Seminar on Axiomatic Metaphysics
Lecture 2
An Exact Science

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Second-Order Logic With Propositions

- In second-order language, we have n-place relations ($F^n, G^n, \ldots$), n-place exemplification predication ($F^n x_1 \ldots x_n$), and Comprehension:
  $\exists F^n \forall x_1 \ldots \forall x_n (F^n x_1 \ldots x_n \equiv \varphi)$,
  where φ has no free F^ns or encoding subformulas.
- 2nd-order logic allows for 0-place relations: $F^0, G^0, \ldots$
  (Abbreviations: $p, q, r, \ldots$)
- Comprehension: $\exists p(p \equiv \varphi)$, where φ has no free ps or encoding subformulas:
  - $\exists p(p \equiv \neg Pa)$
  - $\exists p(p \equiv Pa \,\&\, Qb)$
  - $\exists p(p \equiv \forall y My)$
- A Simple Derivation:
  1. $\exists p(p \equiv \neg Fx)$ — Instance of 0-place Comprehension
  2. $\forall x \exists p(p \equiv \neg Fx)$ — By GEN on (1)
  3. $\forall F \forall x \exists p(p \equiv \neg Fx)$ — By GEN on (2)
- Propositional properties exist:
  - $\exists F \forall x(Fx \equiv p)$ — Instance of 1-place Comprehension
  - $\forall p \exists F \forall x(Fx \equiv p)$ — By GEN on an instance

The Simplest Quantified Modal Logic

- Simplest S5 Modal Logic (only necessary truths are axioms):
  - Language includes the formulas $\Box\varphi$, where φ is any formula
  - Add the definition: $\Diamond\varphi =_{df} \neg\Box\neg\varphi$
  - Assert propositional axioms:
    - K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
    - T: $\Box\varphi \to \varphi$
    - 5: $\Diamond\varphi \to \Box\Diamond\varphi$
  - No special quantifier axioms needed
  - Rule of Necessitation: from φ, infer $\Box\varphi$
- Theorems:
  - B: $\varphi \to \Box\Diamond\varphi$
  - 4: $\Box\varphi \to \Box\Box\varphi$
  - Derived Rule: If $\vdash \varphi \to \psi$, then $\vdash \Box\varphi \to \Box\psi$
  - Derived Rule: If $\vdash \Diamond\varphi \to \psi$, then $\vdash \varphi \to \Box\psi$
  - 1st-, 2nd-order Barcan Formula: $\forall\alpha\Box\varphi \to \Box\forall\alpha\varphi$
  - 1st-, 2nd-order Converse BF: $\Box\forall\alpha\varphi \to \forall\alpha\Box\varphi$
  - Necessary Existence: $\forall\alpha\Box\exists\beta(\beta = \alpha)$
- Interpretation: fixed domains, no accessibility relation

Actuality Operator

- Where φ is any formula, so is $\mathcal{A}\varphi$
- Semantics requires a distinguished actual world (w_0):
  $\mathcal{A}\varphi$ is true w.r.t. w iff φ is true at w_0
- The Logic of Actuality:

  Many of the principles govern interaction of $\mathcal{A}$ with the other connectives.

  In a modal context, there are subtleties to keep in mind

  Two key principles

  ⋆Axiom 1: $\mathcal{A}\varphi \rightarrow \varphi$

  Axiom 6: $\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi$
- It follows from ⋆Axiom 1 that $\varphi \rightarrow \mathcal{A}\varphi$.
- In a modal context, Rule RN would let you infer $\Box(\mathcal{A}\varphi \rightarrow \varphi)$ from ⋆Axiom 1. But this isn't valid!
- ⋆Axiom 1 is a contingent logical truth, i.e., not necessary.
- We will mark any theorem derived from ⋆Axiom 1 with an asterisk, to indicate that RN can't be applied to such theorems!

Classical (Relational) λ-Calculus

- λ-expressions $[\lambda x_1 \ldots x_n\, \varphi]$: being objects $x_1, \ldots, x_n$ such that φ.
- So λ-expressions are interpreted *relationally*.
- Examples: $[\lambda x\, \neg Rx]$, $[\lambda x\, Px \,\&\, Qx]$, etc.
- In the standard λ-calculus, all λ-expressions denote and the background is classical logic.
- (By contrast, in object theory, some λ-expressions fail to denote, e.g., $[\lambda x\, \exists F(xF \,\&\, \neg Fx)]$. And we'll use a free logic. More later.)
- Three main principles:
  - λ- or β-conversion: $[\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi$
    - $[\lambda x\, \neg Rx]x \equiv \neg Rx$
    - $[\lambda x\, Px \,\&\, Qx]x \equiv Px \,\&\, Qx$
  - α-conversion: $[\lambda x_1 \ldots x_n\, \varphi] = [\lambda x_1 \ldots x_n\, \varphi]'$, for alphabetic variants
    - $[\lambda x\, \neg Fx] = [\lambda z\, \neg Fz]$
  - η-Conversion: $[\lambda x_1 \ldots x_n\, F^n x_1 \ldots x_n] = F^n$, for *elementary* λ-expressions.
    - $[\lambda xy\, Rxy] = R$

