The Metaphysics of Possibility Semantics

Uri Nodelman and Edward N. Zalta
Philosophy Department, Stanford University
{nodelman,zalta}@stanford.edu

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What are Possibilities in Possibility Semantics?

- Normally, a possibility is a proposition p such that $\Diamond p$
- But Humberstone 1981:

> Here we have a motivation for the pursuit of modal logic against a semantic background in which less determinate entities than possible worlds, things which I am inclined for want of a better word to call simply *possibilities*, are what sentences (or formulae) are true or false with respect to.

- Edgington 1985:

> ... we can understand talk about *possibilities*, or *possible situations* ... [P]ossibilities differ from possible worlds in leaving many details unspecified.

- See also: Humberstone 2011; van Benthem 1981, 2016; Holliday 2014, forthcoming; and Ding & Holliday 2020.

What Principles Govern Possibilities?

- Humberstone 1981, 318; van Benthem 1981, 3–4; 2016, 3–4; Holliday 2014, 3; forthcoming, 5, 15; and Ding & Holliday 2020, 155): [not necessarily independent]
    - *Ordering*: a *refinement relation* ($\trianglerighteq$) partially orders the possibilities.
    - *Persistence*: every proposition true in a possibility is true in every refinement of that possibility.
    - *Refinement*: if a possibility x doesn't determine the truth value of a proposition p, then (a) there is a possibility which is a refinement of x where p is true, and (b) there is a possibility which is a refinement of x where p is false.
    - *Cofinality*: if, for every possibility x' that is a refinement of possibility x there is a possibility x'' that refines x' and makes p true, then x makes p true.
    - *Negation*: the negation of p is true in a possibility x if and only if p fails to be true in every refinement of x.
    - *Conjunction*: the conjunction p *and* q is true in x if and only if both p and q are true in x.

Our Goals

- All of these authors take possibilities to be *primitive* entities *in the semantics* and stipulate that these are semantic principles that govern these entities.
- They then interpret a propositional (modal) language with various connectives in terms of the semantic domain of possibilities and the principles that govern them.
- Our goals: (1) define *possibilities* in OT, (2) *derive* the above principles as theorems, and thereby prove what others stipulate, and (3) develop a limitation for the *purely semantic* conception.
- Desideratum: Achieve (1) and (2) without assuming set theory or modeling possibilities as mathematical objects.
- Strategy: Identify possibilities in OT as *situations* that are *consistent* and *modally closed*.
- In OT, situations are not primitive, and we can derive the fact that they are partially ordered.

Basic Principles

- OT = 2nd-order, S5 QML (without identity) extended by atomic formula 'xF' ('x encodes F'), '$E!$', free logic for ι (rigid) and λ:
  - $O! =_{df} [\lambda x\, \Diamond E!x]$
  - $A! =_{df} [\lambda x\, \neg\Diamond E!x]$
  - $x = y \equiv_{df}$
    $(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$
  - $F = G \equiv_{df} \Box\forall x(xF \equiv xG)$
- Axiom: Ordinary objects don't encode properties:
  - $O!x \rightarrow \neg\exists F xF$
- Axiom: If x encodes F, then necessarily x encodes F:
  - $xF \rightarrow \Box xF$
- Axiom (Theorem): For any condition φ on properties, there is a (unique) abstract object that encodes just the properties satisfying φ:
  - $\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$, provided x isn't free in φ
  - $\vdash \exists! x(A!x \,\&\, \forall F(xF \equiv \varphi))$, provided x isn't free in φ
- Comprehension for n-ary relations is derived and identity for n-ary relations is defined.

Definitions and Basic Theorems About Situations

- Zalta 1993: Situations are abstracta that encode only propositional properties:
  - $Situation(x) \ \equiv_{df} \ A!x \ \& \ \forall F(xF \rightarrow \exists p(F = [\lambda y\, p]))$
- Situations are necessarily situations:
  - $\vdash Situation(x) \rightarrow \Box Situation(x)$

  We use $s, s', \ldots$ as *rigid*, restricted variables.
- *p is true in s* (i.e., *s makes p true*) iff s encodes *being such that p*:
  - $s \models p \ \equiv_{df} \ s[\lambda y\, p]$
- Situations are identical whenever they make the same propositions true:
  - $\vdash s = s' \equiv \forall p(s \models p \equiv s' \models p)$
- s is a *part of* s' iff s' makes true every proposition s makes true:
  - $s \trianglelefteq s' \ \equiv_{df} \ \forall p(s \models p \rightarrow s' \models p)$
- Parthood ($\trianglelefteq$) is provably reflexive, anti-symmetric, and transitive on the situations:
  - $\vdash s \trianglelefteq s'$

    $\vdash (s \trianglelefteq s' \ \& \ s' \neq s) \rightarrow \neg s' \trianglelefteq s$

    $\vdash s \trianglelefteq s' \ \& \ s' \trianglelefteq s'' \rightarrow s \trianglelefteq s''$

(Modal) Logic of Situations

- Zalta 1993: A situation is *actual* iff every proposition true in it is true:
  - $Actual(s) \ \equiv_{df} \ \forall p(s \models p \rightarrow p)$
- A *possible* situation is one that might be actual:
  - $Possible(s) \ \equiv_{df} \ \Diamond Actual(s)$
- A *consistent* situation is one in which no proposition and its negation are both true:
  - $Consistent(s) \ \equiv_{df} \ \neg\exists p(s \models p \ \& \ s \models \neg p)$
- $\vdash Possible(s) \rightarrow Consistent(s)$, but the converse doesn't hold.
- Usual definition of necessary implication and equivalence:
  - $\varphi \Rightarrow \psi \equiv_{df} \Box(\varphi \rightarrow \psi)$
  - $\varphi \Leftrightarrow \psi \equiv_{df} \varphi \Rightarrow \psi \ \& \ \psi \Rightarrow \varphi$
- Truth in *s* not subject to modal distinctions:
  - $\vdash s \models p \Leftrightarrow \Box s \models p$
  - $\vdash \Diamond s \models p \Leftrightarrow s \models p$

Possible World Theory

- Zalta 1993: A *possible world* is any situation s that might be such that all and only true propositions are true in s:

  $$PossibleWorld(s) \equiv_{df} \Diamond\forall p(s \models p \equiv p)$$

  Given our convention, the subformula $s \models p \equiv p$ is to be parsed as $(s \models p) \equiv p$.
- The basic principles of possible world theory are derivable from the definition of *possible world* given above (Zalta 1993, 414–419). These include formal versions of the following principles:
  - Every possible world is maximal, consistent, and modally closed.
  - There is a unique actual world.
  - Possibly p iff there is a possible world in which p is true.
  - Necessarily p iff p is true in every possible world.
- $\vdash PossibleWorld(s) \equiv Maximal(s)\ \&\ Possible(s)$

