Essence and Modality*

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1. Introduction

In the course of research on modal logic over the past 60 years, it has become traditional to define an essential property in modal terms as follows:

(E) F *is essential to* x $=_{df}$ $\Box(E!x \rightarrow Fx)$,

where '$E!x$' asserts existence and abbreviates '$\exists y(y = x)$'. Kit Fine has developed an intriguing counterexample to (E). He offers the following intuition, concerning Socrates and the singleton set containing just Socrates, to set up the counterexample (Fine, 1994a, 5):

> ... it lies in the nature of the [set-theoretic] singleton [of Socrates] to have Socrates as a member even though it does not lie in the nature of Socrates to belong to the singleton.

Fine notes that if we take on board the usual, uncontroversial principles of modal set theory, then the asymmetry in natures is not preserved, for it can be shown that if singleton Socrates has Socrates as a member essentially, then Socrates has the property of being a member of singleton Socrates essentially. That is, using (E) and modal set theory, one can prove that the property of being an element of singleton Socrates ('$[\lambda y\, y \in \{s\}]$') is essential to Socrates ('s') from the assumption that having Socrates as an element ('$[\lambda y\, s \in y]$') is essential to singleton Socrates ('$\{s\}$'):

> *Proof*: Suppose $[\lambda y\, s \in y]$ is essential to $\{s\}$. Then, by (E) above, $\Box(E!\{s\} \rightarrow [\lambda y\, s \in y]\{s\})$, and by λ-conversion, it follows that $\Box(E!\{s\} \rightarrow s \in \{s\})$. But, it is a principle of modal set theory that necessarily, singleton Socrates exists iff Socrates exists, i.e., $\Box(E!\{s\} \equiv E!s)$. So, $\Box(E!s \rightarrow s \in \{s\})$.[1] And by λ-conversion, $\Box(E!s \rightarrow [\lambda y\, y \in \{s\}]s)$. Thus, by (E) again, $[\lambda y\, y \in \{s\}]$ is essential to Socrates.

Thus, (E) and modal set theory lead to a result contrary to the stated intuition. One cannot accept (E), modal set theory, and that singleton Socrates essentially has Socrates as an element without also accepting that Socrates is essentially an element of singleton Socrates.

Although one might conclude that the problem here is with modal set theory, Fine suggests that the problem goes deeper, and has more to do with (E) than with modal set theory. He develops a second counterexample to (E) (1994a, 5):

> Consider two objects whose natures are unconnected, say Socrates and the Eiffel Tower. Then it is necessary that Socrates and the Tower be distinct. But it is not essential to Socrates that he be distinct from the Tower; for there is nothing in his nature which connects him in any special way to it.

To be more explicit about what the problem is, start with the intuition that there is nothing in the nature of Socrates ('s') which connects him in any way with the Eiffel Tower ('t'). Fine assumes that if Socrates were to have the property of being distinct from the Eiffel Tower ('$[\lambda y\, y \neq t]$') essentially, then his nature would be connected in some special way with

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[1]This is a consequence of the following inference rule of S5 modal logic: from $\Box(\phi \rightarrow \psi)$ and $\Box(\phi \equiv \chi)$, we may infer $\Box(\chi \rightarrow \psi)$. Clearly this is valid, for suppose ψ is true at every world ϕ is true and that ϕ and χ are true at exactly the same worlds. Then pick an arbitrary world w and assume χ is true there. Then, clearly, ϕ is true there, and so ψ is true there. Since w was arbitrary, we have established that $\Box(\chi \rightarrow \psi)$ from our two premises.

the Eiffel Tower. He then concludes that the property of being distinct from the Eiffel Tower is not essential to Socrates. But one can show, using (E) and the necessity of identity, that being distinct from the Eiffel Tower is a property that is essential to Socrates, contrary to intuition:

> Let t = Eiffel Tower and s = Socrates. Let $H = [\lambda y\, y \neq t]$. Now in S5, from the theorem that $x = y \rightarrow \Box x = y$, one can derive that $x \neq y \rightarrow \Box x \neq y$.[2] So, given $s \neq t$, it follows that $\Box(s \neq t)$. By λ-conversion, it follows that: $\Box Hs$. *A fortiori*, $\Box(E!s \rightarrow Hs)$. And so by (E), H is essential to Socrates.

Thus we can use (E) to prove something counterintuitive.

In response to these puzzles, Fine (1995, 2000) develops an interesting language, logic, and semantics of essence. The language involves special 1-place rigid predicates, a 2-place dependence predicate, an essentialist operator symbol, and new formulas to express the idea that a formula ϕ is true in virtue of the nature of objects which F. The logic is an extension of first-order logic with new axioms and rules to govern the new predicates and formulas. Just how much of this apparatus is needed to address the two specific puzzles discussed above, as opposed to other intuitions concerning the logic of essence which Fine brings to bear in those papers, is unclear.

In any case, these counterexamples reveal an important disconnect between the notions of essence and modality, as woven together by definition (E). I think Fine is quite justified in developing a response to the puzzles which redefines the relationship between these fundamental notions. In the present paper, however, I diagnose the puzzles surrounding (E) differently, and propose an alternative set of distinctions that avoid the unintuitive consequences.[3] The system I use for the analysis was not developed specifically for these puzzles, but was developed independently, as a general metaphysical framework.

[2]Assume it is a theorem that $x = y \rightarrow \Box x = y$. So by the Rule of Necessitation, it is a theorem that: $\Box[x = y \rightarrow \Box x = y]$. Now by the modal theorem, $\Box[\phi \rightarrow \psi] \rightarrow (\Diamond\phi \rightarrow \Diamond\psi)$, it follows that (A) $\Diamond x = y \rightarrow \Diamond\Box x = y$. Now we want to show: $x \neq y \rightarrow \Box x \neq y$, i.e., by contraposition and modal negation, $\Diamond x = y \rightarrow x = y$. So assume the antecedent $\Diamond x = y$ (to show $x = y$). Then by (A), it follows that $\Diamond\Box x = y$. But, in S5, $\Diamond\Box x = y$ implies $\Box x = y$, and this in turn implies $x = y$, which is what we had to show.

[3]I do not, however, try to account for the other intuitions Fine brings to bear in the later papers (Fine 1995, 2000), when he develops the full, new apparatus in the logic of essence.

Moreover, the present analysis rejects one of the principles which Fine accepts. Recall that Fine proposes counterexamples only to the right-to-left direction of the biconditional resulting from (E). That is, he rejects the idea that $\Box(E!x \rightarrow Fx)$ implies that F is essential to x, but accepts the idea that if F is essential to x then $\Box(E!x \rightarrow Fx)$. He says (1994a, 4):

> I accept that if an object essentially has a certain property, then it is necessary that it has the property (or has the property if it exists)...

But the present framework offers distinctions which suggest this claim should be qualified. That is, the distinctions drawn in what follows suggest that it is a mistake to suppose that *x has F essentially* always implies *x has F necessarily*. We will discover notions of essence and modality on which they completely come apart.

In what follows, we shall cast our modal definitions of 'essential property' within an axiomatic metaphysics, namely, the theory of abstract objects developed and applied in Zalta 1983, 1988a, 1993, and elsewhere. This theory, hereafter labeled $\mathcal{O}$, will provide us with a conception of abstract objects, and will be instrumental in the discussion of Fine's first counterexample. However, we shall see that Fine's second counterexample can be addressed solely within the quantified modal logic in which $\mathcal{O}$ has been couched. The central elements of that logic have been explicitly defended by Linsky & Zalta (1994, 1996), and more recently by Williamson (1998, 1999).

2. Metaphysical Foundations[4]

So as to make the present paper self-contained, we review here the most important elements of the theory of abstract objects $\mathcal{O}$. In this section, we present and discuss the language, logic, proper axioms/theorems, and applications of $\mathcal{O}$. In terms of an overview, it should be said that $\mathcal{O}$ quantifies over two domains: a domain of objects consisting of abstract and ordinary objects, and a domain of n-place relations. The most important principles of $\mathcal{O}$ assert existence (comprehension) and identity conditions

[4]This section has been greatly expanded at the request of the editors of this journal. Readers already familiar with the theory may wish to skip to the next section. However, many of the technical terms used in subsequent sections will be introduced and defined here.

for abstract objects, existence and identity conditions for n-place relations, and identity conditions for ordinary objects.

2.1 The Language and Logic of $\mathcal{O}$

The theory of abstract objects $\mathcal{O}$ is couched in a second-order modal language in which there are variables and constants for individuals as well as variables and constants for n-place relations ($n \geq 0$). To state $\mathcal{O}$, we need to modify the language of the standard second-order modal predicate calculus only by including a second kind of atomic formula. In addition to the usual atomic formulas of the form '$F^n x_1 \ldots x_n$' (with the 1-place case being formulas of the form $F^1 x$), we require formulas of the form 'xF^1'. (We henceforth suppress the superscript on relation terms when the arity is clear.) These latter express the idea that an (abstract) object x *encodes* property F. The notion of encoding derives from the notion of 'determination' used in Mally 1912, where it is suggested that abstract objects have their properties in one of two ways: an abstract object x may have a property F either by exemplifying F or by being determined by F. Mally's idea is that every group of properties determines an abstract object, but that such an abstract object need not exemplify the properties which determine it. For example, the properties goldenness and mountainhood determine an abstract object which exemplifies neither of these two properties. The intuition here is that the properties determining an abstract object are part of its nature and govern the conception of that object. Indeed, for Mally, there is nothing more to the nature of an abstract object than the properties by which it is to be conceived. In what follows, we shall say that an abstract object *encodes* property F instead of saying that F determines x.