Classical Theory of Definite Descriptions $\iota x\varphi$

- $\iota x\varphi$ is governed by a single axiom (Hintikka 1959):
  - $y = \iota x\varphi \equiv \forall x(\varphi \equiv x = y)$
- The standard logic of definite descriptions is a (negative) free logic:
  - $\forall\alpha\varphi \rightarrow (\exists\beta(\beta = \tau) \rightarrow \varphi^{\tau}_{\alpha})$, where τ is any (individual) term substitutable for α in φ.
  - $\exists\beta(\beta = \tau)$, where τ is a constant or a variable (i.e., not a description).
  - If an atomic formula with a description is true, then the description denotes, e.g., $G\iota xFx \rightarrow \exists y(y = \iota xFx)$.
- One can derive instances of the Russell principle:

$$G\iota xFx \equiv \exists x(Fx \,\&\, \forall y(Fy \rightarrow y = x) \,\&\, Gx)$$

- Our changes: (a) we'll replace $\exists\beta(\beta = \tau)$ with $\tau\downarrow$; (b) our free logic will apply to both descriptions and λ-expressions; (c) we'll interpret $\iota x\varphi$ rigidly, and use an actuality operator in the Hintikka axiom; and (d) we'll derive Russell's analysis, though it becomes a contingent logical theorem.

The Language of Object Theory and its Primitives Notions

- Object variables and constants: $x, y, z, \ldots \quad a, b, c, \ldots$
- Relation variables and constants: $F^n, G^n, H^n, \ldots;$
  $P^n, Q^n, R^n, \ldots$ (for $n \geq 0$); $p, q, r, \ldots$ (when $n = 0$)
- Distinguished 1-place relation: $E!$ (being concrete)
- Atomic formulas:
  $F^n x_1 \ldots x_n$ ($x_1, \ldots, x_n$ exemplify F^n)
  $x_1 \ldots x_n F^n$ ($x_1, \ldots, x_n$ encodes F^n)
- Complex Formulas: $\neg\varphi$, $\varphi \rightarrow \psi$, $\forall\alpha\varphi$, $\Box\varphi$, $\mathcal{A}\varphi$ (α any variable)
- Complex Terms:
  Descriptions: $\imath x\varphi$ (rigid)
  λ-expressions: $[\lambda x_1 \ldots x_n\, \varphi]$
  (interpreted relationally, not functionally)
- From these primitives, we'll define: Truth-values, Classes (Extensions of Properties), Numbers, Possible (Impossible) Worlds, Forms, Fictions, Leibnizian Concepts, and Senses.

A Semantics: If You Want One

- A Semantics for Second-order Object Theory

Modal Object Theory: A BNF for the Language

δ	primitive individual constants
ν	individual variables
Σ^n	primitive n-ary relation constants ($n \geq 0$)
Ω^n	n-ary relation variables ($n \geq 0$)
α	variables
κ	individual terms
Π^n	n-ary relation terms ($n \geq 0$)
φ	formulas
τ	terms

$$
\begin{array}{rcl}
\delta & ::= & a_1, a_2, \ldots \\
\nu & ::= & x_1, x_2, \ldots \\
(n \geq 0)\ \Sigma^n & ::= & P^n_1, P^n_2, \ldots \quad \text{(with } P^1_1 \text{ distinguished and written as } E!\text{)} \\
(n \geq 0)\ \Omega^n & ::= & F^n_1, F^n_2, \ldots \\
\alpha & ::= & \nu \mid \Omega^n\ (n \geq 0) \\
\kappa & ::= & \delta \mid \nu \mid \iota\nu\varphi \\
(n \geq 1)\ \Pi^n & ::= & \Sigma^n \mid \Omega^n \mid [\lambda\nu_1 \ldots \nu_n\ \varphi] \quad (\nu_1, \ldots, \nu_n \text{ are pairwise distinct}) \\
\varphi & ::= & \Sigma^0 \mid \Omega^0 \mid \Pi^n\kappa_1 \ldots \kappa_n\ (n \geq 1) \mid \kappa_1 \ldots \kappa_n\Pi^n\ (n \geq 1) \mid \\
 & & [\lambda\ \varphi] \mid (\neg\varphi) \mid (\varphi \rightarrow \varphi) \mid \forall\alpha\varphi \mid (\Box\varphi) \mid (\mathcal{A}\varphi) \\
\Pi^0 & ::= & \varphi \\
\tau & ::= & \kappa \mid \Pi^n \quad (n \geq 0)
\end{array}
$$