Identifying Situations Uniquely

- Comprehension for situations: for every condition on propositions, there is a unique situation that makes true all and only the proposition satisfying φ:
  - $\vdash \exists! s \forall p(s \models p \equiv \varphi)$, provided s isn't free in φ
- Canonical descriptions for situations are well-defined:
  - $\vdash \exists y(y = \iota s \forall p(s \models p \equiv \varphi))$
- If s is the situation that makes true just the propositions satisfying φ, then s makes true just the propositions satisfying φ:
  - $\vdash (s = \iota s' \forall p(s' \models p \equiv \varphi)) \rightarrow \forall p(s \models p \equiv \varphi)$, provided s' isn't free in φ and φ is modally collapsed

Modally Closed Situations

- A situation s is modally closed just in case it makes true every proposition p necessarily implied by s's being actual:
  - $ModallyClosed(s) \equiv_{df} \forall p((Actual(s) \Rightarrow p) \rightarrow s \models p)$
- If s is modally closed then if s makes p true and p necessarily implies q, then s makes q true:
  - $\vdash ModallyClosed(s) \rightarrow \forall p \forall q(s \models p \;\&\; (p \Rightarrow q) \rightarrow s \models q)$
- If s is modally closed and consistent, then s is possible:
  - $\vdash (ModallyClosed(s) \;\&\; Consistent(s)) \rightarrow Possible(s)$
- If s is modally closed and p is necessary, then s makes p true:
  - $\vdash (ModallyClosed(s) \;\&\; \Box p) \rightarrow s \models p$

Definition of a Possibility

- A possibility is a situation that is consistent and modally closed:
  - $\mathit{Possibility}(s) \equiv_{df} \mathit{Consistent}(s) \;\&\; \mathit{ModallyClosed}(s)$
  - Cf. $\mathit{Possible}(s) \equiv_{df} \Diamond \mathit{Actual}(s)$, i.e., $\Diamond\forall p(s \models p \rightarrow p)$
- Possible worlds are possibilities:
  - $\vdash \mathit{Possibility}(w)$ (since w is modally closed and consistent)
- A possibility is necessarily a possibility:
  - $\vdash \Box\forall s(\mathit{Possibility}(s) \rightarrow \Box\mathit{Possibility}(s))$

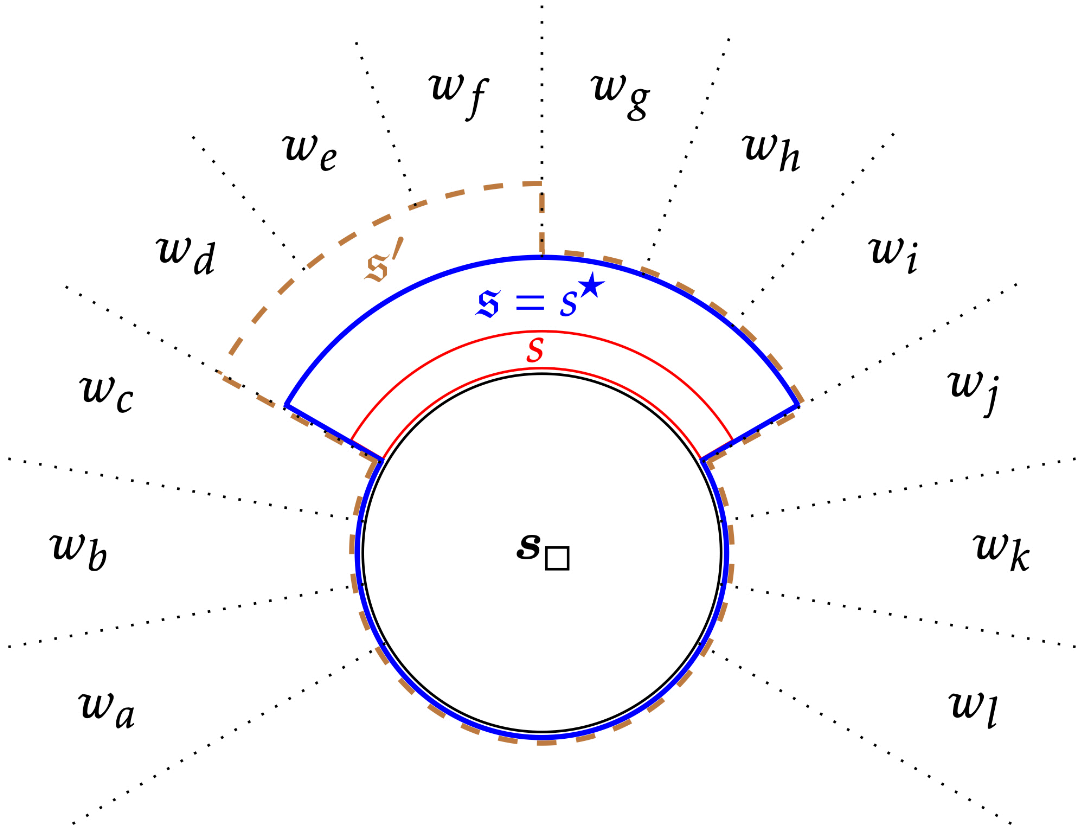
  In what follows we use $\mathfrak{s}, \mathfrak{s}', \ldots$ as rigid restricted variables ranging over possibilities.
- Possibilities are possible:
  - $\vdash \mathit{Possible}(\mathfrak{s})$, i.e., $\vdash \mathit{Possibility}(s) \rightarrow \mathit{Possible}(s)$

  (Expand the definition and apply a previous theorem.)
- A situation is possible just in case it is part of some possible world:
  - $\vdash \mathit{Possible}(s) \equiv \exists w(s \trianglelefteq w)$
- Possibilities are therefore parts of some possible world:
  - $\vdash \exists w(\mathfrak{s} \trianglelefteq w)$

Absolute Necessity, Possibilities, and Gaps

- s has a *gap on p* if s makes neither p nor $\overline{p}$ (i.e., $\neg p$) true:
  - $GapOn(s,p) \equiv_{df} \neg s \models p \;\&\; \neg s \models \overline{p}$,
- "Absolute necessity": $\boldsymbol{s}_{\Box} =_{df} \imath s \forall p(s \models p \equiv \Box p)$
- Absolute necessity has a gap on contingent propositions:
  $\vdash Contingent(p) \equiv GapOn(\boldsymbol{s}_{\Box}, p)$
- Absolute necessity is a possibility: $\vdash Possibility(\boldsymbol{s}_{\Box})$
- No proper part of absolute necessity is a possibility:
  $\vdash \forall s((s \trianglelefteq \boldsymbol{s}_{\Box} \;\&\; s \neq \boldsymbol{s}_{\Box}) \rightarrow \neg Possibility(s))$
- Every possibility is a refinement of absolute necessity:
  $\vdash \forall \mathfrak{s}(\mathfrak{s} \trianglerighteq \boldsymbol{s}_{\Box})$
- Possibilities and Gaps:
  - If a possibility has a gap on p, p is contingent:
    $\vdash GapOn(\mathfrak{s}, p) \rightarrow Contingent(p)$
  - If a possibility has a gap on p, it has a gap on $\neg p$:
    $\vdash \forall p(GapOn(\mathfrak{s}, p) \rightarrow GapOn(\mathfrak{s}, \neg p))$

