Seminar on Axiomatic Metaphysics
Lecture 3
Logical Objects

Edward N. Zalta

Philosophy Department, Stanford University

zalta@stanford.edu, https://mally.stanford.edu/zalta.html

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Logic and Logical Objects

- Frege thought that there are logical objects (logical individuals).
- Fregean logical objects:
  1. truth-values
  2. courses-of-values (extensions)
  3. directions, shapes, etc.
  4. natural numbers
- Frege thought he could reduce everything to courses-of-values:
  - Extensions: courses-of-values of concepts.
  - Truth-values (Gg., §10) are identified with extensions.
  - Directions: $\vec{a} = \grave{\epsilon}(\epsilon \parallel a])$
  - Numbers: $\#G = \epsilon[\lambda x\, \exists F(x = \epsilon F\ \&\ F \approx G)]$
- This reduction failed because the main principle governing courses of values, Basic Law V [$\epsilon f = \epsilon g = \forall x(f(x) = g(x))$] engendered a contradiction when added to his second-order predicate logic.

Problems with Attempts to Reconstruct Frege

- Wright and Hale 2001, Boolos 1986, Fine 2002
- Fregean biconditionals collapse existence and identity conditions. These, however, should be kept separate.
- The Julius Caesar problem: '$\#F = x$' isn't defined for arbitrary x. And so on, for other abstracts.
- Bad-company (Field 1984, 168, [1993], 286): many Fregean biconditionals are contradictory or false. Embarassment of riches (Weir 2003): indefinitely many consistent, but pairwise inconsistent, biconditionals.
- Fine 2002. (1) Burgess (2003) and Shapiro (2004): significant parts of mathematics aren't captured; (2) no solution to the Caesar problem; (3) no abstractions over equivalence relations on individuals (so, no directions, shapes, etc.); and (4) existence of two ordinary individuals required.
- These aren't general theories of abstract objects: each kind of abstract object is governing by a separate principle.

The Theory of Truth Values

- $TruthValueOf(x,p) \ \equiv_{df} \ A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$
- $\forall p \exists ! x TruthValueOf(x,p)$
- x *encodes* p (‘$x\Sigma p$’) $\ \equiv_{df} \ x[\lambda y\, p]$
- $T\text{-}value(x) \ \equiv_{df} \ \exists p TruthValueOf(x,p)$
- Theorem: There are exactly two truth-values:

  $\exists x, y[T\text{-}value(x) \,\&\, T\text{-}value(y) \,\&\, x \neq y \,\&\, \forall z(T\text{-}value(z) \rightarrow z = x \ \vee \ z = y)]$

Proof Sketch: There Are Exactly Two Truth Values

$\exists x, y[T\text{-}value(x) \; \& \; T\text{-}value(y) \; \& \; x \neq y \; \& \; \forall z(T\text{-}value(z) \rightarrow z = x \vee z = y)]$

Proof Sketch. Consider two objects:

- $\exists x(A!x \; \& \; \forall F(xF \equiv \exists q(q \; \& \; F = [\lambda z \, q])))$ ‘a’
- $\exists x(A!x \; \& \; \forall F(xF \equiv \exists q(\neg q \; \& \; F = [\lambda z \, q])))$ ‘b’

(1) To show $T\text{-}value(a)$ and $T\text{-}value(b)$, we have to show $\exists p TruthValueOf(a, p)$ and $\exists p TruthValueOf(b, p)$. Choose any truth, e.g., $\forall x(E!x \rightarrow E!x)$ (‘p_0’) as a witness for the first, and any falsehood, say $\neg p_0$, for the second. Then show a and b satisfy the definition (exercise). (E.g., since a encodes all the truths, it encodes all the propositions materially equivalent to p_0.) It remains only to show (2) a and b are distinct, and (3) that every truth value is identical to either a or b. (2) Reason by disjunctive syllogism from $p \vee \neg p$ (p any proposition). If p, then $a\Sigma p \; \& \; \neg(b\Sigma p)$, so $a \neq b$ (they encode different properties). If $\neg p$, $b\Sigma p \; \& \; \neg(a\Sigma p)$, so $a \neq b$. (3) Assume $T\text{-}Value(z)$, to show $z = a \vee z = b$. So for some proposition, say p_1, $TruthValueOf(z, p_1)$. Hence by definition:

$$A!z \; \& \; \forall F(zF \equiv \exists q((q \equiv p_1) \; \& \; F = [\lambda y \, q]))$$

