The Metaphysics of Plural Quantification

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In a widely-cited paper, George Boolos concluded that "neither the use of plurals nor the employment of second-order logic commits us to the existence of extra items beyond those to which we are already committed" (1984, 449 [= 1998, 72]). This 'ontological innocence' of plural quantification was further defended in Lewis 1991. This conclusion about the ontological innocence of plural idioms seems to imply that the semantic notion of 'plural reference', whereby one can refer plurally to some things, is sufficiently well-understood and precise enough to provide us with clear, ontologically-inert truth conditions for sentences involving such idioms. To that end, philosophical logicians have used set theory and model theory to formulate logics and semantics for pluralities and plural quantification that validate the conclusion in Boolos' paper.

But one can question whether plural reference is as ontologically innocent as it is often claimed to be and whether it offers the best analysis of the plural idioms. Compositionality requires that the meaning of a complex expression be a function of the meaning of each part and the way in which the parts are arranged. No matter whether you think the assignment of meaning is a denotation function or a sense function, it is a function and so we might expect plural terms ('some apples', 'the Cheerios in my bowl', 'they', 'them', etc.), which form parts of sentences,

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to be assigned unique meanings. It is at least a little strange to suggest that *the* meaning of the expression 'some apples' is some apples, since this claim implies that there is an identity between "the meaning of 'some apples'" (which involves a singular reference) and some apples referenced plurally. Identity statements don't seem to work that way. So we might wonder whether we can preserve the assignment of meaning as a function and thereby avoid a non-standard understanding of identity and avoid revising the principle of compositionality.[1]

We might also wonder about the notion of a *plurality*, which is often used to discuss the logic of plural quantification. How is it to be understood so as not to commit us to "the existence of extra items beyond those to which we are already committed". At first glance, it looks like it commits us to a new kind of object. But even if it is to be regarded as eliminable (as a *façon de parler*), the key primitive notion in the logic of plural quantification, "is among" (or "is one of"), is formalized as $x \prec yy$ and thereby introduces a new class of terms (i.e., variables $xx, yy, \ldots$ and constants $aa, bb, \ldots$). But these aren't assigned singular values in the manner of standard variables and constants. Instead, plural idioms are *used* in the semantics to interpret the new terms, in much the same way that we use 'and' in the metalanguage to interpret the primitive conjunction symbol & in a formal language. Of course, this use of the plural idioms in the semantics is based on an independent axiomatization of $x \prec yy$, but any doubts one might have about whether the plural idioms are ontologically neutral aren't really addressed once the same idioms are used semantically to interpret themselves.

By all appearances, the symbol $\prec$ in the sentence $x \prec yy$ looks just like a normal binary relation, but we can't *model* such relations as sets of ordered pairs. One can't use ordinary (singular) existential generalization to infer from $x \prec yy$ that there is something z to which x is related. So at the very least, plural quantification can't remain neutral about the need for a new kind of relation. We have to suppose not only that there are relations R such that xRy but also relations R' such that $xR'yy$. The latter are needed not only in the language of plural quantification, but also in its semantics, since an assignment of meaning to the terms of the language would have to include a relation that relates singular things (e.g., the expression 'some apples') to things plurally (e.g., some apples).

[1] For a more complete discussion of this question see Section 5.4 of Linnebo 2008 [2026].

So a new type of relation seems to be needed in both the object language and metalanguage.

Moreover, it isn't exactly clear how to understand the second argument of the $\prec$ relation when formalized. The value of the variable yy in the claim $x \prec yy$ doesn't *characterize* x in any way, just as the y in $x \in y$ doesn't characterize x. By contrast, it is the nature of a property F to *characterize* the object x in true predications of the form Fx. So, for example, doesn't the property of *being an apple* still have to figure somehow into a plurality yy of some apples, so that we get a characterization of the members of yy as apples?

There are at least some reasons, then, to wonder whether plural quantification is as ontologically neutral as it is supposed to be. In this paper, we explore the intuition that pluralities are *abstractions*, conceived as objects that can be referenced singularly. For suppose that:

- one can define pluralities as abstractions of a sort delineated by a precise background theory of abstract objects,
- one can derive the axioms of the logic of plural quantification from that background theory of abstract objects,
- the background theory of abstract objects was not specifically designed for the analysis of pluralities but instead has numerous other applications and would thereby unify the theory of pluralities with those other applications,
- the background theory has a naturalized (if not nominalized) epistemology so that even those who worry about ontological commitments would have little to complain about quantifying over pluralities so analyzed, and
- all of the known models of the background theory are general, Henkin models, so that even though the theory is most naturally expressed in second-order quantified modal logic, it can be expressed, without loss, in a multi-sorted, first-order, quantified modal logic.

If these suppositions could be realized, one might have a viable alternative to the semantics of plural reference in the form of a metaphysics requiring only singular reference.

So our goal is to reduce plural notions to singular ones by using a system having only singular reference to *define* the primitive notions of plural quantification theory and *derive* the axioms assumed by that theory. The reduction and analysis will be done from the standpoint of a background logic and metaphysics that offers analyses of many other abstract objects and that doesn't require any mathematical entities or a second-order consequence relation.

We can specify the key element of this goal more exactly if we first give a precise characterization of plural quantification. According to a standard characterization (Linnebo 2004, Florio & Linnebo 2021), to formulate the language and theory of plural quantification one would typically extend the language of (first-order) predicate logic without identity by adding:

- primitive plural variables $xx, yy, \ldots$, and plural constants $aa, bb, \ldots$,
- primitive formulas of the form $\kappa_1 = \kappa_2$, where the κ_i are any individual terms (identity for pluralities is, by contrast, defined),
- primitive formulas of the form $\kappa \prec \sigma\sigma$, where κ is any individual term and $\sigma\sigma$ is any plural term, and
- primitive formulas of the form $\exists\nu\nu(\varphi)$, where $\nu\nu$ is any plural variable.

The (non-modal) theory of plural quantification then requires the stipulation of three axioms, all of which are expressible in terms of the above new primitives:

> Comprehension Schema for Pluralities:
> $\exists u\varphi(u) \rightarrow \exists xx\forall u(u \prec xx \equiv \varphi(u))$
>
> Pluralities are Nonempty:
> $\forall xx\exists u(u \prec xx)$
>
> Extensionality/Indiscernibility Schema:
> $\forall xx\forall yy\big(\forall u(u \prec xx \equiv u \prec yy) \rightarrow (\varphi(xx) \equiv \varphi(yy))\big)$

Cf. Linnebo 2004 (Sections 1.1–1.2), and Florio & Linnebo 2021 (19–20). When plural quantification theory is formulated in a modal context, then two modal axioms are typically asserted, namely, that $u \prec xx \rightarrow \Box u \prec xx$ and $\neg u \prec xx \rightarrow \Box\neg u \prec xx$.

In what follows, we analyze the above in terms of *object theory* (OT). OT is expressible in a second-order quantified modal language enhanced with one new kind of atomic formula. It has been applied to analyze abstract objects of numerous kinds. In Section 1, we introduce only the principles of OT needed for the analysis. Then, in Sections 2–3, we show that in OT, one may:

- define the notion of a *plurality*,
- introduce restricted variables and constants to range over, and denote, pluralities as defined, so that the formulas of plural quantification theory become analyzed as formulas of OT without such variables;
- define $u \prec xx$ instead taking it as a primitive;
- prove that the identity conditions for pluralities (which are stipulated in plurality theory) are derivable as a theorem;
- derive the three axioms of non-modal plurality theory as theorems; and
- derive the two axioms of modal plurality theory as theorems.

This will show the extent to which the metaphysics of abstract objects can analyze pluralities, without an appeal to the mechanism of plural reference. In Section 4, we examine how pluralities of propositions can be understood object-theoretically. In Section 5, we analyze an observation about the analysis of the plural quantifiers in terms of non-plural quantification in higher-order logic (specifically, in the typed version of OT that is expressible in relational type theory). As part of this section, we examine how pluralities of properties arise naturally in second-order OT. Finally, in Section 6, we summarize the results obtained in the previous sections and make a few final observations.

1 Object Theory Preliminaries

Let us use $x, y, z, \ldots$ as variables, and $a, b, c, \ldots$ as constants for objects, and use $F^n, G^n, \ldots$ ($n \geq 0$) as variables, and $P^n, Q^n, \ldots$ as constants for n-ary

relations. Object theory (OT) is then naturally expressed in the language of 2nd-order, quantified modal logic (2nd-order QML), extended with:[2]

- encoding formulas of the form $x_1 \ldots x_n F^n$ ('$x_1, \ldots, x_n$ *encode* F^n'),
- a distinguished, unary relation (i.e., property) term, $E!$, which we may read as *being concrete*,
- an actuality operator $\mathcal{A}$ ('*it is actually the case that*'), and
- complex term-forming operators: (rigid) definite descriptions ($\iota x \varphi$) are complex individual terms, and λ-expressions ($[\lambda x_1 \ldots x_n \varphi]$) are complex n-ary relation terms ($n \geq 0$) and are to be interpreted *relationally* (i.e., not as denoting functions); both descriptions and λ-expressions may fail to denote and so both are governed by negative free logic.

OT then distinguishes ordinary objects (i.e., objects that are possibly concrete) from abstract objects (i.e., objects that couldn't be concrete), by defining the predicates $O!$ ($= [\lambda x \Diamond E!x]$) and $A!$ ($= [\lambda x \neg\Diamond E!x]$), respectively. The three main principles of object theory are:

- Ordinary objects don't encode properties.
- If x encodes F, then necessarily x encodes F.
- For any condition φ on properties, there exists an abstract object that encodes all and only the properties such that φ.

These principles may be expressed formally, respectively, as follows:

$$O!x \rightarrow \neg\exists F(xF)$$

$$xF \rightarrow \Box xF \tag{1}$$

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)), \quad \text{for any } \varphi \text{ in which } x \text{ doesn't occur free} \tag{2}$$

[2] In what follows, we often use object language variables as metavariables when it improves readability.

Since these axioms and their applications have been discussed in numerous other papers, familiarity with them will be presupposed in what follows.[3]

Since descriptions and λ-expressions both may fail to denote, we define conditions in the object language under which such terms are considered significant, i.e., have a denotation (and so substitutable into universal claims). Using object language variables instead of metavariables to improve readability (and using p instead of F^0 as a variable for 0-ary relations), we define:

$$x\downarrow \equiv_{df} \exists F(Fx)$$

$$F^n\downarrow \equiv_{df} \exists x_1 \ldots \exists x_n(x_1 \ldots x_n F^n) \qquad (n \geq 1)$$

$$p\downarrow \equiv_{df} [\lambda x\, p]\downarrow \qquad (3)$$

So, as an instance of the first, we have:

$$\iota x Gx\downarrow \equiv_{df} \exists F(F\iota x Gx)$$

This asserts that the x that exemplifies G *exists* just in case the x that exemplifies G exemplifies a property. As an instance of the second, we have:

$$[\lambda z\, \neg Pz]\downarrow \equiv_{df} \exists x(x[\lambda z\, \neg Pz])$$

This asserts that the property *being an object that fails to exemplify P* exists just in case some object encodes it. Whenever τ is any term (i.e., individual variable, individual constant, description, relation variable, relation constant or λ-expression), then it is often convenient to abuse language and render $\tau\downarrow$ in the metalanguage by saying that τ has a denotation, or by saying that τ is significant. Indeed, formulas are defined to be 0-ary relation terms in OT, and so $\varphi\downarrow$ is defined by (3) above. Indeed, $\varphi\downarrow$ is a theorem, though we won't prove it here; intuitively, it guarantees that every formula φ denotes a proposition and can be substituted for p in universally quantified claims of the form $\forall p\psi$ (provided no variable free in φ is captured when φ is substituted for p in ψ).

Furthermore, identity is not primitive in OT but defined by cases for individuals and for n-ary relations ($n \geq 0$). The definition of identity for individuals stipulates that x and y are identical just in case either they are both ordinary objects that necessarily exemplify the same properties or they are both abstract objects that necessarily encode the same properties:

$$x = y \equiv_{df} (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)) \qquad (4)$$

Properties (i.e., unary relations) are defined to be identical just in case, necessarily, they both exist and are encoded by the same objects:

$$F = G \equiv_{df} F\downarrow \,\&\, G\downarrow \,\&\, \Box\forall x(xF \equiv xG) \qquad (5)$$

Identity for propositions (i.e., 0-ary relations) is defined in terms of property identity. If we now use p, q instead of F^0, G^0 as variables ranging over 0-ary relations, then the definition stipulates that propositions p and q are identical just in case they both exist and the property *being such that p* is identical to the property *being such that q*:

$$p = q \equiv_{df} [\lambda x\, p] = [\lambda x\, q] \qquad (6)$$

Similarly, identity for n-ary relations ($n \geq 2$) defined in terms of property identity, but since the definition won't play a significant role in what follows, we leave it to a footnote.[4]

[3] In this paper, we shall prefix the symbol $\vdash$ to theorems of OT. Strictly speaking, though, OT has two proof systems: $\vdash \varphi$ and $\vdash_{\Box} \varphi$. Both are used in the full development of OT because the system includes axioms that are contingent and so anything derived from a contingent axiom isn't subject to the Rule of Necessitation. Specifically, one of the logical axioms for the actuality operator $\mathcal{A}$ is $\mathcal{A}\varphi \rightarrow \varphi$, and this is not a necessary truth (see Zalta 1988). Consequently, OT distinguishes *φ is a modally strict theorem*, written $\vdash_{\Box} \varphi$, when φ is derivable without appealing either to the axiom $\mathcal{A}\varphi \rightarrow \varphi$ or to any theorem derived from this axiom. Instead, $\vdash \varphi$ has its usual meaning of derivability from any of the (combination of) axioms by means of the rules of inference. However, for the purposes of this paper, we'll use the notation $\vdash \varphi$ to indicate that φ is derivable without appeal to any contingencies, since we shall only be proving theorems that are modally strict. And we'll append $\star$ to the label of any claim that is, or derived from, a contingency. As mentioned above, and later in footnote 16, the Rule of Necessitation can't be applied to theorems derived from contingencies.

[4] We define:

$$\begin{aligned} F = G \equiv_{df}\ & F\downarrow \,\&\, G\downarrow \,\& \\ & \forall y_1 \ldots \forall y_{n-1}([\lambda x\, Fxy_1 \ldots y_{n-1}] = [\lambda x\, Gxy_1 \ldots y_{n-1}] \,\& \\ & \quad [\lambda x\, Fy_1 xy_2 \ldots y_{n-1}] = [\lambda x\, Gy_1 xy_2 \ldots y_{n-1}] \,\& \ldots \& \\ & \quad [\lambda x\, Fy_1 \ldots y_{n-1} x] = [\lambda x\, Gy_1 \ldots y_{n-1} x]) \end{aligned}$$

Intuitively, this tells us that n-ary relations F and G ($n \geq 2$) are identical if and only if they both exist and each way of plugging $n-1$ objects into the same positions of F and G result in identical properties.

In each of these cases, the definitions for identity imply that identity is reflexive, and that fact and the axiom for the substitution of identicals jointly imply that identity is also symmetric and transitive.

But more importantly, (2) and (4) jointly imply that for any condition φ on properties, there is a *unique* abstract object that encodes just the properties such that φ:

$$\vdash \exists!x(A!x \,\&\, \forall F(xF \equiv \varphi)), \quad \text{for any } \varphi \text{ in which } x \text{ doesn't occur free} \tag{7}$$

(For the proof of some of the theorems in this section, and of selected other preliminary theorems in Sections 2.1 and 2.2 needed for the derivation of the axioms for plural quantifications, see the Appendix.) Given (7), the definite description, *the* abstract object that encodes all and only the properties satisfying φ, is always guaranteed to have a denotation, for any condition φ on properties. So, for any given φ, $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ is a *canonical* description and may always be used as the definiens for a new constant introduced as definiendum.[5]

OT includes a theory of n-ary relations. To state this theory, we first define:

- x *occurs in encoding position* in φ just in case x occurs as the argument of an encoding formula in φ', where φ' is the formula that results by (recursively) replacing all of the defined terms in φ by their definientia.