A Picture



$s_\square$	=	the smallest possibility ('absolute necessity')
s	=	a *possible* situation
$s^\star$	=	the smallest possibility $\mathfrak{s}$ that contains s
		$\iota s' \forall p(s' \models p \equiv (Actual(s) \Rightarrow p))$
$\mathfrak{s}'$	=	a refinement of $\mathfrak{s}$

What We Have to Show

- We have to show:
  - $\vdash$ Ordering Principle
  - $\vdash$ Persistence Principle
  - $\vdash$ Refinability Principle
  - $\vdash$ Cofinality Principle
  - $\vdash$ Negation Principle
  - $\vdash$ Conjunction Principle

The Ordering Principle

- Definition: a situation s' *contains* situation s, written $s' \trianglerighteq s$, just in case s is a part of s':

  $$s' \trianglerighteq s \equiv_{df} s \trianglelefteq s'$$

- When the situations are possibilities, we read $\mathfrak{s}' \trianglerighteq \mathfrak{s}$ as: $\mathfrak{s}'$ is a *refinement of* $\mathfrak{s}$.
- Since $\trianglelefteq$ is reflexive, anti-symmetric, and transitive on the situations, it follows that *refinement of* is reflexive, anti-symmetric, and transitive on the possibilities:

  (a) $\vdash \mathfrak{s} \trianglerighteq \mathfrak{s}$

  (b) $\vdash (\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \neq \mathfrak{s}) \rightarrow \neg \mathfrak{s} \trianglerighteq \mathfrak{s}'$

  (c) $\vdash (\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \ \& \ \mathfrak{s}' \trianglerighteq \mathfrak{s}) \rightarrow \mathfrak{s}'' \trianglerighteq \mathfrak{s}$

- These facts validate the principle of *Ordering*; cf. Humberstone 1981 (318); 2011 (899); van Benthem 1981 (3); 2016 (3); Holliday 2014 (3); Ding & Holliday 2020 (155); and Holliday forthcoming (Definition 2.1 and 2.21).

The Persistence Principle

- Humberstone: where π is a proposition, X and Y are possibilities, $\geq$ is the refinement condition corresponding to $\trianglerighteq$, and $V(\pi, X)$ is the truth-value of π with respect to X (1981, 318):
  - If $V(\pi, X)$ is defined and $Y \geq X$, then $V(\pi, Y) = V(\pi, X)$

  "Further delimitation of a possible state of affairs should not reverse truth-values, but only reduce indeterminancies" (1981, 318).
- In OT, this Persistence Principle can be represented as the *theorem* that if a proposition p is true in a possibility $\mathfrak{s}$ and $\mathfrak{s}'$ is a refinement of $\mathfrak{s}$, then p is true in $\mathfrak{s}'$:
  - $\vdash (\mathfrak{s} \models p \ \& \ \mathfrak{s}' \trianglerighteq \mathfrak{s}) \rightarrow \mathfrak{s}' \models p$

  Cf. van Benthem 1981, 3 ('Heredity'), 2016, 3; Restall 2000, Definition 1.2 (Heredity Condition); Holliday 2014, 315; forthcoming, 15; Berto 2015, 767 (HC); Berto & Restall 2019, 1128 (HC); and Ding & Holliday 2020, 155.
- Cf. Barwise 1989a (265):

$$Persistent(p) \equiv_{df} \forall s(s \models p \rightarrow \forall s'(s \trianglelefteq s' \rightarrow s' \models p))$$

The Modal Closure of a Situation: I

- The modal closure of s is the situation that makes true all and only those propositions p such that s's being actual necessarily implies p:
  - $s^{\star} =_{df} \iota s' \forall p(s' \models p \equiv (Actual(s) \Rightarrow p))$
- The modal closure of s makes p true iff s's being actual necessarily implies p:
  - $\vdash \forall p(s^{\star} \models p \equiv (Actual(s) \Rightarrow p))$
- A situation is a part of its modal closure:
  - $\vdash s \trianglelefteq s^{\star}$

Interlude: The p-Extension of a Situation

- The p extension of a situation s is that situation that makes all the propositions in s true and also makes q true:
  - $s^{+p} =_{df} \iota s' \forall q(s' \models q \equiv (s \models q \vee q = p))$
- The p-extension of s is a part of a possible world w iff s is a part of w and p is true in w
  - $\vdash s^{+p} \trianglelefteq w \equiv s \trianglelefteq w \ \& \ w \models p$
- p is true in every world of which s is a part iff s's being actual necessarily implies p
  - $\vdash \forall w(s \trianglelefteq w \rightarrow w \models p) \equiv (Actual(s) \Rightarrow p)$

The Modal Closure of a Situation: II

- A situation is a part of a possible world iff its modal closure is:
  - $\vdash s \trianglelefteq w \equiv s^{\star} \trianglelefteq w$
- A situation is possible iff its modal closure is:
  - $\vdash \mathit{Possible}(s) \equiv \mathit{Possible}(s^{\star})$
- The modal closure of a situation is modally closed:
  - $\vdash \mathit{ModallyClosed}(s^{\star})$

The Refinability Principle

- Humberstone (1981, 318) (T and F are truth-values):
  - For any π and any X, if $V(\pi, X)$ is undefined, then $\exists Y(Y \geq X$ with $V(\pi, Y) = T)$ and $\exists Z(Z \geq X$ with $V(\pi, Z) = F)$
- Use p for π, $\mathfrak{s}$ for X, and $GapOn(\mathfrak{s}, p)$ for $V(\pi, X)$ is undefined.
- Refinability: if $\mathfrak{s}$ has a gap on p, then there is a possibility that refines $\mathfrak{s}$ in which p is true and there is a possibility that refines $\mathfrak{s}$ in which $\neg p$ is true:

  $$GapOn(\mathfrak{s}, p) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models p) \,\&\, \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg p)$$

  Cf. Holliday 2014, 315; forthcoming, 15; and D&H 2020, 155.
- But this can be strengthened to a biconditional:

  $$GapOn(\mathfrak{s}, p) \equiv \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models p) \,\&\, \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \,\&\, \mathfrak{s}' \models \neg p)$$

Proof Sketch of Refinability

Proof Sketch: Let r be an arbitrary, but fixed, proposition.
($\rightarrow$) Since $GapOn(\mathfrak{s}, r)$ implies $GapOn(\mathfrak{s}, \neg r)$, it suffices to show only:

$$GapOn(\mathfrak{s}, r) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models r)$$

So assume $GapOn(\mathfrak{s}, r)$ and find a witness to $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models r)$. Consider $(\mathfrak{s}^{+r})^\star$; abbreviate this as $\mathfrak{s}^{+r\star}$. We have to show all of the following: (a) $\mathfrak{s}^{+r\star} \trianglerighteq \mathfrak{s}$, (b) $\mathfrak{s}^{+r\star} \models r$, and (c) $Possibility(\mathfrak{s}^{+r\star})$. And by definition, the last of the above requires us to show (d) $Consistent(\mathfrak{s}^{+r\star})$ and (e) $ModallyClosed(\mathfrak{s}^{+r\star})$. ...