Definitions: Operators, Terms, Existence, Identity

- &, ∨, ≡, ∃, and ◇ are all defined in the usual way
- **Existence** (↓)

$$x{\downarrow} \equiv_{df} \exists F\, Fx$$

$$F^n{\downarrow} \equiv_{df} \exists x_1 \ldots \exists x_n (x_1 \ldots x_n F^n) \qquad (n \geq 1)$$

$$p{\downarrow} \equiv_{df} [\lambda x\, p]{\downarrow}$$

- $O! =_{df} [\lambda x\, \Diamond E!x]$ (‘ordinary’)
- $A! =_{df} [\lambda x\, \neg\Diamond E!x]$ (‘abstract’)
- **Identity** (=)

$$x = y \equiv_{df} (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

$$F^1 = G^1 \equiv_{df} \Box\forall x(xF^1 \equiv xG^1)$$

$$F^n = G^n \equiv_{df} \text{(where } n > 1\text{)}$$

$$\forall x_1 \ldots \forall x_{n-1}([\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, G^n y x_1 \ldots x_{n-1}] \,\&\, [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, G^n x_1 y x_2 \ldots x_{n-1}] \,\&\ldots\&\, [\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, G^n x_1 \ldots x_{n-1} y])$$

$$p = q \equiv_{df} [\lambda y\, p] = [\lambda y\, q]$$

Modal Object Theory: Axioms I

- A *closure* of a formula φ is the result of prefacing any string of $\forall\alpha$, $\Box$, or $\mathcal{A}$ to φ. We take the closures of all of the following:
- **Propositional Logic**: Classical Axioms
- **Predicate Logic** (free logic for complex terms):
  - $\forall\alpha\varphi \rightarrow (\tau\!\downarrow\ \rightarrow \varphi^{\tau}_{\alpha})$, provided τ is substitutable for α in φ
  - $\tau\!\downarrow$, provided τ is primitive constant, a variable, or λ-expression in which the λ doesn't bind a variable that occurs as a primary term in an encoding formula subterm of the matrix
  - $\forall\alpha(\varphi \rightarrow \psi) \rightarrow (\forall\alpha\varphi \rightarrow \forall\alpha\psi)$
  - $\varphi \rightarrow \forall\alpha\varphi$, provided α doesn't occur free in φ
  - $\Pi^n\kappa_1 \ldots \kappa_n \rightarrow (\Pi^n\!\downarrow\ \&\ \kappa_1\!\downarrow\ \&\ \ldots\ \&\ \kappa_n\!\downarrow) \quad (n \geq 0)$
    $\kappa_1 \ldots \kappa_n\Pi^n \rightarrow (\Pi^n\!\downarrow\ \&\ \kappa_1\!\downarrow\ \&\ \ldots\ \&\ \kappa_n\!\downarrow) \quad (n \geq 1)$
- **Substitution of Identicals** (unrestricted):
  $\alpha = \beta \rightarrow (\varphi \rightarrow \varphi')$, whenever β is substitutable for α in φ, and φ' is the result of replacing zero or more free occurrences of α in φ with occurrences of β

Modal Object Theory: Axioms II

- **Axioms for Actuality:**
  - $\mathcal{A}\varphi \rightarrow \varphi$ (★-axiom, only universal closures)
  - $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$
  - $\mathcal{A}(\varphi \rightarrow \psi) \equiv (\mathcal{A}\varphi \rightarrow \mathcal{A}\psi)$
  - $\mathcal{A}\forall\alpha\varphi \equiv \forall\alpha\mathcal{A}\varphi$
  - $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$
- **Axioms for Necessity:**
  - $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ (K axiom)
  - $\Box\varphi \rightarrow \varphi$ (T axiom)
  - $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ (5 axiom)
  - $\Diamond\exists x(E!x \;\&\; \neg\mathcal{A}E!x)$ (new)
- **Axioms for Necessity and Actuality:**
  - $\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi$
  - $\Box\varphi \equiv \mathcal{A}\Box\varphi$