So when we say that x *occurs free* in encoding position in φ, we mean that at least one of the occurrences of x in encoding position in φ is a free (i.e., unbound) occurrence.

Now given definitions of relation identity in (5), (6) and in footnote 4 above, the other key principles for the theory of relations are:[6] The first asserts that $[\lambda x_1 \ldots x_n \, \varphi]$ has a denotation if none of the x_i occur free in encoding position in φ:

$$[\lambda x_1 \ldots x_n \, \varphi]\downarrow, \quad \text{provided none of the } x_i \text{ occur free in encoding position in } \varphi$$

The second asserts that if a λ-expression has a denotation, then it is subject to β-Conversion:

$$[\lambda x_1 \ldots x_n \, \varphi]\downarrow \rightarrow ([\lambda x_1 \ldots x_n \, \varphi]x_1 \ldots x_n \equiv \varphi)$$

These two axioms imply a comprehension principle for n-ary relations ($n \geq 0$). Where φ is a condition on objects (i.e., the n-ary relation variable F doesn't occur free) and x doesn't occur free in encoding position in φ, there is a relation that, necessarily, is exemplified by all and only the n objects satisfying the condition. Formally:[7]

$$\exists F^n \Box \forall x_1 \ldots \forall x_n (F^n x_1 \ldots x_n \equiv \varphi), \quad \text{provided } F \text{ doesn't occur free in } \varphi \text{ and none of the } x_i \text{ occur free in encoding position in } \varphi. \tag{8}$$

[5] In OT, all of individual terms (including descriptions) are interpreted in a single, fixed domain of individuals, and all of the relation terms (including λ-expressions) are interpreted in fixed domains of relations. (OT includes S5 quantified modal logic, with fixed domains.) So the denotations of terms are not relativized to any possible world. This simplifies the denotation function in the semantics and allows one to take any formal sentence of OT, find the proposition it denotes, and then consider the truth value of that proposition in any modal context without worrying that the terms in the sentence expressing the proposition have a different denotation in such contexts.

Definite descriptions are therefore axiomatized with the help of the actuality operator. The one axiom needed is a variant of the axiom developed in Hintikka 1959:

$$y = \imath x \varphi \equiv \forall x(\mathcal{A}\varphi \equiv x = y)$$

In other words: y is the x (in fact) such that φ if and only if all and only the objects actually such that φ are identical to y. Note here that the classical axioms for the actuality operator include an axiom that is a contingent logical truth: $\mathcal{A}\varphi \rightarrow \varphi$ is true at the actual world of every interpretation (since $\mathcal{A}\varphi$ is true (in any context) iff φ is true at the actual world. But $\mathcal{A}\varphi \rightarrow \varphi$ isn't a necessary truth. If φ is true at the actual world but false at w_1, then $\mathcal{A}\varphi \rightarrow \varphi$ is false at w_1. where $\mathcal{A}\varphi$ is true (because φ is true at the actual world) and φ is false. So in using definite descriptions in OT, one has to be careful not to apply the Rule of Necessitation to any theorem whose justification depends on this contingent axiom.

[6] These are not the only axioms governing λ-expressions. OT also asserts the identity of denoting λ-expressions that are simply alphabetic variants (= α-Conversion). And OT endorses η-Conversion, which identifies an *elementary* λ-expression of the form $[\lambda x_1 \ldots x_n \, F^n x_1 \ldots x_n]$ with F^n.

[7] Suppose that none of the x_i occur free in encoding position in φ. Then by the first axiom, $[\lambda x_1 \ldots x_n \, \varphi]\downarrow$, and so by the second axiom, $[\lambda x_1 \ldots x_n \, \varphi]x_1 \ldots x_n \equiv \varphi$. Since this biconditional doesn't depend on any assumptions about the x_i, the Rule of Generalization lets us infer:

$$\forall x_1 \ldots \forall x_n ([\lambda x_1 \ldots x_n \, \varphi]x_1 \ldots x_n \equiv \varphi)$$

And since we derived this last result without appealing to contingencies, Rule RN lets us infer:

$$\Box \forall x_1 \ldots \forall x_n ([\lambda x_1 \ldots x_n \, \varphi]x_1 \ldots x_n \equiv \varphi)$$

Then, provided F doesn't occur free in φ, we may existentially generalize to obtain $\exists F^n \Box \forall x_1 \ldots \forall x_n (F^n x_1 \ldots x_n \equiv \varphi)$. This is just (8).

(8) yields all of the complex properties asserted by the comprehension schema for properties formulable in classical second-order (modal) logic (with only exemplification formulas). Given the definition of property identity in (5) above, the resulting theory of properties is hyperintensional. There are models of OT on which necessarily equivalent properties are not identical, and in the system itself, $\Box\forall x(Fx \equiv Gx)$ does *not* imply $F = G$.

Though we won't prove it here, (2) and (8) jointly imply that there are indiscernible abstract objects, i.e., that there are distinct abstract objects (i.e., abstract objects that differ by at least one encoded property) that exemplify the same properties:[8]

$$\vdash \exists x\exists y(A!x \,\&\, A!y \,\&\, x \neq y \,\&\, \forall F(Fx \equiv Fy)) \tag{9}$$

Since none of the literature on pluralities assumes that there are indiscernibles in the domain, our study will focus on pluralities of discernible objects. The property of *being discernible* is the property of being an object x such that, necessarily, for any distinct object y, there is a property that y exemplifies that x doesn't, or vice versa:

$$D! =_{df} [\lambda x \,\Box\forall y(y \neq x \rightarrow \exists F\neg(Fy \equiv Fx))]$$

The definiens in the above provably exists – see the proof in Zalta ms. (274.1). So $D!$ is well-defined. And OT implies that discernibles exist:[9]

$$\vdash \exists x D!x \tag{10}$$

[8] We can see why this is so if we temporarily take on board some model theory. In the models of OT, (2) is made true by taking abstract objects to be sets of properties. But if so, one can't take an arbitrary relation R and, for each distinct abstract object a, construct a distinct property of the form $[\lambda x\, Rxa]$, such that if $a \neq b$, then $[\lambda x\, Rxa] \neq [\lambda x\, Rxb]$. For that would yield a one-to-one correspondence from the power set of the set of properties into a subset of the set of properties, in violation of Cantor's theorem. Thus, if we take R to be $[\lambda xy\, \forall F(Fx \equiv Fy)]$, one can show that there are indiscernible abstract objects, as described in the following theorem.

[9] This follows from the fact that ordinary objects exist and are discernible. The existence of ordinary objects is guaranteed by an axiom that asserts that there might be a concrete object that isn't actually concrete. This implies that there is a possibly concrete object. And by the definition of identity for ordinary objects (i.e., they are identical whenever they necessarily exemplify the same properties), one can prove that if ordinary objects simply exemplify the same properties, they are identical. From this one can show that every ordinary object x is such that necessarily, whenever something y is distinct from it, then there is a property y exemplifies that x doesn't, thereby implying that x is discernible.

It is also a theorem of OT that the haecceities of discernible objects exist, i.e., that:

$$\vdash \forall x(D!x \rightarrow [\lambda y\, y = x]\downarrow) \tag{11}$$

Since OT also yields $\vdash D!x \rightarrow \Box D!x$, we may introduce $u, v, \ldots$ as (rigid) restricted variables ranging over discernible objects,[10] then we can express (11) as follows:

$$\vdash \forall u([\lambda x\, x = u]\downarrow) \tag{12}$$

Moreover, where $\varphi(x)$ is a formula in which x may or may not be free, it is a theorem schema that:

$$\vdash [\lambda x\, D!x \,\&\, \varphi(x)]\downarrow, \text{ for any formula } \varphi(x) \tag{13}$$

So, using a restricted variable for discernible objects, we can express (13) as:

$$\vdash [\lambda u\, \varphi(u)]\downarrow, \text{ for any formula } \varphi(u) \tag{14}$$

In the next section, we will define $u \prec xx$ and thereby make it clear that the members of a plurality are discernible. But it is important to note that all of the known models of OT are general, Henkin models, and so the theory doesn't require a second-order consequence relation. So though the theory is stated axiomatically in the language of 2nd-order QML, one could translate the axioms into a 1st-order, multi-sorted QML.

2 Pluralities of Individuals: Non-Modal

2.1 Pluralities Defined

With these preliminaries, we may define: *x is a plurality of Gs* if and only if some discernible object exemplifies G and x is an abstract object that encodes all and only those properties F that are the haecceities of those discernible objects that exemplify G:

$$PluralityOf(x, G) \equiv_{df} \exists u Gu \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(Gu \,\&\, F = [\lambda y\, y = u])) \tag{15}$$

[10] A *rigid* restricted variable is one whose value meets the restriction condition even when reasoning with that variable in modal contexts.

For those who don't object to the existence of an empty plurality, the first conjunct of the definiens could be eliminated. If we omit the conjunct $\exists u Gu$, we would have:

$$PluralityOf(x,G) \equiv_{df} A!x \,\&\, \forall F(xF \equiv \exists u(Gu \,\&\, F = [\lambda y\, y = u]))$$

Then when G is unexemplified, no F would satisfy the condition: being the haecceity of a discernible object that exemplifies G. So the null object would be a plurality of any such G. Since plurality theory typically excludes empty pluralities, the conjunct $\exists u Gu$ is needed. But one could simplify plurality theory if we allow for a null plurality by dropping the first conjunct of the definiens in (15). Such a move would eliminate the need for the antecedent in some of the conditional theorems that follow, including Comprehension Schema for Pluralities proved below as (22).

Given definition (15), it follows that for every property G exemplified by at least one discernible object, there is a unique plurality of Gs:

$$\vdash \forall G\big(\exists u Gu \rightarrow \exists! x PluralityOf(x,G)\big) \tag{16}$$

Given definition (15), we may define a *plurality* to be any object x such that, for some property G, x is a plurality of Gs:

$$Plurality(x) \equiv_{df} \exists G(PluralityOf(x,G)) \tag{17}$$

It immediately follows that there are pluralities:

$$\vdash \exists x Plurality(x) \tag{18}$$

Note that $Plurality(x)$ is a well-defined restriction condition:

- it is a formula with a single free, unrestricted variable;
- it is provable, by modally strict means, that it is non-empty, as (18) just established; and
- it has *strict existential import* in the sense that, for any object term κ substitutable for x, $\vdash_{\Box} Plurality(\kappa) \rightarrow \kappa\downarrow$, i.e., it is provable by modally strict means that $Plurality(\kappa)$ implies that κ has a denotation.

We'll see in Section 3 that $\vdash Plurality(x) \rightarrow \Box Plurality(x)$, and so *Plurality*(x) is a rigid restriction condition. But in this section, we'll be reasoning with pluralities in non-modal contexts. So following standard

notational conventions in the literature, we use $xx, yy, zz, \ldots$, and if we need (arbitrary) names we use $aa, bb, cc, \ldots$. Given these variables, we may analyze $\exists xx\varphi$ in the language of plural quantification theory as $\exists x(Plurality(x) \,\&\, \varphi^x_{xx})$, analyze $\forall xx\varphi$ as $\forall x(Plurality(x) \rightarrow \varphi^x_{xx})$. And we can analyze $\iota xx\varphi$ ("the plurality xx such that φ") as $\iota x(Plurality(x) \,\&\, \varphi^x_{xx})$.

2.2 Some Key Theorems and Definitions

To show that our definitions imply the Comprehension Schema for Pluralities, we first have to:

- prove a Corollary to (16) and (18),
- prove a Lemma on Haecceities, and
- define the notion *is one of*.

The Corollary to (16) and (18) is just a more specific version of (16) and asserts that if something is G, then there is a *unique plurality* that is a plurality of Gs:

$$\vdash \forall G\big(\exists u Gu \rightarrow \exists! xx(PluralityOf(xx,G))\big) \tag{19}$$

The Lemma on Haecceities asserts that if the haecceities of discernible objects u and v are identical, then u and v are identical:

$$\vdash [\lambda y\, y = u] = [\lambda y\, y = v] \rightarrow u = v \tag{20}$$

So, by the contrapositive: distinct discernible objects have distinct haecceities.

The definition we need is: u *is one of* (or *is among*) the plurality xx just in case xx encodes the haecceity of u, i.e.,

$$u \prec xx \equiv_{df} xx[\lambda y\, y = u] \tag{21}$$

Clearly, the definiens is not always false: given that $[\lambda y\, y = u]$ exists, by (12), it follows that there are abstract objects that encode only such haecceities. So it is provable that there are truths of the form $u \prec xx$. This is not a result by fiat or that holds by stipulation.

Thus, we can now preserve the formal representation of the following natural language sentence in OT:

There are some apples on the table.

Where A is the property of being an apple and T is the property of being on the table, we can analyze the above as in the texts on plural quantification, namely as:[11]

$$\exists xx \forall u(u \prec xx \rightarrow (Au \,\&\, Tu))$$

However, we can simplify the representation of the natural language sentence as:

$$\exists xx PluralityOf(xx, [\lambda u\, Au \,\&\, Tu])$$

When we expand the restricted variable, this becomes:

$$\exists x\big(Plurality(x) \,\&\, PluralityOf(x, [\lambda u\, Au \,\&\, Tu])\big)$$

And we can represent the difference between:

Some apples on the table are going bad.

The apples on the table are going bad.

as follows, where we make use of the quantifier $\exists$ and the definite description operator $\imath$, respectively, to bind plural variables:

$$\exists xx(\forall u(u \prec xx \rightarrow (Au \,\&\, Tu)) \,\&\, GBxx)$$

$$GB\imath xx(PluralityOf(xx, [\lambda u\, Au \,\&\, Tu]))$$

In these last two examples, we've used the property constants A and T to denote ordinary properties (*being an apple* and *being on the table*) and the constant GB denotes a distributive property, where the notion of a distributive property F is defined as: $\forall xx(Fxx \equiv \forall u(u \prec xx \rightarrow Fu))$. We'll say more about this below.

[11] See, for example, Section 1.1 of Linnebo 2004, where it is suggested that:

(2) There are some apples on the table.

can be formalized as:

(2)′ $\exists xx \forall u(u \prec xx \rightarrow Au \,\&\, Tu)$

We followed Linnebo's numbering scheme in this example.

2.3 A Proof of the Comprehension Schema for Pluralities

We may now state and prove the Comprehension Schema for Pluralities, as formulated in Linnebo 2004, Florio & Linnebo 2021, and elsewhere. Where $\varphi(u)$ is any condition on discernibles, this schema asserts that if there is a discernible object such that φ, then there is a plurality xx such that for any discernible object u, u is one of the xx if and only if u is such that φ:

$$\vdash \exists u\varphi(u) \rightarrow \exists xx \forall u(u \prec xx \equiv \varphi(u)), \quad (22)$$
where φ is any condition on discernibles in which xx doesn't occur free.

A proof can be found in the Appendix, followed by an interestingly different proof produced by Claude Code.

As noted previously, if we eliminate the first conjunct $\exists u Gu$ from (15), then we can derive the variant of (22) that omits the antecedent. We leave the proof of this variant as an exercise.

2.4 A Proof That Pluralities Aren't Empty

A second axiom of plurality theory is also derivable, namely, that for any (plurality of) things, there is something that is one of them (Linnebo 2004, Florio & Linnebo 2021):

$$\vdash \forall xx \exists u(u \prec xx) \quad (23)$$

A proof is in the Appendix.