Then reason from $p_1 \vee \neg p_1$ to $z = a \vee z = b$. (Exercise) $\bowtie$

The Truth Value of Proposition p

- Intuitive background fact: the equivalence classes of materially equivalent propositions vary from world to world.
- The truth value of p ('p°') $=_{df}$ $\iota x TruthValueOf(x,p)$
- $\star \vdash p^\circ \Sigma q \equiv q \equiv p$ (⋆Lemma)

  Proof: ($\rightarrow$) Assume $p_1^\circ \Sigma q_1$, i.e., $p_1^\circ[\lambda y\, q_1]$. Then by definition of p_1° and description theory, there is a proposition, say r_1, such that $r_1 \equiv p_1$ & $[\lambda y\, q_1] = [\lambda y\, r_1]$ (exercise). The right conjunct implies $q_1 = r_1$ (by df $=$), i.e., $r_1 = q_1$. So, $q_1 \equiv p_1$. ($\leftarrow$) Exercise.
- $\star \vdash p^\circ = q^\circ \equiv p \equiv q$ (⋆Theorem)

  Proof: ($\rightarrow$) Assume $p_1^\circ = q_1^\circ$. By $p_1 \equiv p_1$ and the previous ⋆Lemma, $p_1^\circ \Sigma p_1$. So $q_1^\circ \Sigma p_1$. So, by the ⋆Lemma, $p_1 \equiv q_1$. ($\leftarrow$) Assume $p_1 \equiv q_1$. To show that $p_1^\circ = q_1^\circ$, we show: $\Box \forall F(p_1^\circ F \equiv q_1^\circ F)$. By GEN and RN, show: $p_1^\circ F \equiv q_1^\circ F$ (a) Assume $p_1^\circ F$. Then by definition of p_1°, there is a proposition, say r_1, such that $r_1 \equiv p_1$ & $F = [\lambda y\, r_1]$. So there is a proposition r (namely r_1) such that $r \equiv q_1$ & $F = [\lambda y\, r]$. So, by the definition of q_1°, it follows that $q_1^\circ F$. (b) Assume $q_1^\circ F$ and show $p_1^\circ F$, by analogous reasoning.

The Theory of Truth Values (cont'd)

- $\top$ ('The True') $=_{df}$ $\iota x(A!x \;\&\; \forall F(xF \equiv \exists r(r \;\&\; F=[\lambda y\; r])))$
- $\bot$ ('The False') $=_{df}$ $\iota x(A!x \;\&\; \forall F(xF \equiv \exists r(\neg r \;\&\; F=[\lambda y\; r])))$
- $\star \vdash p \equiv (p^\circ = \top)$ ($\star$Lemma)
- *Proof*. ($\rightarrow$) Assume p_1. To show $p_1^\circ = \top$, we have to show $\Box\forall F(p_1^\circ F \equiv \top F)$. So we show $p_1^\circ Q \equiv \top Q$, where Q is an arbitrarily chosen property.
  ($\rightarrow$) Assume $p_1^\circ Q$. By definition of p_1°, it follows that $\exists r(r \equiv p_1 \;\&\; Q=[\lambda y\; r])$. Let r_1 be such a proposition, so that we know $r_1 \equiv p_1 \;\&\; Q=[\lambda y\; r_1]$. But since we know p_1, it follows that r_1. So, we have established: $r_1 \;\&\; Q=[\lambda y\; r_1]$. From which it follows that $\exists r(r \;\&\; Q=[\lambda y\; r])$. But we know, by definition of $\top$ (appeal to $\star$-theorem), that $\forall F(\top F \equiv \exists r(r \;\&\; F=[\lambda y\; r]))$. So in particular, $\top Q \equiv \exists r(r \;\&\; Q=[\lambda y\; r])$. But we've established the right side. So $\top Q$.
  ($\leftarrow$) Assume $\top Q$. Then, by definition of $\top$ (and appeal to $\star$-theorem), $\exists r(r \;\&\; Q=[\lambda y\; r])$. Let r_1 be such a proposition, so that we know $r_1 \;\&\; Q=[\lambda y\; r_1]$. So we know r_1 and we also know p_1 (by assumption). So $r_1 \equiv p_1$. Hence $r_1 \equiv p_1 \;\&\; Q=[\lambda y\; r_1]$. So, $\exists r(r \equiv p_1 \;\&\; Q=[\lambda y\; r])$, from which it follows $p_1^\circ Q$, by definition of p_1°.
  By GEN and RN, we're done. ($\leftarrow$) Exercise.

