The Metaphysics of Possibility Semantics*

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Abstract

Works by (Humberstone 1981, 2011), van Benthem (1981, 2016), Holliday 2014, forthcoming, and Ding & Holliday 2020 attempt to develop a semantics of modal logic in terms of "possibilities", i.e., "less determinate entities than possible worlds" (Edgington 1985). These works take possibilities as semantically primitive entities, stipulate a number of semantic principles that govern these entities (namely, Ordering, Persistence, Refinement, Cofinality, Negation, and Conjunction), and then interpret a modal language via this semantic structure. In this paper, we *define* possibilities in object theory (OT), and *derive*, as theorems, the semantic principles stipulated in the works cited. We then raise a concern for the semantic investigation of possibilities without a primitive modal operator, and show that no such concerns apply to the metaphysics of possibilities as developed in OT.

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1 Introduction

The term *possibility* is often used in philosophy to denote a proposition that might be true. If we focus solely on the sense of *metaphysical* possibility, then where 'p' ranges over propositions and $\Diamond$ represents 'it is possible that', then p is considered a (metaphysical) possibility if $\Diamond p$ is true. But there is another, technical notion of *possibility* in the literature. It has been used instead of *possible world* to develop a formal semantics of modality (Humberstone 1981, 2011; van Benthem 1981, 2016; Edgington 1985; Holliday 2014, forthcoming; and Ding & Holliday 2020). Humberstone introduces this technical notion by considering two observations by Davies about Lewis's (1973, 84) conception of a possible world as a 'way things might have been'. Davies (1975, 57) observed: (1) when we specify a *way things might have been*, we typically do so with a *single* sentence (e.g., "I might have had straight hair"); and (2) if Lewis is to conceive of a *way things might have been* as a possible world, then such things cannot be specified by a single sentence. These observations led Humberstone to write (1981, 314–315):

> Here we have a motivation for the pursuit of modal logic against a semantic background in which less determinate entities than possible worlds, things which I am inclined for want of a better word to call simply *possibilities*, are what sentences (or formulae) are true or false with respect to.

That is, Humberstone is not conceiving of *ways things might have been* as propositions p whose metaphysical possibility implies that there are possible worlds in which p is true. Instead, he is thinking of possibilities as partial (i.e., not maximal) entities, such as proper parts of possible worlds. As Edgington 1985 (564) puts it:

> ...[P]ossibilities differ from possible worlds in leaving many details unspecified.

In the early developments of this theoretical notion of a *possibility*, philosophers and modal logicians have been content to regard them as primitive, undefined entities. The notion of a *possible world* had an analogous history. But the goal of the present paper is to develop a *theory* of possibilities in which we define the notion in more general terms and derive the principles governing possibilities that some of the philosophers cited above stipulate.

1.1 What Principles Govern Possibilities?

In one or the other of Humberstone 1981, 318; van Benthem 1981, 3–4; 2016, 3–4; Holliday 2014, 3; forthcoming, 5, 15; and Ding & Holliday 2020, 155), we find the following principles stipulated in their semantics:

- *Ordering*: a *refinement relation* ($\trianglerighteq$) partially orders the possibilities.
- *Persistence*: every proposition true in a possibility is true in every refinement of that possibility.
- *Refinement*: if a possibility x doesn't determine the truth value of a proposition p, then (a) there is a possibility which is a refinement of x where p is true, and (b) there is a possibility which is a refinement of x where p is false.[1]
- *Cofinality*: if, for every possibility x' that is a refinement of possibility x there is a possibility x'' that refines x' and makes p true, then x makes p true.
- *Negation*: the negation of p is true in a possibility x if and only if p fails to be true in every refinement of x.
- *Conjunction*: the conjunction p *and* q is true in x if and only if both p and q are true in x.

These semantic principles are then used to interpret a propositional (modal) language.

Our goals, however, are to use *object theory* (OT) to: (a) *define* the notion of possibility governed by these principles; (b) *derive* the above principles as theorems from the axioms of OT; and (c) demonstrate that the purely semantic conception of possibilities governed only by non-modal principles such as the above doesn't quite play the role envisioned for such entities. In pursuing goals (a) and (b), we shall *neither* assume that situations are primitive nor stipulate that the domain of situations is partially ordered. Moreover, we shall neither require any of the mathematics (e.g., set theory) that is typically assumed in the semantic characterization of possibilities, nor model possibilities as mathematical objects.

[1] In what follows it should be remembered that when the condition "x' is a refinement of x" is defined, one still has prove that, as defined, it obeys the *Refinement* principle.

Instead, our strategy is to use OT (with its primitive modal operator, as developed in Zalta 1993, 1997, and elsewhere) to identify possibilities as *situations* that are *consistent* and *modally closed*. We'll discuss the importance of having a primitive modal operator in the final section. We hope it will be clear, therefore, that this is *not* an essay in semantics, but rather a systematic metaphysical investigation of the principles adopted in the semantics of modal logic.

2 Object Theory

2.1 Basic Principles

For those unfamiliar with OT, the basics are as follows. OT is formulated in a 2nd-order, quantified modal language (without identity) extended with an *additional* atomic formula '$x_1 \ldots x_n F^n$' ('$x_1, \ldots, x_n$ *encode* F^n') for $n \geq 1$; a distinguished unary predicate '$E!$' ('being concrete'); an actuality operator ($\mathcal{A}$), complex individual terms (rigid definite descriptions) of the form $\imath x\varphi$; and complex n-ary relation terms (λ-expressions) of the form $[\lambda x_1 \ldots x_n\, \varphi]$. The language includes 0-ary relation variables ($p, p', \ldots, q, q', \ldots$) and constants ($p_i, q_i, \ldots$, for $i \geq 0$) and, in what follows, we say that 0-ary relations are *propositions*, whereas unary relations are *properties*.

The definitions that partition the domain of individuals into *ordinary* objects ($O!$) and *absract* objects ($A!$) are:

- $O! =_{df} [\lambda x\, \diamond E!x]$
- $A! =_{df} [\lambda x\, \neg\diamond E!x]$

The definitions that assert the conditions under which individuals and n-ary relations exist are:

- $x\!\downarrow\, \equiv_{df} \exists F(Fx)$
- $F^n\!\downarrow\, \equiv_{df} \exists x_1 \ldots \exists x_n(x_1 \ldots x_n F^n)$

In what follows, we'll regard the free variables in definitions as metavariables, so that they can be instanced even by non-denoting, complex terms. So, for example, '$\imath xTx\!\downarrow$' is defined by the above and asserts "the-x-such-that-Tx exists". The definition tells us that this holds if and only if the

x such that Tx exemplifies some property.[2] When these conditions hold for an individual or relation term τ, we often say, in the metalanguage, that τ has a denotation.

The definitions for identity are:

- $x = y \equiv_{df}$
  $(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$ (1)
- $F = G \equiv_{df} \Box\forall x(xF \equiv xG)$ (2)
- $p = q \equiv_{df} [\lambda x\, p] = [\lambda x\, q]$ (3)

We omit the definition of n-ary relation identity for $n \geq 2$, since it will play no role in what follows. However, we note that the definition reduces n-ary relation identity for $n \geq 2$ to property identity. Similarly, in (3), identity for propositions is defined in terms of identity for properties. Both the definition of identity for individuals and the definitions of identity for properties and propositions imply that identity is reflexive, i.e., one can prove, from these definitions, that $x = x$, $F = F$, and $p = p$. Together with the axiom for the substitution of identicals, this is sufficient to derive that identity is symmetric and transitive, both w.r.t. identity for individuals and identity for n-ary relations ($n \geq 0$). Moreover, the *necessity of identity*, for both individuals and relations, follows from the definitions.[3]

Note that $\Box\forall x_1 \ldots \forall x_n(Fx_1 \ldots x_n \equiv Gx_1 \ldots x_n)$ doesn't imply $F = G$ and $\Box(p \equiv q)$ doesn't imply $p = q$. So OT's theory of relations is hyperintensional. In particular, properties necessarily exemplified by the same objects need not be identical, but properties necessarily encoded by the same objects are.[4] And propositions that are necessarily equivalent need not be identical.

[2]Though we won't go into any detail in this paper, note also that the inferential role of a definition by identity (using the symbol $=_{df}$, as opposed to $\equiv_{df}$) can be easily stated as follows, which can be formulated as a metarule governing OT: if the definiens exists, then the identity holds, and if the definiens doesn't exist, then the definiendum doesn't either.

[3]Clearly, $F = G \rightarrow \Box F = G$ follows from definition (3), by the 4 principle of modal logic. The proof that $x = y \rightarrow \Box x = y$ goes by way of a disjunctive syllogism, reasoning from $O!x \vee A!x$, and makes use of S5 theorems.

[4]If we assume primitive possible worlds for the moment and think semantically, then properties, understood as primitive entities, have two extensions. (The same holds for relations generally but we're simplifying and focusing only on properties.) Properties whose exemplification extensions are the same from world to world need not be identical. But if their encoding extensions are the same from world to world, then they are identical. We'll see below that the modal logic of encoding will require that if properties have the same encoding extension at any world, then they have the same encoding extension at every world. (Intuitively, if F is distinct from G, then OT guarantees that there is an abstract object that encodes one without encoding the other.) But though this suggests we could eliminate the $\Box$ from the definiens of $F = G$ given in the text, we've kept it in to make it clear that identity is a modal notion.

OT asserts *classical* axioms for propositional logic; for quantificational reasoning with primitive constants and variables; for the unrestricted substitution of identicals; for S5 modal logic; and for the logic of actuality.[5] However, a negative free logic governs the *complex* terms (i.e., descriptions and λ-expressions may fail to denote). The axioms for λ-expressions are interpreted *relationally* and are conditional on the λ-expression having a denotation.[6] The distinctive new axioms of OT are the axioms of encoding:

- Ordinary objects necessarily fail to encode properties:
  $O!x \rightarrow \Box\neg\exists F xF$
- If x encodes F, then necessarily x encodes F:
  $xF \rightarrow \Box xF$ (4)
- For any condition φ on properties, there is an abstract object that encodes just the properties satisfying φ:
  $\exists x\big(A!x \,\&\, \forall F(xF \equiv \varphi)\big)$, provided x isn't free in φ (5)

Note that for each appropriate φ, any witness to any instance of (5) is unique: there couldn't be two distinct abstract objects that encode all

[5]This logic includes one axiom that is not a necessary truth, namely, $\mathcal{A}\varphi \rightarrow \varphi$ (Zalta 1988). Note that in a logic with such an axiom, the Rule of Necessitation has to be restricted – it cannot be applied to any line of a proof that depends on this axiom. See the digression at the end of Section 2.5 below, for a discussion of an important consequence of this contingent axiom for the actuality operator.

[6]To see how the logic of λ-expressions works, note first that it is an axiom of OT that $[\lambda x_1 \ldots x_n\, \varphi]\!\downarrow$, provided none of the x_i bound by the λ occur as the arguments of an encoding formula somewhere in φ. So the axiom for β-Conversion has the following conditional formulation:

$$[\lambda x_1 \ldots x_n\, \varphi]\!\downarrow \rightarrow ([\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi)$$

We'll see how this yields a comprehension principle for relations shortly. But it is worth also mentioning now that it is axiomatic that if a λ-expression with matrix φ denotes, and φ and ψ are necessarily and universally equivalent, then the λ-expression with ψ as matrix denotes:

$$[\lambda x_1 \ldots x_n\, \varphi]\!\downarrow \,\&\, \Box\forall x_1 \ldots \forall x_n(\varphi \equiv \psi) \rightarrow [\lambda x_1 \ldots x_n\, \psi]\!\downarrow$$

So even if ψ has encoding formulas in which a free x_i occurs in encoding position, one can embed ψ in a denoting λ-expression if it is provable that ψ necessarily has the same exemplification extension as φ.

and only the properties satisfying φ, since distinct abstract objects have to differ by one of their encoded properties. So it is a theorem schema of OT that, for any φ without free xs, there is a unique abstract object that encodes just the properties such that φ, i.e.,

$$\vdash \exists! x(A!x \,\&\, \forall F(xF \equiv \varphi)), \text{ provided } x \text{ isn't free in } \varphi$$

Thus, definite descriptions of the form $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ are *guaranteed* to have a denotation, for any such formula φ; they are *canonical*.