($\leftarrow$) Assume: $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models r) \ \& \ \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models \neg r)$ (ϑ)
For reductio, suppose $\neg GapOn(\mathfrak{s}, r)$. Then either $\mathfrak{s} \models r$ or $\mathfrak{s} \models \neg r$. Wlog, suppose $\mathfrak{s} \models r$. By Persistence Principle, every refinement of s makes r true. So there can't be a refinement that makes $\neg r$ true, contradicting the right conjunct of (ϑ).

The Cofinality Principle

- Van Benthem (1981, 4; 2016, 3) adds the principle labeled *Cofinality*. In 2016, he formulates this principle as follows, where $P\mathbf{d}$ is any atomic fact and $\geq$ is the the partial order on possibilities:
  - If for all $v \geq w$, there exists a $u \geq v$ with $P\mathbf{d}$ true at u, then $P\mathbf{d}$ is already true at w.
- This can be derived, without restriction to 'atomic facts', as the theorem: if, for every possibility $\mathfrak{s}'$ that refines $\mathfrak{s}$, there is a possibility $\mathfrak{s}''$ that refines $\mathfrak{s}'$ in which p is true, then p is true in $\mathfrak{s}$:

  $$\vdash \forall \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \ \&\ \mathfrak{s}'' \models p)) \rightarrow \mathfrak{s} \models p$$
- Cf. Humberstone's (2011, 900) restatement of the Refinement Principle.
- The proof appeals to Refinability. But Refinability isn't implied by Cofinality unless the notion of *possibility* obeys the Negation Constraint:

  $$\mathfrak{s} \models \neg p \equiv \neg \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \&\ \mathfrak{s}' \models p)$$

  Ours does (next slide); others have to stipulate it.

Negation, Conjunction and Fundamental Theorems

- Humberstone 1981 (319–320) and 2011 (900) adds the Negation and Conjunction Principles.
- Negation Principle: the negation of p is true in $\mathfrak{s}$ if and only if p fails to be true in every refinement of s:

  $$\vdash \mathfrak{s} \models \neg p \equiv \forall \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \neg \mathfrak{s}' \models p)$$

- Conjunction Principle: the conjunction p *and* q is true in $\mathfrak{s}$ if and only if both p and q are true in $\mathfrak{s}$:

  $$\vdash \mathfrak{s} \models (p \,\&\, q) \equiv (\mathfrak{s} \models p \,\&\, \mathfrak{s} \models q)$$

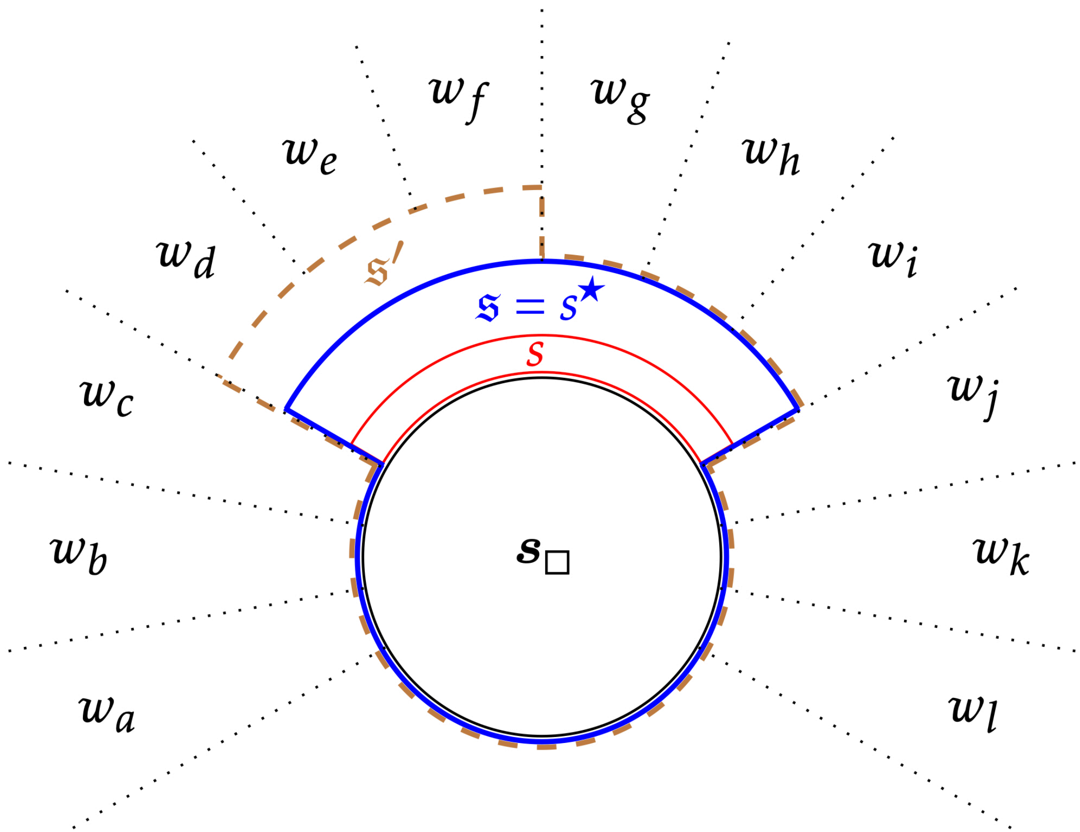
- Fundamental Theorems:
  - p is possibly true if and only if there is a possibility in which p is true:

    $$\vdash \Diamond p \equiv \exists \mathfrak{s}(\mathfrak{s} \models p)$$

  - p is necessarily true if and only if p is true in every possibility:

    $$\Box p \equiv \forall \mathfrak{s}(\mathfrak{s} \models p)$$

The Picture Again



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Consistency Doesn't Imply Possibility

- Fundamental Theorems guarantee that technically-conceived possibilities line up with the possibly true propositions: $\Diamond p \equiv \exists \mathfrak{s}(\mathfrak{s} \models p)$
- The right-to-left direction doesn't necessarily hold in Humberstone, van Benthem, Holliday, Ding & Holliday, and others (putting aside the fact that they don't have modal operators in the semantics).
- What is going wrong in the pure semantic study of possibilities: without a primitive modal operator, the closures of possibilities are deductive closures and not modal closures.
- $\vdash Possible(s) \equiv Consistent(s^{\star})$
- $Maximal(s)\ \&\ Consistent(s) \nvdash Possible(s)$
- $\vdash Maximal(s)\ \&\ Consistent(s^{\star}) \rightarrow PossibleWorld(s)$
- Absolute necessity makes true (encodes) more than just logical truths – it encodes all the metaphysically necessary truths.