Note that if we had omitted the first conjunct from the definiens of (15), then (23) would no longer be derivable, since the null abstract object, $\boldsymbol{a}_{\emptyset}$, which encodes no properties (and, hence, no haecceities), would satisfy the resulting definition of $PluralityOf(x, G)$, for any G that fails to be exemplified. In such cases, we would nevertheless have $Plurality(\boldsymbol{a}_{\emptyset})$, and yet $u \prec \boldsymbol{a}_{\emptyset}$ would be false for all values of u.

2.5 A Proof That Pluralities Behave Extensionally

To complete the reduction of non-modal plural quantification theory to OT, we derive the third axiom governing pluralities by deriving three lemmas. The first asserts that if abstract objects x and y encode the same properties, then they are identical:

$$\vdash (A!x \,\&\, A!y) \rightarrow (\forall F(xF \equiv yF) \rightarrow x = y) \tag{24}$$

This follows from the definition of identity (4).

The second lemma implies that pluralities are extensional entities; it asserts that if all and only the discernible objects among the xx are among the yy, then xx is identical to yy. Formally:

$$\vdash \forall xx \forall yy \big(\forall u(u \prec xx \equiv u \prec yy) \rightarrow xx = yy\big) \tag{25}$$

As a corollary to (25), the definition of identity for pluralities becomes a theorem:

$$\vdash xx = yy \equiv \forall u(u \prec xx \equiv u \prec yy) \tag{26}$$

We leave the proof as a simple exercise. Cf. Florio & Linnebo 2021 (18), who stipulate (26) as a definition.[12]

Finally, from the foregoing lemmas, we may derive the third axiom of plural quantification, which asserts that formulas of the language can't distinguish extensionally equivalent pluralities, i.e., for any formula $\varphi(xx)$ in which xx may or may not be free, if every discernible object is one of the xx iff it is one of the yy, then xx satifies φ if and only if yy satisfies φ (cf. Linnebo 2004, and Florio & Linnebo 2021, who call this the axiom of indiscernibility):

$$\vdash \forall xx \forall yy \big(\forall u(u \prec xx \equiv u \prec yy) \rightarrow (\varphi(xx) \equiv \varphi(yy))\big) \tag{27}$$

Proof: By GEN, it suffices to show $\forall u(u \prec xx \equiv u \prec yy) \rightarrow (\varphi(xx) \equiv \varphi(yy))$. So assume $\forall u(u \prec xx \equiv u \prec yy)$. Without loss of generality, we only prove that $\varphi(xx) \rightarrow \varphi(yy)$. So also assume $\varphi(xx)$. Then by a previous lemma (25), our first assumption implies $xx = yy$. But from this and our second assumption, we can substitute yy for xx for every free occurrence of xx in φ and thereby infer $\varphi(yy)$. ⋈

We have now derived non-modal axioms of plural quantification theory. Theorem schema (22) constitutes existence (comprehension) conditions and (25) constitutes identity conditions for pluralities. As a corollary to (25), theorem (27) tells us that no formulas of the language can distinguish extensionally equivalent pluralities. Moreover, these consequences have all been derived from the definition of a *plurality* (17). So

[12] Florio & Linnebo, however, introduce a special symbol ($\approx$) for the identity of pluralities.

we have defined and derived, respectively, the notions and principles of plural quantification theory without invoking any form of plural reference. One can now extend OT with predicates that apply to pluralities distributively, and ones that apply to pluralities collectively. As we saw earlier, when introducing a distributive predicate, say P, one would stipulate, in addition to the definition of P (if P is not primitive), that xx are some things that exemplify P if and only if each u among the xx exemplifies P:

$$\forall xx(Pxx \equiv \forall u(u \prec xx \rightarrow Pu))$$

And for collective predicates, one would assert the negation of this principle.

3 Pluralities in a Modal Context

3.1 A Preliminary Theorem

Before we turn to the derivation of the two modal axioms of plural quantification theory, we must establish one important lemma, namely, that $\Box(Plurality(x) \rightarrow \Box Plurality(x))$. This shows that $Plurality(x)$ is a rigid restriction condition on objects and that our restricted variables $xx, yy, \ldots$ are rigid restricted variables. This is essential for reasoning with these variables in modal contexts, since we must be assured that any entity that has been assigned to the variable xx still meets the restriction condition even if we reason about xx in a modal context.

The key to showing that $Plurality(x)$ is rigid is Gallin's Extensional Comprehension Principle, which is stipulated as an axiom in Gallin 1975 (77), but which is derivable in OT either as a theorem or as a theorem schema.[13] We'll identify and discuss Gallin's principle in more detail later, in Section 5.1. But for now, it is important to know that with his principle as a theorem, we can now establish:

$$\vdash \Box(Plurality(x) \rightarrow \Box Plurality(x)) \tag{28}$$

A proof is in the Appendix.

[13] Indeed, Gallin's Extensional Comprehension Principle (1975, 77) is derivable in OT in both second-order and type-theoretic forms. Credit goes to Daniel Kirchner for its derivation, which relies on a theorem Kirchner contributed. See the discussion in Section 5.1.

3.2 Derivation of the Modal Axioms

With (28) in place, we can now derive the two modal axioms of plural quantification theory:

- If u is one of xx, then necessarily u is one of xx.
- If u fails to be one of xx, it necessarily fails to be one of xx.

Versions of these principles have been endorsed in a number of places, including Forbes 1989 (93–102), Bricker 1989 (386–390), Williamson 2003 (457), Rumfitt 2005 (Section VII), Williamson 2010 (699–700), Uzquiano 2011, Williamson 2013 (245–249), Linnebo 2016, and Florio and Linnebo 2021 (ch. 10).

Since OT operates in a fixed domain, S5 environment, we can formally represent these principles in the following, simple manner:

$$\vdash u \prec xx \rightarrow \Box u \prec xx \tag{29}$$

$$\vdash \neg u \prec xx \rightarrow \Box \neg u \prec xx \tag{30}$$

The first is derivable from the definition of $\prec$ in (21); the second is derivable from the first in S5 modal logic. Proofs of both modal claims are in the Appendix.

One issue that arises concerns pluralities of contingently existing objects and what such pluralities are like at other possible worlds. Accordingly, in the later versions of Linnebo 2008 [2026], and in Florio & Linnebo 2021 (205), these principles are reformulated by adding one or more existence claims as an antecedent.[14] However, the issues that give rise to such reformulations are handled differently in OT, as we'll see in the next section.

[14] The two principles discussed in the 2026 version of Linnebo 2008 [2026] (Section 2.3), are:

- $u \prec x \rightarrow \Box(Exx \rightarrow u \prec x)$
- $\neg(u \prec x) \rightarrow \Box(Eu \,\&\, Exx \rightarrow \neg(u \prec x))$

But in Florio & Linnebo 2021 (205), they are formulated as:

- $\Box\forall x\forall yy(x \prec yy \rightarrow \Box(Eyy \rightarrow x \prec yy))$
- $\Box\forall x\forall yy(x \not\prec yy \rightarrow \Box x \not\prec yy)$

3.3 Pluralities and Contingency

As just noted, one might be tempted to accommodate pluralities of contingent objects by reformulating (29), and possibly (30), in the manner described in footnote 14. But in the S5, fixed domain setting of OT, this is unnecessary, since all objects exist necessarily. Contingent objects are those that are concrete at some worlds but not at others.[15] These facts underlie theorem (28); if something is a plurality, then it is necessarily a plurality. Consequently, (29) and (30) suffice as the principles asserting the rigidity of $\prec$. Although Florio & Linnebo 2021 (Sections 10.5–10.9) outline a number of arguments for either deriving or endorsing the rigidity of $\prec$, these arguments involve additional premises. By contrast, no additional premises are needed to derive (29) and (30).

To fully understand how contingency and pluralities are related in OT, note first that when G is a property that holds contingently of some object u, and xx is introduced as $PluralityOf(xx, G)$, then the proof of $u \prec xx$ will fail to be modally strict. For if u exemplifies G contingently, then when the claim Gu is introduced into OT, by asserting $Gu \,\&\, \neg\Box Gu$ as an axiom or hypothesis, then no consequence derived from such a claim can be necessitated by the Rule of Necessitation (RN)! As noted in footnote 3, we add a $\star$ to the label on contingent axioms and hypotheses, as well as on theorems derived from such contingencies, so as to make it clear that such facts aren't subject to RN.[16] An example of the foregoing

[15] Abstract objects, which are necessarily nonconcrete, are not considered contingent beings, though it is important to recognize that though they necessarily encode any property they encode, they may exemplify properties contingently (e.g., it is a contingent fact that Zero was thought about by the Mayans, and that Holmes is more famous than Poirot). But among the ordinary objects, i.e., among the objects that might be concrete, there are two subcategories that are to be considered contingent beings. In OT, we may divide the ordinary objects into three groups: (a) objects that are concrete but possibly not ($E!x \,\&\, \Diamond\neg E!x$), such as the concrete things at our world; (b) objects that are not concrete but possibly are ($\neg E!x \,\&\, \Diamond E!x$), such as the objects that might be million carat diamonds and talking donkeys; and (c) necessarily concrete beings ($\Box E!x$), such as Spinoza's God. The objects in subcategories (a) and (b) are contingent beings, whereas any objects in subcategory (c) are not.

[16] As also noted in footnote 3, OT already includes one such axiom, namely, the contingent axiom for the actuality operator $\mathcal{A}$ (namely, $\mathcal{A}\varphi \rightarrow \varphi$). The modal closures of all the axioms except for this axiom are then taken as axioms. So the Rule of Necessitation may be derived in the form: If φ can be derived from a premise set Γ without appeal to a contingent axiom or theorem, then $\Box\varphi$ is derivable from the necessitations of the premises in Γ, i.e., If $\Gamma \vdash_{\Box} \varphi$, then $\Box\Gamma \vdash \Box\varphi$. When Γ is empty, the rule says: if $\vdash_{\Box} \varphi$, then $\vdash \Box\varphi$. But as noted in footnote 3, we'll use $\vdash$ instead of $\vdash_{\Box}$ and indicate theorems that aren't modally

should help.

Although OT doesn't assert any contingent facts about particular ordinary objects, we can consider what happens if we apply and extend OT so as to assert such a contingency. Suppose we take on board the fact that Socrates is an ordinary object who is contingently wise, by asserting, either as axioms or as hypotheses, the following:

- $O!s$
- Ws (S)★
- $\Diamond\neg Ws$

From $O!s$ and (S)★, it follows that $\exists u Wu$. Note that this conclusion depends on (S)★ and so isn't subject to RN. This applies to the next inference as well, since by (19), it follows that there is a unique plurality, say xx, that is a plurality of the wise. Given $PluralityOf(xx, W)$, definition (15) implies:

$$\exists u Wu \,\&\, A!xx \,\&\, \forall F(xxF \equiv \exists u(Wu \,\&\, F = [\lambda y\, y = u])) \qquad \text{(A)}\star$$

We've starred this conclusion given its derivation history. Now to prove that Socrates is one of the wise ('$s \prec xx$'), we have to prove $xx[\lambda y\, y = s]$. So, from (S)★ and the reflexivity of identity, we know:

$$Ws \,\&\, [\lambda y\, y = s] = [\lambda y\, y = s]$$

which yields, by the fact that in OT ordinary objects are discernibles:

$$\exists u(Wu \,\&\, [\lambda y\, y = s] = [\lambda y\, y = u])$$

So by the third conjunct of (A)★, we may infer:

$$xx[\lambda y\, y = s] \qquad \text{(B)}\star$$

Again, (B)★ has been marked with a ★ because it has been derived from a contingency. So RN can't be applied to (B)★ to obtain its necessitation.

But now notice that since it is axiomatic in OT that $xF \rightarrow \Box xF$, we may derive the following from (B)★:

$$\Box xx[\lambda y\, y = s] \qquad \text{(C)}\star$$

So, by definition (21), it follows from (C)★ that:

strict with a ★.

$$\Box s \prec xx \qquad \text{(D)}\star$$

In (D)★, we have derived a necessary truth whose proof depends on a contingency! The presence of such truths, and how they are derived, may explain some of the debate in the texts referenced earlier, concerning the question of whether $\prec$ is, or should be, a rigid condition.[17]

The foregoing discussion also helps us to understand a related result. Though (28) shows that $Plurality(x)$ is a rigid condition on objects, one can show, without too much effort, that $PluralityOf(x, G)$ is not a rigid condition on objects and properties. To show this, however, we must be able to reason in a modal context in which there is contingency, for otherwise, if $\varphi \equiv \Box\varphi$ is generally true, then every open formula is a rigid condition. OT rules out general modal collapse (i.e., rules out models in which there is a single possible world) by including an axiom that asserts: there might be a concrete object that isn't actually concrete, i.e., $\Diamond\exists x(E!x \,\&\, \neg\mathcal{A}E!x)$.[18] This axiom is defended at length in Zalta forthcoming, and the reader may look there for a justification. It is shown there, and elsewhere, that this axiom implies the existence of contingently true propositions, contingently false propositions, and contingently exemplified properties.[19] Given these facts, it is a theorem that:

$$\vdash \Diamond\exists x\exists G(PluralityOf(x, G) \,\&\, \neg\Box PluralityOf(x, G)) \qquad (31)$$

Note that (31) is equivalent to the claim that it is not necessary that for every object x and property G, if x is a plurality of G, then necessarily x is a plurality of G:

$$\vdash \neg\Box\forall x\forall G(PluralityOf(x, G) \rightarrow \Box PluralityOf(x, G)) \qquad (32)$$

Though the equivalence of (31) and (32) can be easily established (by modal negation, interdefinability of the existential and universal quantifiers and contraposition), it should be mentioned that (32) can be inde-

[17] See Zalta 2000, where the notion of necessary truths derived from contingencies is used to help us understand Leibniz's *hypothetical necessities* in his reply to Arnauld's objection that the concept-containment theory of truth turns contingencies into necessities.

[18] This axiom forces models of the theory to include at least one possible world that is not actual. By contrast, Kripke 1959 and 1963 allowed for interpretations of modal logic in which there is only one possible world, and in such models, the claim $\varphi \equiv \Box\varphi$ holds, for every φ.

[19] See the proofs of the theorems labeled (13) and (14) in Appendix C of Zalta forthcoming, and items (218) and (220) in Zalta ms.

pendently derived from the fact that there are contingently exemplified properties.[20]

4 An Application: Pluralities of Propositions

4.1 Preliminary Definitions

One simple application of the foregoing is the OT analysis of pluralities of propositions as object-theoretically defined situations. Quantification involving plural propositional variables (i.e., $\exists pp\varphi$ and $\forall pp\varphi$) has been used in Fritz, Lederman, & Uzquiano 2021 (§4), Hall 2021 (§2), Kment 2022 (§2), Fritz 2023 (Ch. 8), and elsewhere. In OT, such variables need not be taken as primitive, but can instead be introduced as restricted variables ranging over *situations*, which are defined to be any abstract object that encodes only properties F such that F is a property of the form $[\lambda y\, p]$, for some proposition p (Zalta 1993, 410):

$$Situation(x) \equiv_{df} A!x \,\&\, \forall F(xF \rightarrow \exists p(F = [\lambda y\, p])) \tag{33}$$

OT then defines: a proposition p *is true in* a situation x just in case x encodes *being such that* p, i.e.,

$$x \models p \equiv_{df} Situation(x) \,\&\, x[\lambda y\, p] \tag{34}$$

It is provable that $Situation(x)$ is a rigid restriction condition on objects:

$$\vdash \Box(Situation(x) \rightarrow \Box Situation(x))$$

So we may introduce $s, s', \ldots$ as rigid restricted variables ranging over situations and be assured that the situation assigned to such a variable remains a situation when reasoning in any modal context. Using these restricted variables, comprehension and identity principles for situations become derivable in OT:

$$\vdash \exists s \forall p(s \models p \equiv \varphi), \text{ provided } s \text{ doesn't occur free in } \varphi \tag{35}$$

[20] We can see this if we reason semantically in terms of *primitive* possible worlds. Suppose G is contingently exemplified at the actual world, i.e., suppose Ga and $\Diamond\neg Ga$ both hold at the actual world, for some object a. Then suppose that $\neg Ga$ is true at w_1. If x is a plurality of Gs at the actual world, it encodes $[\lambda z\, z = a]$. But any object, say y, that is a plurality of Gs at w_1 encodes all and only the haecceities of objects that are G at w_1. So y won't encode $[\lambda z\, z = a]$ at w_1. Since encoded properties are necessarily encoded, x can't be y, and so x fails to be a plurality of Gs at w_1.

$$\vdash \forall p(s \models p \equiv s' \models p) \rightarrow s = s' \tag{36}$$

(35) follows from the comprehension principle for abstract objects (2). For the proof, see either Zalta 2024 (pp. 147, 165), or the proof of (23) in §2.5 of Nodelman & Zalta 2026, or Zalta under review (§3.2.1). (36) follows from the definition of identity for abstract objects (4). For the proof, see Zalta 1993 (Theorem 2) and 1991.