To supplement these axioms, the theory of hyperintensional relations is embodied by (a) a comprehension principle for (hyperintensional) n-ary relations ($n \geq 0$), derivable as a theorem schema,[7] and (b) the definitions for relation identity described earlier.

2.2 Definitions and Basic Theorems About Situations

In Zalta 1993 (410), situations were defined as abstracta that encode only propositional properties, i.e., only properties of the form $[\lambda y\, p]$ ("being such that p"), where y is vacously bound by the λ:

- $Situation(x) \equiv_{df} A!x \,\&\, \forall F(xF \rightarrow \exists p(F = [\lambda y\, p]))$ (6)

Given the modal logic of encoding and the necessity of identity, it follows that situations are necessarily situations:

$$\vdash Situation(x) \rightarrow \Box Situation(x)$$

So we may use $s, s', \ldots$ as *rigid*, restricted variables that range over situations; we can reason with these variables in any modal context and rest assured that the objects serving as the values of the variables meet the restriction condition (i.e., are situations) in that context.

We say that a proposition p is *true in* s (or *makes* p *true*), written $s \models p$, just in case s encodes *being such that* p:

[7] In footnote 6, we identified a body of λ-expressions that denote, and we formulated a conditional β-Conversion principle. So one can prove from claims of the form $[\lambda x_1 \ldots x_n\, \varphi]{\downarrow}$ that $[\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi$. Then by applying Rule GEN n times, the Rule of Necessitation to the result, and then Existential Introduction, we have a derivation of the following comprehension principle for n-ary relations ($n \geq 0$):

$\exists F \Box \forall x_1 \ldots \forall x_n (Fx_1 \ldots x_n \equiv \varphi)$, provided F isn't free in φ, and none of the x_i occur as the arguments of an encoding formula somewhere in φ

Note that this yields all of the properties, relations, and propositions asserted to exist by the classical relation comprehension schema of second-order logic, since the classical principle doesn't include any encoding formulas.

- $s \models p \equiv_{df} s[\lambda y\, p]$ (7)

It now follows from the definition of identity (1) that situations are identical whenever they make the same propositions true:

$$\vdash s = s' \equiv \forall p(s \models p \equiv s' \models p) \quad (8)$$

We next say that s is a *part of* s' iff s' makes true every proposition s makes true:

- $s \trianglelefteq s' \equiv_{df} \forall p(s \models p \rightarrow s' \models p)$ (9)

It follows that parthood ($\trianglelefteq$) is a reflexive, anti-symmetric, and transitive condition on situations:

$$\begin{aligned} &\vdash s \trianglelefteq s' \\ &\vdash (s \trianglelefteq s' \,\&\, s' \neq s) \rightarrow \neg s' \trianglelefteq s \\ &\vdash s \trianglelefteq s' \,\&\, s' \trianglelefteq s'' \rightarrow s \trianglelefteq s'' \end{aligned} \quad (10)$$

2.3 (Modal) Logic of Situations

In Zalta 1993 (413), a situation was defined to be *actual* just in case every proposition true in it is true:

- $Actual(s) \equiv_{df} \forall p(s \models p \rightarrow p)$ (11)

And a *possible* situation is one that might be actual:

- $Possible(s) \equiv_{df} \Diamond Actual(s)$ (12)

A *consistent* situation is one in which no proposition and its negation are both true:

- $Consistent(s) \equiv_{df} \neg\exists p(s \models p \,\&\, s \models \neg p)$ (13)

It then follows that possible situations are consistent:

$$\vdash Possible(s) \rightarrow Consistent(s) \quad (14)$$

Note that the converse doesn't hold.[8]

In what follows we deploy the usual definition of necessary implication:

[8] For example, consider a situation, say s_1, which makes true only the following three propositions: p, q, and $p \rightarrow \neg q$, where $q \neq \neg p$. Then s_1 is consistent (there is no proposition r such that $s \models r$ and $s \models \neg r$), but not possible (it is not possible that every proposition true in s_1 is true, for otherwise some contradiction would be possibly true).

- $\varphi \Rightarrow \psi \equiv_{df} \Box(\varphi \to \psi)$ (15)

Thus, one can prove that *true in s* is not subject to modal distinctions:

$$\vdash s \models p \Rightarrow \Box s \models p \quad (16)$$

$$\vdash \Diamond s \models p \Rightarrow s \models p \quad (17)$$

We also say that a formula φ is *modally collapsed* whenever it can be established that $\Box(\varphi \to \Box\varphi)$, i.e., $\varphi \Rightarrow \Box\varphi$. As we'll see, $s \models p$ and other modally collapsed formulas will play an important role.

2.4 Possible World Theory

In Zalta 1993 (414), a *possible world* was defined to be any situation s that might be such that all and only true propositions are true in s:

- $PossibleWorld(s) \equiv_{df} \Diamond\forall p(s \models p \equiv p)$ (18)

Given our convention, the $s \models p \equiv p$ is to be parsed as $(s \models p) \equiv p$, not as $s \models (p \equiv p)$.

From this definition, the basic principles of possible world theory are derivable (Zalta 1993, 414–419). These include formal versions of the following principles:

- Every possible world is maximal, consistent, and possible, where $Maximal(s) \equiv_{df} \forall p(s \models p \vee s \models \neg p)$ (19)
- There is a unique actual world.
- Possibly p iff there is a possible world in which p is true.
- Necessarily p iff p is true in every possible world.

In what follows, we shall appeal to more recent theorems about possible worlds (developed after the publication of Zalta 1993); these will be needed for the proofs of the theorems about possibilities. One frequently cited theorem is that a situation s is possible if and only if it is a part of some possible world:

$$Possible(s) \equiv \exists w(s \trianglelefteq w) \quad (20)$$

A proof sketch of this theorem is in the Appendix. Other recent theorems of world theory will be cited (along with where their proofs can be found) as the need arises.

2.5 Identifying Situations Uniquely

Our comprehension principle for abstract objects (5) now yields a comprehension for situations, namely, for every condition on propositions, there is a situation that makes true all and only the proposition satisfying φ:

$$\vdash \exists s\forall p(s \models p \equiv \varphi), \text{ provided } s \text{ isn't free in } \varphi \quad (21)$$

A proof is provided in the Appendix. Since situations are identical whenever they make the same propositions true (8), any witness to the above will be unique – there couldn't be two situations that make true exactly the propositions such that φ. So it follows that:

$$\vdash \exists! s\forall p(s \models p \equiv \varphi), \text{ provided } s \text{ isn't free in } \varphi$$

Thus, by the principles of free logic, we know that descriptions for situations of the form $\imath s\forall p(s \models p \equiv \varphi))$ are always well-defined:

$$\vdash \imath s\forall p(s \models p \equiv \varphi)\downarrow, \text{ provided } s \text{ isn't free in } \varphi$$

Since descriptions having this form are guaranteed by the theory to have a denotation, they are *canonical* descriptions for situations.

In the remainder of this section, we discuss a subtle issue that arises when rigid definite descriptions are deployed in a modal context. (A rigid definite description, $\imath x\varphi$, semantically denotes a unique object, that satisfies φ at the distinguished actual world, if there is one.) Our digression will conclude with (a) an explanation of why this won't affect the present paper and (b) a theorem schema (25) that governs a class of canonical descriptions for situations.

In a modal logic with rigid definite descriptions, one can produce logical theorems that are not necessary. For example, the conditional $y = \imath xGx \to Gy$ will be false at a world, say w_1, when y (is assigned an object that) fails to be G at w_1 but is the unique G at the actual world w_0 (in such a case, the the antecedent is true at w_1 but the consequent false at w_1). More generally, where φ^y_x is the result of substituting y for all the free occurrences of x in φ, the claim $y = \imath x\varphi \to \varphi^y_x$ is not a necessary truth, though it is logically true (i.e., true at the distinguished actual world of every model, for every assignment to y) given the semantics of rigid definite descriptions (Zalta 1988).

In a fuller presentation of OT, we could axiomatize rigid definite descriptions by introducing an actuality operator $\mathcal{A}$ and asserting, as an axiom:

$$y = \iota x\varphi \equiv \forall x(\mathcal{A}\varphi \equiv x = y) \tag{22}$$

This is a form of the Hintikka principle (1959); it is a necessary truth and it immediately implies the following as a necessary truth, in which $\mathcal{A}\varphi^y_x = (\mathcal{A}\varphi)^y_x = \mathcal{A}(\varphi^y_x)$:

$$\vdash y = \iota x\varphi \rightarrow \mathcal{A}\varphi^y_x, \text{ provided } y \text{ is substitutable for } x \text{ in } \varphi \tag{23}$$

So by placing the actuality operator appropriately in our original example, we obtain $y = \iota xGx \rightarrow \mathcal{A}Gy$, and this is a necessary truth, not contingent. But though (22), (23), and their instances are necessary truths, the axiomatization of the actuality operator includes an axiom, namely $\mathcal{A}\varphi \rightarrow \varphi$, that is a logical truth which isn't necessary, as noted in footnote 5.[9] So the Rule of Necessitation has to be applied carefully; one may not apply the rule to necessitate a theorem whose proof depends on the axiom $\mathcal{A}\varphi \rightarrow \varphi$.

In what follows, though, we won't need to worry about illicit applications of the Rule of Necessitation since all of the definite descriptions we'll deploy involve a special class of formulas for which we can derive the conditional $y = \iota x\varphi \rightarrow \varphi^y_x$ without appealing to the contingent axiom for actuality. The formulas in question are modally collapsed, which we defined earlier as those φ for which (it is provable that) $\varphi \Rightarrow \Box\varphi$. When this condition holds for a formula φ, one can prove that $\mathcal{A}\varphi \equiv \varphi$ is necessarily true without appealing to the contingent axiom $\mathcal{A}\varphi \rightarrow \varphi$.[10] If φ is modally collapsed, then $y = \iota x\varphi \rightarrow \varphi^y_x$ is a necessary truth:

$$\vdash y = \iota x\varphi \rightarrow \varphi^y_x, \tag{24}$$
provided φ is modally collapsed and y is substitutable for x in φ

[9] To see why the formula schema $\mathcal{A}\varphi \rightarrow \varphi$ can't be necessitated, note that the conditional is true at the actual world: if φ is true at the actual world, then the conditional is true at the actual world (by truth of the consequent), and if φ is false at the actual world, then the conditional is true at the actual world (by failure of the antecedent). However, the conditional is false at any world w_1 whenever φ is true at the actual world but false at w_1.

[10] Assume $\varphi \Rightarrow \Box\varphi$, i.e., $\Box(\varphi \rightarrow \Box\varphi)$. Then by the K$\Diamond$ principle, i.e., $\Box(\psi \rightarrow \chi) \rightarrow (\Diamond\psi \rightarrow \Diamond\chi)$, it follows that $\Diamond\varphi \rightarrow \Diamond\Box\varphi$. But in S5, $\Diamond\Box\varphi \rightarrow \Box\varphi$. So by hypothetical syllogism, we've established:

(θ) $\Diamond\varphi \rightarrow \Box\varphi$

Now to see that $\Box(\mathcal{A}\varphi \equiv \varphi)$, we first prove both directions of $\mathcal{A}\varphi \equiv \varphi$. ($\rightarrow$) Assume $\mathcal{A}\varphi$. Then $\Diamond\varphi$. So by (θ), $\Box\varphi$. Hence φ, by the T schema. ($\leftarrow$) Assume φ. Then $\Diamond\varphi$. But again by (θ), it follows that $\Box\varphi$. Hence $\mathcal{A}\varphi$. Since we've now established $\mathcal{A}\varphi \equiv \varphi$ without appealing to any contingencies, it follows by Rule RN that $\Box(\mathcal{A}\varphi \equiv \varphi)$.