Since encoding is a way of having a property, it constitutes a kind of predication. That is why we introduce 'xF' as an atomic mode of predication, to express the fact that x encodes F. We rigorously distinguish this from the traditional form of predication, namely, that x *exemplifies* F ('Fx'). (More generally, we read '$F^n x_1 \ldots x_n$' as $x_1, \ldots, x_n$ exemplify or stand in the relation F^n.) For example, on this view, Sherlock Holmes encodes the properties of being a detective, living in London, etc. These are the properties by which we conceive of him, and thus are part of his nature, but on the present view, he doesn't exemplify these properties. He exemplifies, by contrast, properties like being fictional, being admired by

modern criminologists, etc., as well as a variety of properties that things have in virtue of being abstract (more on this below). In general, fictional objects will be said to encode the properties attributed to them in their respective stories. To take another class of examples, mathematical objects will encode the mathematical properties attributed to them in their respective theories. By contrast, they exemplify properties like being abstract, not having mass, not having a texture, being conceived by Euler, etc. Note that by thinking of encoding as a second mode of predication, predication in ordinary language becomes ambiguous relative to a logic that distinguishes xF and Fx.

The principal axiom for abstract objects, described in more detail below, is a comprehension principle that asserts the conditions under which abstract objects exist and encode properties: for any expressible condition ϕ that is satisfiable (in Tarski's sense) by properties F, there exists an abstract object that encodes exactly the properties F satisfying ϕ.

Consider, then, the second-order, modal language that can be formed with $F^n x_1 \ldots x_n$ and xF^1 as a basis, and where the other logical notions are $\neg$ (not), $\rightarrow$ (if-then), $\forall$ (every), and $\Box$ (necessarily). Identity is not primitive in this language, but will instead be defined below for both objects and relations. The language is further enhanced with rigid definite descriptions (complex object terms) and λ-expressions (complex relation terms). The latter are formulated in the usual way, but with the proviso that encoding subformulas may not appear in any formula appearing in these λ-expressions. The reason for this will be discussed more fully, but for now, it should suffice to note that λ-expressions with encoding subformulas, such as $[\lambda x\, \exists F(xF \,\&\, \neg Fx)]$, lead to paradox in the presence of the strong existence axiom for abstract objects. The loss of encoding formulas from λ-expressions is not a serious one; we may formulate all the first-order and second-order definable λ-expressions that are available in classical second-order logic.

Using this background language, a distinguished 1-place relation $E!x$ ('x is concrete') is used to define the properties of being ordinary ('$O!$') and being abstract ('$A!$'):

$$O! \;=_{df}\; [\lambda x\, \Diamond E!x]$$
$$A! \;=_{df}\; [\lambda x\, \neg\Diamond E!x]$$

In other words, ordinary objects are possibly concrete, while abstract objects are not the kind of thing that could be concrete. Clearly, this pair of properties partitions the domain of objects into two mutually exclusive, and jointly exhaustive, subdomains.

We define identity separately for these two subdomains. Identity for ordinary objects ('$=_E$') may be defined as follows:

$$x =_E y \ =_{df} \ O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)$$

In other words, objects x and y are identical$_E$ whenever both x and y are ordinary objects that necessarily exemplify the same properties. Identity for abstract objects, and a more general notion of identity ('$x = y$'), can be defined disjunctively:

$$x = y \ =_{df} \ x =_E y \vee A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)$$

Given our definition of $=_E$, this implies that abstract objects are identical whenever they necessarily encode the same properties.

Identity for n-place relations is also definable. Consider the following definition of property identity (i.e., identity for n-place relations where $n = 1$):

$$F = G \ =_{df} \ \Box\forall x(xF \equiv xG)$$

Identity conditions for n-place relations ($n \geq 2$) and for propositions ($n = 0$) can both be defined in terms of this definition, but since they play no role in what follows, they will not be repeated here. It should suffice to say that all our definitions for n-place relation identity ($n \geq 0$) are consistent with the idea that necessarily equivalent relations (properties, propositions) may be distinct.

Our system is governed by classical S5 quantified modal logic, including the first- and second-order Barcan formulas. This logic is modified only so as to admit the two kinds of complex terms we've added to the language: it includes the classical logical axiom governing rigid definite descriptions and the logical axioms governing λ-expressions. The logical axiom for definite descriptions is just the standard analysis from Russell 1905 configured for the formal descriptions appearing in the system as complex individual terms:[5]

[5]This axiom, which asserts the usual truth conditions for an atomic formula ψ containing a definite description of the form $\imath x\phi$, should be regarded as a non-modal axiom and therefore not subject to the Rule of Necessitation, given that the descriptions are to be interpreted rigidly. It is an example of a logical truth which is not necessary. See Zalta 1988b.

Where ψ is any atomic formula or identity formula containing one free occurrence of the variable y, ϕ is any formula in which x may or may not be free, and χ^{τ}_{ν} is the result of replacing the variable ν by the term τ everywhere in χ, the following is an axiom:

$$\psi^{\imath x\phi}_y \equiv \exists x(\phi \,\&\, \forall z(\phi^z_x \to z = x) \,\&\, \psi^x_y),$$

To see some simple examples, let ψ be either Rby or $b = y$ and ϕ be Gx. Then the following are instances of the logical axiom for descriptions:

$$Rb\imath xGx \equiv \exists x(Gx \,\&\, \forall z(Gz \to z = x) \,\&\, Rbx)$$

$$b = \imath xGx \equiv \exists x(Gx \,\&\, \forall z(Gz \to z = x) \,\&\, b = x)$$

The logic for λ-expressions is also classical, and the principal axiom governing these expressions is:[6]

$$[\lambda x_1 \ldots x_n \, \phi]y_1 \ldots y_n \equiv \phi^{y_1,\ldots,y_n}_{x_1,\ldots,x_n} \qquad (\phi \text{ free of descriptions})$$

This principle, also known as λ-conversion, was used in the reasoning that was invoked in developing Fine's counterexamples to the definition of essence in terms of modality. Note that the usual second-order comprehension schema for relations is easily derivable from λ-conversion, by n applications of the Rule of Universal Generalization, an application of the Rule of Necessitation, and Existential Generalization:

$\exists F^n \Box\forall y_1 \ldots \forall y_n(F^n y_1 \ldots y_n \equiv \phi)$, where ϕ has no free F^ns and no encoding subformulas.

This axiom ensures that there is a rich algebra of properties for abstract objects to encode, once one's favorite primitive properties and relations are added to the system.

One interesting group of properties will play an important role in what follows, namely, those governed by the following instance of λ-conversion, where p is a variable ranging over 0-place relations:

$$[\lambda y \, p]x \equiv p$$

[6]There are two other logical axioms for λ-expressions. They are:

$$[\lambda x_1 \ldots x_n \, F^n x_1 \ldots x_n] = F^n \qquad (\text{for 'atomic' } \lambda\text{-expressions})$$

$$[\lambda x_1 \ldots x_n \, \phi] = [\lambda x'_1 \ldots x'_n \, \phi'] \qquad (\phi, \phi' \text{ alphabetic variants in } x, x')$$

This asserts that for any proposition p and object x, x exemplifies the propositional property *being such that* p ($[\lambda y\, p]$) if and only if p.

Our logic is completed by two axioms. One is a new logical axiom included to handle the modal logic of encoding: $\Diamond xF \rightarrow \Box xF$. This captures the idea that the properties an abstract object encodes are rigidly encoded. Since the properties an abstract object encodes make up the nature of that object, this axiom ensures that each abstract object has a nature which doesn't vary from world to world. The second and final axiom of the logic is substitution of identicals. Though $\alpha = \alpha$ is derivable as a theorem (where α is any object variable x or relation variable F), unrestricted substitution of identicals must be asserted as an axiom. This axiom, and the definition of '$=$' for individual terms and predicates, form the theory of identity that is available as part of $\mathcal{O}$.[7]

2.2 Features of the Logic

It would serve well to mention a few facts about the above logic that will play a role in what follows. The modal logic is the simplest possible formulation of S5, with a fixed domain of objects and no accessibility relation needed in the semantics. Thus, it is a theorem of this logic that everything necessarily exists, i.e., $\forall x \Box \exists y (y = x)$. Ordinary and abstract objects exist at all worlds, given the one fixed domain of objects; objects (both ordinary and abstract) therefore exemplify properties at every world. However, as shown in Linsky & Zalta 1994, this is consistent with the idea of there being contingent ordinary objects. Instead of defining the contingency of ordinary objects in terms of their existing at some worlds but not at others, the contingency of ordinary objects is defined in terms of their being concrete at some worlds and not at others. Thus, the present system is to be contrasted with those systems in which objects disappear from the range of the quantifiers whenever they disappear from physical space. In the present system, nothing of the sort happens. The quantifiers range over everything whatsoever.

To make this even more vivid, consider what happens when, for some property F, both $\neg\exists x Fx$ and $\Diamond\exists x Fx$ are true. For example, suppose there are no aliens but there might have been. Then, the Barcan formula guarantees that something is possibly an alien ($\exists x \Diamond Fx$). Consider an arbitrary such object. Note that it is not required that this object be an alien, but only that it have the modal property of possibly being an alien. One may consistently assert that this object is not in fact an alien, that it is non-concrete, but that in worlds where it is an alien, it is concrete. Thus, although it is an (existing, actual) ordinary object, it is a contingently non-concrete one. So, whereas Lewis (1986) might argue that there exist concrete but non-actual (possible) aliens, in the present system, such 'mere possibilia' are treated as existing, actual objects which are not concrete but which might have been concrete. Note that ordinary concrete objects, such as the rocks and tables of this world, are concrete here but nonconcrete at other worlds. Our system therefore offers an actualist interpretation of the simplest quantified modal logic. As previously mentioned, this view was defended by Linsky & Zalta (1994, 1996) and more recently, by Williamson (1998, 1999).