Modal Object Theory: Axioms III

- **Axioms for Definite Descriptions:**
  - $y = \iota x\varphi \;\equiv\; \forall x(\mathcal{A}\varphi \equiv x = y)$
- **Axioms for Relations** (λ-Calculus for Relations):
  - $[\lambda v_1 \ldots v_n\, \varphi]\!\downarrow\; \rightarrow [\lambda v_1 \ldots v_n\, \varphi] = [\lambda v_1 \ldots v_n\, \varphi]'$ $\quad (n \geq 0)$
    ($[\lambda v_1 \ldots v_n\, \varphi]'$ an alphabetic variant)
  - $[\lambda x_1 \ldots x_n\, \varphi]\!\downarrow\; \rightarrow ([\lambda x_1 \ldots x_n\, \varphi]x_1 \ldots x_n \equiv \varphi)$ $\quad (n \geq 1)$
  - $[\lambda x_1 \ldots x_n\, F^n x_1 \ldots x_n] = F^n$ $\quad (n \geq 0)$
  - $([\lambda x_1 \ldots x_n\, \varphi]\!\downarrow \;\&\; \Box\forall x_1 \ldots \forall x_n(\varphi \equiv \psi)) \;\rightarrow\; [\lambda x_1 \ldots x_n\, \psi]\!\downarrow$ $\quad (n \geq 0)$
- **Axioms for Encoding:**
  - $x_1 \ldots x_n F^n \;\equiv$
    $x_1[\lambda y F^n y x_2 \ldots x_n] \;\&\; x_2[\lambda y F^n x_1 y x_3 \ldots x_n] \;\&\; \ldots \;\&\; x_n[\lambda y F^n x_1 \ldots x_{n-1} y]$
  - $xF \rightarrow \Box xF$
  - $O!x \rightarrow \neg\exists F\, xF$
  - $\exists x(A!x \;\&\; \forall F(xF \equiv \varphi))$, provided x doesn't occur free in φ

Modal Object Theory: Deductive System

- One rule of inference: Modus Ponens
- Two derivation systems: $\Gamma \vdash \varphi$ and $\Gamma \vdash_{\Box} \varphi$.
- Derive Rule GEN: If $\Gamma \vdash \varphi$, then $\Gamma \vdash \forall\alpha\varphi$, provided α doesn't occur free in any premise in Γ.
    - Proof by Induction on Length of Proof: Assume $\Gamma \vdash \varphi$ and α doesn't occur free in Γ.
    - Base Case: $\Gamma \vdash \varphi$ is one-element sequent. Then (A) φ is an axiom or (B) φ is a premise in Γ. (A) Then $\forall\alpha\varphi$ is an axiom, since we took the universal closures as axioms. Then $\Gamma \vdash \forall\alpha\varphi$, since an axiom follows from any set of premises. (B) Then α doesn't occur free in φ. By an axiom of predicate logic, i.e., $\varphi \rightarrow \forall\alpha\varphi$ (when α isn't free in φ), it follows that $\forall\alpha\varphi$. So the sequence $\varphi, \varphi \rightarrow \forall\alpha\varphi, \forall\alpha\varphi$ is a witness to $\Gamma \vdash \forall\alpha\varphi$ (every member of the sequence is either an axiom, a premise, or is a direct consequence of two previous members by MP).
    - Inductive case: (Exercise)
- RN is derived (next slide).

A Derivation of the Rule RN

- Rule RN (with premises): If $\Gamma \vdash_{\Box} \varphi$, then $\Box\Gamma \vdash \Box\varphi$, where $\Box\Gamma$ is obtained from Γ by putting a $\Box$ in front of every formula in Γ. (Exercise)
- Rule RN (no premises): If $\vdash_{\Box} \varphi$ (i.e., there is a proof of φ that doesn't appeal to any contingent $\star$axiom), then $\vdash_{\Box} \Box\varphi$ and $\vdash \Box\varphi$.
- *Proof.* Suppose we're given a proof of φ that doesn't appeal to any contingent axiom. We show by induction on the length of the proof that there is a (modally strict) proof of $\Box\varphi$. If the proof of φ is one line, φ must be a non-contingent axiom. So its modal closure $\Box\varphi$ is a necessary axiom, and hence $\vdash_{\Box} \varphi$. If the modally strict proof of φ is more than one line, then φ was derived by MP from previous lines ψ and $\psi \to \varphi$ by MP. Since the proof of φ is modally strict, we know $\vdash_{\Box} \psi$ and $\vdash_{\Box} (\psi \to \varphi)$. Hence, by the IH, $\vdash_{\Box} \Box\psi$ and $\vdash_{\Box} \Box(\psi \to \varphi)$. But then, since the K axiom is also a theorem, $\vdash_{\Box} \Box\varphi$.