The Theory of Truth Values (cont'd)

- $\star\vdash$ *T-value*$(\top)$ (★Theorem)
  Proof. By a ★-theorem of description theory, $\top$ encodes all and only the truths. Then consider the proposition $\forall x(E!x \rightarrow E!x)$ ('p_0'). Since p_0 is provably a truth, it follows that $\top$ encodes all and only the propositions materially equivalent to p_0.Hence *T-value*$(\top)$.
- ★Lemma: $\star\vdash \neg p \equiv (p^\circ = \bot)$ (Exercise)
- ★Theorem: $\star\vdash$ *T-value*$(\bot)$ (Exercise)
- ★Lemmas: (Exercises)
  $\star\vdash p \equiv (\top\Sigma p)$ $\quad$ $\star\vdash p \equiv \neg(\bot\Sigma p)$
  $\star\vdash \neg p \equiv \neg(\top\Sigma p)$ $\quad$ $\star\vdash \neg p \equiv (\bot\Sigma p)$

Extensions = Natural Classes = Sets Logically Conceived

- $\left.\begin{array}{c} \mathit{ExtensionOf}(x, G) \\ \mathit{ClassOf}(x, G) \end{array}\right\} \equiv_{df} A!x \mathbin{\&} G{\downarrow} \mathbin{\&} \forall F(xF \equiv \forall z(Fz \equiv Gz))$
- $\left.\begin{array}{c} \mathit{Class}(x) \\ \mathit{LogicalSet}(x) \end{array}\right\} \equiv_{df} \left\{\begin{array}{c} \exists G(\mathit{ExtensionOf}(x, G)) \\ \exists G(\mathit{ClassOf}(x, G)) \end{array}\right.$
- $\forall G \exists ! x(\mathit{ExtensionOf}(x, G))$
- Pre-Law V: $(\mathit{ExtensionOf}(x, G) \mathbin{\&} \mathit{ExtensionOf}(y, H)) \rightarrow$
  $(x = y \equiv \forall z(Gz \equiv Hz))$
- Membership: $y \in x \equiv_{df} \exists G(\mathit{ExtensionOf}(x, G) \mathbin{\&} Gy)$
- Law of Extensions/Classes:
  $\mathit{ExtensionOf}(x, H) \rightarrow \forall y(y \in x \equiv Hy)$
- Fundamental Theorem of Classes/Logical Sets:
  $\forall F \exists x(\mathit{Class}(x) \mathbin{\&} \forall y(y \in x \equiv Fy))$

Reconstructing Frege's Conception

- Since extensions are intuitively abstracted from equivalence classes of materially equivalent properties, and these latter vary from world to world, different natural classes arise at other possible worlds. Moreover, if F is contingent, the extension of F at one world won't be the same as that of another world.
- *The extension of G* ('ϵG') $=_{df}$ $\iota x ExtensionOf(x, G)$
- ϵGG (Lemma 1)
- $\star \vdash \epsilon FG \equiv \forall x(Gx \equiv Fx)$ ($\star$Lemma 2)
- $\star \vdash \epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx)$ ($\star$Basic Law V)

  Proof: ($\rightarrow$) Suppose $\epsilon A = \epsilon B$. By $\star$Lemma 2, $\epsilon AG \equiv \forall y(Gy \equiv Ay)$. Since $\epsilon A = \epsilon B$, then $\epsilon BG \equiv \forall y(Gy \equiv Ay)$. In particular, $\epsilon BB \equiv \forall y(By \equiv Ay)$. Since ϵBB (Lemma 1), it follows that $\forall y(By \equiv Ay)$. ($\leftarrow$) Suppose $\forall y(Ay \equiv By)$. (a) Assume ϵAQ (to show ϵBQ). Then by $\star$Lemma 2, $\forall y(Qy \equiv Ay)$. So $\forall y(Qy \equiv By)$. But $\star$Lemma 2 also implies: $\epsilon BQ \equiv \forall y(Qy \equiv By)$. So ϵBQ. (b) Assume ϵBQ (to show ϵAQ). Reverse the reasoning. $\bowtie$