There is one other important feature of the logic to note. Since our logic is classical, abstract and ordinary objects both must *exemplify* a complete complement of properties at every world. In other words, given any property F and its negation $\bar{F}$ ($= [\lambda x\, \neg Fx]$), and any object x, the laws of classical logic and λ-conversion ensure $\Box(Fx \vee \bar{F}x)$. By contrast, an abstract object x may be incomplete with respect to its encoded properties: there may be properties F for which neither xF nor $x\bar{F}$. For example, suppose there is an abstract object that encodes

[7]The semantics for the language and logic just described is pretty much what one would expect for an S5 modal system having both first- and second-order Barcan formulas and in which all the terms are rigid. There is a fixed domain of objects and a fixed domain of n-place relations (for each n). An 'exemplification extension' function maps each n-place relation, at each world, to the set of n-tuples of objects exemplifying that relation at that world. An 'encoding extension' function maps each 1-place relation to the set of objects which encode that relation. (Encoding is independent of the worlds.) If we ignore the rigidly denoting constants and complex terms, then truth (i.e., satisfaction relative to an assignment to the variables) can be defined in a very simple way, without an accessibility relation. Given some assignment f to the variables:

1. f satisfies $F^n x_1 \ldots x_n$ at world $\mathbf{w}$ iff the n-tuple of objects f assigns to $x_1, \ldots, x_n$ is in the exemplification extension, at $\mathbf{w}$, of the relation f assigns to F^n.
2. f satisfies xF at $\mathbf{w}$ iff the object f assigns to x is in the encoding extension of the property f assigns to F.
3. etc. (for $\neg\phi$, $\phi \rightarrow \psi$, and $\forall\alpha\phi$)
4. f satisfies $\Box\phi$ at $\mathbf{w}$ iff for every world $\mathbf{w}'$, f satisfies ϕ at $\mathbf{w}'$.

Of course, the notion of denotation must be simultaneously defined with that of satisfaction, so as to handle formulas with constants, primitive predicates, descriptions, and λ-expressions, but we omit these definitions here for the sake of simplicity.

exactly the properties F such that according to the Conan Doyle novels, Sherlock Holmes is F. Such an object will encode the properties of being a detective, living at 221B Baker Street, being brilliant, etc. But it won't encode either the property of having a mole on one's left foot or the negation of this property. Despite its incompleteness with respect to encoded properties, this abstract object is complete with respect to its exemplified properties.

2.3 The Proper Axioms of $\mathcal{O}$

$\mathcal{O}$ has two proper axioms:

$$O!x \rightarrow \Box\neg\exists F\, xF$$

$$\exists x(A!x \,\&\, \forall F(xF \equiv \phi)), \text{ where } \phi \text{ has no free } x\text{s}$$

The first tells us that ordinary objects necessarily fail to encode properties. The second is a comprehension schema for abstract objects which asserts the existence of an abstract object corresponding to any condition on properties expressible in the language.

To get some sense of the variety of abstract objects which exist given the comprehension schema, consider the following instances, in which 'a', 'R', 'S', and 'h' denote respectively, say, Aristotle, being round, being square, and Holmes:

$$\exists x(A!x \,\&\, \forall F(xF \equiv Fa))$$

$$\exists x(A!x \,\&\, \forall F(xF \equiv F{=}R \vee F{=}S))$$

$$\exists x(A!x \,\&\, \forall F(xF \equiv \text{According to the Conan Doyle stories}, Fh))$$

The first asserts the existence of an abstract object which encodes exactly the properties Aristotle exemplifies. The second asserts an object that encodes exactly two properties: being round and being square. The third asserts an object that encodes all the properties F which satisfy the open sentence 'According to the Conan Doyle stories, Holmes exemplifies F'. (Assume for now that 'According to the Conan Doyle stories, p' can be given an analysis in $\mathcal{O}$.)

Note that for each instance of comprehension, we may formulate a proper description of the form:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \phi))$$

Such descriptions are 'proper' in the sense that no matter what ϕ (with no free xs) is chosen, $\mathcal{O}$ guarantees that the resulting description has a denotation. For (a) the comprehension schema guarantees there is an abstract object encoding exactly the properties satisfying ϕ, and (b) there there couldn't be two distinct such objects, for the identity condition on abstract objects requires that distinct abstract objects differ with respect to at least one encoded property. So each of the above instances of comprehension convert into the following proper descriptions:

$$\imath x(A!x \,\&\, \forall F(xF \equiv Fa))$$

$$\imath x(A!x \,\&\, \forall F(xF \equiv F{=}R \vee F{=}S))$$

$$\imath x(A!x \,\&\, \forall F(xF \equiv \text{According to the Conan Doyle stories}, Fh))$$

It should also be easy to see that the object denoted by a description of the form $\imath x(A!x \,\&\, \forall F(xF \equiv \phi))$ encodes a property G if and only G satisfies ϕ. This in fact is a proper theorem schema of $\mathcal{O}$.

Though we've now seen how $\mathcal{O}$ asserts existence and identity conditions for abstract objects, as well as conditions under which abstract objects encode properties, little has been said thus far as to what properties abstract objects exemplify, other than the property of being abstract ($[\lambda x\, \neg\Diamond E!x]$). Let us therefore digress momentarily to describe the kinds of properties that abstract objects exemplify. First, abstract objects exemplify the (modal) negations of properties that they (necessarily) fail to exemplify. It seems reasonable to suppose that abstract objects necessarily fail to exemplify the properties of having a shape, being colored, having a texture, having mass, having a length, being a planet, being a table, etc. The idea here is clear enough: these are all concreteness-entailing ('CE') properties.

$$CE(F) \;=_{df}\; \Box\forall x(Fx \rightarrow E!x)$$

Given this definition, we may prove as a theorem that abstract objects necessarily fail to exemplify concreteness-entailing properties:[8]

$$\vdash_{\mathcal{O}} \forall F\forall x[(CE(F) \,\&\, A!x) \rightarrow \Box\neg Fx]$$

[8]For the proof of the theorem, suppose P is concreteness-entailing and that a is abstract. Suppose, for reductio, that in some world, say w_1, a exemplifies P. So, by the definition of concreteness-entailing, a exemplifies being concrete at w_1. But, by definition of abstractness, a necessarily fails to exemplify being concrete, so a fails to exemplify being concrete in w_1. Contradiction.

It therefore follows that abstract objects all necessarily exemplify the negations of these concreteness-entailing properties, and exemplify their modal negations as well. In formal terms:

$$\vdash_{\mathcal{O}} \forall F \forall x((CE(F) \,\&\, A!x) \rightarrow \Box[\lambda z\, \neg Fz]x)$$

$$\vdash_{\mathcal{O}} \forall F \forall x((CE(F) \,\&\, A!x) \rightarrow [\lambda z\, \Box\neg Fz]x)$$

Thus, if being John's sister, being a talking donkey, being a million carat diamond, etc. are concreteness-entailing, then we know that abstract objects fail to exemplify such properties as possibly being John's sister, possibly being a talking donkey, etc., and so exemplify such properties as not possibly being John's sister, not possibly being a talking donkey, etc. Of course, we cannot offer a complete list of concreteness-entailing properties, but in what follows, we shall often assert what we take to be reasonable claims to the effect that certain properties are concreteness-entailing, and that consequently, abstract objects exemplify the modal negations of these properties.

In addition to these facts, there is another important group of properties that abstract objects exemplify. As noted above, abstract objects contingently exemplify various intentional properties, such as being thought about by person y, being admired by person z, being worshipped by society z, etc. Such claims may be consistently added to our metaphysics.

We conclude this subsection by discussing some logical subtleties of $\mathcal{O}$ which arise in the context of our two strong existence principles: the comprehension principle for abstract objects and the comprehension principle for relations. The consistency of $\mathcal{O}$ has been established by two different kinds of models, Scott-models (Zalta 1983, Appendix) and Aczel-models (Zalta 1999). Consistency is secured by the one restriction on the formation of λ-expressions noted earlier, namely, that encoding subformulas may not appear in the matrix of such expressions. To see why, suppose the predicate $[\lambda x\, \exists F(xF \,\&\, \neg Fx)]$ were formulable in the system and denoted a property. For simplicity, call this property K. Then consider the instance of comprehension for abstract objects that asserts the existence of an object that encodes just K and no other properties. Call such an object a. Now ask the question, does a exemplify K? If we suppose it does, then (by λ-conversion) there is a property it encodes which it fails to exemplify. Since it encodes only K, it must therefore fail to exemplify K, contrary to hypothesis. If we suppose it doesn't, then there is a property it encodes, namely, K, which it fails to exemplify. So (by λ-conversion),

it does exemplify K, again contrary to assumption. This contradiction, and others like it, are avoided by banishing encoding subformulas from λ-expressions.[9]

Note that as a result of this constraint on λ-expressions, the general notion of identity $(x = y)$, may not appear in λ-expressions. When the defined notion of identity is expanded into primitive notation, it contains encoding subformulas.[10] This turns out to be a fortunate result, since we cannot suppose that there is a distinct property $[\lambda x\; x = b]$ for each distinct abstract object b. The reason why can be sketched informally. Intuitively, the comprehension axiom for abstract objects ensures that there is an abstract object in the domain for each (expressible) set of properties. In standard models of $\mathcal{O}$, this principle can be made true by requiring that the domain of abstract objects be equivalent in size to the power set of the set of properties. But by an argument similar to that used in Cantor's theorem, there cannot be a distinct property, $[\lambda x\; x = b]$, for each distinct abstract object b in the domain, for otherwise there would be a one-to-one mapping from the power set of the set of properties into a subset of the set of properties. So though we may assert that various abstract objects satisfy the condition $x = y$, the system does not guarantee that they stand in a relation thereby.

However, $[\lambda xy\; x =_E y]$ is a well-formed expression and denotes a relation.[11] It is worth remembering that identity$_E$ is a classical, Leibnizian notion of identity, as it applies to ordinary objects. This relation will play a role in what follows, in the discussion of Fine's second counterexample.

[9]See Zalta 1983, and 1999, for a fuller discussion. A paradox of this kind was first noted in Clark 1978, and was discussed in Rapaport 1978.

[10]This lends some validity to Kant's claim that existence is not a property or predicate, assuming existence is given its usual definition in terms of the formula $\exists y(y = x)$. The λ-expression $[\lambda x \exists y(y = x)]$ is *not* well-formed—the identity sign is defined in terms of encoding subformulas. Of course, there is a restricted notion of existence that is a property, namely, $[\lambda x\; \exists y(y =_E x)]$. But this is not a general notion of existence, since it doesn't apply to abstract objects. The fact is that the general notion of existence can't be turned into a predicate.

