Seminar on Axiomatic Metaphysics
Lecture 5
Possible Worlds and Possibilities

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Possible World Theory

- A *possible world* is a situation that might be such that it make true all and only the truths:
  - $PossibleWorld(x) \equiv_{df} Situation(x) \,\&\, \Diamond\forall p(x \models p \equiv p)$
- $\vdash PossibleWorld(x) \rightarrow \Box PossibleWorld(x)$
  - Let $w, w', \ldots$ be rigid, restricted variables over possible worlds.
  - $\forall w \phi^w_x \equiv_{df} \forall x(PossibleWorld(x) \rightarrow \phi)$
  - $\exists w \phi^w_x \equiv_{df} \exists x(PossibleWorld(x) \,\&\, \phi)$
- Truth at a world is already defined (worlds are situations):
  - p is true at w (or w makes p true) iff $w \models p$
- A situation s is *maximal* iff for every proposition p, either s makes p true or s makes the negation of p true:
  - $Maximal(s) \equiv_{df} \forall p(s \models p \vee s \models \neg p)$
- Theorem: Possible worlds are maximal.

  $\vdash \forall w Maximal(w)$, i.e.,

  $\vdash \forall s(PossibleWorld(s) \rightarrow Maximal(s))$, i.e.,

  $\vdash \forall x(PossibleWorld(x) \rightarrow Maximal(x))$
- The proof is on an Appendix slide.

Possible World Theory II

- $\vdash \forall w\, Consistent(w)$ (exercise)
- $p \Rightarrow q \equiv_{df} \Box(p \rightarrow q)$
- $ModallyClosed(s) \quad \equiv_{df} \quad \forall p((Actual(s) \Rightarrow p) \rightarrow s \models p)$
- $\vdash \forall w ModallyClosed(w)$
- $\vdash ModallyClosed(s) \rightarrow$
  $\forall p_1 \ldots \forall p_n \forall q[(s \models p_1 \;\&\; \ldots \;\&\; s \models p_n \;\&\; (p_1 \;\&\; \ldots \;\&\; p_n) \Rightarrow q) \rightarrow s \models q]$
- $\vdash \exists! w Actual(w)$, i.e., $\exists! x(PossibleWorld(x) \;\&\; Actual(x))$

Proof: There is a Unique Actual World

- By Comprehension: $\exists x(A!x \;\&\; \forall F(xF \equiv \exists p(p \;\&\; F = [\lambda y\, p])))$
- Let a be such an object: $A!a \;\&\; \forall F(aF \equiv \exists p(p \;\&\; F = [\lambda y\, p]))$ $\quad(\theta)$
- Strategy:
  - Show: $World(a)$
  - Show: $Actual(a)$
  - Show: $\forall x(World(x) \;\&\; Actual(x) \rightarrow x = a)$
- Show: $World(a)$, i.e., $Situation(a) \;\&\; \Diamond\forall p(a \models p \equiv p)$
- Show: $Situation(a)$:
  - Show: $A!a$ $\quad$ by (θ)
  - Show: $\forall F(aF \rightarrow \exists p(F = [\lambda y\, p]))$
    Proof: By GEN, assume aF. By (θ), $\exists p(p \;\&\; F = [\lambda y\, p])$. *A fortiori*, $\exists p(F = [\lambda y\, p])$. By CP, $aF \rightarrow \exists p(F = [\lambda y\, p])$.
- Appendix slides have a proof of:
  - $World(a)$
  - $Actual(a)$,
  - $\forall x(World(x) \;\&\; Actual(x) \rightarrow x = a)$

Facts About Actual Situations and the Actual World

- $\boldsymbol{w}_\alpha \; =_{df} \; \imath w Actual(w)$, i.e.,
  $\boldsymbol{w}_\alpha \; =_{df} \; \imath x(PossibleWorld(x) \;\&\; Actual(x))$
- $\vdash \forall s(Actual(s) \equiv s \trianglelefteq \boldsymbol{w}_\alpha)$
- $\star \vdash p \equiv \boldsymbol{w}_\alpha \models p$
- $\star \vdash (\boldsymbol{w}_\alpha \models p) \equiv [\lambda y\, p]\boldsymbol{w}_\alpha$
- $\star \vdash p \equiv \boldsymbol{w}_\alpha \models [\lambda y\, p]\boldsymbol{w}_\alpha$
- If given any true proposition (i.e., fact), say p, the last theorem implies something of the form: $s \models \phi(s)$. This suggests that $\boldsymbol{w}_\alpha$ is a constituent of the facts that it makes factual. In situation theory, statements of the form $s \models \phi(s)$ constitute the defining characteristic of 'nonwellfounded' situations. So the actual world $\boldsymbol{w}_\alpha$ seems to be nonwellfounded in the sense that it makes factual states of affairs p of which it is a constituent.