Given this background, we may say that an abstract object x is a *plurality of propositions* if and only if x is a situation:

$$PropositionPlurality(x) \equiv_{df} Situation(x) \tag{37}$$

If we want to exclude proposition pluralities that are empty, then we simply add a conjunct asserting $\exists p(x \models p)$ to the definiens. (The null abstract object, which encodes no properties, satisfies the definition of a situation, by failure of the antecedent of the second conjunct of the definiens in (33); so the null object qualifies as a proposition plurality as they are defined in (37). But if we add the conjunct just described, we rule out the null object as a plurality of propositions). Now since $PropositionPlurality(x)$ is a rigid condition on objects (exercise), we may introduce $pp, qq, \ldots$ as rigid, restricted variables ranging over proposition pluralities. Thus, if we extend our use of the symbol $\prec$ defined in (21), so that it may also stand between a proposition variable and a plural proposition variable, we may define q *is one of* pp as q is true in pp:

$$q \prec pp \equiv_{df} pp \models q \tag{38}$$

Moreover, since any formula φ can be substituted for the propositional variable q (provided no variable free in φ becomes captured by a variable-binding operator when φ is substituted for q), the schema $\varphi \prec pp$ is defined for any formula φ.[21]

[21] Recall that the definiens of (38), $pp \models q$, is defined as $pp[\lambda x\, q]$. So we can't, for example, substitute Rx for q, since λx in $[\lambda x\, q]$ would capture the free x in Rx; the result, $pp[\lambda x\, Rx]$, violates the condition on pp, namely, that it is a situation that encodes only propositional properties of the form $[\lambda x\, \varphi]$ in which the x is vacuously bound by the λ. Of course, $\varphi \prec pp$ is still defined for any formula φ. In the case where we want to substitute Rx for q in $pp \models q$, we simply expand $pp \models q$ to an alphabetic variant of the λ-expression, i.e., expand $pp \models q$ to $pp[\lambda y q]$. OT includes α-Conversion as an axiom governing λ-expressions, and one of its instances is $[\lambda x q] = [\lambda y q]$. So we may substitute Rx for q in $pp[\lambda y q]$ to obtain $pp[\lambda y\, Rx]$. The latter asserts that pp encodes the propositional property *being such that* Rx, since the y is vacuously bound.

4.2 Summary of Fritz 2023

These developments help us to show that the principles discussed in Fritz 2023 (Chapters 7–8) can be derived instead of stipulated. These chapters are interesting because they not only lay out the principles of possible world theory, but also show that there are two ways to satisfy the principles of possible world theory, namely, (a) by conceiving worlds as propositions (and using plural quantification over propositions to prove that there are maximal and possible propositions), or (b) by conceiving worlds as pluralities of propositions of a certain kind. As we'll demonstrate below, these results are already captured in OT, without any further stipulations. To see how this comes about, we first lay out the elements of Fritz's discussion and then show how the conclusions he develops can be derived in OT without adopting the principles he uses as premises.

The key principles that play a role in Fritz's discussion are:

- *p strictly implies q* if and only if necessarily, if p, then q (2023, 126):

  $$p \dashv q \;:=\; \Box(p \rightarrow q)$$

- q is a *maximal* proposition if and only if for every proposition p, q strictly implies $\neg p$ iff q fails to strictly imply p (2023, 131):

  $$\mathrm{M}q \;:=\; \forall p((q \dashv \neg p) \leftrightarrow \neg(q \dashv p)) \tag{M1}$$

  Fritz shows that this definition implies:

  $$\mathrm{M}q \equiv \Diamond q \wedge \forall p(q \dashv p \vee q \dashv \neg p) \tag{M2}$$

- If p is possible, then there is a maximal proposition that strictly implies it (2023, 133):

  $$\Diamond p \rightarrow \exists q(\mathrm{M}q \wedge (q \dashv p)) \tag{W0§}$$

- Necessarily, there is a maximal proposition that is true (2023, 134).

  $$\Box\exists p(\mathrm{M}p \wedge p) \tag{Atomicity}$$

The other principles that play a role in Fritz's discussion will be formalized in footnotes.

Now there are five important steps in Fritz 2023 that lead him to use pluralities of propositions to justify the necessary existence of a maximal proposition that is true and to then conclude that world talk is validated either by appeal to such maximal propositions *or* by appeal to certain pluralities of propositions.

Step 1: Using a language with primitive variables $w, v, \ldots$ ranging over possible worlds, and a primitive formula of the form $w \rhd \varphi$, state the basic principles of possible world theory (2023, 127–128).[22]

Step 2: Show that possible world talk can be interpreted as talk about maximal propositions, since the basic principles of possible world theory in Step 1 are satisfiable by maximal propositions (2023, 130–132).

Step 3: Show that the principles (W0§) and Atomicity, as introduced above, are equivalent. Then argue that since talk of worlds can be reduced to talk of maximal propositions, the truth of Atomicity would show that talk of possible worlds is legitimate (2023, 134).

Step 4: Prove Atomicity by introducing primitive notions and axioms for pluralities of propositions (2023, 142–146),[23] and then conclude that "Plural propositional quantifiers therefore vindicate appeals to possible wolds [sic], and more specifically show that talk of possible worlds may be understood as talk of maximal propositions" (2023, 148).

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[22] Where the notion *w is actual* ('Aw') is defined as $\forall p(p \leftrightarrow w \rhd p)$, Fritz states these principles formally as:

(WN) $\Box p \leftrightarrow \forall w(w \rhd p)$
(WP) $\Diamond p \leftrightarrow \exists w(w \rhd p)$
(W1) $\forall w \Diamond \mathrm{A}w$
(W2) $\forall w(\Diamond w \rhd p \rightarrow \Box w \rhd p)$
(W3) $\forall w \forall v(\forall p(w \rhd p \equiv v \rhd p) \rightarrow w = v)$

He points out that these are not all independent and so, as a minimal world theory, he takes the left-to-right direction of (WP), and W1 – W3, to be axioms.

[23] Fritz takes the following as axioms governing pluralities of propositions:

(UI^p) $\forall pp\varphi \rightarrow \varphi[qq/pp]$, when qq is substitutable for pp in φ
(R$\prec$) $p \prec pp \rightarrow \Box(p \prec pp)$
(PC) $\exists pp \forall p(p \prec pp \equiv \varphi)$, provided pp not free in φ

and takes the following as a rule:

(UG^p) $\varphi \rightarrow \psi \vdash \varphi \rightarrow \forall pp\psi$, provided pp not free in φ

He then notes that if one were to restrict comprehension to non-empty conditions φ (i.e., conditions φ that are satisfied), then axiom (PC) would be revised and another axiom would be added:

Step 5: Finally, as an alternative to analyzing possible worlds as maximal propositions, analyze them as pluralities of propositions that satisfy the axioms of possible world theory. Fritz concludes "Maybe surprisingly, this section has shown that it is immaterial whether one thinks of worlds as propositions or as pluralities of propositions: there are propositions playing the role of worlds if and only if there are pluralities of propositions playing the role of worlds" (2023, 149).

We can now show how the elements of this discussion are preserved, indeed, derivable, in OT.

4.3 Deriving the Principles in OT

We begin with Step 1. Note that the 5 principles of possible world theory that Fritz introduces in Step 1 (2023, 127–128) and reproduced in footnote 22 above, are all theorems of OT. In Zalta 1993, possible worlds were identified as situations that *might* be such that they make true all and only the truths (1993, 414):

$$\mathit{PossibleWorld}(s) \equiv_{df} \Diamond\forall p(s \models p \equiv p) \tag{39}$$

If we then replace Fritz's primitive notion $\triangleright$ with OT's defined notion $\models$ (34), then Fritz's axioms (WN) and (WP) correspond to Theorems 24 and 25 in Zalta 1993 (418).[24]

$$\vdash \Box p \equiv \forall w(w \models p) \tag{40}$$

$$\vdash \Diamond p \equiv \exists w(w \models p) \tag{41}$$

Moreover, once Fritz defines (2023, 128) an actual world to be one that makes true all and only the truths, a version of his principle (W1) ("every world might have been actual") becomes a theorem of OT. That's because in OT, an actual situation is defined as one such that every proposition true in it is true (Zalta 1993, 413):

$$\mathit{Actual}(s) \equiv_{df} \forall p(s \models p \rightarrow p)$$

So from the very definition of a possible world in (39) above it follows *a fortiori* that every world is possibly actual. And the final two principles that Fritz puts forward for possible world theory, namely W2 ("if it is possible that p is true in w, then it is necessary that p is true in w") and W3 ("possible worlds are identical if same propositions are true in them"), are both theorems of OT. Given the fact that possible worlds are identified as situations, then W2 is captured by Lemma 2 in Zalta 1993 (415) and W3 is captured by Theorem 2 in Zalta 1993 (412).

Therefore, OT's theory of possible worlds implies the axioms Fritz introduces as the main principles of possible world theory. Moreover, by defining pluralities of propositions as situations in (37) above, we've started the process of deriving the suggestion, in Step 5, that pluralities can play the role of possible worlds. We'll complete the derivation below, when we also derive the principles Fritz introduces to govern proposition pluralities.

In Step 2, Fritz translates the principles governing possible worlds into principles governing propositions and maximal propositions, as he's defined the latter in (M1) and (M2) above. Intuitively the idea is that when a possible world w is represented as maximal proposition p_w, then what is true in w is represented by what is necessarily implied by p_w. A claim quantifying over all possible worlds becomes a claim quantifying over all maximal propositions; and worlds w and v are identical whenever p_w and p_v are identical.

In OT, we use $p \Rightarrow q$ instead of $p \,\square\!\!\!\rightarrow\, q$ for the definiens $\Box(p \rightarrow q)$. So (an alphabetic variant of) definition (M1) is expressed as follows:

$$\mathit{Max}(p) \equiv_{df} \forall q((p \Rightarrow \neg q) \equiv \neg(p \Rightarrow q)) \tag{M1$'$}$$

and theorem (M2) is preserved in the form:

$$\vdash \mathit{Max}(p) \equiv \Diamond p \,\&\, \forall q(p \Rightarrow q \vee p \Rightarrow \neg q) \tag{42}$$

Moreover, the fact that a maximal proposition is necessarily maximal is preserved:

$$\vdash \mathit{Max}(p) \rightarrow \Box\mathit{Max}(p) \tag{43}$$

The proofs of (42) and (43) are in the Appendix.

There is, however, one proviso to reconstructing Step 2 in OT. Fritz relies on the 'intensionalism' principle, which asserts that propositions

(PC$'$) $\exists p\varphi \rightarrow \exists pp\forall p(p \prec pp \equiv \varphi)$, provided pp isn't free in φ
(PE) $\exists p(p \prec pp)$

We'll see below that these axioms and rules can be derived as theorems of OT.

[24] The proofs of these principles were provided in an early version of Zalta 1993, namely Zalta 1991. Menzel & Zalta 2014 review and explore the proof of (41) and discuss its ontological commitments and epistemological implications.

are identical when necessarily equivalent. He adopts this principle in Sections 7.2 (2023, 127. It is central to his subsequent work in Chapters 7 and 8, since it guarantees that necessarily equivalent maximal propositions are identical, and so guarantees that distinct possible worlds are represented by distinct and unique maximal propositions. But OT doesn't endorse this principle and we'll discuss the consequences of this when we get to Step 5.

In Step 3, Fritz establishes that (W0§) and ATOMICITY are equivalent; this fact is derivable in OT, since the reasoning takes place in S5 QML with quantification over propositions:

$$\vdash \forall p\big(\Diamond p \rightarrow \exists q(Max(q) \,\&\, (q \Rightarrow p))\big) \equiv \Box\exists p(Max(p) \,\&\, p) \tag{44}$$

The proof in the Appendix is much less compressed than the one developed in Fritz 2023 (134).

Now to reconstruct Step 4, recall (37) and the discussion immediately following, where we defined a proposition plurality as a situation, introduced rigid, propositional plurality variables $pp, qq, \ldots$ by definition rather than as primitive, and defined the formula $\varphi \prec pp$ rather than taking it as primitive (38). From these definitions, *all* of the axioms and rules for pluralities of propositions stipulated in Step 4 above become theorems and derived rules of OT. The first group of axioms in Fritz 2023 (144), which are reproduced in footnote 23 above, become the following theorems:

$$\vdash \forall pp\varphi \rightarrow \varphi[qq/pp], \text{ when } qq \text{ is substitutable for } pp \text{ in } \varphi \tag{45}$$

$$\vdash p \prec pp \rightarrow \Box(p \prec pp) \tag{46}$$

$$\vdash \exists pp\forall p(p \prec pp \equiv \varphi), \text{ provided } pp \text{ not free in } \varphi \tag{47}$$

And the primitive rule (UG^{p}) becomes the derived rule:

$$\varphi \rightarrow \psi \vdash \varphi \rightarrow \forall pp\psi, \text{ provided } pp \text{ not free in } \varphi \tag{48}$$

Moreover, if we want to enforce the existence of pluralities corresponding only to conditions φ that are satisfied, then when we add the conjunct $\exists p(s \models p)$ to the definiens of (37), the two alternative axioms Fritz proposes (2023, 145), (PC′) and (PE), become the theorems:

$$\vdash \exists p\varphi \rightarrow \exists pp\forall p(p \prec pp \equiv \varphi), \text{ provided } pp \text{ isn't free in } \varphi \tag{49}$$

$$\vdash \exists p(p \prec pp) \tag{50}$$

The proofs of all of the above can be found in the Appendix. However, in each case, when we eliminate the defined terms, the result can be readily identified as a theorem of OT.[25]

Now, as previously noted, the main result in Step 4 is to derive ATOMICITY, i.e., $\Box\exists p(\mathrm{M}p \wedge p)$, from the facts about pluralities of propositions. But ATOMICITY is already a theorem of OT. If we use OT notation, i.e., use & instead of $\wedge$, $\Rightarrow$ instead of $\preccurlyeq$ and $Max(p)$ instead of $\mathrm{M}(p)$, then ATOMICITY becomes the following theorem:

$$\vdash \Box\exists p(Max(p) \,\&\, p) \tag{51}$$

The proof in the Appendix appeals to theorems of OT that were first developed in Zalta 1993 and then extended in Zalta ms. In particular, it relies on a modally strict proof of the Kaplan-Fine axiom (Kaplan 1970, Fine 1970), which asserts the existence of a proposition that necessarily implies every proposition, i.e., $\exists p(p \,\&\, \forall q(q \rightarrow \Box(p \rightarrow q)))$. A derivation of this principle in OT was contributed by Daniel West.[26] The proof derives the necessitation of the axiom, i.e., $\Box\exists p(p \,\&\, \forall q(q \rightarrow \Box(p \rightarrow q)))$, but it begins by noting that by the fundamental theorem (40) above, it suffices to show:

- $\forall w\big(w \models \exists p(p \,\&\, \forall q(q \rightarrow \Box(p \rightarrow q)))\big)$

The proof then establishes this for an arbitrary world w, and by exporting the quantifier (which is valid in OT), proceeds to show:

[25] When we eliminate the defined terms and restricted variables $pp, qq, \ldots$ by using (33), (34), and (38), then using OT's restricted variables s, s', theorems (45) – (47) become:

$\vdash \forall s\varphi \rightarrow \varphi^{s'}_{s}$, provided s' is substitutable for s in φ (instance of quantifier theorem)

$\vdash s[\lambda x\, p] \rightarrow \Box s[\lambda x\, p]$ (instance of (1), i.e., $xF \rightarrow \Box xF$)

$\vdash \exists s\forall p(s \models p \equiv \varphi)$, provided s doesn't occur free in φ (35)

Rule (48) becomes: $\varphi \rightarrow \psi \vdash \varphi \rightarrow \forall s\psi$ (provided s not free in φ), which is derivable from OT's quantification theory. And, if (as previously mentioned) we revise definition (37) by adding the conjunct $\exists p(s \models p)$ to the definiens, the alternatives (49) and (50) become the following theorems, respectively, expressed without restricted variables:

$\vdash \exists p\varphi \rightarrow \exists x(Situation(x) \,\&\, \exists p(x \models p) \,\&\, \forall p(s \models p \equiv \varphi))$, provided s isn't free in φ (by weakening (35))

$\vdash (Situation(x) \,\&\, \exists p(x \models p)) \rightarrow \exists p(x \models p)$ (logical truth)

If it isn't clear how the above were obtained by expanding definitions, then see the full proofs in the Appendix.