[11]This is not to imply that all well-formed λ-expressions denote relations. The only exceptions are those containing improper, non-denoting definite descriptions. For example, if $\imath xGx$ is a description that doesn't denote, then the expression $[\lambda y Ry\imath xGx]$ will not have a denotation either.

2.4 Applications of $\mathcal{O}$

$\mathcal{O}$ has been applied in a variety of contexts of philosophical analysis. Before turning to the application most relevant to the present paper, namely, the analysis of mathematical language, we mention five applications that may be of interest to readers encountering this material for the first time; three of these play a brief role later in this section.

(1) The most important principles of the theory of possible worlds may be derived in $\mathcal{O}$ from the following two definitions, where the variable 'w' used in the second definition ranges over the objects defined by the first (Zalta 1993):

$$PossibleWorld(x) \ =_{df} \ \Diamond\forall p(x \models p \ \equiv \ p)$$

$$p \ is \ true \ at \ w \ (\text{'}w \models p\text{'}) \ =_{df} \ w[\lambda y\, p]$$

(2) The Fregean notion 'the number of (ordinary) Gs' ($\#G$) may be analyzed in terms of the following definition, where $F \approx_E G$ is the equivalence condition asserting that there is a one-to-one correspondence between the ordinary objects exemplifying F and G:

$$\#G =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv F \approx_E G))$$

This definition plays a role in reconstructing Frege's definition of *predecessor* within $\mathcal{O}$. Such reconstructions of Fregean definitions lead to the recovery of a consistent fragment of Frege 1893/1903, which includes the Peano-Dedekind axioms for number theory as theorems (Zalta 1999). (3) Where $G \Rightarrow F$ indicates that G necessarily implies F (i.e., $\Box\forall x(Gx \rightarrow Fx)$), the Platonic Form of G (Φ_G) may be identified as:

$$\Phi_G =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv G \Rightarrow F))$$

From this definition, the main principles of Plato's theory of Forms become derivable in $\mathcal{O}$ (Pelletier & Zalta 2000). (4) The Leibnizian individual concept c_a of a given ordinary object a (e.g., Aristotle), may be identified as:

$$c_a =_{df} \ \imath x(A!x \,\&\, \forall F(xF \equiv Fa))$$

This is one of the definitions by which Leibniz's modal metaphysics can be assimilated to his nonmodal calculus of concepts (Zalta 2000b). (5) Finally, a consistent analysis of the Fregean notion 'the extension of the concept G' (ϵG) may be given in terms of the following definitions (Anderson & Zalta 2004):

$$\epsilon G =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv \forall y(Fy \equiv Gy)))$$

$$y \in x \ =_{df} \exists F(x = \epsilon F \,\&\, Fy)$$

From these definitions, a consistent version of Frege's Basic Law V becomes derivable in $\mathcal{O}$.

Since a proper treatment of Fine's first counterexample, concerning Socrates and singleton Socrates, requires a philosophical understanding of mathematical objects, it is important to turn now to a discussion of how mathematical theories and mathematical objects have been analyzed in $\mathcal{O}$. On the approach taken here, the data to be represented consists of the ordinary (though often formal) mathematical claims made by either mathematicians or nonmathematicians. From the present perspective, ordinary mathematical claims occur either (a) explicitly or implicitly in the context of some mathematical theory T, or (b) in the context of 'natural, naive mathematics', such as the ordinary naive geometrical claims, ordinary number statements appealing to the 'natural' numbers, and ordinary naive statements about sets or classes (i.e., extensions of ordinary properties). Thus, whenever we attempt to analyze some ordinary mathematical claim, we must decide whether we have a case of (a) or (b). We shall assume that (a) and (b) are exclusive possibilities, and that any ambiguity must be resolved in one way or the other.

In $\mathcal{O}$, we represent mathematical claims of type (b) by way of the applications (2), (3), and (5) mentioned above. For example, The Triangle of natural, naive geometry would be represented by the Platonic Form of triangularity Φ_T (which encodes all and only the properties necessarily implied by the property of being a triangle). The number of planets would be represented as the Fregean number $\#P$ (which encodes all and only the properties in one-to-one correspondence on the ordinary objects with the property of being a planet). And the class of humans, if discussed naively (and *not* in the context of some modern set theory), would be represented as ϵH, i.e., as the abstract object which encodes all and only the properties which are materially equivalent to the property of being human ('H'). Thus, singleton Socrates, if discussed naively in this same way, would be represented as $\epsilon[\lambda x\, x =_E s]$, i.e., as the abstract object which encodes all and only the properties which are materially equivalent to the property *being identical*$_E$ *to Socrates*. Although it is nearly certain that Fine wasn't talking about this object when discussing his counterexample to (E), we will nevertheless briefly mention it at the end of the paper, insofar as it constitutes one possible reconstruction of Fine's counterexample.

However, $\mathcal{O}$ offers a general method for representing mathematical claims of type (a). In what follows, we shall concern ourselves only with (the representation of) the well-defined individual terms appearing in mathematical theories, i.e., names and proper definite descriptions (or function terms) which appear in contexts presupposing some theoretical mathematics and which uniquely denote mathematical individuals.[12] Now when we encounter mathematical terms such as '2', '2^3', '2+3', '3/4', 'π', '$\emptyset$', $\aleph_0$, etc., within ordinary mathematical contexts, we have to identify the mathematical theory that is assumed in that context, whether it is Peano Number Theory (N), Rational Number Theory (Q), Real Number Theory ($\Re$), Zermelo-Fraenkel set theory (ZF), Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), NBG, etc. Thus, ordinary, theoretical mathematical terms such as '2', 'ω', '3/4', '$\{\emptyset\}$' etc., are assumed to be ambiguous until the relevant mathematical theory is identified (or, at the very least, until the principles which govern the terms are identified). These ordinary terms are to be represented within $\mathcal{O}$ by importing them into the language of $\mathcal{O}$ and indexing them to their respective mathematical theories. $\mathcal{O}$, therefore, will include such expressions as 2_{N}, $2_{\Re}$, $\emptyset_{\mathrm{ZF}}$, $\emptyset_{\mathrm{ZFC}}$, $\{\emptyset_{\mathrm{ZF}}\}$, etc. Strictly speaking, functional notation and operators, such as $(\,) + (\,)$, $(\,)^{(\,)}$, $\{\ldots\}$, etc., should be indexed as well, but we sometimes omit this for purposes of readability when the context is clear. Thus, 'singleton Socrates' ('$\{s\}$'), if used within the context that assumes a modern set theory including urelements, would be represented in $\mathcal{O}$ by indexing it to the relevant set theory.

Now these representations of ordinary mathematical expressions in $\mathcal{O}$ are governed by a very general principle. This principle was put forward as part of the analysis of mathematical theories developed in Linsky & Zalta 1995, Zalta 2000a, and Linsky & Zalta 2006. In those works, mathematical theories are analyzed not syntactically but rather as abstract entities encoding propositional content. In particular, a mathematical theory T is identified as an abstract object that encodes propositional properties of the form *being such that p* ('$[\lambda y\, p]$'). (Propositional properties were discussed briefly in Section 2.1.) We therefore say that a proposition p *is*

[12] $\mathcal{O}$ has also been applied to the analysis of mathematical relations, such as the predecessor relation of Peano Number Theory, the membership relation of ZF, the membership relation of ZFC, etc. See Zalta 2000a. But this further application of $\mathcal{O}$ will play no role in what follows.

true according to mathematical theory T ('$\mathrm{T} \models p$') just in case T encodes $[\lambda y\, p]$:

$$\mathrm{T} \models p \ =_{df} \mathrm{T}[\lambda y\, p]$$

In what follows, we assume that mathematical theories are to be formulated in a classical second-order (modal) predicate calculus with descriptions and λ-expressions, i.e., in a language just like the one developed here but without encoding formulas. Therefore, all the λ-expressions formulable in the language of T are consistent with $\mathcal{O}$'s requirement that λ-expressions have no encoding subformulas. Moreover, sentences of T become sentences of $\mathcal{O}$ simply by indexing the terms of T as outlined above. We indicate below how these sentences of T become *assertible* in $\mathcal{O}$ when so indexed.

We turn next to the principle which guarantees that every well-defined term of a mathematical theory is represented in $\mathcal{O}$ as denoting a unique abstract object. Where T is any mathematical theory, κ is any name or proper description (or function term) appearing in T, and κ_{T} is the representation of κ in $\mathcal{O}$, the following theoretical identification principle is asserted to hold in $\mathcal{O}$:

Theoretical Identification Principle:
$$\kappa_{\mathrm{T}} = \imath x(A!x \,\&\, \forall F(xF \equiv \ \mathrm{T} \models F\kappa_{\mathrm{T}}))$$

This asserts that the object κ of theory T is the abstract object which encodes just the properties F exemplified by κ_{T} according to theory T. To take an example instance of this principle, the empty set of ZF is identified as the abstract object that encodes just the properties F that the empty set of ZF exemplifies according to ZF:

$$\emptyset_{\mathrm{ZF}} = \imath x(A!x \,\&\, \forall F(xF \equiv \ \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}))$$

A few words of explanation are in order.

Clearly, the Theoretical Identification Principle is not a definition of κ_{T}, since that term appears on both sides of the identity sign. Rather, the idea is that it gives us a principled way to identify a mathematical object x relative to a fixed group of sentences of the form $\mathrm{T} \models Fx$. These latter sentences become assertible in $\mathcal{O}$ in the presence of the following importation principle:

Metatheoretic Importation Principle:
Where ϕ^* is the result of replacing κ by κ_{T} everywhere in ϕ, then:

$$\text{If } \vdash_{\mathrm{T}} \phi, \text{ then } \vdash_{\mathcal{O}} \mathrm{T} \models \phi^*$$

In other words, $\mathrm{T} \models \phi^*$ becomes an assertible sentence of $\mathcal{O}$ when ϕ is a theorem of T.[13] Note that $\mathrm{T} \models \phi^*$ may be reasonably seen as an *analytic* truth of $\mathcal{O}$, when ϕ is a theorem of T.