(See the Appendix for the proof.) In this paper, we shall appeal only to definite descriptions $\iota x\varphi$ in which φ is modally collapsed, and so we won't need to worry about mistakenly applying the Rule of Necessitation to theorems derived from a logical truth that is not necessary.

In particular, when we identify (e.g., introduce by definition) a situation using a canonical description as definiens, then as a special case of (24) for modally collapsed φ, it is a theorem that if a situation s is (identical to) the situation s' such that s' makes true just the propositions satisfying φ, then s makes true just the propositions satisfying φ:

$$\vdash \big(s = \iota s'\forall p(s' \models p \equiv \varphi)\big) \rightarrow \forall p(s \models p \equiv \varphi) \tag{25}$$
provided s' isn't free in φ and φ is modally collapsed

The proof in the Appendix appeals to the fact that $s' \models p \Rightarrow \Box s' \models p$ is an instance of (16), by definition (15), and the fact that φ is modally collapsed. So we may validly infer that the formula $\forall p(s' \models p \equiv \varphi)$ is modally collapsed. To derive (25), then, one can simply instantiate s and the description $\iota s'\forall p(s' \models p \equiv \varphi)$, for y and $\iota x\varphi$, respectively, in (24).

(25) is crucial to the theorems that follow. All the descriptions of the form $\iota s'\forall p(s' \models p \equiv \varphi)$ used below will be constructed in terms of formulas φ that are modally collapsed. This should forestall any concerns about the fact that we shall be working within a modal context in which definite descriptions are interpreted rigidly.

3 New Definitions and Theorems

3.1 Modally Closed Situations

A situation s is *modally closed* just in case it makes true every proposition p necessarily implied by s's being actual:

- $\mathit{ModallyClosed}(s) \equiv_{df} \forall p\big((\mathit{Actual}(s) \Rightarrow p) \rightarrow s \models p\big)$ (26)

It follows that if s is modally closed then if s makes p true and p necessarily implies q, then s makes q true:

$$\vdash \mathit{ModallyClosed}(s) \rightarrow \forall p\forall q\big(s \models p \,\&\, (p \Rightarrow q) \;\rightarrow\; s \models q\big) \tag{27}$$

Also, if s is modally closed and consistent, then s is possible:

$$\vdash (\mathit{ModallyClosed}(s) \,\&\, \mathit{Consistent}(s)) \rightarrow \mathit{Possible}(s) \tag{28}$$

Another fact that will play a role is that if s is modally closed and p is necessary, then s makes p true:

$$\vdash (\mathit{ModallyClosed}(s) \,\&\, \Box p) \to s \models p \tag{29}$$

Finally, it should be easy to see that possible worlds are modally closed:

$$\vdash \forall w \mathit{ModallyClosed}(w) \tag{30}$$

3.2 Definition of a Possibility

We plan to show that the notion of a possibility, as introduced by Humberstone and the other cited theorists, can be defined as a situation that is consistent and modally closed:

- $\mathit{Possibility}(s) \equiv_{df} \mathit{Consistent}(s) \,\&\, \mathit{ModallyClosed}(s)$ (31)

Be sure in what follows to distinguish this notion from *Possible*(s), as the latter is defined in (12). It is now provable that possible worlds are possibilities:

$$\vdash \mathit{Possibility}(w) \tag{32}$$

This follows from the facts that possible worlds are modally closed and consistent. The former was established above and the latter was established in Zalta 1993 (415).

It is also a theorem that possibilities are necessarily possibilities:

$$\vdash \Box\forall s(\mathit{Possibility}(s) \to \Box\mathit{Possibility}(s))$$

Given this fact, we may introduce $\mathfrak{s}, \mathfrak{s}', \ldots$ as rigid restricted variables ranging over possibilities.

It follows from (31) and (28) that possibilities are possible:

$\vdash \mathit{Possible}(\mathfrak{s})$, i.e.,
$$\vdash \mathit{Possibility}(s) \to \mathit{Possible}(s) \tag{33}$$

Finally, it proves handy to know that possibilities are parts of some possible world:

$$\vdash \exists w(\mathfrak{s} \trianglelefteq w)$$

This follows from (33) and (20). Note that possibilities need not be proper parts of possible worlds; cf. Humberstone 1981 (315).[11]

[11] Humberstone entertains the idea that possible worlds aren't possibilities, but decides not to pursue it, though leaving it an open question for consideration. If one wants to require that possible worlds fail to be possibilities, one could conjoin the definiens of (31) with the clause $\neg\mathit{Maximal}(s)$. Though that rules out the theorem labeled (32), i.e., that possible worlds are possibilities, one can still prove the existence of possibilities from theorem stated in (37), namely, that the particular situation defined later as $\boldsymbol{s}_\Box$ is a possibility.

3.3 Absolute Necessity, Possibilities, and Gaps

Where $\overline{p}$ is defined as $\neg p$, we say a situation s has a gap on p just in case s doesn't make p true and doesn't make $\overline{p}$ true:

- $\mathit{GapOn}(s,p) \equiv_{df} \neg s \models p \,\&\, \neg s \models \overline{p}$ (34)

We now introduce a special situation named *absolute necessity*; it is the situation that makes true all and only necessary truths:

- $\boldsymbol{s}_\Box =_{df} \imath s \forall p(s \models p \equiv \Box p)$

Since the definiens is a canonical description constructed from a modally collapsed formula, we therefore know:

$$\forall p(\boldsymbol{s}_\Box \models p \equiv \Box p) \tag{35}$$

Absolute necessity has a number of interesting features. The first is that it has gaps on all and only contingent propositions:

$$\vdash \mathit{Contingent}(p) \equiv \mathit{GapOn}(\boldsymbol{s}_\Box, p) \tag{36}$$

Moreover, absolute necessity is a possibility:

$$\vdash \mathit{Possibility}(\boldsymbol{s}_\Box) \tag{37}$$

It can also be established that no proper part of absolute necessity is a possibility:

$$\vdash \forall s\big((s \trianglelefteq \boldsymbol{s}_\Box \,\&\, s \neq \boldsymbol{s}_\Box) \to \neg\mathit{Possibility}(s)\big) \tag{38}$$

Consequently, $\boldsymbol{s}_\Box$ is the smallest possibility. Moreover, if *any* possibility has a gap on p, then p is contingent:

$$\vdash \mathit{GapOn}(\mathfrak{s}, p) \to \mathit{Contingent}(p) \tag{39}$$

We're now in a position to derive the principles governing possibilities stipulated in the literature. We have to show:

⊢ Ordering Principle	⊢ Persistence Principle
⊢ Refinability Principle	⊢ Cofinality Principle
⊢ Negation Principle	⊢ Conjunction Principle

3.4 The Ordering Principle

We now say that a situation s' *contains* situation s, written $s' \trianglerighteq s$, just in case s is a part of s':

$$s' \trianglerighteq s \equiv_{df} s \trianglelefteq s' \tag{40}$$

However, when this definition is instanced by possibilities, we read $\mathfrak{s}' \trianglerighteq \mathfrak{s}$ as: $\mathfrak{s}'$ is a *refinement of* $\mathfrak{s}$. So we may now establish that every possibility is a refinement of absolute necessity:

$$\vdash \forall \mathfrak{s}(\mathfrak{s} \trianglerighteq \boldsymbol{s}_{\Box}) \tag{41}$$

Since $\trianglelefteq$ is reflexive, anti-symmetric, and transitive on the situations (10), the Ordering Principle becomes a theorem – *refinement of* is reflexive, anti-symmetric, and transitive on the possibilities:

Ordering Principle
$\vdash \mathfrak{s} \trianglerighteq \mathfrak{s}$
$\vdash (\mathfrak{s}' \trianglerighteq \mathfrak{s} \mathbin{\&} \mathfrak{s}' \neq \mathfrak{s}) \rightarrow \neg \mathfrak{s} \trianglerighteq \mathfrak{s}'$
$\vdash (\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \mathbin{\&} \mathfrak{s}' \trianglerighteq \mathfrak{s}) \rightarrow \mathfrak{s}'' \trianglerighteq \mathfrak{s}$

Cf. Humberstone 1981 (318); 2011 (899); van Benthem 1981 (3); 2016 (3); Holliday 2014 (3); Ding & Holliday 2020 (155); and Holliday forthcoming (Definition 2.1 and 2.21).

3.5 The Persistence Principle

Humberstone (1981, 318) introduces the Persistence Principle as follows. Where π is a proposition, X and Y are possibilities, $\geq$ is the refinement relation on possibilities, and $V(\pi, X)$ is the truth-value of π with respect to X:

- If $V(\pi, X)$ is defined and $Y \geq X$, then $V(\pi, Y) = V(\pi, X)$

He takes this to intuitively assert that "[f]urther delimitation of a possible state of affairs should not reverse truth-values, but only reduce indeterminancies" (1981, 318).

In OT, the Persistence Principle can be represented as the *theorem* that if a proposition p is true in a possibility $\mathfrak{s}$ and $\mathfrak{s}'$ is a refinement of $\mathfrak{s}$, then p is true in $\mathfrak{s}'$:

Persistence Principle
$\vdash (\mathfrak{s} \models p \mathbin{\&} \mathfrak{s}' \trianglerighteq \mathfrak{s}) \rightarrow \mathfrak{s}' \models p$

Cf. van Benthem 1981, 3 ('Heredity'), 2016, 3; Restall 2000, Definition 1.2 (Heredity Condition); Holliday 2014, 315; forthcoming, 15; Berto 2015, 767 (HC); Berto & Restall 2019, 1128 (HC); and Ding & Holliday 2020, 155.[12] (We have omitted the proof from the Appendix since it is an immediate consequence of definition (9).)

3.6 The Refinability Principle

To formulate and prove the Refinability Principle, we have to introduce a number of definitions and prove a number of theorems.

3.6.1 The Modal Closure of a Situation: I

We begin with a definition of *the modal closure* $s^\star$ of a situation s. The modal closure of s is the situation that makes true all and only those propositions p such that s's being actual necessarily implies p:

- $s^\star =_{df} \imath s' \forall p\big(s' \models p \equiv (Actual(s) \Rightarrow p)\big)$

So in what follows, we shall carefully distinguish the formula expressing the condition that s is modally closed, i.e., $ModallyClosed(s)$, from the term that denotes the modal closure of a situation s, i.e., $s^\star$. By (25) and the fact that the formula $Actual(s) \Rightarrow p$ is modally collapsed, it follows that $s^\star$ makes p true iff s's being actual necessarily implies p:

$$\vdash \forall p\big(s^\star \models p \equiv (Actual(s) \Rightarrow p)\big) \tag{42}$$

It also follows that a situation is a part of its modal closure:

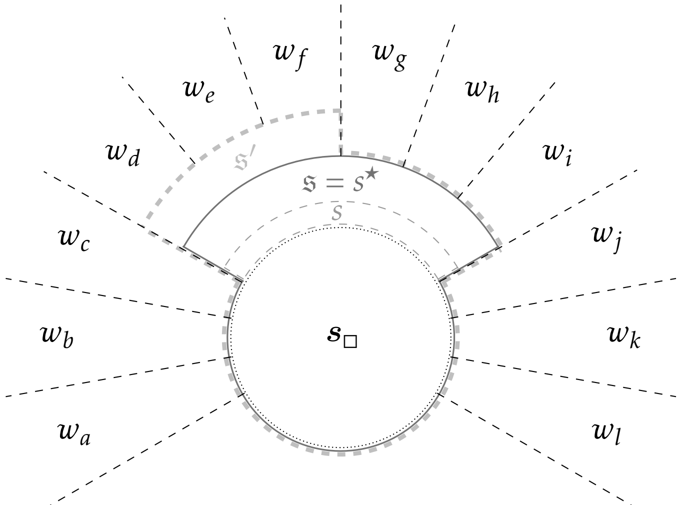
$$\vdash s \trianglelefteq s^\star \tag{43}$$

To better understand the foregoing definitions, think about $\boldsymbol{s}_{\Box}$. It is the modal closure of the empty situation (i.e., the situation that makes no propositions true, which one might define as $\imath s \forall p(s \models p \equiv p \neq p)$). Also, for a picture of the main constructions defined thus far, see Figure 1.