The Paradoxical Properties and Extensions Don't Exist

- The properties and extensions that lead to paradox don't exist:

$\neg[\lambda x\, \exists G(x = \epsilon G \;\&\; \neg Gx)]\!\downarrow$ $\quad$ $\neg\epsilon[\lambda x\, \exists G(x = \epsilon G \;\&\; \neg Gx)]\!\downarrow$

$\neg[\lambda x\, x \in x]\!\downarrow$ $\quad$ $\neg\epsilon[\lambda x\, x \in x]\!\downarrow$

$\neg[\lambda x\, x \notin x]\!\downarrow$ $\quad$ $\neg\epsilon[\lambda x\, x \notin x]\!\downarrow$

$\neg[\lambda x\, \exists F(xF \;\&\; \neg Fx)]\!\downarrow$ $\quad$ $\neg\epsilon[\lambda x\, \exists F(xF \;\&\; \neg Fx)]\!\downarrow$

Extension/Natural Class/Logical Set Theory

- $\forall c \forall c'[\forall z(z \in c \equiv z \in c') \rightarrow c = c']$ (Extensionality)

  Proof: Suppose $\forall z(z \in c \equiv z \in c')$. So there are properties, say P and Q, such that $ExtensionOf(c, P)$ and $ExtensionOf(c', Q)$. Then by Law of Extensions, our assumption implies $\forall z(Pz \equiv Qz)$ Then, by the Pre-Law V, $c = c'$. ⋈

- $\exists! c \forall y(y \notin c)$ (Null Extension)

  Proof: Consider $[\lambda z\, E!z \,\&\, \neg E!z]$ $(= P)$. Then by Fundamental Theorem, $\exists x(Class(x) \,\&\, \forall y(y \in x \equiv Py))$, say a. Then $Class(a) \,\&\, \forall y(y \in a \equiv Py)$. But $\forall y \neg Py$. So $\forall y(y \notin a)$. For uniqueness, suppose, for reductio, there exists class c', where $c' \neq c$, such that $\forall y(y \notin c')$. Then $\forall y(y \in c' \equiv y \in a)$ and so by Extensionality, $c = c'$. Contradiction. ⋈

- $\forall c' \forall c'' \exists c \forall y(y \in c \equiv y \in c' \vee y \in c'')$ (Unions)

  Proof: Consider arbitrarily chosen classes c' and c''. Then there are properties P and Q such that $ExtensionOf(c', P)$ and $ExtensionOf(c'', Q)$. Consider $[\lambda z\, Pz \vee Qz]$ $(= H)$, which exists axiomatically. By Fundamental Theorem, there is a class, say a, such that $\forall y(y \in a \equiv Hy)$. But $\forall y(Hy \equiv (Py \vee Qy))$ (by λ-Conversion), and $\forall y((Py \vee Qy) \equiv (y \in c' \vee y \in c''))$ (by Law of Extensions). So $\forall y(y \in a \equiv (y \in c' \vee y \in c''))$.

Extension Theory/Natural Class/Logical Set Theory

- $\forall c' \exists c \forall y (y \in c \equiv x \notin c')$ (Complements)
  - Fix c'; then *ExtensionOf*(c', P) (P arbitrary). The witness for c is given by $\exists x$*ExtensionOf*$(x, [\lambda z\, \neg Pz])$.
- $\forall c' \forall c'' \exists c \forall y (y \in c \equiv y \in c' \;\&\; y \in c'')$ (Intersections)
  - Fix c' and c''; then *ExtensionOf*(c', P) and *ExtensionOf*(c'', Q) (P, Q arbitrary). The witness for c is given by $\exists x$*ExtensionOf*$(x, [\lambda z\, Pz \;\&\; Qz])$.
- $[\lambda y\, \varphi]\!\downarrow \to \exists c \forall y (y \in c \equiv \varphi)$ (Conditional Comprehension)
  - Assume $[\lambda y\, \varphi]\!\downarrow$. The witness to c is given by $\exists x$*ExtensionOf*$(x, [\lambda x\, \varphi])$.
- $[\lambda y\, \varphi]\!\downarrow \to \forall c' \exists c \forall y (y \in c \equiv y \in c' \;\&\; \varphi)$ (Separation)
  - Fix c'. And let *ExtensionOf*$(x, [\lambda z\, \varphi])$. Then there is an intersection of c' and x. Show any such class is a witness to c.