To take some simple examples. Consider the following theorems of ZF and $\Re$:

$$\vdash_{\mathrm{ZF}} \emptyset \in \{\emptyset\}$$

$$\vdash_{\Re} 2 \leq \pi$$

So by the Metatheoretic Importation Principle, the following are theorems of $\mathcal{O}$:

$$\mathrm{ZF} \models \emptyset_{\mathrm{ZF}} \in \{\emptyset_{\mathrm{ZF}}\}$$

$$\Re \models 2_{\Re} \leq \pi_{\Re}$$

From the point of view of $\mathcal{O}$, the claims of mathematics which are not prefixed by the theory operator are not true. That is, the following two claims, and the others like them, are neither assertible nor assumed true in $\mathcal{O}$:

$$\emptyset_{\mathrm{ZF}} \in \{\emptyset_{\mathrm{ZF}}\}$$

$$2_{\Re} \leq \pi_{\Re}$$

Indeed, we regard these representations as false, for the present view is that mathematical objects encode rather than exemplify their mathematical properties. It should be noted, however, that $\mathcal{O}$ does offers *true* readings of ordinary theoretical mathematical claims such as the claim of ZF that "$\emptyset$ is an element of $\{\emptyset\}$" and the claim of $\Re$ that "2 is less than or equal to π." Given a background logic with two modes of predication, these ordinary theoretical claims of mathematics become ambiguous. Though we shall not go into the matter in here, there is a procedure for formulating the encoding readings on which these ordinary claims are true (Zalta 2000a). Some of these true readings will mentioned below.

[13] Strictly speaking, this imports $\mathrm{T} \models \phi^*$ as an axiom of $\mathcal{O}$ (though see the next remark in the text concerning its analyticity). In previous work (Linsky & Zalta 1995, Zalta 2000a), we imported only the axioms of T instead of all the theorems of T. We used a special principle to derive claims of the form $\mathrm{T} \models \phi^*$ for those ϕ which were derived theorems of T. That is, instead of the Metatheoretic Importation Principle, we employed a Rule of Closure, which allowed us to infer (in $\mathcal{O}$) that $\mathrm{T} \models q$ whenever $\mathrm{T} \models p_1, \ldots, \mathrm{T} \models p_n$ and $\{p_1, \ldots, p_n\} \vdash_{\mathcal{O}} q$. However, the present importation procedure is simpler and has an added virtue: it guarantees that mathematical objects will encode only properties expressible in their governing theory, for only claims expressible in T get imported. Given our current procedure, it is clear that the following claim doesn't become assertible in $\mathcal{O}$: according to T, κ_{T} exemplifies the property of being such that either John is happy or John is not happy. Though κ_{T} is conceived through all the properties attributed to it in T (including purely logical ones), it is not conceived by way of properties which can't be expressed in T.

But now we must focus on an important group of claims which are derivable from the above theorems of ZF and $\Re$ and whose transformations can be asserted $\mathcal{O}$. Note that the following are also theorems of those theories, respectively:

$$\vdash_{\mathrm{ZF}} [\lambda x\, x \in \{\emptyset\}]\emptyset$$

$$\vdash_{\Re} [\lambda x\, x \leq \pi]2$$

It therefore follows, by the Metatheoretic Importation Principle, that the following two claims are theorems of $\mathcal{O}$:

$$\mathrm{ZF} \models [\lambda x\, x \in \{\emptyset_{\mathrm{ZF}}\}]\emptyset_{\mathrm{ZF}} \tag{η}$$

$$\Re \models [\lambda x\, x \leq \pi_{\Re}]2_{\Re} \tag{θ}$$

The first sentence asserts that according to ZF, the ZF-empty set exemplifies the property of being an element of the (ZF-)singleton of the ZF-empty set. The second asserts that according to real number theory, the real number 2 exemplifies the property of being less than or equal to the real number π.

Finally, we may infer from these results some facts about the properties that $\emptyset_{\mathrm{ZF}}$ and $2_{\Re}$ encode. Note that the following is an immediate consequence of the Theoretical Identification Principle:

Equivalence Theorem:
$$\kappa_{\mathrm{T}}F \equiv \mathrm{T} \models F\kappa_{\mathrm{T}}$$

In other words, mathematical objects encode all and only the properties they exemplify according to their governing mathematical theory. Here are two instances of this theorem:

$$\emptyset_{\mathrm{ZF}}F \equiv \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}$$

$$2_{\Re}F \equiv \Re \models F2_{\Re}$$

From these two instances, and from (η) and (θ), we may conclude:

$$\emptyset_{\mathrm{ZF}}[\lambda x\, x \in \{\emptyset_{\mathrm{ZF}}\}]$$

$$2_{\Re}[\lambda x\, x \leq \pi_{\Re}]$$

The first asserts that the ZF-empty set encodes the property of being an element of the (ZF-)singleton of the ZF-empty set. The second asserts that the real number 2 encodes the property of being less than or equal to the real number π. The Equivalence Theorem therefore guarantees that mathematical objects encode all and only the mathematical properties attributed to them in their governing theories.

Reasoning analogous to the above will play a role in what follows. We shall analyze 'singleton Socrates' as a theoretical term of Modal Set Theory with Urelements (= M), but where 'Socrates' is a nonmathematical term not subject to indexing or the above Theoretical Principle of Identification when represented in $\mathcal{O}$. We shall then be able to deduce that the M-singleton of Socrates encodes the property of having Socrates as an element. But Socrates provably neither (a) encodes any properties, nor (b) exemplifies any properties in virtue of the properties encoded by the M-singleton of Socrates, nor (c) exemplifies any properties abstracted from the properties attributed to the M-singleton of Socrates in M. This will be discussed in more detail in Section 5.

3. Essence, Modality, and Ordinary Objects

It is clear from the foregoing that in our metaphysical foundations, there are two fundamentally different kinds of objects, abstract objects and ordinary objects, constituting mutually exclusive domains. Whereas ordinary objects exemplify their properties in the classical way, abstract objects are the kind of object which can both encode and exemplify properties. Such a basic distinction in kinds of objects merits a distinction in the notion of 'essential property' that applies to each kind. It is therefore natural to suppose that the notion of 'essential property' that is definable for abstract objects differs from the notion definable for ordinary objects. Accordingly, we shall divide our discussion of essential properties into two parts. In this section, we investigate the notions of essential property which are appropriate for ordinary objects, and in the subsequent sections, the notion apppropriate for abstract objects. Consequently, we address Fine's second counterexample first, since it concerns the identity

and essential properties of ordinary objects. The analysis of this example requires only an appeal to the *logic* underlying $\mathcal{O}$. When we move on to discuss Fine's first counterexample, however, we shall need to appeal to the entire theory $\mathcal{O}$ to ground a definition of essential property that applies to abstract objects.

Let us introduce the special variables u and v to range only over ordinary objects (we continue to use the variables x, y, z as unrestricted). Let us also adopt the following convention: even though the two modes of predication in our formal framework suggest an ambiguity in predication in natural language, the fact that ordinary objects only exemplify (and don't encode) properties suggests that when we use the ordinary predicative copula 'is' or 'has' to informally read or assert claims about the properties of ordinary objects, it should be clear that this is intended to be analyzed in terms of our formal notion of exemplification.

What, then, are essential properties of ordinary objects? From the discussion in Section 2.3 (and the extended discussion in Linsky & Zalta 1994, 447), it should be clear that since ordinary objects exist in every world, their 'essential' properties are *not* the ones they have in every world in which they exist, but rather ones they have in every world in which they are concrete objects. If the intuition we wish to capture is that Socrates is essentially human, the representation of this intuition in the present system is to assert that *being human* is a property that Socrates exemplifies in every world in which he is concrete. In those worlds where Socrates is not concrete, he will not exemplify being human or any of the properties that humans typically exemplify (though he will exemplify the negations of those properties).

Given this basic understanding of the notion of essential property as it relates to ordinary objects, several further distinctions can be drawn. The notion of 'essential property', in the present framework, can be analyzed into the following three distinct notions:

$$\mathit{Necessary}(F,u) \ =_{df} \ \Box Fu$$

$$\mathit{WeaklyEssential}(F,u) \ =_{df} \ \Box(E!u \rightarrow Fu)$$

$$\mathit{StronglyEssential}(F,u) \ =_{df}$$
$$\mathit{WeaklyEssential}(F,u) \,\&\, \neg\mathit{Necessary}(F,u)$$

Note that the form of the definition of *WeaklyEssential* is only superficially identical to that of (E). In our system, '$E!x$' is not defined as

$\exists y(y{=}x)$ and so the antecedent of the modal conditional in the definiens is not an existence claim. We read this definition as: *F is weakly essential to u* iff necessarily, u exemplifies F whenever u exemplifies being concrete. Moreover, it will soon become apparent why the third definition defines an interesting notion of essential property. Let's see how these definitions work.

Some properties F are provably such that $Necessary(F, u)$. Consider, for example, the property of being self-identical ($= [\lambda z\, z{=}_E z]$). One can prove in $\mathcal{O}$ both that necessarily every ordinary object is self-identical$_E$ and that every ordinary object is necessarily self-identical$_E$.[14] From these facts, one can establish that the following are theorems of $\mathcal{O}$, given the above definitions:[15]

$$\vdash_{\mathcal{O}} Necessary([\lambda z\ z =_E z], u)$$
$$\vdash_{\mathcal{O}} WeaklyEssential([\lambda z\ z =_E z], u)$$
$$\vdash_{\mathcal{O}} \neg StronglyEssential([\lambda z\ z =_E z], u)$$

Notice here that being self-identical$_E$ is 'essential' to ordinary objects in

[14]To see this, first consider the proof of: $\Box\forall x(O!x \rightarrow x{=}_E x)$. Suppose $O!a$, where a is arbitrary. Note that by logic alone, one can establish $\Box\forall F(Fa \equiv Fa)$. So, we have established: $O!a\ \&\ O!a\ \&\ \Box\forall F(Fa \equiv Fa)$, which by definition, is $a =_E a$. So, $O!a \rightarrow a{=}_E a$, by conditional proof. And by generalization, we have $\forall x(O!x \rightarrow x{=}_E x)$. Finally, by necessitation, we have $\Box\forall x(O!x \rightarrow x{=}_E x)$.