[12] A version of this principle also appeared in Barwise 1989a (265), prefaced by the definition:

$$Persistent(p) \equiv_{df} \forall s(s \models p \rightarrow \forall s'(s \trianglelefteq s' \rightarrow s' \models p))$$

In Zalta 1993 (Theorem 8), it was noted that OT implies $\forall p Persistent(p)$, which settled Alternative 6.1 at choice point 6 in Barwise 1989a, 265.



$w_a - w_l$	=	possible worlds
$\boldsymbol{s}_\Box$	=	absolute necessity
	=	the smallest possibility
	=	$\imath s' \forall p(s \models p \equiv \Box p)$
s	=	a *possible* situation
$s^\star$	=	the modal closure of s
	=	the smallest possibility $\mathfrak{s}$ that contains s
	=	$\imath s' \forall p(s' \models p \equiv (Actual(s) \Rightarrow p))$
$\mathfrak{s}'$	=	a refinement of $\mathfrak{s}$

Figure 1: In this figure, $\boldsymbol{s}_\Box$ ('absolute necessity') is the smallest possibility. The regions labeled as possible worlds $w_a - w_l$ all *overlap* with $\boldsymbol{s}_\Box$. The region s is some possible situation that is a part of possible worlds $w_d - w_i$, but it need not be a possibility. The *modal closure of* s, namely $s^\star$, will provably be a possibility (i.e., consistent and modally closed): it makes true all of the necessary consequences of propositions true in s, including all of the necessary propositions true in $\boldsymbol{s}_\Box$. $s^\star$ will therefore be a refinement of absolute necessity. Note that $s^\star$ is part of the same possible worlds as s. However, any refinement $\mathfrak{s}'$ of possibility $\mathfrak{s}$ will not be a part of all the possible worlds of which s is a part.

3.6.2 Interlude: The p-Extension of a Situation

We define the p extension of a situation s to be that situation that makes all the propositions in s true and also makes p true:

- $s^{+p} =_{df} \imath s' \forall q\big(s' \models q \equiv (s \models q \lor q = p)\big)$

Since the definiens is a canonical description with a modally collapsed formula, we know:

$$\vdash \forall q\big(s^{+p} \models q \equiv (s \models q \lor q = p)\big) \tag{44}$$

It follows that the p-extension of s is a part of a possible world w iff s is a part of w and p is true in w:

$$\vdash s^{+p} \trianglelefteq w \equiv s \trianglelefteq w \;\&\; w \models p \tag{45}$$

Theorems (44) and (45) also help us to prove that p is true in every world of which s is a part iff s's being actual necessarily implies p:

$$\vdash \forall w(s \trianglelefteq w \rightarrow w \models p) \equiv (Actual(s) \Rightarrow p) \tag{46}$$

Consider Figure 2. In Figure 2, the p-extension of the situation $\boldsymbol{s}_\Box$ makes true everything in $\boldsymbol{s}_\Box$ and makes p true as well. However, the modal closure of the p-extension of $\boldsymbol{s}_\Box$ will be a possibility that refines $\boldsymbol{s}_\Box$, as will the modal closure of the $\overline{p}$-extension of $\boldsymbol{s}_\Box$. So if p is contingent, then the modal closure of the p-extension of $\boldsymbol{s}_\Box$ will be a possibility, as will the modal closure of the $\overline{p}$-extension of $\boldsymbol{s}_\Box$.

3.6.3 The Modal Closure of a Situation: II

A few final facts about the modal closure of a situation will put us in position to prove the Refinability Principle. First is the fact that a situation is a part of a possible world if and only if its modal closure is:

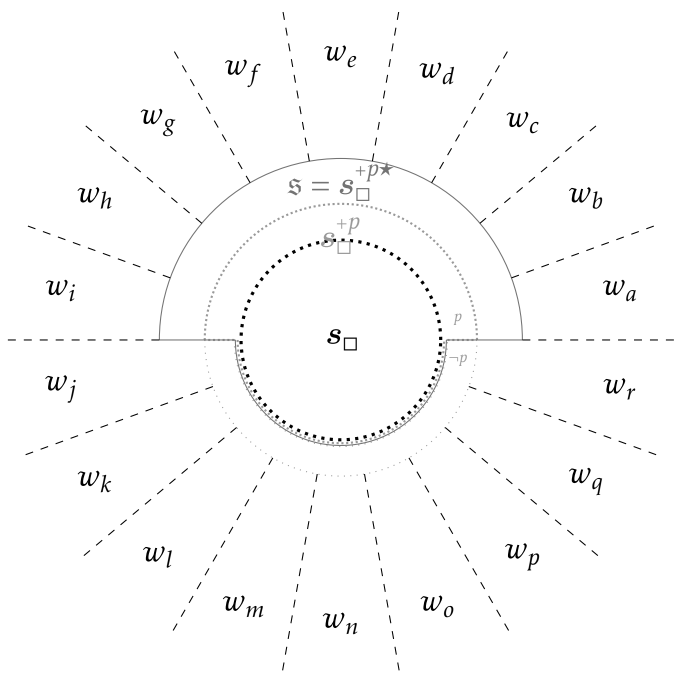
$$\vdash s \trianglelefteq w \equiv s^\star \trianglelefteq w \tag{47}$$

Moreover, a situation is possible iff its modal closure is:

$$\vdash Possible(s) \equiv Possible(s^\star) \tag{48}$$

Finally, the modal closure of a situation is modally closed:

$$\vdash ModallyClosed(s^\star) \tag{49}$$



$w_a - w_r$	=	possible worlds
$\boldsymbol{s}_\Box$	=	absolute necessity
	=	$\imath s\forall p(s \models p \equiv \Box p)$
$\boldsymbol{s}_\Box^{+p}$	=	the p extension of absolute necessity
	=	$\imath s\forall q(s \models q \equiv \boldsymbol{s}_\Box \models q \vee q = p)$
$\boldsymbol{s}_\Box^{+p\star}$	=	the modal closure of $\boldsymbol{s}_\Box^{+p}$
	=	$\imath s\forall q(s \models q \equiv (Actual(\boldsymbol{s}_\Box^{+p}) \Rightarrow q))$
$\mathfrak{s}$	=	a refinement of $\boldsymbol{s}_\Box = \boldsymbol{s}_\Box^{+p\star}$

Figure 2: In this figure, absolute necessity ($= \boldsymbol{s}_\Box$) makes true all and only necessary truths. But proposition p is a contingent truth; it is true in some worlds ($w_a - w_i$) and not in others ($w_j - w_r$). The p-extension of $\boldsymbol{s}_\Box$ makes true the propositions that are true in $\boldsymbol{s}_\Box$ and also makes p true. But that is not yet a possibility or a refinement of $\boldsymbol{s}_\Box$. Instead the modal closure of the p-extension of $\boldsymbol{s}_\Box$ is a possibility and is a refinement of $\boldsymbol{s}_\Box$; it makes true all of the necessary consequences of propositions true in the p-extension of $\boldsymbol{s}_\Box$.

3.6.4 The Refinability Principle and its Proof

Humberstone (1981, 318) formulates the Refinability Principle as follows, where π ranges over propositional variables, X, Y range over possibilities, and T, F are truth-values:

- For any π and any X, if $V(\pi, X)$ is undefined, then $\exists Y(Y \geq X$ with $V(\pi, Y) = T)$ and $\exists Z(Z \geq X$ with $V(\pi, Z) = F)$

If we use p for π, $\mathfrak{s}$ for X, and $GapOn(\mathfrak{s}, p)$ for $V(\pi, X)$ is undefined, then we may formulate the above in OT as the claim: if $\mathfrak{s}$ has a gap on p, then there is a possibility that refines $\mathfrak{s}$ in which p is true and there is a possibility that refines $\mathfrak{s}$ in which $\neg p$ is true. Formally:

- $GapOn(\mathfrak{s}, p) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models p) \,\&\, \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg p)$

Cf. Holliday 2014, 315; forthcoming, 15; and D&H 2020, 155.

However, this principle can be derived even when strengthened to a biconditional:

Refinability Principle
$$\vdash GapOn(\mathfrak{s}, p) \equiv \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models p) \,\&\, \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg p) \tag{50}$$

Here is proof sketch, where r is a arbitrary, but fixed, proposition.

($\rightarrow$) Since $GapOn(\mathfrak{s}, r)$ implies $GapOn(\mathfrak{s}, \neg r)$,[13] it suffices to show only:

$$GapOn(\mathfrak{s}, r) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models r)$$

So assume $GapOn(\mathfrak{s}, r)$. As our witness to $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models r)$, consider $(\mathfrak{s}^{+r})^\star$, i.e., the modal closure of the r extension of $\mathfrak{s}$, which we may write more simply as $\mathfrak{s}^{+r\star}$. We have to show all of the following: (a) $\mathfrak{s}^{+r\star} \trianglerighteq \mathfrak{s}$, (b) $\mathfrak{s}^{+r\star} \models r$, and (c) $Possibility(\mathfrak{s}^{+r\star})$. Since, by definition, $\mathfrak{s}^{+r\star}$ must contain $\mathfrak{s}^{+r}$, the proof of (a) and (b) are trivial. The proof of (c) requires us to show (d) $ModallyClosed(\mathfrak{s}^{+r\star})$ and (e) $Consistent(\mathfrak{s}^{+r\star})$. But (d) follows from the fact that $\mathfrak{s}^{+r\star}$ is a modal closure and, hence, modally closed. To show (e), we start by showing $Possible(\mathfrak{s}^{+r})$. That just means there must be a world containing $\mathfrak{s}$ that makes r true. But if no such world were to exist,

[13] Assume $GapOn(\mathfrak{s}, p)$ and for reductio, $\neg GapOn(\mathfrak{s}, \neg p)$. Then, by definition of $GapOn$ (34), either $\mathfrak{s} \models \neg p$ or $\mathfrak{s} \models \neg\overline{p}$, i.e., either $\mathfrak{s} \models \neg p$ or $\mathfrak{s} \models \neg\neg p$. But the former contradicts $GapOn(\mathfrak{s}, p)$. The latter, by a consequence (27) of the fact that $\mathfrak{s}$ is modally closed, implies $\mathfrak{s} \models p$, which also contradicts $GapOn(\mathfrak{s}, p)$.

then $\neg r$ would necessarily follow from $Actual(\mathfrak{s})$ which means $\mathfrak{s}$ (being modally closed) would make $\neg r$ true and $GapOn(\mathfrak{s}, r)$ would be false (contradiction). Now, any world containing $\mathfrak{s}^{+r}$ already contains the modal closure of $\mathfrak{s}^{+r}$ so it follows that $Possible(\mathfrak{s}^{+r\star})$. But then we are done because, as noted above, every possible situation is consistent.[14]

($\leftarrow$) Assume:

(ϑ) $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \;\&\; \mathfrak{s}' \models r) \;\&\; \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \;\&\; \mathfrak{s}' \models \neg r)$

For reductio, suppose $\neg GapOn(\mathfrak{s}, r)$. Then either $\mathfrak{s} \models r$ or $\mathfrak{s} \models \neg r$. Without loss of generality, suppose $\mathfrak{s} \models r$. By the Persistence Principle, every refinement of s makes r true. So there can't be a refinement that makes $\neg r$ true, contradicting the right conjunct of (ϑ). $\bowtie$

A full proof is in the Appendix.

3.7 The Cofinality Principle

In van Benthem (1981, 4; 2016, 3), we find the principle labeled *Cofinality*. In 2016, he formulates this principle as follows, where $P\mathbf{d}$ is any atomic fact and $\geq$ is the the partial order on possibilities:

- If for all $v \geq w$, there exists a $u \geq v$ with $P\mathbf{d}$ true at u, then $P\mathbf{d}$ is already true at w.