[26] Personal communication, January 2, 2023. Though the theorem is now item (551) of Zalta ms; the proof was first presented as item (549), in the May 25, 2023 version of Zalta ms. See https://mally.stanford.edu/principia-2023-05-25.pdf.

- $\exists p(w \models (p \,\&\, \forall q(q \rightarrow \Box(p \rightarrow q))))$

The proof concludes by showing that the formula $\forall p(p \equiv w \models p)$ ("all and only the truths are true in w") denotes a proposition that is verifiably a witness to this claim.

We report these details about the proof because Fritz derives Atomicity along somewhat similar lines. The proposition just described that serves as the witness can be expressed, in terms of the above definitions of proposition pluralities and $\prec$, as $\forall p(p \equiv p \prec pp)$. And whereas this formula is the key to West's derivation of Atomicity, Fritz used the weaker $\forall p(p \prec pp \rightarrow p)$ for the derivation in his system $\vdash_{pp}$ (2023, 146). The key difference, however, is that in OT, the pluralities required for the proof are analyzable as situations and no additional principles are required. Moreover, the very definition of a maximal plurality of propositions is the exact analogue of the definition of a possible world in Zalta 1983 and 1993. He says (2023, 148):

> Call a plurality of propositions *pp maximal* if it is possible that *pp* are all and only the truths.

This would be formalized as $Maximal(pp) \equiv_{df} \Diamond\forall p(p \prec pp \equiv p)$. Given our analysis of pluralities of propositions and $\prec$, this is an exact analogue of the definition of a possible world in Zalta 1993 (414), introduced above as (39) above.[27]

In Step 5, Fritz concludes that we can analyze possible worlds either in terms of maximal propositions or in terms of pluralities of propositions. As noted above, the analysis of worlds as maximal propositions is underwritten by the principle of intensionalism. However, OT does not imply, and is not committed to this principle; it doesn't imply $\Box(p \equiv q) \rightarrow p = q$. Identity for propositions is defined in (6), as part of OT's hyperintensional theory of n-ary relations ($n \geq 0$); one may consistently assert the existence of necessarily equivalent propositions that are distinct. So OT does not *identify* possible worlds as propositions, since that would undermine the proof of the uniqueness of the actual world and the uniqueness of each possible world. Their uniqueness is guaranteed by theorem (36), which asserts that situations that make the same propositions true are identical. By contrast, if worlds were maximal propositions, then their uniqueness would fail in OT. For suppose, for example, that there exists a proposition, say p_0, that necessarily implies every true proposition. Such a proposition would then be an actual world. But, then, in any model in which there is a proposition that is necessarily equivalent but distinct from p_0, say $p_0 \,\&\, (q \vee \neg q)$, for some proposition q, both p_0 and $p_0 \,\&\, (q \vee \neg q)$ would be distinct actual worlds.

Of course, if one wanted to preserve intensionalism and the idea that worlds are maximal propositions, then there is a simple expedient that would preserve his analysis in OT, namely, add the intensionalism principle as an axiom. This would yield his conclusion, in Step 5, that possible worlds could be analyzed either as propositions or as pluralities of propositions. And pluralities of propositions would still be needed to prove the existence of the maximal propositions that serve as possible worlds, as in Step 4. But officially, the preferred, hyperintensional formulation of OT doesn't endorse the conclusion that possible worlds are to be identified as maximal propositions.[28]

[27] Indeed, the definition in Zalta 1993 is equivalent to the one in Zalta 1983 (79).

5 Some Observations and Extensions

5.1 The Status of Gallin's Principle

Fritz says, about his argument in Steps 4 and 5 (2023, 147):

> Variants of this argument have been given by a number of other authors. In particular, Gallin (1975, §11, see esp. theorem 11.5) shows in a higher-order modal logic that Atomicity follows from an axiom of comprehension for rigid properties. Fine (2005 [1977], pp. 136–137) informally presents a version of the argument using sets, which is worked out more formally in Fine (1980, pp. 192–195, see esp. theorem 37). A similar argument in a different formal setting can be found in Menzel and Zalta (2014). Gallin and Fine both argue model-theoretically, in the case of Gallin using a completeness result for a class of general ("Henkin") models; Menzel

[28] Another argument as to why possible worlds should be analyzed as situations and not as propositions is based on a fact about the notion of encoding. The formula xF (x encodes F) is a mode of *predication*. It is a way for F to *characterize* x. So when possible worlds are identified as situations and p is true in w is defined as $w \models p$, i.e., $w[\lambda x\, p]$, the propositions true at a world w *characterize* w by way of a predication. By contrast, if worlds were propositions and p is true at w is defined as $\Box(w \rightarrow p)$, w is not *characterize* by p via a predication. OT preserves Wittgenstein' opening Tractarian claim (1921), that "the world is all that is the case', as the theorem that the actual world encodes every true proposition.

> and Zalta reason deductively. The novelty of the present formulation is the appeal to plural propositional quantification.

And he follows up in the next chapter by noting how one could "[eliminate the] plural quantifiers using restricted higher-order quantifiers", saying (2023, 162):

> The resulting modal theory of propositions and their properties could be formulated in greater generality, and its formal presentation could be investigated in more detail. For example, it would be natural to consider a fuller higher-order language, such as the language of relational type theory $\mathcal{L}^*$ motivated in Chapter 1. However, this has effectively already been done by Gallin (1975, part II). Gallin considers a system which he calls $ML_p + C + EC$, defined in his book on p. 77. This system includes a principle of "extensional comprehension," which postulates the existence of a rigid property—a property which applies necessarily to everything it possibly applies to—for every defining condition. Higher-order quantification over such rigid properties can be used to simulate plural quantification, in the sense that an extension of Gallin's system by higher-order plural quantifiers proves that the proposition expressed by any formula involving plural quantifiers is identical to the proposition expressed by any corresponding formula in which plural quantifiers are replaced by the relevant higher-order quantifiers restricted to rigid properties.

This is a high-level observation, which doesn't seem to be limited just to pluralities of propositions, but rather to pluralities of any higher type. Indeed, the observation has a counterpart in OT. Here is why.

OT has been developed in relational type theory from its early conception (see Zalta 1982; 1983 (Chapters V, VI); 1988 (Chapters 9–12); and 2020). In the most refined typing scheme for OT, i is the type for individuals and $\langle t_1,\ldots,t_n\rangle$ is the type for n-ary relations among objects having types $t_1,\ldots,t_n$, respectively. It is then straightforward to type the language of OT, and then assert typed versions of the axioms. The typed version of the comprehension axiom for abstract objects asserts, for any type t and any formula φ placing a condition on properties having type $\langle t\rangle$, the existence of an abstract object of type t that encodes exactly the properties satisfying φ. Given this formulation, two facts about typed

OT become relevant: (i) Gallin's axiom schema for extensional comprehension is derivable as a theorem in both second-order OT and in full generality in typed OT, and (ii) in typed OT's reconstruction of plural quantification at higher types, the extensional comprehension principle plays a more limited role. We discuss these in turn.

First, Gallin's extensional comprehension principle has been derived in both second-order and typed OT, both as a theorem and as a schema. The full description of this derivation is the subject of another research project, but here is a overview of this work and its current documentation. Recall that Gallin (1975, 77) started with the following definition of a *rigid* n-ary relation f of arbitrary relational type (suppressing type indices):

- $\mathrm{Rn(f)} : \forall x^0 \ldots \forall x^{n-1}[\Box f x^0 \ldots x^{n-1} \vee \Box\neg f x^0 \ldots x^{n-1}]$

It can be shown, in both second-order OT and typed OT, that this definition is equivalent to saying that a rigid relation is one that applies necessarily to everything it possibly applies to (cf. the quote from Fritz above), and equivalent to saying that a rigid relation is one that necessarily, applies necessarily to anything it applies to. These facts are captured in the following theorems of OT using notation in which F can be regarded either as a second-order variable (with the x_i being individual variables) or as a higher-order relation variable of any type $\langle t_1,\ldots,t_n\rangle$ (with the x_i being variables of types $t_1,\ldots,t_n$, respectively):[29]

$$\vdash \Box\forall x_1 \ldots \forall x_n(Fx_1 \ldots x_n \rightarrow \Box Fx_1 \ldots x_n) \equiv \forall x_1 \ldots \forall x_n(\Diamond Fx_1 \ldots x_n \rightarrow \Box Fx_1 \ldots x_n)$$

$$\vdash \Box\forall x_1 \ldots \forall x_n(Fx_1 \ldots x_n \rightarrow \Box Fx_1 \ldots x_n) \equiv \forall x_1 \ldots \forall x_n(\Box Fx_1 \ldots x_n \vee \Box\neg Fx_1 \ldots x_n)$$

In what follows, we shall use the definition:[30]

$$Rigid(F) \equiv_{df} \Box\forall x_1 \ldots \forall x_n(Fx_1 \ldots x_n \rightarrow \Box Fx_1 \ldots x_n)$$

Gallin then asserts, as an axiom schema, the following comprehension principle, where f is again a variable of any relational type and the x_i are variables of the right type:

[29] For the proofs of these theorems, see the Appendix to Zalta ms. for items (578.1) and (578.2). The type-theoretic counterparts are described in Section 15.5.1.

[30] Strictly speaking, the definiens should include the conjunct $F\!\downarrow$, to allow for instances of the definition involving non-denoting λ-expressions, but we can omit this nicety for simplicity, now that it has been flagged.

- $\Box\exists f[Rn(f) \wedge \forall x^0 \ldots \forall x^{n-1}[fx^0 \ldots x^{n-1} \equiv A]]$

In OT, this axiom schema is captured, in the first instance, as a theorem and not as a theorem schema, as follows. Say that *F rigidifies G* just in case *F* is rigid and is materially equivalent to *G*:

$$Rigidifies(F,G) \equiv_{df} Rigid(F) \,\&\, \forall x_1 \ldots \forall x_n(Fx_1 \ldots x_n \equiv Gx_1 \ldots x_n)$$

Then we may formulate and prove a principle expressible as a formal sentence (again using variables of second-order OT or typed OT) that asserts that necessarily, for every n-ary relation G ($n \geq 0$), there is a relation F that rigidifies it, as follows:

$$\vdash \Box\forall G\exists F(Rigidifies(F,G))$$

As mentioned in (3.1), proof of this theorem was developed by Daniel Kirchner.[31] Though expressed here as a proposition, a schematic version can be established as well, though given the presence of indiscernibles in OT, Gallin's schema is derivable for any formula φ that places a condition on *discernible* objects.[32]

So OT preserves the reduction of plural quantifiers to non-plural, higher-order quantification *without* taking Gallin's principle as an axiom. But, more importantly, OT doesn't require Gallin's principle for every part of the reduction. Since encoding is a rigid notion, some of the principles governing higher-order pluralities in typed OT can be derive without his principle. To see why, simply reformulate the definitions and theorems in Sections 2 and 3 above in typed OT, as its been presented most recently in Zalta 2020. In particular, using (15) as a model, we may define, where x is an entity of any type t and G is a property having type $\langle t\rangle$: x is a plurality of G just in case there are some discernible Gs and x is an abstract object that encodes just those properties that are the haecceities of some G object, i.e.,

[31] Personal communication, 5 March 2020, and first reported in 05 April 2020 version of Zalta ms. See item (550.3) at https://mally.stanford.edu/principia-2020-04-05.pdf. The second-order theorem is now numbered (577.3) in the current version of Zalta ms. and the type-theoretic version is is item (975.5).

[32] If we let $u_1, \ldots, u_n$ be rigid, restricted variables ranging over discernible objects, then Gallin's axiom schema is derivable in the form of the following theorem schema:

$\exists F(Rigid(F) \,\&\, \forall u_1 \ldots \forall u_n(Fu_1 \ldots u_n \equiv \varphi))$,
where F is a variable of any n-ary relational type.

The proof in Zalta ms. (975.6) was left as an exercise.

$$PluralityOf(x,G) \equiv_{df} \exists u Gu \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(Gu \,\&\, F = [\lambda y\, y = u]))$$

And using (17) as a model, higher-order pluralities x of any type t become defined, where G has type $\langle t\rangle$, as:

$$Plurality(x) \equiv_{df} \exists G(PluralityOf(x,G))$$

So one can reconstruct higher-order pluralities in typed OT. Note that this doesn't yield any advantages in the case of pluralities of propositions – those already exist in second-order OT, as we saw in the previous section. The advantages come if one takes on board Fritz's claim about analyzing or reducing higher-order plural quantifiers in terms of higher-order relational type theory. For then, given the above two definitions, the axioms of plural quantification theory for any type t become derivable, in the manner described in Sections 2 and 3 above. Note that the rigidification that Gallin's axiom offers isn't necessary to prove $u \prec xx \rightarrow \Box u \prec xx$ (29) in typed OT, since this becomes a modally strict theorem that holds for any discernible u and plurality xx of any type t. However, the derivation of Gallin's principle in typed OT does play a role in showing that pluralities are necessarily pluralities, as one can see once the proof of theorem (28) is reconstructed in typed OT.

5.2 Pluralities of Properties in 2nd-order OT

In this section, we show that when we focus only on second-order OT, then in addition to the analysis of pluralities of propositions described in the previous section, one may also analyze pluralities of properties. Clearly, in the first instance, one may analyze pluralities of properties, if such are needed, as non-null abstract objects:

$$PropertyPlurality(x) \equiv_{df} \exists F(xF) \tag{52}$$

If one wants to allow for empty property pluralities, then the definiens in the above definition can simply be replaced by the condition $A!x$. Then every abstract object, including the null object, would become a property plurality.