Now consider the proof of: $x =_E x \rightarrow \Box x =_E x$. Let a be arbitrary and suppose $a{=}_E a$ (to show $\Box a{=}_E a$). Then, by definition,

$$O!a\ \&\ O!a\ \&\ \Box\forall F(Fa \equiv Fa),$$

i.e.,

$$\Diamond E!a\ \&\ \Diamond E!a\ \&\ \Box\forall F(Fa \equiv Fa)$$

Since each conjunct is a modal claim, in S5, each is necessary:

$$\Box\Diamond E!a\ \&\ \Box\Diamond E!a\ \&\ \Box\Box\forall F(Fa \equiv Fa)$$

Now a conjunction of necessary truths is a necessary conjunction of truths:

$$\Box[\Diamond E!a\ \&\ \Diamond E!a\ \&\ \Box\forall F(Fa \equiv Fa)]$$

And, given the definition of ordinary object, we have:

$$\Box[O!a\ \&\ O!a\ \&\ \Box\forall F(Fa \equiv Fa)],$$

i.e., $\Box a{=}_E a$. Since a was arbitrary, we've established $x{=}_E x \rightarrow \Box x{=}_E x$.

Clearly, given the proofs of these two theorems, we can immediately infer that every ordinary object is necessarily self-identical$_E$, i.e., $\forall x(O!x \rightarrow \Box x{=}_E x)$.

[15]Given the proof of $\forall x(O!x \rightarrow \Box x{=}_E x)$ in the previous footnote, it follows immediately that $\forall x(O!x \rightarrow \Box[\lambda z\, z{=}_E z]x)$, i.e., $Necessary([\lambda z\, z{=}_E z], u)$. The second and third theorems follow straightforwardly from this.

one sense but not in another. Since ordinary objects are self-identical$_E$ in every world whatsoever, they are self-identical$_E$ in every world in which they are concrete. We have defined a notion of essential property which focuses on those properties of ordinary objects that are exemplified in every world in which they are concrete and that are *not* properties that they exemplify in every world.

As another example, consider the property of *not being a stone*, or *not being a sea urchin* ($= [\lambda z\, \neg Sz]$). Many philosophers have the intuition that Socrates couldn't have been a stone or a sea urchin. Although this could be taken to mean that Socrates is a stone/sea urchin in no possible world, let us for now represent this intuition in the present system by the claim that Socrates is such that in every world in which he is concrete, he fails to be stone/sea urchin. However, note that in every world in which Socrates is not concrete, he also fails to be a stone/sea urchin. This latter claim follows from the reasonable assumption that being a stone/sea urchin is a concreteness-entailing property.

Since Socrates fails to be a stone/sea urchin in every world in which he is concrete and in every world in which he is non-concrete, we've established that necessarily, Socrates exemplifies $[\lambda z\, \neg Sz]$. We therefore have another case of a property F which Socrates exemplifies necessarily, but for which there is a sense in which F is, and a sense in which F isn't, 'essential' to Socrates:

$$Necessary([\lambda z\, \neg Sz], s)$$
$$WeaklyEssential([\lambda z\, \neg Sz], s)$$
$$\neg StronglyEssential([\lambda z\, \neg Sz], s)$$

Though these claims are not *theorems* of $\mathcal{O}$, we can extend the theory to accomodate these essentialist claims.

We turn next to those properties with which we began our discussion of 'essential property'. These will be essential to Socrates in both senses of 'essential'. Consider the property of being human. Many philosophers have the intuition that Socrates is essentially human. Of course, one might just represent this intuition in the present framework by saying that necessarily, Socrates exemplifies being human whenever he is concrete. But, as yet, that doesn't distinguish this property from those properties which Socrates has in every world, since those, too, are properties F such that necessarily Socrates has F whenever he is concrete. But we can establish something further, if given the following auxiliary hypotheses:

(a) Socrates is contingent, i.e., $\Diamond E!s \,\&\, \Diamond\neg E!s$

(b) Being human is concreteness-entailing, i.e., $\Box\forall x(Hx \rightarrow E!x)$

From these claims, one can prove that it is not necessary that Socrates is human. For by (a), there is a world, say w_1, where Socrates fails to exemplify being concrete, and so by (b), he fails to exemplify being human at w_1.

Summarizing, then, we have the following claims, where being human $= H$, and Socrates $= s$:

$$\neg\mathit{Necessary}(H, s)$$
$$\mathit{WeaklyEssential}(H, s)$$
$$\mathit{StronglyEssential}(H, s)$$

Again, these are essentialist claims which can be added to our theory, though the first and third can be proved from the second, with the help of the auxiliary hypotheses (a) and (b).

Our auxiliary hypotheses (a) and (b) are relatively straightforward and uncontroversial. They preserve familiar intuitions in terms of the language of the present theory. Socrates' contingency, given that he is an ordinary object, lies in the fact that he is not concrete in every world rather than in the fact that he exists in some worlds and not in others. Moreover, what philosophers have elsewhere called 'existence-entailing' properties are in the present theory conceived as 'concreteness-entailing' properties, as defined in Section 2.3. It is uncontroversial to claim that being human is concreteness-entailing.

Of course, the usual claims concerning properties that are not essential to Socrates can be represented and reanalyzed along the above lines. Being snub-nosed ($= S$) is neither exemplified necessarily nor 'essential' to Socrates in either sense:

$$\neg\mathit{Necessary}(S, s)$$
$$\neg\,\mathit{WeaklyEssential}(S, s)$$
$$\neg\mathit{StronglyEssential}(S, s)$$

The first claim would follow from the facts that Socrates is contingent and that necessarily, anything snub-nosed is concrete, by now familiar reasoning.

We are now in a position to give a straightforward analysis of the second counterexample which Fine develops for (E). Recall that he says:

> Consider two objects whose natures are unconnected, say Socrates and the Eiffel Tower. Then it is necessary that Socrates and the Tower be distinct. But [intuitively] it is not essential to Socrates that he be distinct from the Eiffel Tower, for there is nothing in his nature which connects him in any special way to it. (Fine 1994a, 5)

But given the above definitions and the assumption that Socrates ('s') is not identical$_E$ to the Eiffel Tower ('t'), we have:

$$\mathit{Necessary}([\lambda z\, z \neq_E t], s)$$
$$\mathit{WeaklyEssential}([\lambda z\, z \neq_E t], s)$$
$$\neg\mathit{StronglyEssential}([\lambda z\, z \neq_E t], s)$$

Thus we have a natural and well-defined sense in which it is not essential to Socrates that he be distinct from the Eiffel Tower.

Indeed, given that Socrates and the Eiffel Tower are ordinary objects, the above claims are provable in object theory from the following theorems, where u, v are variables ranging over ordinary objects:[16]

$$\vdash_{\mathcal{O}} u \neq_E v \rightarrow \mathit{Necessary}([\lambda z\, z \neq_E v], u)$$
$$\vdash_{\mathcal{O}} u \neq_E v \rightarrow \mathit{WeaklyEssential}([\lambda z\, z \neq_E v], u)$$
$$\vdash_{\mathcal{O}} u \neq_E v \rightarrow \neg\mathit{StronglyEssential}([\lambda z\, z \neq_E v], u)$$

These results preserve Fine's suggestion that this counterexample shows that (E) is too simplistic. However, (a) we do not abandon the idea that essence and modality are connected in an intimate way, since the superficial form of (E) is preserved as a conjunct of *StronglyEssential*; (b) we

[16]For the proofs of the theorems which follow in the text, note first that the following are theorems:

$$u =_E v \rightarrow \Box u =_E v \qquad (=_E 1)$$
$$\Diamond u =_E v \rightarrow u =_E v \qquad (=_E 2)$$
$$u \neq_E v \rightarrow \Box u \neq_E v \qquad (=_E 3)$$
$$\Diamond u \neq_E v \rightarrow u \neq_E v \qquad (=_E 4)$$

($=_E 1$) is the basic one; it was proved in footnote 14. ($=_E 2$) and ($=_E 3$) then become derivable by a proof similar to that in footnote 2. ($=_E 4$) is a consequence of ($=_E 1$) by modal negation.

From these theorems, the claims in the text are straightforwardly derivable. The first follows by ($=_E 3$), λ-conversion, and the definition of *Necessary*(F,u). The second by ($=_E 3$), λ-conversion, the S5 theorem that $\Box\phi \rightarrow \Box(\psi \rightarrow \phi)$, and the definition of *WeaklyEssential*(F,u). The third by the first theorem and the definition of *StronglyEssential*(F,u).

do not require a special logic of essence to understand what has gone wrong but rather the simplest quantified modal logic; and (c) it becomes an interesting (and surprising) fact that not every notion of 'F is essential to x' has 'x exemplifies F necessarily' as a necessary condition. Neither *WeaklyEssential* nor *StronglyEssential* has 'x exemplifies F necessarily' as a necessary condition, and *StronglyEssential* explicitly rejects it as a necessary condition. All of this suggests reasons for qualifying the classic view, which Fine accepts, that if an object essentially has a certain property, then it is necessary that it has the property.

Before we turn to the notion of 'essential property' that is appropriate for abstract, as opposed to ordinary, objects, let us consider a potential counterexample. Fine might object as follows: consider the conjunctive property of *being human and not identical*$_E$ *to the Eiffel Tower*, i.e., $[\lambda z\, Hz \,\&\, z \neq_E t]$. Fine might note that given the above definitions, this property is strongly essential to Socrates, since he has it in every world in which he is concrete but doesn't have it necessarily. Yet this consequence runs counter to his intuition that "nothing in [Socrates'] nature connects him in any special way to it [the Eiffel Tower]".

