzouni is committed to the existence of something called "mathematical practice", which consists of a variety of large-scale patterns of behavior (patterns of speech, language use, etc.) that mathematicians engage in when they do number theory, linear algebra, analysis, etc. He has to accept this much if he wants to give an analysis that is faithful to mathematical practice. But if this is correct, then it simply remains to point out that instead of interpreting (A) and (C) above as quantifying over, and identifying mathematical objects within, a domain of Platonic objects, one can reconceive (A) and (C) simply as principles that systematize our mathematical practice and thereby ground various parts of that practice. This can be spelled out as follows.

[6] A similar principle can be constructed for identifying the abstract properties and relations denoted by the well-defined predicates of T. This would require the third-order or type-theoretic formulation of object theory described in Linsky & Zalta 1995 or Zalta 2000. However, for the purposes of this paper, we shall not need to pursue the matter further, since Azzouni 2004 doesn't focus on the question of mathematical properties and relations.

The interpretation of (A) and (C) we are considering is one which starts with the mathematician's practice of stating principles, introducing new terms using the language of those principles, and proving new theorems from those principles. For example, a mathematician might start with some principles of number theory. A mathematician like Peano might come along and suppose that 0 is a number, that 0 doesn't succeed any number, that no two distinct numbers have the same successor, that every number has a successor, and the principle of mathematical induction. Call this theory PNT. Now Peano will use the term '0' in formulating consequences of these principles, such as that "0 has a unique successor which is also a number." He will assert that this claim is true in PNT. He takes himself to be referring to the number 0 when he formulates and proves this truth from the axioms of PNT. He uses the referential pronoun 'it' to say, of 0, that given PNT, it has a unique successor. He introduces a new term '1' for the object which, according to PNT, is the successor of 0. And so on. This is all part of mathematical practice. For the most part, mathematicians will omit the explicit relativization of all their claims to the scope of the theory operator "In PNT", but it is understood that their claims are all relative to some initial starting principles, such as the axioms of PNT. Whether these starting principles of PNT are true simpliciter is a question that mathematicians hardly ever bother about. (In the final section, we show the ways in which mathematicians can correctly take their starting principles to be true, and the ways in which they cannot.)

Now our comprehension principle (A) and identification principle (C) simply systematize this practice by justifying (a) the move from claims of the form "In PNT, p" to the use of the existential quantifier to quantify over the objects of PNT, (b) the practice of using referential pronouns such as 'it' to refer to well-defined mathematical objects, (c) the practice of introducing new terms and taking them to be referential within the context of a theory. Object theory justifies these practices because (A) always provides a larger context of existential quantification whenever mathematicians develop judgments of the form "In theory T, Fx". It does that by linking these judgments to existential claims through the following instances, all of which are axioms:

$$\exists x(A!x \,\&\, \forall F(xF \equiv T \models Fx)).$$

These instances of (A) systematize the mathematician's practice of taking herself to be introducing and quantifying over a domain of mathematical "objects".

To see this, note how easily the above instances introduce "objects" into our domain of discourse. On this interpretation of object theory, such objects are nothing like the conception of objects used by the Platonist, according to which they are outside spacetime and mind-independent in the sense that they would have existed even if no humans had existed. Instead, this deflationary view of the "objects" is perfectly consistent with Azzouni's idea, described above, that if we have just made something up through our linguistic practices or psychological processes, there's no need for us to be committed to the existence of the objects in question, though we can legitimately quantify over them. On our view, (A) and its special instances like the above are simply objectifying the patterns of talk and manners of speaking that the mathematicians actually engage in, and justify the introduction of referential pronouns like 'it' to refer to them. Mathematical objects are no longer "self-subsistent" and "transcendental", but rather objectified patterns the existence of which depends upon the activities of mathematicians. Surely, this is a deflationary ('ultrathin') conception of objects that Azzouni could accept, given that he accepts the existence of something called "mathematical practice" and that there are 'posits'.

Moreover, our view also justifies Azzouni's claim that "a mathematical subject with its accompanying posits can be created *ex nihilo* by simply writing down a set of axioms" (op cit.), given that an explicit comprehension principle sits in the background and readily provides an instance of comprehension whenever mathematicians do any positing.

Our new interpretation of object theory and its application to mathematics is also consistent with Azzouni's conception of having ultra-thin epistemic access to mathematical "objects". For there is nothing more to acquaintance with, and reference to, a mathematical object than introducing/understanding its identifying instance (C). Surely, this is an ultra-thin method of acquiring epistemic access to mathematical objects, and we see this as another reason for thinking that we are offering a friendly amendment to Azzouni's view.