From (52) it follows that $PropertyPlurality(x)$ is a rigid condition on objects:[33]

[33] Assume $PropertyPlurality(x)$. Then by (52), $\exists F(xF)$. Suppose G is such an F, so that xG. Then by axiom (1), $\Box xG$. Hence $\exists F\Box xF$. So by the Buridan formula, $\Box\exists F(xF)$. Hence $\Box PropertyPlurality(x)$, by (52).

$$\vdash \mathit{PropertyPlurality}(x) \rightarrow \Box \mathit{PropertyPlurality}(x)$$

So, one may introduce rigid, restricted variables $FF, GG, \ldots$ to range over property pluralities. And if we further extend our use of the symbol $\prec$ defined in (21), so that it may also stand between a property variable and a plural property variable, we may define G is one of FF as FF encodes G, i.e.,

$$G \prec FF \equiv_{df} FF G$$

Then we immediately obtain existence conditions (a comprehension schema) and identity conditions as theorems:

$$\vdash \exists G \varphi \rightarrow \exists FF \forall G (G \prec FF \equiv \varphi), \quad \text{provided } FF \text{ doesn't occur free in } \varphi \tag{53}$$

$$\vdash FF = GG \equiv \forall H (H \prec FF \equiv H \prec GG) \tag{54}$$

And if we allow for empty pluralities, by revising definition (52) as indicated above, we can drop the antecedent of the comprehension schema (53).

Theorems (53) and (54) follow straightforwardly from (2) and (4), respectively, by applying definitions. (We omit the proofs.) For more specificity, consider any condition $\psi(G)$ on properties. Then one may introduce, as a definition schema, *FF is a plurality of ψ-properties* just in case FF is a property plurality and encodes all and only those properties G such that $\psi(G)$:

$$\mathit{PropertyPluralityOf}(FF, \psi) \equiv_{df} \exists G \psi \,\&\, \mathit{PropertyPlurality}(FF) \,\&\, \forall G (FF G \equiv \psi(G))$$

In the usual way, the first conjunct of the definiens can be omitted if we allow for empty property pluralities, as described above. But given this general analysis, second-order OT constitutes a theory of property pluralities, without invoking typed OT, i.e., without invoking the version of OT that is formulable in relational type theory.

6 Conclusion

The foregoing effort shows that we have a choice. On the one hand, we may (a) assume the primitive notions and primitive axioms of plural quantification theory, (b) take on board the notion of plural reference, (c) accept the existence of special relations between individuals and pluralities, and (d) attempt to solve the puzzling issue concerning compositionality described in the Introduction. Alternatively, in light of the foregoing, we may analyze the plural idioms in terms of abstractions (singularly-conceived) that are systematized by a very general theory that has been widely applied. Moreover, if pluralities of propositions, relations, and properties, and their governing axioms, are needed, they can be defined and derived, respectively, as described in Sections 4, 5.1, and 5.2, respectively. OT, which wasn't designed with pluralities in mind, nevertheless unifies the theory of pluralities with a number of other theories that have been derived within its axiomatic framework.

Nor can one legitimately charge that OT has a large ontology in comparison to the ontological innocence of the plural idioms. For one thing, if there are lingering questions about the new primitive notions, axioms, plural reference, and special relations of plural quantification theory, then one may legitimately question the alleged ontological innocence of this theory. But no matter how that issue is resolved, it should be noted that OT doesn't require a large ontology; indeed, if one omits the axiom that forestalls the one-world models in which there is a general modal collapse of truth and necessity, OT has extremely small models. Without that axiom, OT has models in which there is one ordinary individual, two properties, and four abstract objects. As such, it is no more problematic than 3rd-order logic, ontologically speaking, since the smallest models of classical 3rd-order logic require only one individual, two properties, and four properties of properties. And OT's ontology doesn't get any larger until one either (a) asserts the modal axiom that forestalls modal collapse (thereby guaranteeing the existence of at least two possible worlds), (b) asserts an axiom guaranteeing the existence of a predecessor relation defined in Fregean terms (see Nodelman & Zalta 2024), or (c) applies the theory by adding distinguished, new primitive properties (at which point OT grows by simply objectifying the expressible patterns of such properties).

Still, some philosophers may take it that even being committed to at least two properties and four abstract objects is too much of a commitment. In that case, one should consider the naturalized epistemology developed for OT in Linsky & Zalta 1995, which is based on the idea that the mind-independence and objectivity of abstract objects is not analogous to the mind-independence and objectivity of physical objects. It is argued there that there is no appearance-reality distinction for abstract

objects, that they are not 'out there' in a sparse way waiting to be discovered *a posteriori*, and that they have a dimension on which they may be incomplete. Given these disanalogies, the comprehension principle for abstract objects can be seen as a plenitude principle that postulates, in a non-arbitrary way, abstract objects that may be incomplete with respect to what they encode. Epistemologically, to become acquainted with an abstract object, you simply have to understand its defining condition (Linsky & Zalta 1995), and this is a sense in which such objects are 'thin', both epistemologically and metaphysically. Metaphysically, abstract objects take up no spacetime and, arguably, are simply 'immanent' forms and patterns, in the Aristotelian sense. In so far as the Aristotelian approach to the form vs. matter is viable, we can understand the foregoing effort as defining each plurality xx as the (objectified) pattern that is constituted solely by those things that are among the xx.

7 Appendix: Proofs of the Theorems

Proof of (7): Consider any formula φ in which x doesn't occur free. Then the fact that there is such an abstract object is established by (2). But there couldn't be two distinct abstract objects that encode exactly the properties satisfying φ, since distinct abstract objects have to differ by one of their encoded properties, by (4). ⋈

Proof of (9): For the proof, see Zalta ms. (= PLM), (270).

Proof of (10): For the proof, see PLM (274.5).

Proof of (11): For the proof, see PLM (274.34).

Proof of (13): For the proof, see PLM (274.13).

Proof of (16): By GEN, it suffices to show: $\exists u Gu \rightarrow \exists! x PluralityOf(x, G)$. So assume $\exists u Gu$. As an instance of (7), in which φ is the formula $\exists u(Gu \,\&\, F = [\lambda y\, y = u])$, we know:

(ϑ) $\exists! x\big(A!x \,\&\, \forall F(xF \equiv \exists u(Gu \,\&\, F = [\lambda y\, y = u]))\big)$

Let us now abbreviate our assumption $\exists u Gu$ as χ, and abbreviate (ϑ) as $\exists! x \psi$. But from $\chi \,\&\, \exists! x \psi$, it follows that $\exists! x(\chi \,\&\, \psi)$.[34] So from $\exists u Gu$ and (ϑ), we may infer:

$$\exists! x\big(\exists u Gu \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(Gu \,\&\, F = [\lambda y\, y = u]))\big)$$

Then by definition (15), $\exists! x PluralityOf(x, G)$. ⋈

Proof of (18): If we expand $Plurality(x)$ by its definiens in (17), we have to show:

(ϑ) $\exists x \exists G(PluralityOf(x, G))$

But now instantiate the property of being discernible ($D!$) into (16), to obtain:

$$\exists u D!u \rightarrow \exists x PluralityOf(x, D!)$$

Since we know, by theorem (10) that $\exists x D!x$, it then follows that:

$$\exists x PluralityOf(x, D!)$$

Hence, $\exists G \exists x PluralityOf(x, G)$, and by the commutativity of the existential quantifier, it follows that (ϑ). ⋈

Proof of (19): By GEN, we prove this for a fixed, but arbitrary G. So assume $\exists u Gu$. Then, if we eliminate the restricted variable xx, we have to show:

$$\exists! x(Plurality(x) \,\&\, PluralityOf(x, G))$$

By (16), our assumption implies that there is a unique individual that is a plurality of Gs. So suppose a is such an object, i.e., that:

(ϑ) $PluralityOf(a, G) \,\&\, \forall z(PluralityOf(z, G) \rightarrow z = a)$

The first conjunct implies $Plurality(a)$, by $\exists$I and definition (17). So we know:

$$Plurality(a) \,\&\, PluralityOf(a, G)$$

It therefore remains to show:

[34] Assume $\chi \,\&\, \exists! x \psi$. By the definition of the uniqueness quantifier, we have to show:

(ξ) $\exists x(\chi \,\&\, \psi \,\&\, \forall z((\chi^z_x \,\&\, \psi^z_x) \rightarrow z = x))$

To find our witness to (ξ), let a be our witness to $\exists! x \psi$, so that we know:

(ζ) $\psi^a_x \,\&\, \forall z(\psi^z_x \rightarrow z = a)$

Since x doesn't occur free in χ, our assumption that χ is the same formula as χ^a_x. Hence, we know $\chi^a_x \,\&\, \psi^a_x$. So to show that a is a witness to (ξ), it remains only to show:

$$\forall z((\chi^z_x \,\&\, \psi^z_x) \rightarrow z = a)$$

So by GEN, assume $\chi^z_x \,\&\, \psi^z_x$. The second conjunct and (ζ) imply $z = a$. ⋈

$$\forall z((\mathit{Plurality}(z) \,\&\, \mathit{PluralityOf}(z, G)) \rightarrow z = a)$$

So, by GEN, assume $\mathit{Plurality}(z)$ & $\mathit{PluralityOf}(z, G)$. Then by the second conjunct of (ϑ), $z = a$. ⋈

Proof of (20): Assume $[\lambda y\ y = u] = [\lambda y\ y = v]$. By the reflexivity of identity, we know $u = u$. So by β-Conversion, $[\lambda y\ y = u]u$. But then, by our assumption, $[\lambda y\ y = v]u$. So again by β-Conversion, $u = v$. ⋈

Proof of (22): Let $\varphi(u)$ be any condition on discernibles in which xx doesn't occur free but where u may or may not occur free. Assume $\exists u\varphi(u)$, for conditional proof. Then consider the property $[\lambda u\ \varphi(u)]$. By theorem (14), this property exists. So, to find a witness to our theorem, we first instantiate this property into Corollary (19), to obtain:

$$\exists u([\lambda u\ \varphi(u)]u) \rightarrow \exists! xx(\mathit{PluralityOf}(xx, [\lambda u\ \varphi(u)]))$$

Since we know, by β-Conversion, that $\varphi(u) \equiv [\lambda u\ \varphi(u)]u$, our assumption, $\exists u\varphi(u)$, implies $\exists u([\lambda u\varphi(u)]u)$, by the Rule of Substitution. Hence, $\exists! xx(\mathit{PluralityOf}(xx, [\lambda u\ \varphi(u)]))$, by (19). Now suppose aa is that object, so that we know $\mathit{PluralityOf}(aa, [\lambda u\ \varphi(u)])$. To show that aa is a witness to the consequent of (22), it remains only to show $\forall u(u \prec aa \equiv \varphi(u))$, and so by the Rule of Generalization (GEN), we show $u \prec aa \equiv \varphi(u)$. $(\rightarrow)$ Take $u \prec aa$ as a local assumption. Then by definition (21), $aa[\lambda y\ y = u]$. Since aa is a plurality of $[\lambda u\varphi(u)]$, it follows *a fortiori* from (an alphabetic variant of) the definition of *PluralityOf* (15), that:

$$\forall F(aaF \equiv \exists v([\lambda u\ \varphi(u)]v \,\&\, F = [\lambda y\ y = v]))$$

We can instantiate this last fact to $[\lambda y\ y = u]$, which we know exists by (12), and thereby obtain the following fact about aa:

(ϑ) $aa[\lambda y\ y = u] \equiv \exists v([\lambda u\ \varphi(u)]v \,\&\, [\lambda y\ y = u] = [\lambda y\ y = v])$

Since the left side of (ϑ) followed from our local assumption, we therefore know:

$$\exists v([\lambda u\ \varphi(u)]v \,\&\, [\lambda y\ y = u] = [\lambda y\ y = v])$$

Suppose b is such an object, so that we know:

(ξ) $[\lambda u\ \varphi(u)]b \,\&\, [\lambda y\ y = u] = [\lambda y\ y = b]$

The second conjunct of (ξ) implies $u = b$, by our Lemma (20). Then by =E, the first conjunct of (ξ) implies $[\lambda u\ \varphi(u)]u$, which by β-Conversion is just $\varphi(u)$.

$(\leftarrow)$ Assume $\varphi(u)$. Then by β-Conversion, $[\lambda u\ \varphi(u)]u$. Now conjoin this with a self-identity claim, to obtain $[\lambda u\varphi(u)]u \,\&\, [\lambda y y = u] = [\lambda y y = u]$. If we existentially generalize on the appropriate occurrences of u, we obtain:

$$\exists v([\lambda u\ \varphi(u)]v \,\&\, [\lambda y\ y = u] = [\lambda y\ y = v])$$

So by (ϑ), $aa[\lambda y\ y = u]$. Hence, by definition (21), $u \prec aa$. ⋈

Proof of (22): (Contributed by Claude Code and human-checked) Let φ be any condition on discernibles in which xx isn't free. Assume $\exists u\varphi(u)$. Now by (14), we know $[\lambda u\ \varphi(u)]\downarrow$. Call this property P. Then by β-Conversion, we know $\forall u(Pu \equiv \varphi(u))$. From this and our assumption $\exists u\varphi(u)$, it follows that $\exists uPu$. Hence, by theorem (16), $\exists! x\mathit{PluralityOf}(x, P)$. Since any such object is a plurality (17), let aa be this unique object. Then by definition (15):

(ϑ) $\exists uPu \,\&\, A!aa \,\&\, \forall F(aaF \equiv \exists u(Pu \,\&\, F = [\lambda y\ y = u]))$

It remains to show that aa is our witness to $\exists xx\forall u(u \prec xx \equiv \varphi(u))$.

$(\rightarrow)$ Assume $u \prec aa$. Then by definition (21): $aa[\lambda y\ y = u]$. So by (ϑ), $\exists v(Pv \,\&\, [\lambda y\ y = u] = [\lambda y\ y = v])$. By Theorem (20), $\exists v(Pv \,\&\, u = v)$. Hence, Pu. Since $Pu \equiv \varphi(u)$, it follows that $\varphi(u)$.

$(\leftarrow)$ Assume $\varphi(u)$. Then Pu. Note that $[\lambda y\ y = u] = [\lambda y\ y = u]$ (by the reflexivity of identity). So, if we combine what we know and existentially generalize on appropriate occurrences of u, we obtain $\exists v(Pv \,\&\, [\lambda y\ y = u] = [\lambda y\ y = v])$. So by (ϑ), $aa[\lambda y\ y = u]$. Then by definition (21), $u \prec aa$.