There are several natural things to say in response, however. The issue turns on how to construe Fine's intuition that Socrates' nature is not connected in any special way to the Eiffel Tower, and why one might think that the fact that *being human and not identical*$_E$ *to the Eiffel Tower* is strongly essential to Socrates connects him in a special way to the Eiffel Tower. It is important to remember that in the present framework, it easy to prove connections between anything whatsoever and the Eiffel Tower. For recall that it is a theorem of $\mathcal{O}$ that necessarily the Eiffel Tower exists (this is not to say that the Eiffel Tower is necessarily concrete). That is, $\vdash_{\mathcal{O}} \Box\exists y(y = t)$, and so every proposition whatsoever, about anything we choose, will imply the existence of the Eiffel Tower.

Given our modal logic, the important worry is not that something in Socrates' nature might imply that the Eiffel Tower exists but rather that something might imply that the Eiffel Tower is concrete. However, the fact that $[\lambda z\, Hz \,\&\, z \neq_E t]$ is strongly essential to Socrates does *not* imply that the Eiffel Tower is concrete. The theory $\mathcal{O}$ as formulated is consistent with the assumption that the Eiffel Tower is not concrete, though of course, we wouldn't want to add such an assumption. So, if something like this is Fine's worry, there is no special problem concerning this conjunctive property.

Of course, Fine might reiterate his worry by saying that the fact that *being human and not identical*$_E$ *to the Eiffel Tower* is strongly essential to Socrates in and of itself is an unintuitive connection between Socrates' nature and the Eiffel Tower. But should we accept this? On the one hand, the nature of Socrates has not been defined rigorously. One could place a constraint on the principles governing that notion so as to exclude any property which necessarily implies a property that Socrates has in every possible world. So even though the *being human and not identical*$_E$ *to the Eiffel Tower* is strongly essential to Socrates, it would not form part of Socrates' nature, since it necessarily implies the property of *being not identical*$_E$ *to the Eiffel Tower* (which Socrates has in every world). On the other hand, if we are assuming that it is in Socrates's nature to be human (in the sense that this is one of his strongly essential properties), then one might suggest that it is similarly in his nature to be human and distinct$_E$ from every other thing. If so, it may be in his nature to be human and distinct$_E$ from every other particular thing, such as the Eiffel Tower.

It is not clear which of the responses just outlined is the best way to proceed. A resolution depends on a more explicit expression of Fine's intuition that nothing in Socrates' nature connects him in any special way to it the Eiffel Tower. However that is spelled out, it would serve well to remember that even Fine's logic of essence has to take extra steps to avoid similar consequences. He says,

> ... the propositions true in virtue of the nature of given objects are taken to be closed under logical implication. ... However, this closure condition is subject to a certain constraint. For we do not allow the logical consequences in question to involve objects which do not pertain to the nature of the given objects. (1995, 242)

Although it is not the point of the present paper to try to build a logic of essence that can do all the work that Fine's logic can do, it should be noted both that (a) the present theory does not require that G is strongly essential to u whenever both F is strongly essential to u and F necessarily implies G (as demonstrated by the case where F = *being human and not identical*$_E$ *to the Eiffel Tower* and G = *being not identical*$_E$ *to the Eiffel Tower*), and (b) many of stipulations in Fine's framework which implement the constraint described above could be specifiable within the present framework and adopted as axioms.

However, given that the present theory reconceives the nature of existence and essence, the case of $[\lambda z\, Hz \,\&\, z \neq_E t]$ may be one where we should let the theory itself help us to refine some of our intuitions about what properties are strongly essential to ordinary objects.[17] For our work so far has an unheralded virtue, namely, it plays a role in understanding Fine's other counterexample, concerning Socrates and singleton Socrates, as we shall see in the next two sections.

4. Essence, Modality, and Abstract Objects

When asking the question, "What properties do abstract objects have essentially?," the overriding consideration in the present framework is the fact that abstract objects both encode and exemplify properties. Both ways of having properties might provide a source of essential properties.

It should be clear from the motivations described in Section 2 that the essential properties of abstract objects are their *encoded* properties. Let's first look at some intuitive examples. What is essential to Sherlock Holmes are the properties by which he is conceived: being a detective, being brilliant, having Dr. Watson as a friend, having Moriarty as his arch-enemy, etc. Thus, Holmes' encoded properties are even more crucial to his identity than properties that he necessarily exemplifies. For example, Holmes necessarily exemplifies abstractness given that he exemplifies abstractness (this follows from the modal theorem $\Box\phi \rightarrow \Box\Box\phi$ and the fact that by definition, abstractness is equivalent to necessary non-concreteness). But abstractness is not part of the conception of Holmes; rather concreteness is part of its conception, for it is relevantly implied in the story that Holmes is a concrete object. Concreteness is therefore one of Holmes' encoded and essential properties. To continue discussion of this first example further, note also that Holmes necessarily exemplifies the property of *not being a detective*, since being a detective is a concreteness-entailing property. Clearly we don't want to identify *not*

[17]Our theory allows us to formulate other distinctions and principles. Consider, for example, a variation on our previous suggestion: define a property H as *vacuously strongly essential to* x whenever H is identical to a conjunctive property of the form $[\lambda x\, Fx \,\&\, Gx]$ such that $\mathit{StronglyEssential}(F, x)$ and $\mathit{Necessary}(G, x)$. Then we could require that the nature of Socrates be defined only in terms of the strongly essential properties which aren't vacuously strongly essential to x. But I think there is no need to do this without a more explicit expression of what Fine's intuition amounts to.

being a detective as one of Holmes' essential properties, and this shows that encoded properties are more closely connected with the identity of Holmes than are the properties he exemplifies necessarily.[18]

To take another intuitive example, consider the golden mountain, which we identified earlier as the object that encodes just the two properties of being golden and being a mountain. These are the two properties essential to this object, and no others. Of course, if you think that being concrete is essential to this object, then maybe you are representing the English description 'the golden mountain' as the (rather different) object that encodes all the properties implied by being golden and being a mountain. In either case, being abstract, being non-golden, not being a mountain, etc., are all properties that these objects necessarily exemplify but which should not be labeled as 'essential' to them.

Finally, consider any of the abstract objects we technically defined or identified: the number of ordinary Gs ($\#G$), the Form of G (Φ_G), the concept of Alexander (c_a), the extension of the concept G (ϵG), the empty set of Zermelo-Fraenkel set theory ($\emptyset_{\mathrm{ZF}}$), etc. In each of these cases, the present analysis suggests that the properties essential to the abstract object in question are the ones it encodes. For example, $\emptyset_{\mathrm{ZF}}$ encodes all and only the mathematical properties that it exemplifies according to the ZF. By claiming that $\emptyset_{\mathrm{ZF}}$'s encoded properties are its essential properties, we are claiming that its mathematical properties are the only ones that are constitutive of its nature as an object. Indeed, one way to interpret the comprehension principle for abstract objects is: it requires that every possible conception of an object define an abstract object with a distinct nature. Thus the properties involved in a given possible conception are the essential properties of the object that might be so conceived. Thus, there are no properties common to the essences of all abstract objects, since there are no properties common to all the various possible conceptions of objects.

Accordingly, we define, where x ranges over abstract objects:

[18]One might object that if being a detective is essential to Holmes, then we can't say, as we must surely be able to say, that Holmes might have been a mathematician instead of a detective. Indeed, there is a way to attribute the property of possibly being a mathematician to Holmes. For we may reasonably suppose that the story implies that Holmes might have embarked on various careers as a lad, and that he might have been a mathematician. If so, then possibly being a mathmatician will be one of Holmes' encoded properties! And in English, where predication is ambiguous, we would read this encoding claim as "Holmes might have been a mathematician".

$$Essential(F, x) \ =_{df} \ xF$$

Although this notion of essential property for abstract objects is not a modal notion, one can derive a modal claim from it, given the logic of encoding ($\Diamond xF \rightarrow \Box xF$). Thus, if x is abstract and F is essential to x, then necessarily F is essential to x.

Our definition of 'F is essential to x' has two consequences for abstract objects x, given the ambiguity in predication resolved by the distinction between Fx and xF. On the one hand, it gives us a sense in which essence implies necessity, since an abstract object x will encode F necessarily whenever F is essential to x. On the other hand, it gives us a sense in which essence doesn't imply necessity, for an an abstract object x need not, and typically does not, exemplify or necessarily exemplify F when F is essential to x. The properties of being a detective, being brilliant, living on 221B Baker Street, etc., are all essential to Sherlock Holmes, but this object does not exemplify these properties, and *a fortiori*, does not necessarily exemplify them.

5. Socrates and Singleton Socrates

On the present theory, the ordinary mathematical expression 'singleton Socrates' or 'the set consisting solely of Socrates' remains ambiguous until we supply a context. There is no singleton Socrates without some conception of sets and the membership relation, and various conceptions can be distinguished. So, to interpret 'singleton Socrates', we need to identify the conception of set in play (if only some minimal principles of modal set theory) or else suppose that some naive, pretheoretical, notion of extension is being used. Now, when Fine developed the counterexample to (E) concerning Socrates and singleton Socrates, he assumed a context of 'modal set theory' plus ordinary *Urelemente* (1994a, 4). But, of course, there are numerous ways of developing modal set theory. Some include all the principles of ZF, others don't; some include the Axiom of Choice, some don't. For each conception of set and the set membership relation, there will be a different singleton Socrates. Since Fine didn't specify the particular modal set theory in play, we shall treat his use of 'singleton Socrates' as a term of minimal modal set theory, i.e., the minimal set of principles required for Fine's counterexample. As we shall see, our analysis is going to require an appeal to the full theory of abstract objects, and not just to its underlying logic.