Propositional and Predicate Logic

- All the usual theorems of propositional logic are preserved.
- Classical quantification theory holds for primitive constants and variables: if τ is a primitive constant or a variable, then $\forall\alpha\varphi \rightarrow \varphi^{\tau}_{\alpha}$.
- Every 0-place term and formula signifies a proposition: pick a variable ν that isn't free in Π^0. Then it is axiomatic that $[\lambda\nu\, \Pi^0]\!\downarrow$. So by definition, $\Pi^0\!\downarrow$, and since formulas are 0-place relation terms, $\varphi\!\downarrow$, for any φ.
- Logical existence is necessary: $\tau\!\downarrow \rightarrow \Box\tau\!\downarrow$
  - Note: This does not imply that $E!x \rightarrow \Box E!x$.
- Identity implies existence: $\tau = \sigma \rightarrow (\tau\!\downarrow \;\&\; \sigma\!\downarrow)$
- $[\lambda\, \varphi] = \varphi$ (substitute φ into 0-place η-Conversion)
- The Theory of Truth: $[\lambda\, \varphi] \equiv \varphi$

Theorems Governing Identity

- $\alpha = \alpha$
- Proof by cases:
  - Case 1: $x = x$
  - Case 2: $F^1 = F^1$
  - Case 3: $p = p$
  - Case 4: $F^n = F^n$
- $\alpha = \beta \rightarrow \beta = \alpha$
- $(\alpha = \beta \,\&\, \beta = \gamma) \rightarrow \alpha = \gamma$
- $\alpha = \beta \rightarrow \Box\alpha = \beta$
- Axioms of Free Logic Derived as Theorems:
  - $\tau\!\downarrow\; \equiv \exists\beta(\beta = \tau)$, provided that β doesn't occur free in τ
  - $\forall\alpha\varphi \rightarrow (\exists\beta(\beta = \tau) \rightarrow \varphi^{\tau}_{\alpha})$, provided τ is substitutable for α in φ and β doesn't occur free in τ
  - $\exists\beta(\beta = \tau)$, provided (a) τ is either a primitive constant, a variable (or a λ-expression compliant with axiom)
  - $(\Pi^n\kappa_1 \ldots \kappa_n \vee \kappa_1 \ldots \kappa_n\Pi^n) \rightarrow \exists\beta(\beta = \tau)$

Actuality and Descriptions

- $\star\vdash \varphi \rightarrow \mathcal{A}\varphi$ (Assume φ and, for reductio, $\neg\mathcal{A}\varphi$; so by logic of actuality, $\mathcal{A}\neg\varphi$; then $\star$-axiom $[\mathcal{A}\psi \rightarrow \psi]$ implies $\neg\varphi$. $\bowtie$)
- Rule of Actualization: If $\vdash \varphi$, then $\vdash \mathcal{A}\varphi$ and If $\vdash_{\Box} \varphi$, then $\vdash_{\Box} \mathcal{A}\varphi$
- Logical of Actuality: $\mathcal{A}$ distributes over conditionals, conjunctions, disjunctions; commutes with universal quantifier ($\mathcal{A}\exists\alpha\varphi \equiv \exists\alpha\mathcal{A}\varphi$); etc.
- Classical description theory is not modally strict:
  - $\star$Hintikka: $\star\vdash y\!=\!\imath x\varphi \;\equiv\; \varphi \,\&\, \forall x(\varphi \rightarrow x\!=\!y)$
  - $\star$Russell: $\star\vdash F\imath xGx \equiv \exists x(Gx \,\&\, \forall z(Gz \rightarrow z\!=\!x) \,\&\, Fx)$ (exercise)
  - $\star\vdash y\!=\!\imath x\varphi \rightarrow \varphi^{y}_{x}$
  - $\vdash_{\Box} y\!=\!\imath x\varphi \rightarrow \mathcal{A}\varphi^{y}_{x}$