So $\forall u(u \prec aa \equiv \varphi(u))$, confirming aa as a witness to $\exists xx\forall u(u \prec xx \equiv \varphi(u))$. ⋈

Proof of (23): It suffices by GEN to show $\exists u(u \prec xx)$, where xx is a fixed, but arbitrary plurality. Since $\mathit{Plurality}(xx)$ by hypothesis, it follows that:

$$\exists G(\mathit{PluralityOf}(xx, G)),$$

by (17). Suppose P is such a property, so that we know, by (15):

$(\vartheta)\ \exists u Pu \,\&\, A!xx \,\&\, \forall F(xxF \equiv \exists u(Pu \,\&\, F = [\lambda y\, y = u]))$

Let a be a witness to the first conjunct of (ϑ), so that we know Pa. Then, to complete our proof, we show that a is a witness to $\exists u(u \prec xx)$, i.e., that $a \prec xx$. By definition (21), it suffices to show: $xx[\lambda y\, y = a]$. If we now instantiate $[\lambda y\, y = a]$ into the third conjunct of (ϑ), then it suffices to show:

$$\exists u(Pu \,\&\, [\lambda y\, y = a] = [\lambda y\, y = u])$$

But this is easy, since it follows from the conjunction of the facts that Pa and $[\lambda y\, y = a] = [\lambda y\, y = a]$, the latter being an instance of the reflexivity of identity. ⋈

Proof of (25): By GEN, it suffices to show:

$$\forall u(u \prec xx \equiv u \prec yy) \rightarrow xx = yy$$

where xx and yy are fixed, but arbitrary, pluralities. Assume $\forall u(u \prec xx \equiv u \prec yy)$. By definition (21) and the Rule of Substitution, we may infer from our assumption that:

$(\vartheta)\ \forall u(xx[\lambda y\, y = u] \equiv yy[\lambda y\, y = u])$

Now by definition (17) and the fact that xx and yy are restricted variables ranging over pluralities, we know, respectively:

$$\exists G(PluralityOf(xx, G))$$

$$\exists G(PluralityOf(yy, G))$$

Suppose P and Q are our witnesses, respectively, so that we know:

$$PluralityOf(xx, P)$$

$$PluralityOf(yy, Q)$$

Then, by (15), we have both:

$(\xi_1)\ A!xx \,\&\, \forall F(xxF \equiv \exists v(Pv \,\&\, F = [\lambda y\, y = v]))$

$(\xi_2)\ A!yy \,\&\, \forall F(yyF \equiv \exists v(Qv \,\&\, F = [\lambda y\, y = v]))$

So xx and yy are both abstract objects. Consequently, to show $xx = yy$, it suffices to show $\forall F(xxF \equiv yyF)$, by the above fact (24) about the identity of abstract objects. By GEN, and without loss of generality, we show only $xxF \rightarrow yyF$. So assume xxF. Then by the 2nd conjunct of (ξ_1), $\exists v(Pv \,\&\, F = [\lambda y\, y = v])$. Suppose it is a, so that we know both Pa and $F = [\lambda y\, y = a]$. By the latter and our assumption that xxF, it follows that $xx[\lambda y\, y = a]$. But then by (ϑ), $yy[\lambda y\, y = a]$. Hence yyF. ⋈

Proof of (27): By GEN, it suffices to show $\forall u(u \prec xx \equiv u \prec yy) \rightarrow (\varphi(xx) \equiv \varphi(yy))$. So assume $\forall u(u \prec xx \equiv u \prec yy)$. Without loss of generality, we only prove that $\varphi(xx) \rightarrow \varphi(yy)$. So also assume $\varphi(xx)$. Then by a previous lemma (25), our first assumption implies $xx = yy$. But then from this and our second assumption, we can substitute yy for xx for every free occurrence of xx in φ and thereby infer $\varphi(yy)$. ⋈

Proof of (28): By Rule RN, it suffices to prove $Plurality(x) \rightarrow \Box Plurality(x)$ without appealing to any contingencies. So assume $Plurality(x)$. Then we know $\exists G(PluralityOf(x, G))$. By definition (15), we therefore know:

$$\exists G\big(\exists u Gu \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(Gu \,\&\, F = [\lambda y\, y = u]))\big)$$

Suppose P is such a property, so that we know:

$(\vartheta)\ \exists u Pu \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(Pu \,\&\, F = [\lambda y\, y = u]))$

Now we want to show $\Box Plurality(x)$, i.e., by expanding definitions (17) and (15), we want to show:

$$\Box\exists G\big(\exists u Gu \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(Gu \,\&\, F = [\lambda y\, y = u]))\big)$$

By the Buridan formula (which is a theorem of S5), it suffices to show:

$(\zeta)\ \exists G\Box\big(\exists u Gu \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(Gu \,\&\, F = [\lambda y\, y = u]))\big)$

So we have to produce a witness to this claim, but we won't use P as the witness. Instead, we'll show that the property that rigidifies P is the witness. So if we instantiate Gallin's principle (which, as noted above, is a theorem of OT) to P, we know $\exists F(Rigidifies(F, P))$. Let P' be such that $Rigidifies(P', P)$. By definition of $Rigidifies(F, G)$ – see footnote 13 – it follows that (a) necessarily, anything that exemplifies P' necessarily exemplifies P', and (b) P' is materially equivalent to P. So, restricting our results to discernible objects, we know:

$(\xi)\ \Box\forall u(P'u \rightarrow \Box P'u) \,\&\, \forall u(P'u \equiv Pu)$

Now to help us show that P' is a witness to (ζ), note that from the second conjunct of (ξ), we can infer that (ϑ) is equivalent to:[35]

$(\vartheta)'$ $\exists u P'u \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u]))$

Now to show that P' is a witness to (ζ), we have to show:

$$\Box\big(\exists u P'u \,\&\, A!x \,\&\, \forall F(xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u]))\big)$$

Now in classical modal logic, $\Box\varphi \,\&\, \Box\psi$ implies $\Box(\varphi \,\&\, \psi)$. So to prove the above formula, it suffices to show:

(a) $\Box\exists u P'u$

(b) $\Box A!x$

(c) $\Box\forall F(xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u]))$

(a) By $(\vartheta)'$, we know $\exists u P'u$. Suppose a is such an object, so that $P'a$. Then since the first conjunct of (ξ) implies, by the T schema, $\forall u(P'u \to \Box P'u)$, it follows that $\Box P'a$. So, $\exists u \Box P'u$. Then by the Buridan formula, $\Box\exists u P'u$.

(b) We know $A!x$, by (ϑ). So by definition of $A!$ and λ-Conversion, $\neg\Diamond E!x$, i.e., $\Box\neg E!x$. Hence, by the 4 schema, $\Box\Box\neg E!x$, which by the definition of $A!$, β-Conversion, and a Rule of Substitution implies $\Box A!x$.

(c) To show this, it suffices, by the Barcan Formula, to show:

$$\forall F\Box(xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u]))$$

So by GEN, we show:

(ω) $\Box(xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u]))$

Our strategy is to show that (ω) is derivable from the following fact in modal logic:

(κ) $(\Box(\varphi \to \Box\varphi) \,\&\, \Box(\psi \to \Box\psi)) \to \Box\big((\varphi \equiv \psi) \to \Box(\varphi \equiv \psi)\big)$

[35] Without loss of generality, we show only $(\vartheta) \to (\vartheta)'$, since the reasoning in the right-to-left direction is analogous. Assume (ϑ). The first conjunct of (ϑ), namely $\exists u Pu$, and the second conjunct of (ξ) imply $\exists u P'u$. So it remains to show $\forall F(xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u]))$. By GEN, we show $xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u])$ $(\to)$ Assume xF. Then by (ϑ), $\exists u(Pu \,\&\, F = [\lambda y\, y = u])$. Suppose a is such a discernible object. Then Pa and $F = [\lambda y\, y = a]$. But, by the second conjunct of (ξ), $P'a$. So $\exists u(P'u \,\&\, F = [\lambda y\, y = u])$. $(\leftarrow)$ Assume $\exists u(P'u \,\&\, F = [\lambda y\, y = u])$. Let b be such an object. Then both $P'b$ and $F = [\lambda y\, y = b]$. But again by the second conjunct of (ξ), Pb. Hence $\exists u(Pu \,\&\, F = [\lambda y\, y = u])$. So xF, by (ϑ).

This tells us that if φ and ψ are modally collapsed, then necessarily, their material equivalence implies their necessary equivalence.[36] To obtain the instance of this theorem that we need, let φ be xF and ψ be $\exists u(P'u \,\&\, F = [\lambda y\, y = u])$. Now we already know $\Box(\varphi \to \Box\varphi)$, by an application of RN to axiom (1). To show $\Box(\psi \to \Box\psi)$, we have to show:

$$\Box\big(\exists u(P'u \,\&\, F = [\lambda y\, y = u]) \to \Box\exists u(P'u \,\&\, F = [\lambda y\, y = u])\big)$$

Proof. By RN, we have to show:[37]

$$\exists u(P'u \,\&\, F = [\lambda y\, y = u]) \to \Box\exists u(P'u \,\&\, F = [\lambda y\, y = u])$$

So assume $\exists u(P'u \,\&\, F = [\lambda y\, y = u])$. Suppose a is such an object, so that we know both $P'a$ and $F = [\lambda y\, y = a]$. But the first implies $\Box P'a$, by the first conjunct of (ξ), and the second implies $\Box(F = [\lambda y\, y = a])$, by the necessity of identity (which easily follows from (5)). Hence, $\Box(P'a \,\&\, F = [\lambda y\, y = a])$. So by $\exists$I, $\exists u\Box(P'u \,\&\, F = [\lambda y\, y = u])$, and by the Buridan formula, $\Box\exists u(P'u \,\&\, F = [\lambda y\, y = u])$.

Hence, by the relevant instance of (κ), we know:

$$\Box\big((\varphi \equiv \psi) \to \Box(\varphi \equiv \psi)\big)$$

And by the T schema:

$$(\varphi \equiv \psi) \to \Box(\varphi \equiv \psi)$$

i.e.,

$$(xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u])) \to \Box(xF \equiv \exists u(P'u \,\&\, F = [\lambda y\, y = u]))$$

Since the consequent of this conditional is (ω), i.e., what we're trying to show, we have to establish the antecedent. But that follows immediately from the third conjunct of $(\vartheta)'$. ⋈

Proof of (29): Assume $u \prec xx$. Then, by definition, $xx[\lambda y\, y = u]$. But this is an encoding claim of the form xF and by axiom (1), we know $xF \to \Box xF$. Hence $\Box xx[\lambda y\, y = u]$, i.e., by definition (21), $\Box u \prec xx$. ⋈

[36] For a proof, see Zalta ms. (172.5).

[37] We may appeal to RN here because in the reasoning that follows in this subproof, the only assumption used, namely *Rigidifies*(P', P), will get discharged by $\exists$E at the end, so that the reasoning relies only on the claim $\exists F(\textit{Rigidifies}(F, P))$, which is derivable without appeal to contingencies in object theory. So Rule RN can be applied once we derive the consequent from the antecedent of the claim that follows in the text.

Proof of (30): In S5, if $\vdash \varphi \rightarrow \Box\varphi$, then $\vdash \Diamond\varphi \rightarrow \varphi$. If we contrapose the consequent of this rule, then we have: if $\vdash \varphi \rightarrow \Box\varphi$, then $\vdash \neg\varphi \rightarrow \neg\Diamond\varphi$, i.e., if $\vdash \varphi \rightarrow \Box\varphi$, then $\vdash \neg\varphi \rightarrow \Box\neg\varphi$. So from the fact that (29) is a theorem, it is also a theorem that $\neg u \prec xx \rightarrow \Box\neg u \prec xx$. ⋈

Proof of (31):[38] By the T$\Diamond$ schema, it suffices to establish:

$$\exists x \exists G(PluralityOf(x, G) \,\&\, \neg\Box PluralityOf(x, G))$$

So we need to find witnesses for the two existential claims and prove that they are witnesses. To find our witness to $\exists G$, we appeal to the fact that there are contingently true propositions, i.e., that $\exists p(p \,\&\, \Diamond\neg p)$. Suppose p_0 is such a proposition, so that we know $p_0 \,\&\, \Diamond\neg p_0$. Then consider the property $[\lambda z\, p_0]$, which is known to exist (no variable bound by the λ occurs in the matrix and so no variable bound by the λ occurs in encoding position in the matrix). This will be our witness to $\exists G$. As to our witness to $\exists x$, consider the following instance of (2):

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists u([\lambda z\, p_0]u \,\&\, F = [\lambda y\, y = u])))$$

Let a be such an object, so that:

(ϑ) $A!a \,\&\, \forall F(aF \equiv \exists u([\lambda z\, p_0]u \,\&\, F = [\lambda y\, y = u]))$

If a and $[\lambda z\, p_0]$ are to be our witnesses, we have to show:

(A) $PluralityOf(a, [\lambda z\, p_0])$

(B) $\neg\Box PluralityOf(a, [\lambda z\, p_0])$

(A) We know, by β-Conversion, that $[\lambda z\, p_0]x \equiv p_0$. Moreover, it is a theorem of OT that $\exists x D!x$; let b serve as a witness, so that $D!b$. Then since we know p_0 by assumption, we know $[\lambda z\, p_0]b$, and hence $\exists u([\lambda z\, p_0]u)$. If we conjoin this last fact to the front of (ϑ) and apply definition (15), we have established $PluralityOf(a, [\lambda z\, p_0])$.

(B) We prove this with the help of the following, modally strict Lemma:

Lemma: $\forall x \forall q \Box(PluralityOf(x, [\lambda z\, q]) \rightarrow q)$

[38] The original proof used the fact that $\Diamond\exists x(\neg E!x \,\&\, \Diamond E!x)$ ('possibly, there is something that isn't concrete but that is possibly concrete'), which is a theorem of OT. The proof used $\overline{E}$ (the negation of $E!$) and the plurality of $\overline{E}$ as the witnesses to the theorem. But the following is a simpler proof suggested by Daniel Kirchner (to which I have made some tweaks, prompted by our personal communications). Kirchner derives the theorem from the existence of contingently true propositions.

Proof. By RN and GEN, it suffices to show $PluralityOf(x, [\lambda z q]) \rightarrow q$. So assume $PluralityOf(x, [\lambda z\, q])$. Then by (15) it follows *a fortiori* that $\exists u([\lambda z\, q]u)$. Suppose c is such an object, so that $[\lambda z\, q]c$. Then by β-Conversion, q.

Now to prove (B), assume, for reductio:

$$\Box PluralityOf(a, [\lambda z\, p_0])$$

But if we instantiate our Lemma to a and $p_0]$, we also know:

$$\Box(PluralityOf(a, [\lambda z\, p_0]) \rightarrow p_0)$$

So by the K axiom, $\Box p_0$, which contradicts our assumption that $\Diamond\neg p_0$. ⋈

Proof of (42): We first prove a helpful lemma, namely that maximal propositions are possibly true:

Lemma: $Max(p) \rightarrow \Diamond p$

Proof. Assume $Max(p)$, i.e., by definition (M1)′, $\forall q((p \Rightarrow \neg q) \equiv \neg(p \Rightarrow q))$. If we instantiate this to p, we know $(p \Rightarrow \neg p) \equiv \neg(p \Rightarrow p)$, i.e., by definition of $\Rightarrow$:

(A) $\Box(p \rightarrow \neg p) \equiv \neg\Box(p \rightarrow p)$

But, clearly, $\Box(p \rightarrow p)$ and $\neg\neg\Box(p \rightarrow p)$ are modally strict theorems of propositional modal logic. The latter and (A) imply $\neg\Box(p \rightarrow \neg p)$. But it is also a modally strict theorem of propositional modal logic that $\neg p \rightarrow (p \rightarrow \neg p)$. So by Rule RM, $\Box\neg p \rightarrow \Box(p \rightarrow \neg p)$, and by contraposition, $\neg\Box(p \rightarrow \neg p) \rightarrow \neg\Box\neg p$. Since we've established the antecedent, it follows that $\neg\Box\neg p$, i.e., $\Diamond p$.

We now prove both directions of our theorem. ($\rightarrow$) Assume $Max(p)$. Then by our Lemma, $\Diamond p$. So it remains to show $\forall q(p \Rightarrow q \vee p \Rightarrow \neg q)$, i.e., show $\forall q(\Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q))$. By GEN, it suffices to prove $\Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q)$, which we do by cases from the tautology $\Box(p \rightarrow q) \vee \neg\Box(p \rightarrow q)$. If $\Box(p \rightarrow q)$, then by $\vee$I, $\Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q)$. If $\neg\Box(p \rightarrow q)$, then note that our assumption $Max(p)$ implies, by definition (M1)′:

(ϑ) $\Box(p \rightarrow \neg q) \equiv \neg\Box(p \rightarrow q)$

So it follows that $\Box(p \rightarrow \neg q)$. So again by $\vee$I, $\Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q)$.