Proof of the Strengthened Lewis Principle

- Lemma: $\vdash \Diamond \mathit{Situation}(x) \rightarrow \mathit{Situation}(x)$
  - Derive this from $\vdash_{\Box} \mathit{Situation}(x) \rightarrow \Box \mathit{Situation}(x)$, or see the direct derivation in an Appendix slide.
- Lewis Principle (1986): For every way a world might be, there is a world which is that way.
- Strengthened Lewis Principle: p is possible if and only if there exists a possible world in which p is true:
  - $\vdash \forall p(\Diamond p \equiv \exists w(w \models p))$
- Proof:
  - Show: $(\rightarrow)$: $\Diamond q \rightarrow \exists w(w \models q)$, where q is arbitrary.

    Stage A: Show: $\Diamond q \rightarrow \Diamond \exists w(w \models q)$

    Stage B: Show $\Diamond \exists w(w \models q) \rightarrow \exists w(w \models q)$.
  - Show $(\leftarrow)$: $\exists w(w \models q) \rightarrow \Diamond q$, where q is arbitrary.
- For the full proof, see the Appendix slides.

The Leibniz/Kripke Principle

- Leibniz/Kripke Principle: A proposition p is necessarily true iff p is true in all possible worlds.
- $\vdash \Box p \equiv \forall w(w \models p)$
- Proof:

1.	$\Diamond\neg p \equiv \exists w(w \models \neg p)$	Instance of Lewis Principle, with $\neg p$ for p.
2.	$\Diamond\neg p \equiv \exists w\neg(w \models p)$	From 1 and Coherence ($w \models \neg p \equiv \neg w \models p$).
3.	$\neg\Diamond\neg p \equiv \neg\exists w\neg(w \models p)$	From 2 by basic propositional logic.
4.	$\Box p \equiv \forall w(w \models p)$	From 3, dfn $\Box/\Diamond$, and dfn $\forall/\exists$.

Some Interesting Consequences

- There are non-actual possible worlds:
  - $\vdash \exists w \neg Actual(w)$
- This follows from the existence of contingently true (false) propositions:
  - $\vdash \exists p(p \ \& \ \Diamond \neg p)$
  - $\vdash \exists p(\neg p \ \& \ \Diamond p)$

  which in turn follows from our axiom:
  - $\Diamond \exists x(E!x \ \& \ \neg \mathcal{A}E!x)$
- Epistemologically: we don't have to justify our knowledge of particular possible worlds. Use Equivalence Principle and modal beliefs.
- Menzel & Zalta 2014 show that the Strengthened Lewis Principle can be derived in a subtheory with tiny models: use monadic object theory with comprehension.

World-Indexed T-Values

- First we examine truth values of propositions and extensions of properties that are world-indexed. Then we examine world-indexed relations.
- $TruthValueAtOf(s, w, p) \;\equiv_{df}\; \forall q(s \models q \equiv w \models (q \equiv p))$
- $\vdash \exists! s TruthValueAtOf(s, w, p)$
- $\circ_w p \;=_{df}\; \imath s TruthValueAtOf(s, w, p)$
- $\vdash \circ_w p = \circ_w q \;\equiv\; w \models (p \equiv q)$
- The True at w and The False at w:
  - $\top_w \;=_{df}\; \imath s \forall p(s \models p \equiv w \models p)$
  - $\bot_w =_{df}\; \imath s \forall p(s \models p \equiv w \models \neg p$
- $\vdash w \models p \;\equiv\; \circ_w p = \top_w$
- $\vdash \Box p \equiv \forall w(\circ_w p = \top_w)$