Classical Quantified S5 Modal Logic

- Proof of K$\Diamond$ ($\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$): (exercise).
- Proof of T$\Diamond$ ($\varphi \rightarrow \Diamond\varphi$):
  - $\Box\neg\psi \rightarrow \neg\psi$ (instance of T); so $\neg\neg\psi \rightarrow \neg\Box\neg\psi$ (by contraposition); so $\psi \rightarrow \Diamond\psi$ (by propositional logic and and definitions)
- Proof of B ($\varphi \rightarrow \Box\Diamond\varphi$)
  - $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ (instance of 5 axiom); $\varphi \rightarrow \Diamond\varphi$ (instance of T$\Diamond$); $\varphi \rightarrow \Box\Diamond\varphi$ (by hypothetical syllogism)
- Proof of 4 ($\Box\varphi \rightarrow \Box\Box\varphi$):
  - $\Diamond\neg\psi \rightarrow \Box\Diamond\neg\psi$ (instance of 5 axiom); $\neg\Box\Diamond\neg\psi \rightarrow \neg\Diamond\neg\psi$ (contraposition); $\Diamond\Box\psi \rightarrow \Box\psi$ (by definitions); $\Box(\Diamond\Box\psi \rightarrow \Box\psi)$ (by RN); $\Box\Diamond\Box\psi \rightarrow \Box\Box\psi$ (by K axiom); $\Box\psi \rightarrow \Box\Diamond\Box\psi$ (instance of B); $\Box\psi \rightarrow \Box\Box\psi$ (hypothetical syllogism)
- Proof of B$\Diamond$ ($\Diamond\Box\varphi \rightarrow \varphi$)
  - $\neg\psi \rightarrow \Box\Diamond\neg\psi$ (instance of B); $\neg\Box\Diamond\neg\psi \rightarrow \neg\neg\psi$ (contraposition); $\Diamond\Box\psi \rightarrow \psi$ (by definitions and logic)

Barcan Formulas

- 1st-order Barcan Formula $\forall x \Box \varphi \to \Box \forall x \varphi$:
  1. $\forall x \Box \varphi \to \Box \varphi$ — quantifier axiom
  2. $\Box(\forall x \Box \varphi \to \Box \varphi)$ — from 1 by RN
  3. $\Box(\forall x \Box \varphi \to \Box \varphi) \to (\Diamond \forall x \Box \varphi \to \Diamond \Box \varphi)$ — theorem of S5
  4. $\Diamond \forall x \Box \varphi \to \Diamond \Box \varphi$ — from 2,3 by MP
  5. $\Diamond \Box \varphi \to \varphi$ — Lemma (B$\Diamond$)
  6. $\Diamond \forall x \Box \varphi \to \varphi$ — from 4,5 by logic
  7. $\forall x(\Diamond \forall x \Box \varphi \to \varphi)$ — from 6 by GEN
  8. $\forall x(\Diamond \forall x \Box \varphi \to \varphi) \to (\Diamond \forall x \Box \varphi \to \forall x \varphi)$ — quantifier theorem
  9. $\Diamond \forall x \Box \varphi \to \forall x \varphi$ — from 7,8 by MP
  10. $\forall x \Box \varphi \to \Box \forall x \varphi$ — from 9 by DR2
- 1st-order Converse Barcan Formula: $\Box \forall x \varphi \to \forall x \Box \varphi$
  1. $\forall x \varphi \to \varphi$ — quantifier axiom
  2. $\Box(\forall x \varphi \to \varphi)$ — from 1 by RN
  3. $\Box(\forall x \varphi \to \varphi) \to (\Box \forall x \varphi \to \Box \varphi)$ — Instance of K axiom
  4. $\Box \forall x \varphi \to \Box \varphi$ — from 2, 3 by MP
  5. $\forall x(\Box \forall x \varphi \to \Box \varphi)$ — from 4 by GEN
  6. $\forall x(\Box \forall x \varphi \to \Box \varphi) \to (\Box \forall x \varphi \to \forall x \Box \varphi)$ — quantifier theorem
  7. $\Box \forall x \varphi \to \forall x \Box \varphi$ — from 5,6 by MP

Unproblematic Modal Collapse

- A modal logic should not have, as theorems, $\varphi \rightarrow \Box\varphi$, or $\Diamond\varphi \rightarrow \varphi$, or $\Diamond\varphi \equiv \Box\varphi$, or $\varphi \equiv \Box\varphi$, for any contingent formula φ.
- This is modal collapse (i.e., modal distinctions fail).
- However, some necessary claims, like identity claims, are modally collapsed.
- We've already seen:
  - $\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi$ (axiom)
  - $xF \rightarrow \Box xF$ (axiom)
  - $\tau\downarrow \rightarrow \Box\tau\downarrow$ (theorem)
  - $\alpha = \beta \rightarrow \Box\alpha = \beta$ (theorem)
- But we also have:
  - $O!x \rightarrow \Box O!x$
  - $A!x \rightarrow \Box A!x$
- Finally, some consequences of modal collapse:
  - $\Box(\varphi \rightarrow \Box\varphi) \rightarrow (\Diamond\varphi \rightarrow \Box\varphi)$
  - $\Box(\varphi \rightarrow \Box\varphi) \rightarrow (\neg\Box\varphi \equiv \Box\neg\varphi)$