Extensions/Natural Classes/Logical Set Theory

- $\forall R \forall c' \exists c \forall y(y \in c \equiv \exists z(z \in c' \,\&\, Rzy))$ (Collections)
  - Fix R and c', and let $ExtensionOf(c', P)$. Then consider $[\lambda x\, Px \,\&\, Rxy]$ and its class c.
- $\exists c \forall y(y \in c \equiv D!y \,\&\, y = x)$ (Singletons)
  - So discernible abstract objects have well-behaved singletons.
- $\exists c \forall y(y \in c \equiv D!y \,\&\, (y = x \vee y = z))$ (Pairs)
  - So distinct, discernible abstract objects have well-behaved pair sets.
- $\forall c' \exists c \forall y(y \in c \equiv y \in c' \vee y =_D x)$ (Adjunction)
  - Fix c', x. So let $ExtensionOf(c', P)$. Consider $[\lambda z\, Pz \vee z =_D x]$ and its class c.
- No power sets, since you can't prove $[\lambda x\, x \subseteq z]\!\downarrow$ for arbitrary z, where $x \subseteq z \equiv_{df} \forall y(y \in x \rightarrow y \in z)$. (This is a flat set theory.)

Directions and Shapes

- Assumptions: $\|$ is an equivalence relation on *ordinary lines*:
  - $Lx \rightarrow x\|x$
    $(Lx \,\&\, Ly) \rightarrow (x\|y \rightarrow y\|x)$
    $(Lx \,\&\, Ly \,\&\, Lz) \rightarrow (x\|y \,\&\, y\|z \rightarrow z\|z)$

  and where we use u, v as restricted variables ranging over ordinary lines, that *being parallel to u* is materially equivalent to *being parallel to u′* iff $u\|u'$:
  - $\forall u \forall u' (\forall z([\lambda v\ v\|u]z \equiv [\lambda v\ v\|u']z) \equiv u\|u')$
- Define and prove:
  - $DirectionOf(x, u) \equiv_{df} ExtensionOf(x, [\lambda v\ v\|u])$
  - $\exists! x DirectionOf(x, u)$
  - $(DirectionOf(x, u) \,\&\, DirectionOf(y, v)) \rightarrow (x = y \equiv u\|v)$
  - $Direction(x) \equiv_{df} \exists u DirectionOf(x, u)$
  - $\vec{u} =_{df} \imath x DirectionOf(x, u)$
- Fregean biconditional: $\star \vdash \vec{u} = \vec{v} \equiv u\|v$

Proof of Fregean Biconditional

- ($\rightarrow$) Assume $\vec{a} = \vec{b}$. Since we know independently $\forall y([\lambda z\, z\|a]y \equiv [\lambda z\, z\|a]y)$, it follows by definition of $\vec{a}$ (by $\star$-theorem) that $\vec{a}[\lambda z\, z\|a]$. Substituting $\vec{b}$ for $\vec{a}$ yields $\vec{b}[\lambda z\, z\|a]$. Then by the definition of $\vec{b}$ (and a $\star$-theorem), we know $\forall y([\lambda z\, z\|a]y \equiv [\lambda z\, z\|b]y)$ and in particular $[\lambda z\, z\|a]b \equiv [\lambda z\, z\|b]b$ which is equivalent, by λ-abstraction, to $b\|b \equiv b\|a$. Since $b\|b$, $b\|a$. So by symmetry of $\|$, $a\|b$.
- ($\leftarrow$) Assume $a\|b$. It suffices to show that for any P, $\vec{a}P \equiv \vec{b}P$. ($\rightarrow$) Suppose $\vec{a}P$. Then by the definition of $\vec{a}$ (and a $\star$-theorem), $\forall y(Py \equiv [\lambda z\, z\|a]y)$. Since $a\|b$ this is equivalent to $\forall y(Py \equiv [\lambda z\, z\|b]y)$. By the definition of $\vec{b}$ this implies $\vec{b}P$. ($\leftarrow$) Exercise.

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