This can be derived, without restriction to atomic facts, as the theorem: if, for every possibility $\mathfrak{s}'$ that refines $\mathfrak{s}$, there is a possibility $\mathfrak{s}''$ that refines $\mathfrak{s}'$ in which p is true, then p is true in $\mathfrak{s}$:

Cofinality Principle
$\vdash \forall \mathfrak{s}'\big(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \;\&\; \mathfrak{s}'' \models p)\big) \rightarrow \mathfrak{s} \models p$ (51)

Cf. Humberstone's (2011, 900) restatement of the Refinement Principle.

Note that the proof appeals to Refinability. But Refinability isn't implied by Cofinality unless the notion of *possibility* obeys the Negation Constraint, to which we now turn.

[14] The proof of (a) appeals to (40), (43), and (44); the proof of (b) appeals to (43) and (44); the proof of (d) is that it is an instance of (49); and the proof of (e) appeals to (14), (20), (26), (31), (45), (46), and (48).

3.8 Negation, Conjunction, and Fundamental Theorems

Humberstone 1981 (319–320) and 2011 (900) include the Negation and Conjunction Principles. It is straightforward to translate these principles into the language of OT and derive them. The Negation Principle asserts that the negation of p is true in $\mathfrak{s}$ if and only if p fails to be true in every refinement of $\mathfrak{s}$:

Negation Principle
$\vdash \mathfrak{s} \models \neg p \equiv \forall \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \neg \mathfrak{s}' \models p)$

The Conjunction Principle asserts that the conjunction p *and* q is true in $\mathfrak{s}$ if and only if both p and q are true in $\mathfrak{s}$:

Conjunction Principle
$\vdash \mathfrak{s} \models (p \;\&\; q) \equiv (\mathfrak{s} \models p \;\&\; \mathfrak{s} \models q)$

We omit the proofs of the Negation and Conjunction Principles since they are straightforward.

Finally, we note that there is a fundamental theorem and corollary that are analogous to the fundamental theorem and corollary governing possible worlds. Just as one can prove that p is possibly true iff there is a possible world in which p is true (Zalta 1993, Theorem 25), one can prove that p is possibly true if and only if there is a possibility in which p is true:

Fundamental Theorem
$\vdash \Diamond p \equiv \exists \mathfrak{s}(\mathfrak{s} \models p)$ (52)

And just as one can prove that p is necessarily true iff true in all possible worlds (Zalta 1993, Theorem 24), one can prove that p is necessarily true if and only if p is true in every possibility:

Corollary to the Fundamental Theorem
$\vdash \Box p \equiv \forall \mathfrak{s}(\mathfrak{s} \models p)$ (53)

These last two theorems validate the intuition (shared by Humberstone, Holliday, and others) that the primitive modal operators can indeed be understood as quantifiers over possibilities. We have shown that that this intuition can be derived as a theorem in OT. But, as we shall see in the next section, it cannot be derived (and arguably does not hold) in a context where the modal operators are governed by semantics that are characterized purely by non-modal semantic principles.

4 Observations

The fundamental theorem (52) guarantees that object-theoretic possibilities line up exactly with the possibly true propositions. Of course, people may disagree about what propositions are possibly true, but we are operating under the assumption that there are at least some propositions that are possibily true. Thus, (52) serves as a kind of representation theorem. But the question arises whether such a theorem can be proved in systems that take possibilities as semantically-primitive and don't include modal operators in the semantics. One might suggest that the question is irrelevant since the semantically-primitive possibilities will delineate the propositions that, in the object language, are 'metaphysically possible'.

But such a response is too fast. First, note that no such theorem can be developed in those systems, either wholly within the object language or wholly within the semantics, given that there is no modal operator present in the semantics and there are no possibilities present in the object language. At best, any such theorem would have to be a metatheorem that establishes that the object language claims $\Diamond p$ holds if and only if the semantic claim $\exists\mathfrak{s}(\mathfrak{s} \models p)$ holds.

But we suggest that even if these systems of possibility semantics stipulate that the propositions that are possibly true in the object language are given by the existence of semantically-primitive possibilities, then they overgenerate metaphysical possibilities. That is, the models described in the works of Humberstone, van Benthem, Holliday, and Ding & Holliday allow the semantic claim $\exists\mathfrak{s}(\mathfrak{s} \models p)$ to be true even when, intuitively, the object-language claim $\Diamond p$ is not. For example, one might hold one of the following, reasonable metaphysical views, the first two of which are discussed in Kripke (1972 [1980, 112–114]):

- No (existing, concrete) object could have been the (biological) sister of sisterless person b.
- Aristotle could not have had different parents,
- Aristotle couldn't have been a rock.

For simplicity, let's use the last of these as a typical example (though, if you think this particular example is not persuasive, switch to one of the other examples or your own). We may represent this as $\neg\Diamond Ra$. Nothing in possibility semantics (formulated with primitive possibilities, without

a modal operator) prevents a model in which $\exists\mathfrak{s}(\mathfrak{s} \models Ra)$ holds semantically (where $\models$ here denotes the semantically defined notion of *truth in a situation* used in possibility semantics). But if Ra is taken to express that Aristotle is a rock and there are strong metaphysical reasons for believing that he couldn't have been a rock, then there will be a model in which a proposition is true at a semantically-primitive possibility but not in fact possibly true. Since the model *just is* the theory of possibilities for those who take possibilities as primitive in the semantics, the theory overgenerates possibilities.[15]

By contrast, the metaphysics of possibilities as developed in OT does not overgenerate possibilities. If Aristotle couldn't have been a rock holds, or couldn't have had different parents, or if no existing, concrete object could have been a sister of sisterless person b, or if any of a number of other modal claims about metaphysical impossibilities hold, then the fundamental theorem of OT ensures that there are no metaphysical possibilties where those claims are true.

The concern here is that the pure semantic study of possibilities seems to be based on the assumption that *absence of contradiction implies possibility*. But without a primitive modal operator, the refinements of a semantically primitive possibility are only deductively closed and not modally closed. In OT, however, the smallest possibility, absolute necessity, makes true (encodes) more than just logical truths – it encodes all the metaphysically necessary truths.

Some final theorems of OT shed light on this issue. We already know that *Consistent*(s) doesn't imply *Possible*(s). However, it is provable that s is possible if and only if the modal closure of s is consistent:

$$\vdash Possible(s) \equiv Consistent(s^{\star}) \qquad (54)$$

These considerations help to explain why a maximal and consistent situation need not satisfy the definition of a possible world, for one can show:

- $Maximal(s)\ \&\ Consistent(s) \nvdash Possible(s)$

[15] One might suggest, as a response, that possibility semanticists could simply add constraints to their models, but then the data that is the source of the constraints would become part of the meaning of modality. By contrast, in OT, the data in the bulleted list above simply become expressed as hypotheses or axioms of the object language, not as meaning postulates governing the modal operators.

Proof. Consider the following instance of theorem schema (21), which asserts the existence of a situation that makes true all and only the propositions that are not true:

$$\exists s \forall p(s \models p \equiv \neg p)$$

Let s_0 be such a situation. Then, clearly, s_0 is maximal: since the negation of every true proposition is true in s_0, then for every proposition, either p or its negation is true in s_0. Moreover, $Consistent(s_0)$ since there is no proposition p such that s_0 makes p and $\neg p$ true. Otherwise, both $\neg p$ and $\neg\neg p$ would be true, which is a contradiction. But, clearly, $\neg Possible(s_0)$, since s_0 makes true the negations of necessary truths. It is not possible that every proposition true in s_0 is true.

Indeed, as constructed in the proof, s_0 is an impossible world that is consistent, though it isn't deductively closed. But beyond s_0, OT allows for deductively-closed impossible worlds (i.e., situations that are deductively closed, maximal, and consistent, but not possible).

The new insight that we can derive from these observations can be developed as follows. In Zalta 1993 (Theorems 15, 17), it was established that possible worlds are both maximal and possible, and with a bit more work, one can show that s is a possible world *if and only if* s is both maximal and possible, i.e.,[16]

[16] Given Theorems 15 and 17 in Zalta 1993, it suffices to prove just the right-to-left direction. So assume $Maximal(s)$ and $Possible(s)$. Then by definitions (19), (12), and (11), we know both:

(ϑ) $\forall p(s \models p \vee s \models \neg p)$

(ξ) $\Diamond \forall p(s \models p \rightarrow p)$

By definition (18), we have to show $\Diamond \forall p(s \models p \equiv p)$. Our strategy is to show:

- show $\forall p(s \models p \rightarrow p) \rightarrow \forall p(s \models p \equiv p)$ without appealing to contingencies,
- apply the Rule of Necessitation to obtain $\Box(\forall p(s \models p \rightarrow p) \rightarrow \forall p(s \models p \equiv p))$, and
- use this last result and (ξ) to infer, by the K$\Diamond$ principle, that $\Diamond \forall p(s \models p \equiv p)$.

Since the final two steps are straightforward, it only remains to show $\forall p(s \models p \rightarrow p) \rightarrow \forall p(s \models p \equiv p)$. So assume:

(ζ) $\forall p(s \models p \rightarrow p)$

Then it suffices to show that $\forall p(p \rightarrow s \models p)$. To avoid a clash of variables, we show $q \rightarrow s \models q$, where q is arbitrary. So assume q and, for reductio, that $\neg s \models q$. Then by (ϑ), it follows that $s \models \neg q$. Then by (ζ), $\neg q$. Contradiction.

$$\vdash PossibleWorld(s) \equiv Maximal(s) \,\&\, Possible(s)$$

But from this and (54) above, it follows that s is a possible world if and only if s is maximal and its modal closure is consistent:

$$\vdash PossibleWorld(s) \equiv Maximal(s) \,\&\, Consistent(s^{\star})$$

So whereas the maximality and consistency of s are not sufficient for s to be a possible world, the maximality of s and the consistency of its modal closure are. These theorems offer further reasons why one should not assume that absence of contradiction implies possibility. And they take on further significance when it is observed that the propositions of OT have predicational and quantificational structure and that there are OT-situations that make true such propositions. (The metaphysics of possibilities in OT has been built on top of a quantified modal logic, with propositions as 0-ary relations.) We claim, therefore, that in the wider domain of situations that make propositions with predicational and quantification structure true, absence of contradiction doesn't imply possibility.

5 Appendix: Proofs of Selected Theorems

(20)[17] ($\rightarrow$) Assume $Possible(s)$. Then by definition, $\Diamond \forall p(s \models p \rightarrow p)$. By the fundamental theorem of possible world theory (Zalta 1993, 418, Theorem 25), $\exists w(w \models \forall p(s \models p \rightarrow p))$. Suppose $w_1 \models \forall p(s \models p \rightarrow p)$. Then by another theorem of world theory (Zalta m.s., currently item (547.5)), we can export the quantifier:

(ϑ) $\forall p(w_1 \models (s \models p \rightarrow p))$

But since $w_1 \models (s \models p \rightarrow p)$ is necessarily equivalent to $w_1 \models (s \models p) \rightarrow (w_1 \models p)$ (Zalta m.s., item (547.2)), it follows from (ϑ) by a Rule of Substitution that:

(ξ) $\forall p(w_1 \models (s \models p) \rightarrow (w_1 \models p))$

It remains to show that w_1 is a witness to $\exists w(s \trianglelefteq w)$, and so we have to show that $\forall p(s \models p \rightarrow w_1 \models p)$. By GEN, assume $s \models p$. Then $\Box s \models p$,

[17] The following proof is by Daniel Kirchner; he reviewed the original proof by Zalta, simplified it, and verified his proof using his implementation of OT in Isabelle/HOL.

by (16). So by the fundamental theorem cited above, $\forall w(w \models (s \models p))$. Hence $w_1 \models (s \models p)$. Instantiating p into (ξ), it follows that $w_1 \models p$.