Derivation: Comprehension for Relations

- Show: $\exists F^n \Box \forall x_1 \ldots \forall x_n (F x_1 \ldots x_n \equiv \varphi)$, $(n \geq 1)$
  provided F^n doesn't occur free in φ and none of $x_1, \ldots, x_n$ occur free as primary terms in an encoding formula subterm of φ.
- *Proof.*
  1. $[\lambda x_1 \ldots x_n\ \varphi]\!\downarrow$ — for appropriate φ, by an axiom of free logic
  2. $[\lambda x_1 \ldots x_n\ \varphi] x_1 \ldots x_n \equiv \varphi$ — Consequence of λ-conversion
  3. $\forall x_1 \ldots \forall x_n ([\lambda x_1 \ldots x_n\ \varphi] x_1 \ldots x_n \equiv \varphi)$ — GEN ($\times n$), 1
  4. $\Box \forall x_1 \ldots \forall x_n ([\lambda x_1 \ldots x_n\ \varphi] x_1 \ldots x_n \equiv \varphi)$ — RN, 2
  5. $\exists F^n \Box \forall x_1 \ldots \forall x_n (F x_1 \ldots x_n \equiv \varphi)$ — EI, 3
- Instances:
  - $\exists F \Box \forall x (Fx \equiv \neg Gx)$ — $[\lambda x\ \neg Gx]$
  - $\exists F \Box \forall x (Fx \equiv Gx \,\&\, Hx)$ — $[\lambda x\ Gx \,\&\, Hx]$
  - $\exists F \Box \forall x (Fx \equiv p)$ — $[\lambda x\ p]$

Basic Object Theory I

- There are ordinary objects and there are abstract objects:
  - $\Box\exists x O!x$
  - $\exists x A!x$
- An identity relation on ordinary objects:
  - $[\lambda xy\ O!x \,\&\, O!y \,\&\, x=y]\downarrow$
  - $=_E \;=_{df}\; [\lambda xy\ O!x \,\&\, O!y \,\&\, x=y]$
  - $x =_E y \rightarrow \Box x =_E y$
  - $O!x \rightarrow x =_E x$ (implies symmetry, transitivity)
- Indiscernibility is necessary: $\forall F(Fx \equiv Fy) \rightarrow \Box\forall F(Fx \equiv Fy)$
- Ordinary objects are logically well-behaved:
  - $O!y \rightarrow [\lambda x\ x=y]\downarrow$
  - $(O!x \vee O!y) \rightarrow (\forall F(Fx \equiv Fy) \rightarrow x=y)$
  - $(O!x \,\&\, O!y) \rightarrow (x \neq y \equiv [\lambda z\ z=x] \neq [\lambda z\ z=y])$

Basic Object Theory II

- $\exists!x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))$, where φ has no free xs
- $\iota x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))\downarrow$
- $\star \vdash \iota x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))F \equiv \varphi$
- *Proof*. $(\rightarrow)$ Assume $\iota x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))F$. Then $\iota x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))\downarrow$ and we can instantiate this description in its own matrix ($\star$-theorem):

  $$A!\iota x(A!x \mathbin{\&} \forall F(xF \equiv \varphi)) \mathbin{\&} \forall F(\iota x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))F \equiv \varphi)$$

  Detach the second conjunct and instantiate to F:

  $$\iota x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))F \equiv \varphi$$
- $\iota x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))F \equiv \mathcal{A}\varphi$

Distinct A-Objects and Relational Properties

- $\forall R \exists x \exists y(A!x \;\&\; A!y \;\&\; x \neq y \;\&\; [\lambda z\, Rzx] = [\lambda z\, Rzy])$

  Proof: Consider an arbitrary R. By **OC**,

  $$\exists x(A!x \;\&\; \forall F(xF \equiv \exists y(A!y \;\&\; F = [\lambda z\, Rzy] \;\&\; \neg yF)))$$

  Call such an object k, so we know:

  $$\forall F(kF \equiv \exists y(A!y \;\&\; F = [\lambda z Rzy] \;\&\; \neg yF))$$

  Now consider $[\lambda z\, Rzk]$. Assume $\neg k[\lambda z\, Rzk]$. Then, by definition of k,

  $$\forall y(A!y \;\&\; [\lambda z\, Rzk] = [\lambda z\, Rzy] \rightarrow y[\lambda z\, Rzk]).$$

  Instantiate to k, and it follows that $k[\lambda z\, Rzk]$, contrary to assumption. So $k[\lambda z\, Rzk]$. So by the definition of k, there is an object, say l, such that

  $$A!l \;\&\; [\lambda z\, Rzk] = [\lambda z\, Rzl] \;\&\; \neg l[\lambda z\, Rzk].$$

  But since $k[\lambda z\, Rzk]$ and $\neg l[\lambda z\, Rzk]$, $k \neq l$. So

  $$\exists x, y(A!x \;\&\; A!y \;\&\; x \neq y \;\&\; [\lambda z\, Rzx] = [\lambda z\, Rzy]).$$

  $\bowtie$

- Why Cantor's Theorem forces this result.