In what follows, then, we shall assume that some minimal modal theory of sets and urelements has been identified and we therefore refer to this theory as M (Modal Set Theory + Urelements). In M, 'Socrates' ('s') names one of the urelements, and 'singleton Socrates' ('$\{s\}$') abbreviates the proper description 'the unit set of Socrates'. Now, as we learned in Section 2.4, we import into $\mathcal{O}$ each theorem ϕ of M by (a) prefacing ϕ with the operator 'According to M' and (b) indexing the well-defined terms in ϕ to M. So if ϕ is a theorem of M, then the following becomes a theorem of $\mathcal{O}$:

$$\mathrm{M} \models \phi^*,$$

where ϕ^* is the result of indexing the terms of ϕ to M. Since many of the theorems of M involve the term '$\{s\}$', there will be corresponding sentences assertible in $\mathcal{O}$ involving the term '$\{s\}_{\mathrm{M}}$'. (The term 's' denoting Socrates is itself exempt from this indexing, since it denotes one of the urelements. We are not here trying to identify Socrates, but rather singleton Socrates. Whereas Socrates is an ordinary object which is given independently of any mathematical theory, singleton Socrates is not.)[19] Thus, there will be theorems of $\mathcal{O}$ which take the following form:

$$\mathrm{M} \models F\{s\}_{\mathrm{M}}$$

This distinguishes a group of properties F which are exemplified by the M-singleton of Socrates according to M. Now sentences of this form play a role in the following instance of the Theoretical Identification Principle:

$$\{s\}_{\mathrm{M}} = \imath x(A!x \,\&\, \forall F(xF \equiv \mathrm{M} \models F\{s\}_{\mathrm{M}}))$$

This asserts (henceforth suppressing the index to M) that the singleton of Socrates is (identical to) the abstract object which encodes exactly the properties F that the singleton of Socrates exemplifies according to M. Given this identification, the Equivalence Theorem (Section 2.4) and the definition of essential properties for abstract objects (Section 4), it follows that the properties essential to singleton Socrates are the properties it exemplifies according to M, since these are its encoded properties.

[19]Of course, the resource of $\mathcal{O}$ give us the means to talk about the mathematical object we might call 'the Socrates of M'. That mathematical object would be theoretically identified in the same way that we are identifying other mathematical objects. It will encode only the properties attributed to Socrates in M. But since that is not the object which plays are role in Fine's counterexample, we do not consider it here.

Now given the theorems of M that (i) Socrates is an element of singleton Socrates, (ii) singleton Socrates exemplifies the property of having Socrates as an element, and (iii) Socrates exemplifies the property of being an element of singleton Socrates, the following claims are theorems in $\mathcal{O}$:

$$\mathrm{M} \models s \in \{s\}_{\mathrm{M}} \tag{ξ_1}$$

$$\mathrm{M} \models [\lambda z\, s \in z]\{s\}_{\mathrm{M}} \tag{ξ_2}$$

$$\mathrm{M} \models [\lambda z\, z \in \{s\}_{\mathrm{M}}]s \tag{ξ_3}$$

(ξ_1) asserts that according to M, Socrates is an element of singleton Socrates; (ξ_2) that according to M, singleton Socrates exemplifies the property of having Socrates as an element; and (ξ_3) that according to M, Socrates exemplifies the property of being an element of singleton Socrates.

Notice that it follows from (ξ_2), given the Equivalence Theorem, that singleton Socrates encodes the property of having Socrates as an element:

$$\{s\}_{\mathrm{M}}[\lambda z\, s \in z] \tag{ρ}$$

And, finally, it follows from (ρ), by the definition of essential properties for abstract objects, that the property of having Socrates as a member ($[\lambda z\, s \in z]$) is essential to singleton Socrates.

It is worth remarking on the fact that we have now proved something which Fine takes as a premise in his counterexample to (E), namely, that the property of having Socrates as an element is essential to singleton Socrates. This premise falls out as a consequence of our theory of abstract objects and analysis of mathematical objects in terms of that theory. And given our discussion about the properties that abstract objects exemplify (Section 2), it should be clear that the mathematical properties singleton Socrates encodes are even more central to its identity than the properties it necessarily exemplifies. Though singleton Socrates necessarily exemplifies the modal negations of concreteness-entailing properties, these are not part of its nature.

By contrast, given that Socrates is an ordinary object, Socrates himself is governed by the axiom that ordinary objects (necessarily) do not encode properties: $O!x \rightarrow \Box\neg\exists F(xF)$. So it is provable that Socrates encodes no properties, and *a fortiori*, does *not* encode the property of being an element of singleton Socrates, i.e., $\neg s[\lambda z\, z \in \{s\}_{\mathrm{M}}]$. Nothing about

Socrates follows from either the Theoretical Identification Principle or the Equivalence Theorem given (ξ_3), since those principles don't apply to Socrates. Moreover, we can't abstract out any properties of Socrates in virtue of the properties exemplified by singleton Socrates according to M, or in virtue of the properties Socrates himself exemplifies according to M, or in virtue of properties encoded by the singleton of Socrates. In particular, since none of the following expressions are well-formed, none follow from (ξ_1), (ξ_2), (ξ_3) and (ρ), respectively, by λ-conversion:

$$[\lambda y\, \mathrm{M} \models y \in \{s\}_{\mathrm{M}}]s$$

$$[\lambda y\, \mathrm{M} \models [\lambda z\, y \in z]\{s\}_{\mathrm{M}}]s$$

$$[\lambda y\, \mathrm{M} \models [\lambda z\, z \in \{s\}_{\mathrm{M}}]y]s$$

$$[\lambda y\, \{s\}_{\mathrm{M}}[\lambda z\, y \in z]]s$$

In each case, the λ-expressions fail the restrictions banishing encoding subformulas. Socrates has no new properties in virtue of our analysis of singleton Socrates.

And, finally, we may consistently assert that the property of being a member of singleton Socrates is not essential to Socrates in any of the senses defined in Section 3:

$$\neg \mathit{Necessary}([\lambda z\, z \in \{s\}_{\mathrm{M}}], s)$$

$$\neg \mathit{WeaklyEssential}([\lambda z\, z \in \{s\}_{\mathrm{M}}], s)$$

$$\neg \mathit{StronglyEssential}([\lambda z\, z \in \{s\}_{\mathrm{M}}], s)$$

Consider the reasons for asserting the first, which amounts to the claim: $\neg\Box[\lambda z\, z \in \{s\}_{\mathrm{M}}]s$. Recall that in Section 2.4 we briefly touched upon the fact (in connection with $\emptyset_{\mathrm{ZF}}$) that an unadorned claim such as $s \in \{s\}_{\mathrm{M}}$ is neither true nor assertible in $\mathcal{O}$; indeed we asserted the negations of such claims. (Although there is an encoding reading of the ordinary M-claim "Socrates is an element of singleton Socrates" which preserves its truth and necessary truth, the exemplification reading, unprefixed by the theory operator, we claim to be false.) Consequently, if we assert in our theory that:

$$\neg\, s \in \{s\}_{\mathrm{M}},$$

then it follows that

$$\neg\Box\, s \in \{s\}_{\mathrm{M}}$$

from which in turn it follows, by λ-conversion, that

$$\neg\Box[\lambda z\; z \in \{s\}_{\mathrm{M}}]s$$

So it is not the case that Socrates necessarily exemplifies the property of being an element of singleton Socrates. Indeed, it is consistent with our theory to claim not only that $s \in \{s\}_{\mathrm{M}}$ is false but that it is necessarily false. (The necessary truths of mathematics are the encoding readings of ordinary mathematical claims, as outlined in some detail in Zalta 2000a.) Thus, it is necessary that Socrates fails to exemplify the property of being an element of singleton Socrates. And so we may, conclude *a fortiori* that the property of being an element of singleton Socrates is neither weakly nor strongly essential to Socrates. These are the second and third formal claims displayed at the beginning of this paragraph.

Clearly, the above facts establish an asymmetry between Socrates and singleton Socrates: we can *prove* that it is essential to singleton Socrates that it has Socrates as an element (in the sense of 'essential' appropriate to abstract objects), and consistently maintain that it is not essential to Socrates (in any of the senses of 'essential' appropriate to ordinary objects) that he is an element of singleton Socrates. So we may say, with Fine:

> . . . it lies in the nature of the singleton [of Socrates] to have Socrates as a member even though it does not lie in the nature of Socrates to belong to the singleton (1994a, 5)

But we account for the asymmetry on theoretical grounds which conceptualize (the natures of) abstract objects and ordinary objects in fundamentally different ways. Though a discussion of the issue will not be undertaken here, this fundamental asymmetry applies (though in a rather different guise) even if we analyze Fine's counterexample concerning 'singleton Socrates' not in the context of a modal set theory, but rather in terms of the extension of the concept *being identical*$_E$ *to Socrates*.[20]

[20] The extensions of concepts were defined in Section 2.4. In particular, $\epsilon[\lambda x\, x =_E s]$ is an abstract object that encodes all and only the properties materially equivalent to $[\lambda x\, x =_E s]$. Recall also that $y \in x$ is *defined* for extensions, in Section 2.4, as: $\exists F(x = \epsilon F \;\&\; Fy)$. As such, when the defined identity sign in the definiens is replaced in terms of primitive notation, we see that no property of the form $[\lambda y\, y \in \epsilon F]$ is formulable in $\mathcal{O}$. So when we analyze 'singleton Socrates' as $\epsilon[\lambda x\, x =_E s]$, the present theory tells us that the questions (a) whether Socrates essentially has the property of being an element of singleton Socrates, and (b) whether singleton Socrates essentially has the property of having Socrates as an element, don't even arise. Though it is tempting to conclude that the matter ends with the metaphysical facts that Socrates and singleton Socrates have different natures, there are interesting questions that arise through further analysis of this example, such as whether the defined notion $y \in x$ leads to a necessary truth of the form $s \in \epsilon[\lambda x\, x =_E s]$.

It is in the nature of abstract objects both to encode and exemplify properties, while it is in the nature of ordinary objects only to exemplify their properties. The asymmetry in natures explains in part why the traditional definition (E) is too simplistic. Though Fine developed insightful counterexamples to (E), there is an equally coherent alternative to his diagnosis which invokes the simplest quantified modal logic and a theory of abstract objects grounded in a distinction concerning the way in which abstract and ordinary objects 'have' their properties.

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