Appendix: Proof of Situation Comprehension

$\vdash \exists s \forall p(s \models p \equiv \varphi)$, provided s isn't free in φ

Proof: We have to show: $\exists x(Situation(x) \,\&\, \forall p(x \models p \equiv \varphi))$, provided x isn't free in φ. Pick φ where x isn't free, and consider a property variable that isn't free in φ, say G. Let ψ be $\exists p(\varphi \,\&\, G = [\lambda z\, p])$. Then $\exists x(A!x \,\&\, \forall G(xG \equiv \psi))$, i.e.,

$$\exists x(A!x \,\&\, \forall G(xG \equiv \exists p(\varphi \,\&\, G = [\lambda z\, p])))$$

Suppose it is a. Then $A!a$ and $\forall G(aG \equiv \exists p(\varphi \,\&\, G = [\lambda z\, p]))$ (A)

Clearly, $Situation(a)$. So, by GEN, we only have to show $a \models p \equiv \varphi$. Instantiate $a[\lambda z\, p]$ into the following alphabetic variant of (A), where q is a variable that is substitutable for p, and doesn't occur free, in φ: $\forall G(aG \equiv \exists q(\varphi^q_p \,\&\, G = [\lambda z\, q]))$ (A′)

to obtain $a[\lambda z\, p] \equiv \exists q((\varphi^q_p)^{[\lambda z\, p]}_G \,\&\, [\lambda z\, p] = [\lambda z\, q])$. But since G isn't free in φ, $(\varphi^q_p)^{[\lambda z\, p]}_G$ is just φ^q_p.

(B) $a[\lambda z\, p] \equiv \exists q(\varphi^q_p \,\&\, [\lambda z\, p] = [\lambda z\, q])$

Now prove $a \models p \equiv \varphi$, using $p = q \equiv_{df} [\lambda z\, p] = [\lambda z\, q]$.

(See Zalta m.s., (484).)

Proof of Fact About Modally Closed Situations

$\vdash \mathit{ModallyClosed}(s) \rightarrow \forall p \forall q(s \models p \,\&\, (p \Rightarrow q) \;\rightarrow\; s \models q)$

Proof: Assume $\mathit{ModallyClosed}(s)$. So

(ϑ) $\forall q((\mathit{Actual}(s) \Rightarrow q) \rightarrow s \models q)$

We want to show: $(s \models p \,\&\, (p \Rightarrow q)) \rightarrow s \models q$. So assume:

(ξ) $s \models p \,\&\, p \Rightarrow q$

If we instantiate (ϑ) to q, it follows that:

(ζ) $(\mathit{Actual}(s) \Rightarrow q) \rightarrow s \models q$

So to show $s \models q$, it remains only to show $\mathit{Actual}(s) \Rightarrow q$. Use the Lemma: $\forall r(\Box s \models r \rightarrow \Box(\mathit{Actual}(s) \rightarrow r))$. Instantiate this to p:

$$\Box s \models p \rightarrow \Box(\mathit{Actual}(s) \rightarrow p)$$

But the first conjunct of (ξ) implies its own necessity, and $\Box s \models p$. Hence:

$$\Box(\mathit{Actual}(s) \rightarrow p)$$

So by definition of $\Rightarrow$:

(θ) $\mathit{Actual}(s) \Rightarrow p$

But (θ) and the second conjunct of (ξ) jointly imply $\mathit{Actual}(s) \Rightarrow q$. $\bowtie$

Proof of: ModallyClosed & Consistent Implies Possible

$\vdash (\mathit{ModallyClosed}(s) \,\&\, \mathit{Consistent}(s)) \to \mathit{Possible}(s)$

Proof: Assume $\mathit{ModallyClosed}(s)$ and $\mathit{Consistent}(s)$. Then we know, respectively:

(ϑ) $\forall q((\mathit{Actual}(s) \Rightarrow q) \to s \models q)$

(ξ) $\neg\exists p(s \models p \,\&\, s \models \neg p)$

For reductio, assume $\neg \mathit{Possible}(s)$. By definition and a Rule of Substitution, this entails $\neg\Diamond \mathit{Actual}(s)$. So $\Box\neg \mathit{Actual}(s)$ and, hence, $\neg \mathit{Actual}(s)$. By the definition of $\mathit{Actual}(s)$, this implies $\exists p(s \models p \,\&\, \neg p)$. Suppose p_1 is such a proposition, so that we know both $s \models p_1$ and $\neg p_1$. The former implies $\neg s \models \neg p_1$, by (ξ). Now, separately, if we instantiate (ϑ) to $\neg p_1$, then we also know:

(ζ) $(\mathit{Actual}(s) \Rightarrow \neg p_1) \to s \models \neg p_1$

But we've established $\neg s \models \neg p_1$, and so by (ζ), $\neg(\mathit{Actual}(s) \Rightarrow \neg p_1)$. By definition of ($\Rightarrow$) and a Rule of Substitution, it follows that $\neg\Box(\mathit{Actual}(s) \to \neg p_1)$. This implies $\Diamond\neg(\mathit{Actual}(s) \to \neg p_1)$, which in turn implies $\Diamond(\mathit{Actual}(s) \,\&\, p_1)$. But this last result implies $\Diamond \mathit{Actual}(s)$. Contradiction. $\bowtie$

Proof: Modally Closed s Make Necessary Truths True

- $\vdash (\mathit{ModallyClosed}(s) \mathbin{\&} \Box p) \rightarrow s \models p$
- Assume $\mathit{ModallyClosed}(s)$ and $\Box p$. The second implies $\Box(\mathit{Actual}(s) \rightarrow p)$. So $\mathit{Actual}(s) \Rightarrow p$, by definition. Then by definition of $\mathit{ModallyClosed}(s)$, $s \models p$. $\bowtie$

Possible Situations are Parts of Worlds

$\vdash Possible(s) \equiv \exists w(s \trianglelefteq w)$

Proof: $(\rightarrow)$ Assume $Possible(s)$. Then by definition, $\Diamond\forall p(s \models p \rightarrow p)$. By Fund. Thm., $\exists w(w \models \forall p(s \models p \rightarrow p))$. Suppose w_1 is such that $w_1 \models \forall p(s \models p \rightarrow p)$. Then by a theorem of world theory, we can export the quantifier:

$$\forall p(w_1 \models (s \models p \rightarrow p)) \qquad (\vartheta)$$

But since:

$$w_1 \models (s \models p \rightarrow p) \text{ is necessarily equivalent to } w_1 \models (s \models p) \rightarrow (w_1 \models p)$$

it follows that:

$$\forall p(w_1 \models (s \models p) \rightarrow (w_1 \models p)) \qquad (\xi)$$

It remains to show that w_1 is a witness to $\exists w(s \trianglelefteq w)$, and so we have to show that $\forall p(s \models p \rightarrow w_1 \models p)$. By GEN, assume $s \models p$. Then $\Box s \models p$. So by Fund. Thm., $\forall w(w \models (s \models p))$. Hence $w_1 \models (s \models p)$. Instantiating p into (ξ), it follows that $w_1 \models p$.
$(\leftarrow)$ Assume $\exists w(s \trianglelefteq w)$ and suppose w_1 is a witness, so that we know $s \trianglelefteq w_1$:

$$\forall p(s \models p \rightarrow w_1 \models p) \qquad (\zeta)$$

It suffices to show $w_1 \models Actual(s)$, since we can then conclude $\exists w(w \models Actual(s))$, then $\Diamond Actual(s)$, and then conclude $Possible(s)$, by definition. Since worlds are modally closed, it suffices to show $w_1 \models \forall p(s \models p \rightarrow p)$. But now it suffices to show $\forall p(w_1 \models (s \models p \rightarrow p))$. So by GEN, we show: $w_1 \models (s \models p \rightarrow p)$. Now it suffices to show $(w_1 \models (s \models p)) \rightarrow (w_1 \models p)$. So assume $w_1 \models (s \models p)$. Then by Fund. Thm., $\Diamond s \models p$. Hence, $s \models p$. So $w_1 \models p$, by (ζ). $\bowtie$