Azzouni might object that, on our view, the number 2 (of Peano Number Theory, say) would have properties, which is something he wanted to avoid. (On his view, once you are committed to saying that $\exists y(y = 2)$, you must allow that 2 has properties.) But note first that the number 2 doesn't exemplify its mathematical properties on our view, it only encodes them. So the view doesn't require that the number 2 exemplify the property of being greater than 1, for example. We think that Azzouni was motivated, in the first instance, to avoid asserting the existence of objects that exemplified mathematical properties. We have avoided this. Note second that, on the interpretation of object theory we are describing, the number 2 (of PNT, say) becomes identified as nothing more than a reified pattern of talk (i.e., Azzouni's posits) within the larger context of mathematical practice. If 2 is a posit, it must still be something and so have properties.[7] On our view, 2 does exemplify properties, but the properties it exemplifies are ones that attach simply to its nature as a pattern or posit within the larger context of mathematical practice. It will therefore exemplify such properties as: not being a number, not being a mathematical object, not being concrete (assuming our existence predicate is coextensive with concreteness), being an artifact of such and such linguistic practice, being thought about by such and such persons, etc. We think that the fact that mathematical objects have these properties is consistent with Azzouni's desire only to have a quantifier commitment to mathematical objects, not an ontological commitment.

5. Final Observations

There is an added benefit to this friendly amendment to Azzouni's view. Before we say what this is, note that our view does take mathematical language literally. As noted, it is part of mathematical practice that mathematical statements are typically made relative to some initial starting principles, such as the axioms of PNT.[8] When mathematical language

[7] That is, it is a consequence of Azzouni's claim that mathematical objects are 'posits' that we can quantify over them. But the point is, we think Azzouni can't simultaneously claim that mathematical terms refer to nothing at all, but then still allow quantifier commitment involving those terms. For once you have quantifier commitment, you are attributing some sort of being to the objects in question and so must allow that they have properties.

[8] It might be objected here that our view will face problems, for instance, identifying the number two of number theory with the number two of analysis. Does this raise a

is relativized to the principles assumed in each mathematical context, sentences of the form, "In theory T, x is F" are taken literally. The copula is analyzed as the familiar, standard exemplification predication. But our view also accommodates unprefixed sentences of mathematics. Since we are treating unprefixed mathematical sentences of the form 'x is F' as ambiguous between 'Fx' and 'xF', they have two literal readings. On the former, they are literally false, while on the latter, they are literally true. In other words, unprefixed sentences of mathematics are literally true when they are analyzed as encoding predications, but literally false when analyzed as exemplification predications.[9] Zalta 2000 explains in some technical detail how these readings can be systematically assigned.

Now the added benefit of our analysis is that we can account for the philosophical controversy concerning the very data from mathematics. Philosophers of mathematics often talk past each other when they don't agree on the data to be explained. Indeed, there is disagreement as to whether unprefixed mathematical statements (i.e., without their theory operators) like "2+2=4" and "2 is algebraic" are true. Platonists, for instance, claim they are, whereas nominalists such as Field claim they aren't. But rarely do philosophers of mathematics try to explain why there is such disagreement. We think we have an explanation, however.

Suppose, as our view implies, that there is an ambiguity in predication that underlies the natural language of mathematics and that this ambiguity can be disambiguated using the distinction between exemplification and encoding. Then, as just noted, philosophical analysis of ordinary, unprefixed mathematical statements yields readings on which they are true and readings on which they are false. Suppose we fix the context to be Peano Number Theory, say. Then there are readings of "2+2=4" and "2 is algebraic" on which they are false (namely, by interpreting the predication as exemplification). This shows that there is something correct about Field's view. He is quite correct to argue that unprefixed mathematical sentences are false, though we think he overlooked the reading on which such sentences are true, namely, by interpreting predication as encoding. But these latter are readings of these sentences on which they are true. So there is something correct about the Platonist view as well. Platonists, too, are correct when they take mathematicians to be saying something true (when asserting unprefixed statements), though they overlook the sense in which ordinary mathematical statements are false. The point is, though, that we have an explanation on why philosophers disagree on the truth-value of the data. A subtle ambiguity in predication has undermined attempts to develop a wider perspective on the philosophy of mathematics. We hope to have provided such a wider perspective here.

difficulty for the nominalist's proposal (Azzouni's or ours)? We don't think so. Note that this style of objection raises a problem for Platonistic accounts of mathematics, which characterize mathematical practice as a true description of existing mathematical objects and relations. No such characterization is presupposed on the nominalist picture (Azzouni's or ours). In any case, in the context of a 'larger' practice (say, of a given set theory), we can identify the number two of number theory with the number two of analysis by stipulation, and explore the theoretical advantages of doing that.

[9] We hope it is clear that we don't require mathematical language to be rewritten; mathematical sentences prefixed by the theory operator can be translated into first-order and second-order logic in just the way one would expect. However, we are offering an understanding of the language on which even simple, unprefixed statements like '2 is prime', which appear to be atomic, can be analyzed as literally true atomic encoding predications or analyzed as literally false atomic exemplification predications.

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