Indiscernible Abstract Objects

- $\exists x \exists y(A!x \,\&\, A!y \,\&\, x \neq y \,\&\, \forall F(Fx \equiv Fy))$

  Proof: Let R_0 be the relation $[\lambda xy\, \forall F(Fx \equiv Fy)]$. By the previous theorem, there exist distinct abstract objects a, b such that $[\lambda z\, R_0 za] = [\lambda z\, R_0 zb]$. By logic alone, it is easily provable that $R_0 aa$, from which it follows that $[\lambda z\, R_0 za]a$. But, now, $[\lambda z\, R_0 zb]a$, from which it follows that $R_0 ab$. Thus, by λ-conversion, $\forall F(Fa \equiv Fb)$. $\bowtie$
- Why our models explain this result.
- Kirchner Theorem:
  - $[\lambda x\, \varphi]\downarrow \equiv \Box \forall x \forall y(\forall F(Fx \equiv Fy) \rightarrow (\varphi \equiv \varphi^y_x))$,
    provided y doesn't occur free in φ.
  - The property $[\lambda x\, \varphi]$ exists if and only if necessarily, φ doesn't distinguish between indiscernible objects x and y.
  - The n-place relation $[\lambda x_1 \ldots x_n\, \varphi]$ exists if and only if necessarily, φ doesn't distinguish among objects $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ that exemplify the same relations.

Discernible Objects

- $[\lambda x\, \Box\forall y(y \neq x \rightarrow \exists F\neg(Fy \equiv Fx))]\downarrow$

  Proof: Let φ be the above matrix; By Kirchner Thm., GEN, and RN, show $\forall F(Fx \equiv Fz) \rightarrow (\varphi \equiv \varphi^z_x)$.
- $D! =_{df} [\lambda x\, \Box\forall y(y \neq x \rightarrow \exists F\neg(Fy \equiv Fx))]$
- $O!x \rightarrow D!x$
- $D!x$ is modally collapsed: $\Box(D!x \rightarrow \Box D!x)$
- Discernible objects are logically well-behaved:
  - $(D!x \vee D!z) \rightarrow (\forall F(Fx \equiv Fz) \rightarrow x = z)$

    Proof. W.l.o.g., assume $D!x$ and $\forall F(Fx \equiv Fz)$. Then $\forall y(y \neq x \rightarrow \exists F\neg(Fy \equiv Fx))$, i.e., $\forall y(\forall F(Fy \equiv Fx) \rightarrow y = x)$. Then $\forall F(Fz \equiv Fx) \rightarrow z = x$. So $z = x$, i.e., $x = z$.
  - $[\lambda x\, D!x \;\&\; \varphi]\downarrow$, any φ (use Kirchner Thm)
- Identity Relation for Indiscernibles:
  - $[\lambda xy\, D!x \;\&\; D!y \;\&\; x = y]\downarrow$
  - $=_D \; =_{df} \; [\lambda xy\, D!x \;\&\; D!y \;\&\; x = y]$
  - $D!x \rightarrow x =_D x$ (symmetry, transitivity follows)
  - $D!y \rightarrow [\lambda x\, x = y]\downarrow$

Consistency Proved in Isabelle/HOL

- Daniel Kirchner (Ph.D. Mathematics, Freie Universität Berlin), extended techniques developed by Christoph Benzmüller (Universität Bamberg), to implement the above system in Isabelle/HOL in his Ph.D. Thesis.
- His implementation constructs a model that in which all the axioms are true and, hence, consistent with one another.
  - Daniel's object theory website: https://aot.ekpyron.org/
  - Daniel's GitHub repository: https://github.com/ekpyron/AOT/
  - Daniel's Ph.D. Thesis:

    https://refubium.fu-berlin.de/bitstream/handle/fub188/35426/dissertation_kirchner.pdf?sequence=3&isAllowed=y
  - Isabelle download page: https://isabelle.in.tum.de/
- The definitions, axioms, and theorems correspond to the current version of *Principia Logico-Metaphysica* https://mally.stanford.edu/principia.pdf

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