Absolute Necessity Has Gaps on Contingencies

$\vdash$ *Contingent*$(p) \equiv$ *GapOn*$(\boldsymbol{s}_\Box, p)$

Proof: ($\rightarrow$) Assume *Contingent*(p). Then $\Diamond p$ and $\Diamond\neg p$. Independently, by definition:

(ϑ) $\forall p(\boldsymbol{s}_\Box \models p \equiv \Box p)$

To show *GapOn*$(\boldsymbol{s}_\Box, p)$, we have to show both $\neg\boldsymbol{s}_\Box \models p$ and $\neg\boldsymbol{s}_\Box \models \overline{p}$. Since we know $\Diamond\neg p$, we know $\neg\Box p$. So if we instantiate (ϑ) to p, then it follows that $\neg\boldsymbol{s}_\Box \models p$. Since we also know $\Diamond p$, we know $\neg\Box\neg p$. So if we instantiate (ϑ) to $\neg p$, then it follows that $\neg\boldsymbol{s}_\Box \models \neg p$. So by logic, $\neg\boldsymbol{s}_\Box \models \overline{p}$.

($\leftarrow$) (Exercise) $\bowtie$

Absolute Necessity is a Possibility

$\vdash Possibility(\boldsymbol{s}_\Box)$

$\boldsymbol{s}_\Box$ is a canonically defined situation. So we have to show:

$$Consistent(\boldsymbol{s}_\Box) \quad \text{(A)}$$

$$ModallyClosed(\boldsymbol{s}_\Box) \quad \text{(B)}$$

Now we know:

$$\forall p(\boldsymbol{s}_\Box \models p \equiv \Box p) \quad (\vartheta)$$

(A) For reductio, suppose $\neg Consistent(\boldsymbol{s}_\Box)$, i.e., $\exists p(\boldsymbol{s}_\Box \models p \ \&\ \boldsymbol{s}_\Box \models \neg p)$. Let q_1 be such a proposition, so that we know $\boldsymbol{s}_\Box \models q_1$ and $\boldsymbol{s}_\Box \models \neg q_1$. By (ϑ), these imply, respectively, $\Box q_1$ and $\Box\neg q_1$. Contradiction, once the T schema is applied to both results.

(B) We have to show: $(Actual(s_\Box) \Rightarrow p) \rightarrow s_\Box \models p$, for arbitrary p. So assume:

$$Actual(s_\Box) \Rightarrow p \quad (\xi)$$

To show $s_\Box \models p$, it suffices, by (ϑ), to show $\Box p$. For reductio, suppose $\neg\Box p$, i.e., $\Diamond\neg p$. But our assumption (ξ) implies $\Box(Actual(s_\Box) \rightarrow p)$. So $\Box(\neg p \rightarrow \neg Actual(s_\Box))$. But from this and $\Diamond\neg p$, it follows by K$\Diamond$ that $\Diamond\neg Actual(s_\Box)$. By definition this implies $\Diamond\neg\forall q(s_\Box \models q \rightarrow q)$. So $\Diamond\exists q\neg(s_\Box \models q \rightarrow q)$, i.e., $\Diamond\exists q(s_\Box \models q \ \&\ \neg q)$. By BF$\Diamond$, $\exists q\Diamond(s_\Box \models q \ \&\ \neg q)$. Suppose p_1 is such a proposition, so that we know $\Diamond(s_\Box \models p_1 \ \&\ \neg p_1)$. Then $\Diamond(s_\Box \models p_1)$ and $\Diamond\neg p_1$. The latter implies $\neg\Box p_1$. The former implies $s_\Box \models p_1$. So by (ϑ), $\Box p_1$. Contradiction. $\bowtie$

No Proper Part of Absolute Necessity Is a Possibility

$\vdash \forall s((s \trianglelefteq \boldsymbol{s}_{\Box} \;\&\; s \neq \boldsymbol{s}_{\Box}) \rightarrow \neg Possibility(s))$

Proof: We have to show $(s \trianglelefteq \boldsymbol{s}_{\Box} \;\&\; s \neq \boldsymbol{s}_{\Box}) \rightarrow \neg Possibility(s)$. So assume $s \trianglelefteq \boldsymbol{s}_{\Box}$ and $s \neq \boldsymbol{s}_{\Box}$. The second implies
$\neg\forall p(s \models p \equiv \boldsymbol{s}_{\Box} \models p)$, i.e.,

$$\exists p((s \models p \;\&\; \neg \boldsymbol{s}_{\Box} \models p) \vee (\boldsymbol{s}_{\Box} \models p \;\&\; \neg s \models p))$$

Suppose q_1 is such a proposition, so that we know:

$$(s \models q_1 \;\&\; \neg \boldsymbol{s}_{\Box} \models q_1) \vee (\boldsymbol{s}_{\Box} \models q_1 \;\&\; \neg s \models q_1)$$

The left disjunct contradicts $s \trianglelefteq \boldsymbol{s}_{\Box}$ (exercise). So we know $\boldsymbol{s}_{\Box} \models q_1$ and $\neg s \models q_1$. The first implies $\Box q_1$, by a modally strict, immediate consequence of the definition of $\boldsymbol{s}_{\Box}$. Now, for reductio, suppose *Possibility*(s). Then, s is modally closed, by definition. So by a previous theorem (modally closed situations make necessary truths true), this last fact and $\Box q_1$ imply $s \models q_1$. Contradiction. $\bowtie$

Every Possibility Refines Absolute Necessity

$\vdash \forall \mathfrak{s}(\mathfrak{s} \trianglerighteq \boldsymbol{s}_{\Box})$

By GEN, it suffices to show $\mathfrak{s} \trianglerighteq \boldsymbol{s}_{\Box}$. So we have to show $\forall p(\boldsymbol{s}_{\Box} \models p \rightarrow \mathfrak{s} \models p)$. So, again, by GEN, we show $\boldsymbol{s}_{\Box} \models p \rightarrow \mathfrak{s} \models p$. Assume $\boldsymbol{s}_{\Box} \models p$. Then by definition of $\boldsymbol{s}_{\Box}$, it follows that $\Box p$. But since possibilities are modally closed and modally closed situations make necessary truths true, it follows that $\mathfrak{s} \models p$. $\bowtie$

Proof of: p is Contingent if $\mathfrak{s}$ has a Gap on p

$\vdash GapOn(\mathfrak{s}, p) \rightarrow Contingent(p)$

Assume $GapOn(\mathfrak{s}, p)$, i.e., both $\neg\mathfrak{s} \models p$ and $\neg\mathfrak{s} \models \overline{p}$. The latter implies $\neg\mathfrak{s} \models \neg p$. Now suppose $\neg Contingent(p)$, for reductio. Then by $\neg(\Diamond p \,\&\, \Diamond\neg p)$, i.e., $\Box\neg p \vee \Box p$. But both disjuncts lead to contradiction. If $\Box\neg p$, then $\mathfrak{s} \models \neg p$ (by a now familiar fact), which contadicts $\neg\mathfrak{s} \models \neg p$; if $\Box p$, then again by familiar reasoning, $\mathfrak{s} \models p$, which contradicts $\neg\mathfrak{s} \models p$. Contradiction full stop. $\bowtie$