($\leftarrow$) Assume $\exists w(s \trianglelefteq w)$ and suppose w_1 is a witness, so that we know $s \trianglelefteq w_1$. Then by (9):

$$(\zeta)\quad \forall p(s \models p \rightarrow w_1 \models p)$$

Note that it suffices to show $w_1 \models \mathit{Actual}(s)$, since from this we can then conclude $\exists w(w \models \mathit{Actual}(s))$, then $\Diamond \mathit{Actual}(s)$, and then $\mathit{Possible}(s)$. But by definition (11) and the fact that possible worlds are modally closed (30), it suffices in turn to show $w_1 \models \forall p(s \models p \rightarrow p)$. But again by exporting the quantifier, it suffices to show $\forall p(w_1 \models (s \models p \rightarrow p))$. So by GEN, we only need to show: $w_1 \models (s \models p \rightarrow p)$. But by a theorem cited earlier, it now suffices to show $(w_1 \models (s \models p)) \rightarrow (w_1 \models p)$. So assume $w_1 \models (s \models p)$. Then by the fundamental theorem of world theory, $\Diamond s \models p$. Hence, $s \models p$. So $w_1 \models p$, by (ζ). ⋈

(21) Consider any formula φ in which x isn't free. Then if we eliminate the restricted variable from the claim to be established, we have to show:

$$\exists x\big(\mathit{Situation}(x) \,\&\, \forall p(x \models p \equiv \varphi)\big)$$

Pick some property variable that isn't free in φ, say G, and let ψ be the formula $\exists p(\varphi \,\&\, G = [\lambda z\, p])$. Then by the comprehension principle for abstract objects (5), we know $\exists x(A!x \,\&\, \forall G(xG \equiv \psi))$, i.e.,

$$\exists x\big(A!x \,\&\, \forall G(xG \equiv \exists p(\varphi \,\&\, G = [\lambda z\, p]))\big)$$

This asserts that there exists an abstract object that encodes just the properties G that are propositional properties constructed out of propositions that satisfy φ. Suppose a is such an object. Then $A!a$ and:

$$(\vartheta)\quad \forall G\big(aG \equiv \exists p(\varphi \,\&\, G = [\lambda z\, p])\big)$$

Clearly, $\mathit{Situation}(a)$, by definition (6). So, by GEN, we only have to show $a \models p \equiv \varphi$. Note that we can't instantiate $[\lambda z\, p]$ into (ϑ); the variable p would get captured by the quantifier $\exists p$. But we can instantiate $[\lambda z\, p]$ into the following alphabetic variant of (ϑ):

$$(\vartheta')\quad \forall G\big(aG \equiv \exists q(\varphi^q_p \,\&\, G = [\lambda z\, q])\big)$$

So if we instantiate $[\lambda z p]$ into (ϑ') remembering that G isn't free in φ, we obtain:

$$a[\lambda z\, p] \equiv \exists q\big(\varphi^q_p \,\&\, [\lambda z\, p] = [\lambda z\, q]\big)$$

From this, one can establish $a \models p \equiv \varphi$ using definitions (7) and (3).[18] ⋈

(25) Suppose s' isn't free in φ and φ is modally collapsed. To show:

$$s = \imath s' \forall p(s' \models p \equiv \varphi) \rightarrow \forall p(s \models p \equiv \varphi)$$

it suffices to show that the formula $\forall p(s' \models p \equiv \varphi)$ is modally collapsed, for then our theorem becomes an instance of (24). So we have to prove:

$$\Box\big(\forall p(s' \models p \equiv \varphi) \rightarrow \Box \forall p(s' \models p \equiv \varphi)\big)$$

By the Rule of Necessitation, it suffices to prove:

$$\forall p(s' \models p \equiv \varphi) \rightarrow \Box \forall p(s' \models p \equiv \varphi)$$

So assume $\forall p(s' \models p \equiv \varphi)$, to show $\Box \forall p(s' \models p \equiv \varphi)$. By the Barcan Formula, it suffices to show $\forall p \Box(s' \models p \equiv p)$. Since p isn't free in our assumption, it remains, by GEN, to show $\Box(s' \models p \equiv p)$. So p is a fixed, but arbitrary proposition, and so our assumption that $\forall p(s' \models p \equiv \varphi)$ implies:

$$\text{(A)}\quad s' \models p \equiv \varphi$$

By hypothesis, φ is modally collapsed, and so we know that the following is a theorem:

$$\text{(B)}\quad \Box(\varphi \rightarrow \Box \varphi)$$

Moreover, the following is an instance of (16):

$$\text{(C)}\quad \Box(s' \models p \rightarrow \Box s' \models p)$$

But it is a theorem of modal logic that if formulas ψ and χ necessarily imply their own necessity, then the material equivalence of ψ and χ necessarily implies their necessary equivalence:

$$(\Box(\psi \rightarrow \Box\psi) \,\&\, \Box(\chi \rightarrow \Box\chi)) \rightarrow \Box((\psi \equiv \chi) \rightarrow \Box(\psi \equiv \chi))$$

Given this theorem and setting ψ to $s \models p$ and χ to φ, (C) and (B) jointly imply:

$$\Box((s' \models p \equiv \varphi) \rightarrow \Box(s' \models p \equiv \varphi))$$

[18] For the full proof, see Zalta m.s., (484).

So by the T schema,

$$(s' \models p \equiv \varphi) \rightarrow \Box(s' \models p \equiv \varphi)$$

Hence, by (A), $\Box(s' \models p \equiv \varphi)$, which is what it remained to show. ⋈

(27) Assume *ModallyClosed*(s). Then by definition (26):

(ϑ) $\forall q\big((Actual(s) \Rightarrow q) \rightarrow s \models q\big)$

We want to show: $\big(s \models p \,\&\, (p \Rightarrow q)\big) \rightarrow s \models q$. So assume:

(ξ) $s \models p \,\&\, p \Rightarrow q$

If we instantiate (ϑ) to q, it follows that:

(ζ) $(Actual(s) \Rightarrow q) \rightarrow s \models q$

So to show $s \models q$, it remains only to show $Actual(s) \Rightarrow q$. But consider the following lemma:

Lemma: $\forall r\big(\Box s \models r \rightarrow \Box(Actual(s) \rightarrow r)\big)$

Proof. By GEN, we have to show $\Box s \models r \rightarrow \Box(Actual(s) \rightarrow r)$. But by Rule RM, it suffices to show $s \models r \rightarrow (Actual(s) \rightarrow r)$ without appealing to any contingencies. So assume $s \models r$ and further assume $Actual(s)$. By definition (11), the latter implies $\forall p(s \models p \rightarrow p)$. Instantiating this last fact to r yields $s \models r \rightarrow r$. Hence r.

If we instantiate this Lemma to p, we have $\Box s \models p \rightarrow \Box(Actual(s) \rightarrow p)$. But we know the antecedent of this last claim, since the first conjunct of (ξ) implies $\Box s \models p$, by (16). Hence, $\Box(Actual(s) \rightarrow p)$. So by definition of $\Rightarrow$ (15), $Actual(s) \Rightarrow p$. But this fact and the second conjunct of (ξ) jointly imply $Actual(s) \Rightarrow q$. ⋈

(28) Assume *ModallyClosed*(s) and *Consistent*(s). Then by definitions (26) and (13), we know, respectively:

(ϑ) $\forall q\big((Actual(s) \Rightarrow q) \rightarrow s \models q\big)$

(ξ) $\neg\exists p(s \models p \,\&\, s \models \neg p)$

For reductio, assume $\neg Possible(s)$. By definition (12) and a Rule of Substitution, this entails $\neg\Diamond Actual(s)$. So $\Box\neg Actual(s)$ and, hence, $\neg Actual(s)$. By the definition of $Actual(s)$ (11), this implies $\exists p(s \models p \,\&\, \neg p)$. Suppose p_1 is such a proposition, so that we know both $s \models p_1$ and $\neg p_1$. The former implies $\neg s \models \neg p_1$, by (ξ). Now, separately, if we instantiate (ϑ) to $\neg p_1$, then we also know:

(ζ) $(Actual(s) \Rightarrow \neg p_1) \rightarrow s \models \neg p_1$

But we've established $\neg s \models \neg p_1$, and so by (ζ), $\neg(Actual(s) \Rightarrow \neg p_1)$. By definition of ($\Rightarrow$) and a Rule of Substitution, it follows that $\neg\Box(Actual(s) \rightarrow \neg p_1)$. By modal logic, this is equivalent to $\Diamond(Actual(s) \,\&\, p_1)$. But this last result implies $\Diamond Actual(s)$. Contradiction. ⋈

(29) Assume *ModallyClosed*(s) and $\Box p$. The second implies $\Box(Actual(s) \rightarrow p)$, since every proposition implies a necessary truth and necessarily so. So $Actual(s) \Rightarrow p$, by definition (15). Then by definition of *ModallyClosed*(s) (26), $s \models p$. ⋈

(30) We first prove the following Lemma:

Lemma: $\forall s \forall q\big((Actual(s) \Rightarrow q) \rightarrow \Box(\forall p(s \models p \equiv p) \rightarrow s \models q)\big)$

Proof. By GEN, it suffices to prove:

$$(Actual(s) \Rightarrow q) \rightarrow \Box(\forall p(s \models p \equiv p) \rightarrow s \models q)$$

To prove this, our strategy is to first prove (ϑ) and then apply Rule RM:

(ϑ) $(Actual(s) \rightarrow q) \rightarrow (\forall p(s \models p \equiv p) \rightarrow s \models q)$

So assume:

(ξ) $Actual(s) \rightarrow q$

(ζ) $\forall p(s \models p \equiv p)$

Now (ζ) implies, *a fortiori*, $\forall p(s \models p \rightarrow p)$. Hence $Actual(s)$, by (11). So by (ξ), q. But q and (ζ) imply $s \models q$. Since we've established (ϑ), we may conclude, by RM:

$$\Box(Actual(s) \rightarrow q) \rightarrow \Box(\forall p(s \models p \equiv p) \rightarrow s \models q)$$

By definition of $\Rightarrow$ (15), this becomes:

$$(Actual(s) \Rightarrow q) \rightarrow \Box(\forall p(s \models p \equiv p) \rightarrow s \models q)$$ ⊠

Now to establish *ModallyClosed*(w), we have to show, by (26):

$$\forall q((Actual(w) \Rightarrow q) \rightarrow w \models q)$$

By GEN, assume $Actual(w) \Rightarrow q$. Since w is, by hypothesis, a possible world, w is a situation, and so we can instantiate w for s in the above Lemma, to obtain:

$$\forall q\big((Actual(w) \Rightarrow q) \rightarrow \Box(\forall p(w \models p \equiv p) \rightarrow s \models q)\big)$$

And if we also instantiate this last fact to q and apply our assumption, we may infer $\Box(\forall p(w \models p \equiv p) \rightarrow w \models q)$. But since w is a possible world, we also know $\Diamond\forall p(w \models p \equiv p)$, by (18). The last two results then imply $\Diamond w \models q$, by the K$\Diamond$ principle. So by (17), $w \models q$. ⋈

(36) ($\rightarrow$) Assume $Contingent(p)$. Then $\Diamond p$ and $\Diamond\neg p$. To show $GapOn(\boldsymbol{s}_\Box, p)$, we have to show both $\neg\boldsymbol{s}_\Box \models p$ and $\neg\boldsymbol{s}_\Box \models \overline{p}$, by (34). Since we know $\Diamond\neg p$, we know $\neg\Box p$. So if we instantiate a fact about absolute necessity, namely (35), to p, then it follows that $\neg\boldsymbol{s}_\Box \models p$. Since we also know $\Diamond p$, we know $\neg\Box\neg p$. So if we instantiate (35) to $\neg p$, then it follows that $\neg\boldsymbol{s}_\Box \models \neg p$. So by definition, $\neg\boldsymbol{s}_\Box \models \overline{p}$.