World-Indexed Extensions

- $ExtensionAtOf(x, w, G) \equiv_{df} A!x \,\&\, G\downarrow \,\&\, \forall F(xF \equiv w \models \forall y(Fy \equiv Gy))$
- $\vdash \exists! x ExtensionAtOf(x, w, G)$
- $\epsilon_w G =_{df} \imath x ExtensionAtOf(x, w, G)$
- $\vdash ExtensionAtOf(\epsilon_w G, w, G)$
- $\vdash (ExtensionAtOf(x, w, G) \,\&\, ExtensionAtOf(y, w, H)) \rightarrow$
  $(x = y \equiv w \models \forall z(Gz \equiv Hz))$
- Proof: Assume antecedent. ($\Rightarrow$) Assume $x = y$. Then by expanding $ExtensionAtOf(x, w, G)$ and substituting y for x, we know: $\forall F(yF \equiv w \models \forall z(Fz \equiv Gz))$. This and $ExtensionAtOf(y, w, H)$ implies:
  $$\forall F[w \models \forall z(Fz \equiv Gz) \equiv w \models \forall z(Fz \equiv Hz)]$$
  So $w \models \forall z(Gz \equiv Gz) \equiv w \models \forall z(Gz \equiv Hz)$. Hence $w \models \forall z(Gz \equiv Hz)$. ($\Leftarrow$) Assume $w \models \forall z(Gz \equiv Hz)$, show: $xF \equiv yF$. ($\rightarrow$) Assume xF. Then $w \models \forall z(Fz \equiv Gz)$. Since w is modally closed: $w \models \forall z(Fz \equiv Hz)$. Hence, by initial assumption and definition of y, yF. ($\leftarrow$) Exercise.
- $\vdash \epsilon_w F = \epsilon_w G \equiv w \models \forall z(Fz \equiv Gz)$ (Modal Law V)

World-Indexed Relations

- $\vdash [\lambda x_1 \ldots x_n\, w \models Fx_1 \ldots x_n]\!\downarrow$ $(n \geq 0)$
- $F^n_w =_{df} [\lambda x_1 \ldots x_n\, w \models F^n x_1 \ldots x_n]$ $(n \geq 0)$
- $\vdash \forall F \forall w (F^n_w\!\downarrow)$ $(n \geq 0)$
- $\vdash F^n_w x_1 \ldots x_n \equiv w \models F^n x_1 \ldots x_n$ $(n \geq 0)$
- So we don't assume the existence of world-indexed relations in the semantics (Williamson 2013, 237), but prove they exist in object theory.
- $Rigid(F^n) \;\equiv_{df}\; F^n\!\downarrow \,\&\, \Box \forall x_1 \ldots \forall x_n (F^n x_1 \ldots x_n \rightarrow \Box F^n x_1 \ldots x_n)$ $(n \geq 0)$
- $\vdash Rigid(G^n_w)$ $(n \geq 0)$
- $Rigidifies(F^n, G^n) \;\equiv_{df}\; Rigid(F^n) \,\&\, \forall x_1 \ldots \forall x_n (F^n x_1 \ldots x_n \equiv G^n x_1 \ldots x_n)$ $(n \geq 0)$
- Every relation has a rigidification (cf. Gallin 1975):
  - $\vdash \forall G \exists F^n (Rigidifies(F^n, G^n))$ $(n \geq 0)$
  - Proof: Fix G and consider $G^n_{w_0}$. Show: $Rigid(G^n_{w_0})$ and $\forall x_1 \ldots \forall x_n (G^n_{w_0} x_1 \ldots x_n \equiv G x_1 \ldots x_n)$ (Exercises)
- Be sure to distinguish ϵF_w from $\epsilon_w F$, and $\circ p_w$ from $\circ_w p$.

Introduction

- This is work coauthored with Uri Nodelman.
- Typically the term 'possibility' is used in philosophy to denote a proposition that might be true ($\Diamond p$).
- A different, technical sense of 'possibility' is in Humberstone 1981, 2011; van Benthem 1981, 2016; Edgington 1985; Holliday 2014, forthcoming; and Ding & Holliday 2020.
- Possibilities are partial (i.e., not necessarily maximal) entities, such as proper parts of possible worlds. Edgington 1985 (564): *possibilities*, or *possible situations* . . . differ from possible worlds in leaving many details unspecified.
- But all of these philosophical logicians takes them as primitive entities governed by axioms stipulated in their semantics. We develop a theory that derives these axioms as theorems.