Gap on p implies Gap on $\neg p$

$$\forall p(\mathit{GapOn}(\mathfrak{s}, p) \rightarrow \mathit{GapOn}(\mathfrak{s}, \neg p))$$

Proof. Assume $\mathit{GapOn}(\mathfrak{s}, p)$ and for reductio, $\neg\mathit{GapOn}(\mathfrak{s}, \neg p)$. Then, by definition, either $\mathfrak{s} \models \neg p$ or $\mathfrak{s} \models \neg\overline{p}$, i.e., either $\mathfrak{s} \models \neg p$ or $\mathfrak{s} \models \neg\neg p$. But the former contradicts $\mathit{GapOn}(\mathfrak{s}, p)$. The latter, by a consequence of the fact that $\mathfrak{s}$ is modally closed, implies $\mathfrak{s} \models p$, which also contradicts $\mathit{GapOn}(\mathfrak{s}, p)$. Contradiction. $\bowtie$

Proof of: s is a Part of its Modal Closure

$\vdash s \trianglelefteq s^\star$

Proof. $s^\star$ is clearly a situation and so it remains to show $\forall p(s \models p \rightarrow s^\star \models p)$. Proof strategy:

(A) Independently show $s \models p \rightarrow (Actual(s) \rightarrow p)$ is a modally strict theorem.

(B) Conclude from (A) that $\Box s \models p \rightarrow \Box(Actual(s) \rightarrow p)$, by Rule RM.

(C) Assume $s \models p$, for conditional proof. To show $s^\star \models p$, we have to show $Actual(s) \Rightarrow p$, by definition.

(D) Our assumption in (C) implies $\Box s \models p$.

(E) From (D) and (B) it follows that $\Box(Actual(s) \rightarrow p)$.

(F) Conclude $Actual(s) \Rightarrow p$, by definition of $\Rightarrow$.

Since (B) – (F) are straightforward, it remains to show (A). So assume both $s \models p$ and $Actual(s)$. The latter implies $\forall q(s \models q \rightarrow q)$, by definition. Instantiating this to p yields $s \models p \rightarrow p$. But then p, since $s \models p$ by assumption. $\bowtie$

Proof of Theorem

The p-extension of s is a part of w iff s is a part of w and w makes p true.

$\vdash s^{+p} \trianglelefteq w \equiv s \trianglelefteq w \;\&\; w \models p$

Proof. Clearly, s^{+p} and w are both situations. Then:

$$
\begin{aligned}
& s^{+p} \trianglelefteq w \\
& \quad \equiv \forall q(s^{+p} \models q \rightarrow w \models q) && \text{by definition } \trianglelefteq \\
& \quad \equiv \forall q((s \models q \vee q = p) \rightarrow w \models q) && \text{by definition of } s^{+}p \\
& \quad \equiv \forall q((s \models q \rightarrow w \models q) \;\&\; (q = p \rightarrow w \models q)) && \text{by logic} \\
& \quad \equiv \forall q(s \models q \rightarrow w \models q) \;\&\; \forall q(q = p \rightarrow w \models q) && \text{by logic} \\
& \quad \equiv \forall q(s \models q \rightarrow w \models q) \;\&\; w \models p && \text{by logic} \\
& \quad \equiv s \trianglelefteq w \;\&\; w \models p && \text{by definition } \trianglelefteq
\end{aligned}
$$

$\bowtie$

Proof of Theorem

- p is true in every world of which s is a part iff s's being actual necessarily implies p:

$$\vdash \forall w(s \trianglelefteq w \rightarrow w \models p) \equiv (Actual(s) \Rightarrow p)$$

- *Proof*:

$\forall w(s \trianglelefteq w \rightarrow w \models p)$

$\equiv \forall w \neg(s \trianglelefteq w \;\&\; \neg w \models p)$	by logic
$\equiv \neg\exists w(s \trianglelefteq w \;\&\; \neg w \models p)$	by logic
$\equiv \neg\exists w(s \trianglelefteq w \;\&\; w \models \neg p)$	by coherency of worlds
$\equiv \neg\exists w(s \trianglelefteq w \;\&\; w \models \overline{p})$	by logic
$\equiv \neg\exists w(s^{+\overline{p}} \trianglelefteq w)$	by previous theorem
$\equiv \neg Possible(s^{+\overline{p}})$	by by previous theorem
$\equiv \neg\Diamond Actual(s^{+\overline{p}})$	by definition
$\equiv \neg\Diamond\forall q(s^{+\overline{p}} \models q \rightarrow q)$	by definition
$\equiv \neg\Diamond\forall q((s \models q \vee q = \overline{p}) \rightarrow q)$	by definition $s^{+\overline{p}}$
$\equiv \neg\Diamond\forall q((s \models q \rightarrow q) \;\&\; (q = \overline{p} \rightarrow q))$	by logic
$\equiv \neg\Diamond(\forall q(s \models q \rightarrow q) \;\&\; \forall q(q = \overline{p} \rightarrow q))$	by logic
$\equiv \neg\Diamond(\forall q(s \models q \rightarrow q) \;\&\; \overline{p})$	by logic
$\equiv \neg\Diamond(\forall q(s \models q \rightarrow q) \;\&\; \neg p)$	by logic
$\equiv \neg\Diamond(Actual(s) \;\&\; \neg p)$	by definition
$\equiv \Box\neg(Actual(s) \;\&\; \neg p)$	by modal logic
$\equiv \Box(Actual(s) \rightarrow p)$	by logic
$\equiv Actual(s) \Rightarrow p$	by definition

$\bowtie$

s is part of w iff $s^\star$ is part of w

$\vdash s \trianglelefteq w \equiv s^\star \trianglelefteq w$

Proof. ($\rightarrow$) Assume $s \trianglelefteq w$, and for reductio, suppose $\neg s^\star \trianglelefteq w$. Then $\exists p(s^\star \models p \;\&\; \neg w \models p)$. Let p_1 be such a proposition, so that we know both $s^\star \models p_1$ and $\neg w \models p_1$. Independently, from the fact that $s^\star \models p_1$ it follows that $\mathit{Actual}(s) \Rightarrow p_1$, by definition. But the following is an instance of a previous theorem:

$$(s \trianglelefteq w \rightarrow w \models p_1) \equiv (\mathit{Actual}(s) \Rightarrow p_1)$$

It follows that $s \trianglelefteq w \rightarrow w \models p_1$. Hence, $w \models p_1$. Contradiction.