($\leftarrow$) (Exercise) ⋈

(37) By definition (31), we have to show:

(A) $Consistent(\boldsymbol{s}_\Box)$

(B) $ModallyClosed(\boldsymbol{s}_\Box)$

(A) For reductio, suppose $\neg Consistent(\boldsymbol{s}_\Box)$, i.e., $\exists p(\boldsymbol{s}_\Box \models p \;\&\; \boldsymbol{s}_\Box \models \neg p)$, by definition (13). Let q_1 be such a proposition, so that we know $\boldsymbol{s}_\Box \models q_1$ and $\boldsymbol{s}_\Box \models \neg q_1$. By (35), these imply, respectively, $\Box q_1$ and $\Box\neg q_1$. Contradiction, once the T schema is applied to both results.

(B) We have to show: $(Actual(\boldsymbol{s}_\Box) \Rightarrow p) \rightarrow \boldsymbol{s}_\Box \models p$, for arbitrary p. So assume:

(ξ) $Actual(\boldsymbol{s}_\Box) \Rightarrow p$

To show $\boldsymbol{s}_\Box \models p$, it suffices, by (35), to show $\Box p$. For reductio, suppose $\neg\Box p$, i.e., $\Diamond\neg p$. But our assumption (ξ) implies $\Box(Actual(\boldsymbol{s}_\Box) \rightarrow p)$. So $\Box(\neg p \rightarrow \neg Actual(\boldsymbol{s}_\Box))$. But from this and $\Diamond\neg p$, it follows by K$\Diamond$ that $\Diamond\neg Actual(\boldsymbol{s}_\Box)$. By definition (11) this implies $\Diamond\neg\forall q(\boldsymbol{s}_\Box \models q \rightarrow q)$. So $\Diamond\exists q(\boldsymbol{s}_\Box \models q \;\&\; \neg q)$. By BF$\Diamond$, $\exists q\Diamond(\boldsymbol{s}_\Box \models q \;\&\; \neg q)$. Suppose p_1 is such a proposition, so that we know $\Diamond(\boldsymbol{s}_\Box \models p_1 \;\&\; \neg p_1)$. Then $\Diamond(\boldsymbol{s}_\Box \models p_1)$ and $\Diamond\neg p_1$. The latter implies $\neg\Box p_1$. The former implies $\boldsymbol{s}_\Box \models p_1$. So by (35), $\Box p_1$. Contradiction. ⋈

(38) Assume $s \trianglelefteq \boldsymbol{s}_\Box$ and $s \neq \boldsymbol{s}_\Box$. The second implies, by the definition of identity for situations (8), $\neg\forall p(s \models p \equiv \boldsymbol{s}_\Box \models p)$, i.e.,

$$\exists p\big((s \models p \;\&\; \neg\boldsymbol{s}_\Box \models p) \vee (\boldsymbol{s}_\Box \models p \;\&\; \neg s \models p)\big)$$

Suppose q_1 is such a proposition, so that we know:

$$(s \models q_1 \;\&\; \neg\boldsymbol{s}_\Box \models q_1) \vee (\boldsymbol{s}_\Box \models q_1 \;\&\; \neg s \models q_1)$$

The left disjunct contradicts our first assumption $s \trianglelefteq \boldsymbol{s}_\Box$, by the definition of $\trianglelefteq$ (9). So we know $\boldsymbol{s}_\Box \models q_1$ and $\neg s \models q_1$. The first of these implies $\Box q_1$, by a a fact about $\boldsymbol{s}_\Box$ (35). Now, for reductio, suppose $Possibility(s)$. Then, by definition (31), s is modally closed and so, by a previous theorem (29), this last fact and $\Box q_1$ imply $s \models q_1$. Contradiction. ⋈

(39) $GapOn(\mathfrak{s}, p)$, i.e., both $\neg\mathfrak{s} \models p$ and $\neg\mathfrak{s} \models \overline{p}$. By definition of $\overline{p}$, the latter implies $\neg\mathfrak{s} \models \neg p$. Now suppose $\neg Contingent(p)$, for reductio. Then by $\neg(\Diamond p \;\&\; \Diamond\neg p)$, i.e., $\Box\neg p \vee \Box p$. But both disjuncts lead to contradiction. If $\Box\neg p$, then $\mathfrak{s} \models \neg p$, by (29) and the fact that $\mathfrak{s}$ is modally closed. This contadicts $\neg\mathfrak{s} \models \neg p$; if $\Box p$, then again by familiar reasoning, $\mathfrak{s} \models p$, which contradicts $\neg\mathfrak{s} \models p$. Contradiction full stop. ⋈

(41) By GEN, it suffices to show $\mathfrak{s} \trianglerighteq \boldsymbol{s}_\Box$. So by definition (9), we have to show $\forall p(\boldsymbol{s}_\Box \models p \rightarrow \mathfrak{s} \models p)$. So, again, by GEN, we show $\boldsymbol{s}_\Box \models p \rightarrow \mathfrak{s} \models p$. Assume $\boldsymbol{s}_\Box \models p$. Then by a fact about $\boldsymbol{s}_\Box$ (35), it follows that $\Box p$. But since possibilities are modally closed (31) and modally closed situations make necessary truths true (29), it follows that $\mathfrak{s} \models p$. ⋈

(43) $s^\star$ is clearly a situation and so it remains to show $\forall p(s \models p \rightarrow s^\star \models p)$. Proof strategy:

(A) Independently show $s \models p \rightarrow (Actual(s) \rightarrow p)$ without appealing to any contingencies.

(B) Conclude from (A) that $\Box s \models p \rightarrow \Box(Actual(s) \rightarrow p)$, by Rule RM.

(C) Assume $s \models p$, for conditional proof. To show $s^\star \models p$, we have to show $Actual(s) \Rightarrow p$, by (42)

(D) Our assumption in (C) implies $\Box s \models p$.

(E) From (D) and (B) it follows that $\Box(Actual(s) \rightarrow p)$.

(F) Conclude $Actual(s) \Rightarrow p$, by definition of $\Rightarrow$.

Since (B) – (F) are straightforward, it remains to show (A). So assume both $s \models p$ and $Actual(s)$. The latter implies $\forall q(s \models q \rightarrow q)$, by definition. Instantiating this to p yields $s \models p \rightarrow p$. But then p, since $s \models p$ by assumption. $\bowtie$

(45) Clearly, s^{+p} and w are both situations. Then we can establish our theorem via the following biconditional chain:

$$
\begin{array}{lll}
s^{+p} \trianglelefteq w & \equiv \forall q(s^{+p} \models q \rightarrow w \models q) & \text{by definition (9)} \\
& \equiv \forall q((s \models q \vee q = p) \rightarrow w \models q) & \text{by (44)} \\
& \equiv \forall q((s \models q \rightarrow w \models q) \,\&\, (q = p \rightarrow w \models q)) & \text{by logic} \\
& \equiv \forall q(s \models q \rightarrow w \models q) \,\&\, \forall q(q = p \rightarrow w \models q) & \text{by logic} \\
& \equiv \forall q(s \models q \rightarrow w \models q) \,\&\, w \models p & \text{by logic} \\
& \equiv s \trianglelefteq w \,\&\, w \models p & \text{by definition (9)}
\end{array}
$$

$\bowtie$

(46) By the following biconditional chain:

$$
\begin{array}{lll}
\multicolumn{3}{l}{\forall w(s \trianglelefteq w \rightarrow w \models p)} \\
& \equiv \forall w \neg(s \trianglelefteq w \,\&\, \neg w \models p) & \text{by logic} \\
& \equiv \neg \exists w(s \trianglelefteq w \,\&\, \neg w \models p) & \text{by logic} \\
& \equiv \neg \exists w(s \trianglelefteq w \,\&\, w \models \neg p) & \text{by maximality of } w \\
& \equiv \neg \exists w(s \trianglelefteq w \,\&\, w \models \overline{p}) & \text{by logic} \\
& \equiv \neg \exists w(s^{+\overline{p}} \trianglelefteq w) & \text{by (45)} \\
& \equiv \neg Possible(s^{+\overline{p}}) & \text{by (20)} \\
& \equiv \neg \Diamond Actual(s^{+\overline{p}}) & \text{by definition (12)} \\
& \equiv \neg \Diamond \forall q(s^{+\overline{p}} \models q \rightarrow q) & \text{by definition (11)} \\
& \equiv \neg \Diamond \forall q\big((s \models q \vee q = \overline{p}) \rightarrow q\big) & \text{by (44)} \\
& \equiv \neg \Diamond \forall q\big((s \models q \rightarrow q) \,\&\, (q = \overline{p} \rightarrow q)\big) & \text{by logic} \\
& \equiv \neg \Diamond \big(\forall q(s \models q \rightarrow q) \,\&\, \forall q(q = \overline{p} \rightarrow q)\big) & \text{by logic} \\
& \equiv \neg \Diamond \big(\forall q(s \models q \rightarrow q) \,\&\, \overline{p}\big) & \text{by logic} \\
& \equiv \neg \Diamond \big(\forall q(s \models q \rightarrow q) \,\&\, \neg p\big) & \text{by logic} \\
& \equiv \neg \Diamond \big(Actual(s) \,\&\, \neg p\big) & \text{by definition (11)} \\
& \equiv \Box \neg (Actual(s) \,\&\, \neg p) & \text{by modal logic} \\
& \equiv \Box (Actual(s) \rightarrow p) & \text{by logic} \\
& \equiv Actual(s) \Rightarrow p & \text{by definition (15)}
\end{array}
$$

$\bowtie$

(47) ($\rightarrow$) Assume $s \trianglelefteq w$. For reductio, suppose $\neg s^\star \trianglelefteq w$. Then $\exists p(s^\star \models p \,\&$ $\neg w \models p)$. Let p_1 be such a proposition, so that we know both $s^\star \models p_1$ and $\neg w \models p_1$. Independently, from the fact that $s^\star \models p_1$ it follows that $Actual(s) \Rightarrow p_1$, by (42). But the following is an instance of (46):

$$(s \trianglelefteq w \rightarrow w \models p_1) \equiv (Actual(s) \Rightarrow p_1)$$

It follows that $s \trianglelefteq w \rightarrow w \models p_1$. Hence, $w \models p_1$. Contradiction.

($\leftarrow$) Assume $s^\star \trianglelefteq w$. But we just established $s \trianglelefteq s^\star$. Since $s^\star$ and w are situations, it follows by the transitivity of $\trianglelefteq$ that $s \trianglelefteq w$. $\bowtie$

(48) By the following biconditional chain:

$$
\begin{array}{llll}
Possible(s) & \equiv \exists w(s \trianglelefteq w) & \text{by (20)} & \\
& \equiv \exists w(s^\star \trianglelefteq w) & \text{by (47)} & \\
& \equiv Possible(s^\star) & \text{by (20)} & \bowtie
\end{array}
$$

(49) We have to show: $\forall p\big((Actual(s^\star) \Rightarrow p) \rightarrow s^\star \models p\big)$. We prove this by hypothetical syllogism, from:

(A) $(Actual(s^\star) \Rightarrow p) \rightarrow (Actual(s) \Rightarrow p)$

(B) $(Actual(s) \Rightarrow p) \rightarrow s^\star \models p$

(B) is just the right-to-left direction of (42). For (A), assume the antecedent:

(ϑ) $Actual(s^\star) \Rightarrow p$

Now, for reductio, assume $\neg(Actual(s) \Rightarrow p)$. Then, $\neg\Box(Actual(s) \rightarrow p)$. Since $\overline{p} = \neg p$, we have $\Diamond(Actual(s) \,\&\, \overline{p})$. But this contradicts ($\vartheta$), by the following conditional chain:

$$
\begin{array}{lll}
\multicolumn{3}{l}{\Diamond(Actual(s) \,\&\, \overline{p})} \\
& \rightarrow \Diamond(\forall q(s \models q \rightarrow q) \,\&\, \overline{p}) & \text{by definition (11)} \\
& \rightarrow \Diamond(\forall q(s \models q \rightarrow q) \,\&\, \forall q(q = \overline{p} \rightarrow q)) & \text{by logic} \\
& \rightarrow \Diamond \forall q\big((s \models q \rightarrow q) \,\&\, (q = \overline{p} \rightarrow q)\big) & \text{by logic} \\
& \rightarrow \Diamond \forall q\big((s \models q \vee q = \overline{p}) \rightarrow q\big) & \text{by logic} \\
& \rightarrow \Diamond \forall q(s^{+\overline{p}} \models q \rightarrow q) & \text{by (44)} \\
& \rightarrow \Diamond Actual(s^{+\overline{p}}) & \text{by definition (11)} \\
& \rightarrow Possible(s^{+\overline{p}}) & \text{by definition (12)} \\
& \rightarrow \exists w(s^{+\overline{p}} \trianglelefteq w) & \text{by (20)} \\
& \rightarrow \exists w(s \trianglelefteq w \,\&\, w \models \overline{p}) & \text{by (45)} \\
& \rightarrow \exists w(s^\star \trianglelefteq w \,\&\, w \models \overline{p}) & \text{by (47)} \\
& \rightarrow \exists w(s^\star \trianglelefteq w \,\&\, w \models \neg p) & \text{by logic} \\
& \rightarrow \exists w(s^\star \trianglelefteq w \,\&\, \neg w \models p) & \text{by coherency of worlds} \\
& \rightarrow \exists w \neg(s^\star \trianglelefteq w \rightarrow w \models p) & \text{by logic} \\
& \rightarrow \neg \forall w(s^\star \trianglelefteq w \rightarrow w \models p) & \text{by logic} \\
& \rightarrow \neg(Actual(s^\star) \Rightarrow p) & \text{by (46)}
\end{array}
$$

This last line contradicts (ϑ). ⋈

(50) To avoid clash of variables, we show that the theorem holds for an arbitrarily chosen proposition. Let r be an arbitrary, but fixed, proposition. ($\rightarrow$) Then we have to show:

$$GapOn(\mathfrak{s},r) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models r) \,\&\, \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg r)$$

In fact, however, it suffices to show only:

(ξ) $GapOn(\mathfrak{s},r) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models r)$

To see why, note that $GapOn(\mathfrak{s},r)$ implies $GapOn(\mathfrak{s},\neg r)$.[19] By universally generalizing (ξ) to every proposition, we can simply instantiate (ξ) to $\neg r$ and conclude $GapOn(\mathfrak{s},\neg r) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg r)$. By hypothetical syllogism, then, $GapOn(\mathfrak{s},r) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg r)$. So it suffices to show (ξ).

Assume $GapOn(\mathfrak{s},r)$. Now we have to find a witness to the claim $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models r)$. But consider the modal closure of the r-extension of $\mathfrak{s}$, i.e., consider $(\mathfrak{s}^{+r})^{\star}$, which we henceforth write more simply as $\mathfrak{s}^{+r\star}$. (We leave it as an exercise to show $\mathfrak{s}^{+r\star}\downarrow$.) To show that $\mathfrak{s}^{+r\star}$ is a witness to $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models r)$, we have to show all of the following:

(a) $\mathfrak{s}^{+r\star} \trianglerighteq \mathfrak{s}$

(b) $\mathfrak{s}^{+r\star} \models r$

(c) $Possibility(\mathfrak{s}^{+r\star})$

And by definition (31), the last of the above requires us to show that:

(d) $ModallyClosed(\mathfrak{s}^{+r\star})$

(e) $Consistent(\mathfrak{s}^{+r\star})$

We prove these in turn, though with the help of a consequence of the fact that $\mathfrak{s}^{+r} \trianglelefteq \mathfrak{s}^{+r\star}$ (43), namely:

[19] Here is the Lemma that shows this:

Lemma: $\forall p\big(GapOn(\mathfrak{s},p) \rightarrow GapOn(\mathfrak{s},\neg p)\big)$

Proof. Assume $GapOn(\mathfrak{s},p)$ and for reductio, $\neg GapOn(\mathfrak{s},\neg p)$. Then, by definition of $GapOn$ (34, either $\mathfrak{s} \models \neg p$ or $\mathfrak{s} \models \neg\overline{p}$, i.e., either $\mathfrak{s} \models \neg p$ or $\mathfrak{s} \models \neg\neg p$, by definition. But the former contradicts $GapOn(\mathfrak{s},p)$. The latter, by a consequence (27) of the fact that $\mathfrak{s}$ is modally closed, implies $\mathfrak{s} \models p$, which also contradicts $GapOn(\mathfrak{s},p)$.

(ϑ) $\forall p(\mathfrak{s}^{+r} \models p \rightarrow \mathfrak{s}^{+r\star} \models p)$

(a) To show that $\mathfrak{s}^{+r\star} \trianglerighteq \mathfrak{s}$, we have to show $\mathfrak{s} \trianglelefteq \mathfrak{s}^{+r\star}$ (40), i.e., by definition (9), that $\forall p(\mathfrak{s} \models p \rightarrow \mathfrak{s}^{+r\star} \models p)$. So by GEN, take $\mathfrak{s} \models p$ as a local assumption. But if we instantiate $\mathfrak{s}$ for s, r for p, and p for q in (44), our local assumption implies $\mathfrak{s}^{+r} \models p$. Hence, by ($\vartheta$), $\mathfrak{s}^{+r\star} \models p$.

(b) To establish that $\mathfrak{s}^{+r\star} \models r$, note that if we instantiate $\mathfrak{s}$ for s and r for p in (44), then we also know $\mathfrak{s}^{+r} \models r$. From this, ($\vartheta$) implies $s^{+r\star} \models r$.

(d) To show that $s^{+r\star}$ is modally closed, we simply note that this is an instance of theorem (49).

(e) To show that $Consistent(\mathfrak{s}^{+r\star})$, our proof strategy is to derive this conclusion by way of the following syllogism:

- $Possible(\mathfrak{s}^{+r})$
- $Possible(\mathfrak{s}^{+r}) \rightarrow Possible(\mathfrak{s}^{+r\star})$
- $Possible(\mathfrak{s}^{+r\star}) \rightarrow Consistent(\mathfrak{s}^{+r\star})$

Note that the second is an instance of the left-to-right direction of (48) and that the third is an instance of (14). So it remains only to show the first. For reductio, suppose $\neg Possible(\mathfrak{s}^{+r})$. But this implies that $Actual(\mathfrak{s}) \Rightarrow \neg r$, by the following conditional chain, in which citations to a Rule of Substitution have been suppressed:

$$\begin{array}{rcll}
\neg Possible(\mathfrak{s}^{+r}) & \rightarrow & \neg\exists w(\mathfrak{s}^{+r} \trianglelefteq w) & \text{by (20)}\\
 & \rightarrow & \neg\exists w(\mathfrak{s} \trianglelefteq w \,\&\, w \models r) & \text{by (45)}\\
 & \rightarrow & \forall w \neg(\mathfrak{s} \trianglelefteq w \,\&\, w \models r) & \text{by quantification theory}\\
 & \rightarrow & \forall w(\mathfrak{s} \trianglelefteq w \rightarrow \neg w \models r) & \text{by propositional logic}\\
 & \rightarrow & \forall w(\mathfrak{s} \trianglelefteq w \rightarrow w \models \neg r) & \text{by coherency of worlds}\\
 & \rightarrow & Actual(\mathfrak{s}) \Rightarrow \neg r & \text{by (46)}
\end{array}$$

Now since $\mathfrak{s}$ is a possibility and, hence, modally closed (31), it follows from the last fact in the biconditional chain that $\mathfrak{s} \models \neg r$, by the definition of modally closed situations (26). But this contradicts our initial assumption that $GapOn(\mathfrak{s},r)$, for that implies, *a fortiori*, that $\neg\mathfrak{s} \models \neg r$. ⋈

($\leftarrow$) Assume:

(ζ) $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models r) \,\&\, \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg r)$

For reductio, suppose $\neg GapOn(\mathfrak{s}, r)$. Then either $\mathfrak{s} \models r$ or $\mathfrak{s} \models \neg r$. Without loss of generality, suppose $\mathfrak{s} \models r$. By the Persistence Principle, every refinement of s makes r true. So there can't be a refinement that makes $\neg r$ true, contradicting the right conjunct of (ζ). ⋈

(51) Assume:

(ϑ) $\forall \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \,\&\, \mathfrak{s}'' \models p))$

For reductio, assume $\neg \mathfrak{s} \models p$. Now $\mathfrak{s} \models \neg p$ or $\neg \mathfrak{s} \models \neg p$, by excluded middle. But both lead to contradiction. For suppose $\mathfrak{s} \models \neg p$. By the Ordering Principle, $\trianglerighteq$ is reflexive and so we independently know $\mathfrak{s} \trianglerighteq \mathfrak{s}$. Instantiating $\mathfrak{s}$ for $\forall \mathfrak{s}'$ in (ϑ), it follows that $\exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}'' \models p)$. Suppose $\mathfrak{s}_1$ is such a possibility, so that we know both (a) $\mathfrak{s}_1 \trianglerighteq \mathfrak{s}$ and (b) $\mathfrak{s}_1 \models p$. But (a) and the assumption $\mathfrak{s} \models \neg p$ imply $\mathfrak{s}_1 \models \neg p$, by the Persistence Principle. But this and (b) contradict the consistency of $\mathfrak{s}_1$.

Alternatively, suppose $\neg \mathfrak{s} \models \neg p$. From this and our reductio assumption that $\neg \mathfrak{s} \models p$, it follows that $GapOn(\mathfrak{s}, p)$, by (34). So, by the Refinability Principle, it follows *a fortiori* that there is a refinement of $\mathfrak{s}$ in which $\neg p$ is true: $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg p)$. Suppose $\mathfrak{s}_2$ is such a possibility, so that we know both $\mathfrak{s}_2 \trianglerighteq \mathfrak{s}$ and $\mathfrak{s}_2 \models \neg p$. The former and ($\vartheta$) imply $\exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s}_2 \,\&\, \mathfrak{s}'' \models p)$. Suppose $\mathfrak{s}_3$ is such a possibility, so that we know both $\mathfrak{s}_3 \trianglerighteq \mathfrak{s}_2$ and $\mathfrak{s}_3 \models p$. But $\mathfrak{s}_3 \trianglerighteq \mathfrak{s}_2$ and the assumption that $\mathfrak{s}_2 \models \neg p$ jointly imply $\mathfrak{s}_3 \models \neg p$, by the Persistence Principle. But this contradicts the consistencty of $\mathfrak{s}_3$. ⋈

(52) ($\rightarrow$)[20] Assume $\Diamond p$. Then by the fundamental theorem of world theory (Zalta 1993, Theorem 25), we know $\exists w(w \models p)$. Let w_1 be such a possible world, so that we know $w_1 \models p$. But by (32), possible worlds are possibilities i.e., $Possibility(w_1)$. Hence $\exists \mathfrak{s}(\mathfrak{s} \models p)$. ($\leftarrow$) Assume $\exists \mathfrak{s}(\mathfrak{s} \models p)$. Suppose $\mathfrak{s}_1$ is such a possibility, so that we know $\mathfrak{s}_1 \models p$. Suppose, for reductio, that $\neg \Diamond p$. Then $\Box \neg p$. So $s_\Box \models \neg p$, by (35). But by (41), we know that $\mathfrak{s}_1$ is a refinement of $s_\Box$. So by by the Persistence Principle, $\mathfrak{s}_1 \models \neg p$, which contradicts the consistency of $\mathfrak{s}_1$. ⋈

(54) ($\rightarrow$) Assume $Possible(s)$. Then by (48), $Possible(s^\star)$. Hence by (14), $Consistent(s^\star)$. ($\leftarrow$) Assume $Consistent(s^\star)$. But independently, by (49), we know $ModallyClosed(s^\star)$. From this and our assumption, it follows by (28) that $Possible(s^\star)$. But then by (48), $Possible(s)$. ⋈

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