($\leftarrow$) Assume $\Diamond p \,\&\, \forall q(p \Rightarrow q \vee p \Rightarrow \neg q)$, i.e.,

$(\xi)\ \ \Diamond p \,\&\, \forall q(\Box(p \to q) \vee \Box(p \to \neg q))$

By GEN and definitions (M1)′ and $\Rightarrow$, we have to show: $\Box(p \to \neg q) \equiv \neg\Box(p \to q)$. Again we show both directions:

($\to$) Assume $\Box(p \to \neg q)$ and, for reductio, assume $\Box(p \to q)$ Now it is a modally strict theorem of propositional modal logic that $(p \to q) \to ((p \to \neg q) \to \neg p)$. So by Rule RM, $\Box(p \to q) \to \Box((p \to \neg q) \to \neg p)$. But as an instance of K, we know $\Box((p \to \neg q) \to \neg p) \to (\Box(p \to \neg q) \to \Box\neg p)$. So by hypothetical syllogism, $\Box(p \to q) \to (\Box(p \to \neg q) \to \Box\neg p)$. But from this and our reductio hypothesis it follows that $\Box(p \to \neg q) \to \Box\neg p$. And from this and our initial assumption, it follows that $\Box\neg p$, i.e., $\neg\Diamond p$, contradicting the first conjunct of (ξ).

($\leftarrow$) Assume $\neg\Box(p \to q)$. From this and the second conjunct of (ξ), it follows $\Box(p \to \neg q)$. ⋈

Proof of (43): Assume $Max(p)$. Then by (42), $\Diamond p \,\&\, \forall q((p \Rightarrow q) \vee (p \Rightarrow \neg q))$. The first conjunct implies $\Box\Diamond p$, by the 5 schema. When we expand the second conjunct to eliminate the defined $\Rightarrow$, we have:

$$\forall q(\Box(p \to q) \vee \Box(p \to \neg q))$$

But since the S4 fact $\Box\varphi \equiv \Box\Box\varphi$ is a modally strict theorem of S5, it follows by a Rule of Substitution that:

$$\forall q(\Box\Box(p \to q) \vee \Box\Box(p \to \neg q))$$

And since it is a modally strict theorem that $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$, it also follows by a Rule of Substitution that:

$$\forall q\Box(\Box(p \to q) \vee \Box(p \to \neg q))$$

So by the Barcan Formula:

$$\Box\forall q(\Box(p \to q) \vee \Box(p \to \neg q))$$

i.e.,

$(\vartheta)\ \ \Box\forall q((p \Rightarrow q) \vee (p \Rightarrow \neg q))$

Since $(\Box\varphi \,\&\, \Box\psi) \to \Box(\varphi \,\&\, \psi)$, it follows from the conjunction of $\Box\Diamond p$ and (ϑ) that:

$$\Box\big(\Diamond p \,\&\, \forall q((p \Rightarrow q) \vee (p \Rightarrow \neg q))\big)$$

i.e., $\Box Max(p)$, by (42). ⋈

Proof of (44): (Claude was used to check this proof.) We prove the directions separately. ($\to$) Assume:

$(\xi)\ \ \forall p\big(\Diamond p \to \exists q(Max(q) \,\&\, \Box(q \to p))\big)$

For reductio, assume $\neg\Box\exists p(Max(p) \,\&\, p)$, i.e., $\Diamond\neg\exists p(Max(p) \,\&\, p)$. Then let p_0 be the proposition, $\neg\exists p(Max(p) \,\&\, p)$, so that we know $\Diamond p_0$. From this fact and (ξ), it follows that $\exists q\big(Max(q) \,\&\, \Box(q \to p_0)\big)$. Let q_0 be a witness to this result, so that we know:

$(\vartheta)\ \ Max(q_0) \,\&\, \Box(q_0 \to p_0)$

By theorems (42) and (43), respectively, the first conjunct of (ϑ) implies $\Diamond q_0$ and $\Box Max(q_0)$. Now note independently, that it is a simple theorem that $(Max(q_0) \,\&\, q_0) \to \exists p(Max(p) \,\&\, p)$. By contraposition and the definition of p_0, this last fact implies that the following is a theorem:

$$p_0 \to (Max(q_0) \to \neg q_0)$$

So by the tautology $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$, the following is also a theorem:

$$(q_0 \to p_0) \to (q_0 \to (Max(q_0) \to \neg q_0))$$

Since this is a modally strict theorem, it follows by Rule RM that:

$$\Box(q_0 \to p_0) \to \Box(q_0 \to (Max(q_0) \to \neg q_0))$$

This and the second conjunct of (ϑ) imply that $\Box(q_0 \to (Max(q_0) \to \neg q_0))$. Then by exporting and importing antecedents using a Rule of Substitution, we have $\Box(Max(q_0) \to (q_0 \to \neg q_0))$. So by the K axiom, $\Box Max(q_0) \to \Box(q_0 \to \neg q_0)$. Since we've previously established $\Box Max(q_0)$, we may infer $\Box(q_0 \to \neg q_0)$. Independently, it is a theorem of propositional modal logic that $\Box((q_0 \to \neg q_0) \to \neg q_0)$. So by the K axiom, it follows that $\Box\neg q_0$, i.e., $\neg\Diamond q_0$. Contradiction.

($\leftarrow$) Assume $\Box\exists p(Max(p) \,\&\, p)$. It suffices to show, for an arbitrary proposition r, that $\Diamond r \to \exists q(Max(q) \,\&\, \Box(q \to r))$. So assume $\Diamond r$. Using the K theorem $(\Diamond\varphi \,\&\, \Box\psi) \to \Diamond(\varphi \,\&\, \psi)$, our two assumptions imply:

(A) $\Diamond(r \,\&\, \exists p(Max(p) \,\&\, p))$

Now, if we appeal to the predicate logic theorem $(\varphi \& \exists \alpha \psi) \rightarrow \exists \alpha(\psi \& \varphi)$, then we independently know:

$$(r \,\&\, \exists p(Max(p) \,\&\, p)) \rightarrow \exists p(Max(p) \,\&\, p \,\&\, r)$$

Since this is a modally strict theorem, it is necessary, and from its necessitation and (A), it follows by the K$\Diamond$ principle that:

$$\Diamond \exists p(Max(p) \,\&\, p \,\&\, r)$$

So by BF$\Diamond$:

$$\exists p \Diamond(Max(p) \,\&\, p \,\&\, r)$$

Let p_0 a witness to this claim, so that we know:

(B) $\Diamond(Max(p_0) \,\&\, p_0 \,\&\, r)$

By grouping the conjunction and distributing the $\Diamond$, it follows both that:

(C) $\Diamond Max(p_0)$

(D) $\Diamond(p_0 \,\&\, r)$

Now (C) implies $Max(p_0)$, by the facts that (a) theorem (43) is a modally strict theorem and so necessary, and (b) in S5, $\Box(\varphi \rightarrow \Box\psi)$ implies $\Diamond\varphi \rightarrow \psi$. From $Max(p_0)$, it follows by theorem (42) that:

$$\forall q((p_0 \Rightarrow q) \vee (p_0 \Rightarrow \neg q))$$

Instantiating to r and eliminating the defined $\Rightarrow$, we know $\Box(p_0 \rightarrow r) \vee \Box(p_0 \rightarrow \neg r)$. But it can't be that $\Box(p_0 \rightarrow \neg r)$, since that is equivalent to $\Box\neg(p_0 \,\&\, r)$, i.e., $\neg\Diamond(p_0 \,\&\, r)$, which contradicts (D). So $\Box(p_0 \rightarrow r)$. Hence $Max(p_0) \,\&\, \Box(p_0 \rightarrow r)$, and thus $\exists q(Max(q) \,\&\, \Box(q \rightarrow r))$. ⋈

Proof of (45): If we can show the universal generalization $\forall qq(\forall pp\varphi \rightarrow \varphi[qq/pp])$, then by universal instantiation to qq we're done. To show $\forall pp\varphi \rightarrow \varphi[qq/pp]$, eliminate the restricted variables and use OT notation for substitutions, so that we have to show:

$$\forall y\big(PropositionPlurality(y) \rightarrow (\forall x(PropositionPlurality(x) \rightarrow \varphi) \rightarrow \varphi^{y}_{x})\big)$$

But this follows by Rule GEN once it is recognized that φ^{y}_{x} follows by conditional proof from the two assumptions:

$$PropositionPlurality(y)$$

$$\forall x(PropositionPlurality(x) \rightarrow \varphi)$$

⋈

Proof of (46): Assume $p \prec pp$. Then by definition (38), $pp \models p$. But this is, by definition (34), the encoding formula $pp[\lambda x\, p]$. Hence by (1), $\Box pp[\lambda x\, p]$. So by reversing the reasoning, $\Box(p \prec pp)$. ⋈

Proof of (47): As noted in the text, the following comprehension principle for situations was derived in Zalta 2024 (147, 165), in Nodelman & Zalta 2026 (proof of (23) in §2.5), and in Zalta under review (§3.2.1) – it follows from the comprehension principle (2) for abstract objects:[39]

$$\exists s \forall p(s \models p \equiv \varphi), \text{ provided } s \text{ doesn't occur free in } \varphi \qquad (35)$$

If we expand the restricted variable, we have:

$$\exists x(Situation(x) \,\&\, \forall p(x \models p \equiv \varphi)), \text{ provided } x \text{ doesn't occur free in } \varphi$$

Since $Situation(x)$ and $PropositionPlurality(x)$ are necessarily equivalent by definition (37), it follows by a Rule of Substitution that:

$$\exists x(PropositionPlurality(x) \,\&\, \forall p(x \models p \equiv \varphi)),$$
$$\text{provided } x \text{ doesn't occur free in } \varphi$$

Hence, using our restricted variables, we may conclude:

$$\exists pp \forall p(pp \models p \equiv \varphi), \text{ provided } pp \text{ not free in } \varphi$$

But by definition (38), $pp \models q$ is necessarily equivalent to $q \prec pp$. So by a Rule of Substitution, we may infer from our last result, that:

$$\exists pp \forall p(p \prec pp \equiv \varphi), \text{ provided } pp \text{ not free in } \varphi$$

⋈

Proof of (48): In what follows, assume that x doesn't occur free in ψ. Then it is an axiom of OT that $\psi \rightarrow \forall x\psi$. Now we first show that the following is a (modally strict) derived rule of inference in OT:

$$\varphi \rightarrow \psi \vdash \varphi \rightarrow \forall x\psi$$

Let $\varphi \rightarrow \psi$ be our global assumption. And for conditional proof, let φ be our local assumption. Then, ψ, by MP. But since x doesn't occur free in ψ, it follows by the axiom stated at the outset, that $\forall x\psi$.

[39] This comprehension principle for situations plays a role in each of these papers, and is one of the advances in OT developed since Zalta 1993.

Now we want to show: $\varphi \rightarrow \psi \vdash \varphi \rightarrow \forall pp\psi$. So assume $\varphi \rightarrow \psi$ (to show that we can validly infer $\varphi \rightarrow \forall pp\psi$). Then by the derived rule we just established, $\varphi \rightarrow \forall x\psi$. But now let χ be *PropositionPlurality*(x). Then it follows *a fortiori* by the necessarily true propositional axiom $\psi \rightarrow (\chi \rightarrow \psi)$, that:

$$\varphi \rightarrow \forall x(\mathit{PropositionPlurality}(x) \rightarrow \psi)$$

Hence, using restricted variables: $\varphi \rightarrow \forall pp\psi$. ⋈

Proof of (49): In the following proof, we assume that definition (37) has been revised to read:

$$\mathit{PropositionPlurality}(x) \equiv_{df} \mathit{Situation}(x) \,\&\, \exists p(x \models p) \qquad (37)'$$

Now, in the proof of (47), we established:

(ϑ) $\exists x(\mathit{Situation}(x) \,\&\, \forall p(x \models p \equiv \varphi))$, provided x doesn't occur free in φ

For conditional proof, assume $\exists p\varphi$ as our global assumption and consider any such φ. Then the following universal biconditional holds:

(ζ) $\forall x\big((\mathit{Situation}(x) \,\&\, \forall p(x \models p \equiv \varphi)) \equiv$
$(\mathit{Situation}(x) \,\&\, \exists p(x \models p) \,\&\, \forall p(x \models p \equiv \varphi))\big)$

Proof. By GEN, it suffices to prove both directions for the free variable x. ($\rightarrow$) Assume $\mathit{Situation}(x) \,\&\, \forall p(x \models p \equiv \varphi)$. It remains to show only $\exists p(x \models p)$. But from the second conjunct, namely $\forall p(x \models p \equiv \varphi)$, and our global assumption, namely $\exists p\varphi$, it follows that $\exists p(x \models p)$. ($\leftarrow$) By &E. ⋈

Now given (ϑ), let a be a witness, so that we know:

$$\mathit{Situation}(a) \,\&\, \forall p(a \models p \equiv \varphi)$$

Then from this and (ζ), it follows that:

$$\mathit{Situation}(a) \,\&\, \exists p(a \models p) \,\&\, \forall p(a \models p \equiv \varphi)$$

Hence, by the revised definition of *PropositionPlurality* (37)′, it follows that:

$$\mathit{PropositionPlurality}(a) \,\&\, \forall p(a \models p \equiv \varphi)$$

So by ∃I:

$\exists x(\mathit{PropositionPlurality}(x) \,\&\, \forall p(x \models p \equiv \varphi))$,
provided x doesn't occur free in φ

The proof now concludes in the manner of the proof of (47). ⋈

Proof of (50): In this proof, we again assume the revised definition of a proposition plurality, formulated as (37)′ in the proof of (49). We have to show, for an arbitrary proposition plurality pp, that there exists a proposition p that is among pp. But, by definition (37)′, an arbitrary proposition plurality pp is such that:

$$\mathit{Situation}(pp) \,\&\, \exists p(pp \models p)$$

So let q be such a proposition, so that we know:

$$\mathit{Situation}(pp) \,\&\, pp \models q$$

By definition (38), the second conjunct implies $q \prec pp$. Hence $\exists p(p \prec pp)$. ⋈

Proof of (51): Again, the proof relies on Daniel West's derivation of the Kaplan-Fine axiom (Kaplan 1970, Fine 1970) as the following theorem:[40]

$$\exists p(p \,\&\, \forall q(q \rightarrow \Box(p \rightarrow q)))$$

Let p_0 be a witness to this fact, so that we know:

(ϑ) $p_0 \,\&\, \forall q(q \rightarrow \Box(p_0 \rightarrow q))$

By theorem (42), it suffices to show:

$$\Box\exists p(\Diamond p \,\&\, \forall q(p \Rightarrow q \vee p \Rightarrow \neg q) \,\&\, p)$$

So by Rule RN, it suffices to produce a modally strict proof of:

$$\exists p\big(\Diamond p \,\&\, \forall q((p \Rightarrow q) \vee (p \Rightarrow \neg q)) \,\&\, p\big)$$

To see that p_0 is a witness, we have to show :

(ξ) $\Diamond p_0 \,\&\, \forall q((p_0 \Rightarrow q) \vee (p_0 \Rightarrow \neg q)) \,\&\, p_0$

But we already know p_0, by the first conjunct of (ϑ). And it follows from this that $\Diamond p_0$, by the T$\Diamond$ principle. So it remains to show the second conjunct of (ξ). And since $\varphi \Rightarrow \psi$ is defined as $\Box(\varphi \rightarrow \psi)$, we have to show $\forall q(\Box(p_0 \rightarrow q) \vee \Box(p_0 \rightarrow \neg q))$. So by GEN, it remains only to show:

[40] Though the proof was first reported in one of the 2023 releases of Zalta ms. – see footnote 26 – it is now item (551) in the latest version of Zalta ms.

$\Box(p_0 \to q) \vee \Box(p_0 \to \neg q)$

We reason by cases from the tautology $q \vee \neg q$. If q, then by instantiating q into the second conjunct of (ϑ), it follows that $\Box(p_0 \to q)$. Hence, $\Box(p_0 \to q) \vee \Box(p_0 \to \neg q)$, by $\vee$I. If $\neg q$, then by analogous reasoning, $\Box(p_0 \to q) \vee \Box(p_0 \to \neg q)$. $\bowtie$

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