($\leftarrow$) Assume $s^\star \trianglelefteq w$. But we just established $s \trianglelefteq s^\star$. Since $s^\star$ and w are situations, it follows by transitivity of $\trianglelefteq$ that $s \trianglelefteq w$. ⋈

A Situation is Possible iff its Modal Closure Is

$\vdash Possible(s) \equiv Possible(s^{\star})$

Proof. $Possible(s)$

$\equiv \exists w(s \trianglelefteq w)$ by previous theorem

$\equiv \exists w(s^{\star} \trianglelefteq w)$ by previous theorem

$\equiv Possible(s^{\star})$ by previous theorem

$\bowtie$

The Modal Closure of s is Modally Closed

$\vdash \mathit{ModallyClosed}(s^\star)$

Proof. We have to show: $\forall p((\mathit{Actual}(s^\star) \Rightarrow p) \rightarrow s^\star \models p)$. Our strategy:

$$(\mathit{Actual}(s^\star) \Rightarrow p) \rightarrow (\mathit{Actual}(s) \Rightarrow p) \quad \text{(A)}$$

$$(\mathit{Actual}(s) \Rightarrow p) \rightarrow s^\star \models p \quad \text{(B)}$$

Since (B) is just an instance of a previous theorem. For (A), assume the antecedent:

$$\mathit{Actual}(s^\star) \Rightarrow p \quad (\vartheta)$$

Now, for reductio, assume $\neg(\mathit{Actual}(s) \Rightarrow p)$. Then, $\neg\Box(\mathit{Actual}(s) \rightarrow p)$. Since $\overline{p} = \neg p$, we have $\Diamond(\mathit{Actual}(s) \,\&\, \overline{p})$. But this contradicts (ϑ):

$$\begin{aligned}
&\Diamond(\mathit{Actual}(s) \,\&\, \overline{p}) \\
&\equiv \Diamond(\forall q(s \models q \rightarrow q) \,\&\, \overline{p}) && \text{by definition} \\
&\equiv \Diamond(\forall q(s \models q \rightarrow q) \,\&\, \forall q(q = \overline{p} \rightarrow q)) && \text{by logic} \\
&\equiv \Diamond\forall q((s \models q \rightarrow q) \,\&\, (q = \overline{p} \rightarrow q)) && \text{by logic} \\
&\equiv \Diamond\forall q((s \models q \vee q = \overline{p}) \rightarrow q) && \text{by logic} \\
&\equiv \Diamond\forall q(s^{+\overline{p}} \models q \rightarrow q) && \text{by definition} \\
&\equiv \Diamond\mathit{Actual}(s^{+\overline{p}}) && \text{by definition} \\
&\equiv \mathit{Possible}(s^{+\overline{p}}) && \text{by definition} \\
&\equiv \exists w(s^{+\overline{p}} \trianglelefteq w) && \text{by previous theorem} \\
&\equiv \exists w(s \trianglelefteq w \,\&\, w \models \overline{p}) && \text{by previous theorem} \\
&\equiv \exists w(s^\star \trianglelefteq w \,\&\, w \models \overline{p}) && \text{by previous theorem} \\
&\equiv \exists w(s^\star \trianglelefteq w \,\&\, w \models \neg p) && \text{by logic} \\
&\equiv \exists w(s^\star \trianglelefteq w \,\&\, \neg w \models p) && \text{by coherency of worlds} \\
&\equiv \exists w\neg(s^\star \trianglelefteq w \rightarrow w \models p) && \text{by logic} \\
&\equiv \neg\forall w(s^\star \trianglelefteq w \rightarrow w \models p) && \text{by logic} \\
&\equiv \neg(\mathit{Actual}(s^\star) \Rightarrow p) && \text{by previous theorem}
\end{aligned}$$

This last line contradicts (ϑ). ⋈

Proof of Cofinality

$\vdash \forall \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \;\&\; \mathfrak{s}'' \models p)) \rightarrow \mathfrak{s} \models p$

Proof. Assume:

$$\forall \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \;\&\; \mathfrak{s}'' \models p)) \qquad (\vartheta)$$

For reductio, assume $\neg \mathfrak{s} \models p$. Now by excluded middle, $\mathfrak{s} \models \neg p$ or $\neg \mathfrak{s} \models \neg p$. But both lead to contradiction. For suppose $\mathfrak{s} \models \neg p$. Since $\trianglerighteq$ is reflexive, we independently know $\mathfrak{s} \trianglerighteq \mathfrak{s}$. Then by (ϑ) it follows that $\exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s} \;\&\; \mathfrak{s}'' \models p)$. Suppose $\mathfrak{s}_1$ is such a possibility, so that we know both $\mathfrak{s}_1 \trianglerighteq \mathfrak{s}$ and $\mathfrak{s}_1 \models p$. Then $\mathfrak{s}_1 \trianglerighteq \mathfrak{s}$ and our assumption that $\mathfrak{s} \models \neg p$ imply $\mathfrak{s}_1 \models \neg p$, by the Persistence Principle. But this and $\mathfrak{s}_1 \models p$ contradict the consistency of $\mathfrak{s}_1$.

Now suppose $\neg \mathfrak{s} \models \neg p$. From this and our reductio assumption that $\neg \mathfrak{s} \models p$, it follows that *GapOn*$(\mathfrak{s}, p)$. So, by the Refinability Principle, it follows *a fortiori* that there is a refinement of $\mathfrak{s}$ in which $\neg p$ is true: $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \;\&\; \mathfrak{s}' \models \neg p)$. Suppose $\mathfrak{s}_2$ is such a possibility, so that we know both $\mathfrak{s}_2 \trianglerighteq \mathfrak{s}$ and $\mathfrak{s}_2 \models \neg p$. The former and (ϑ) imply $\exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s}_2 \;\&\; \mathfrak{s}'' \models p)$. Suppose $\mathfrak{s}_3$ is such a possibility, so that we know both $\mathfrak{s}_3 \trianglerighteq \mathfrak{s}_2$ and $\mathfrak{s}_3 \models p$. But $\mathfrak{s}_3 \trianglerighteq \mathfrak{s}_2$ and the assumption that $\mathfrak{s}_2 \models \neg p$ jointly imply $\mathfrak{s}_3 \models \neg p$, by the Persistence Principle. But now we've contradicted the consistencty of $\mathfrak{s}_3$. $\bowtie$

Proof of Fundamental Theorem

$\vdash \Diamond p \equiv \exists \mathfrak{s}(\mathfrak{s} \models p)$

($\rightarrow$) Assume $\Diamond p$. Then by the fundamental theorem of world theory, we know $\exists w(w \models p)$. Let w_1 be such a possible world, so that we know $w_1 \models p$. But possible worlds are possibilities i.e., $Possibility(w_1)$. Hence $\exists \mathfrak{s}(\mathfrak{s} \models p)$.

($\leftarrow$) Assume $\exists \mathfrak{s}(\mathfrak{s} \models p)$. Suppose $\mathfrak{s}_1$ is such a possibility, so that we know $\mathfrak{s}_1 \models p$. Suppose, for reductio, that $\neg\Diamond p$. Then $\Box\neg p$. So $\boldsymbol{s}_\Box \models \neg p$, by an immediate consequence of the definition of $\boldsymbol{s}_\Box$. But by a previous theorem and the definition of $\trianglerighteq$, we know that $\boldsymbol{s}_\Box$ is a part of $\mathfrak{s}_1$. So by definition of $\trianglelefteq$, $\mathfrak{s}_1 \models \neg p$, which contradicts the consistency of $\mathfrak{s}_1$. $\bowtie$

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