The Semantic Axioms Stipulated

- Humberstone 1981 (318), 2011 (900); van Benthem 1981 (3–4), 2016 (3–4); Holliday 2014 (3), forthcoming (5, 15); and Ding & Holliday 2020 (155):
  - *Ordering*: a relation $\trianglerighteq$ partially orders the possibilities,
  - *Persistence*: every proposition true in a possibility is true in every refinement,
  - *Refinement*: if a possibility x has a gap on p, then (a) there is a refinement of x where p is true, and (b) there is refinement where p is false.
  - *Cofinality*: if, for every x' that is a refinement of x there is an x'' that refines x' and makes p true, then x makes p true.
- They must also satisfy negation and conjunction:
  - *Negation*: a possibility x makes the negation of p true if and only if every refinement of x fails to make p true.
  - *Conjunction*: a possibility x makes the conjunction p & q true iff x makes both p true and makes q true.

Prerequisites: I

- We've seen: comprehension for situations, canonical situations, and:

  $\vdash (s = \imath s' \forall p(s' \models p \equiv \phi)) \rightarrow \forall p(s \models p \equiv \phi),$
  provided s' isn't free in ϕ and ϕ is modally collapsed.
- $\vdash \mathit{Possible}(s) \equiv \exists w(s \trianglelefteq w)$
- $\mathit{ModallyClosed}(s) \equiv_{df} \forall p((\mathit{Actual}(s) \Rightarrow p) \rightarrow s \models p)$
- $\mathit{ModallyClosed}(s) \rightarrow \forall p_1 \ldots \forall p_n \forall q((s \models p_1 \;\&\; \ldots \;\&\; s \models p_n \;\&\; ((p_1 \;\&\; \ldots \;\&\; p_n) \Rightarrow q)) \rightarrow s \models q)$
- $\vdash (\mathit{ModallyClosed}(s) \;\&\; \mathit{Consistent}(s)) \rightarrow \mathit{Possible}(s)$
- $\vdash (\mathit{ModallyClosed}(s) \;\&\; \Box p) \rightarrow s \models p$
- $s^{+p} =_{df} \imath s' \forall q(s' \models q \equiv (s \models q \vee q = p))$
- $\vdash s^{+p} \trianglelefteq w \equiv s \trianglelefteq w \;\&\; w \models p$
- $\vdash \forall w(s \trianglelefteq w \rightarrow w \models p) \equiv (\mathit{Actual}(s) \Rightarrow p)$

Prerequisites: II

- $s^\star$ = the modal closure of s
- $s^\star =_{df} \iota s' \forall p(s' \models p \equiv (Actual(s) \Rightarrow p))$
- $\vdash \forall p(s^\star \models p \equiv (Actual(s) \Rightarrow p))$
- $\vdash s \trianglelefteq s^\star$
- $s \trianglelefteq w \equiv s^\star \trianglelefteq w$
- $\vdash Possible(s) \equiv Possible(s^\star)$
- $\vdash ModallyClosed(s^\star)$
- $\vdash Possible(s) \equiv Consistent(s^\star)$

Definition of a Possibility

- A situation s is a *possibility* if and only if s is both consistent and modally closed:

  $$Possibility(s) \equiv_{df} Consistent(s) \,\&\, ModallyClosed(s)$$

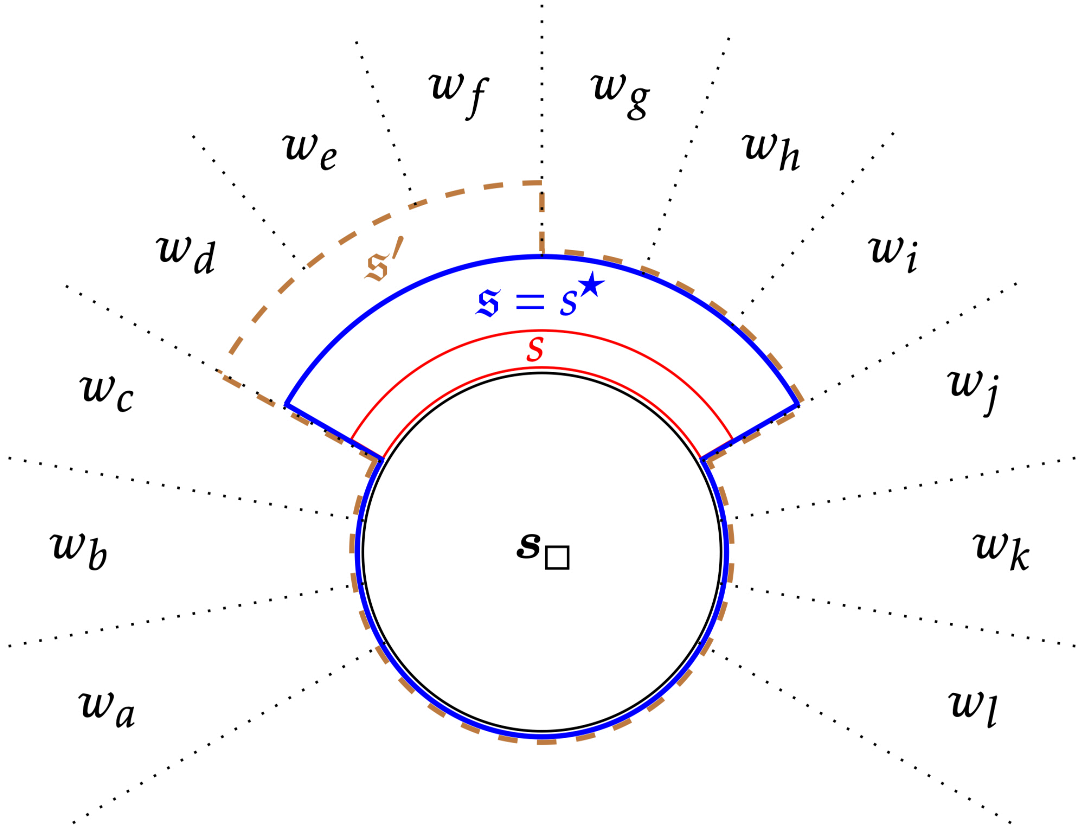
- Theorem: possible worlds are possibilities:

  $$\vdash Possibility(w)$$

- $Possibility(s)$ is a rigid restriction condition, since:
  - $Possibility(s)$ contains a single free variable.
  - $Possibility(s)$ is strictly non-empty, i.e., $\vdash_{\Box} \exists s Possibility(s)$
  - $Possibility(s)$ has strict existential import, i.e., $\vdash_{\Box} Possibility(\kappa) \rightarrow \kappa\downarrow$
  - $Possibility(s) \rightarrow \Box Possibility(s)$
- We henceforth use the variables $\mathfrak{s}, \mathfrak{s}', \mathfrak{s}'', \ldots$ as rigid, restricted variables for possibilities.
- Theorem: Necessary truths are true in every possibility:

  $$\vdash \Box p \rightarrow \forall \mathfrak{s}(\mathfrak{s} \models p)$$

A Picture



$\boldsymbol{s}_\Box$	=	the smallest possibility ('absolute necessity')
s	=	a *possible* situation
$s^\star$	=	the smallest possibility $\mathfrak{s}$ that contains s
$\mathfrak{s}'$	=	a refinement of $\mathfrak{s}$

The Ordering Principle

- We say a situation s' *contains* situation s, written $s' \trianglerighteq s$, just in case s is a part of s':

  $$s' \trianglerighteq s \equiv_{df} s \trianglelefteq s'$$

- Then, when s' and s are *possibilities* and $s' \trianglerighteq s$, we say that s' *is a refinement of* s, i.e., we read $\mathfrak{s}' \trianglerighteq \mathfrak{s}$ as: $\mathfrak{s}'$ is a refinement of $\mathfrak{s}$.
- Since *part of* ($\trianglelefteq$) is reflexive, anti-symmetric, and transitive on the situations, it follows that *refinement of* is a reflexive, anti-symmetric, and transitive condition on the possibilities:

  (a) $\vdash \mathfrak{s} \trianglerighteq \mathfrak{s}$
  (b) $\vdash (\mathfrak{s}' \trianglerighteq \mathfrak{s} \;\&\; \mathfrak{s}' \neq \mathfrak{s}) \rightarrow \neg \mathfrak{s} \trianglerighteq \mathfrak{s}'$
  (c) $\vdash (\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \;\&\; \mathfrak{s}' \trianglerighteq \mathfrak{s}) \rightarrow \mathfrak{s}'' \trianglerighteq \mathfrak{s}$

- These jointly validate the principle of *Ordering*; cf. Humberstone 1981 (318), Ding & Holliday 2020 (155), and Holliday forthcoming (Definition 2.1 and 2.21).

The Persistence Principle

- *Persistence*: for every proposition p, (a) if p is true in $\mathfrak{s}$ and $\mathfrak{s}'$ is a refinement of $\mathfrak{s}$, then p is true in $\mathfrak{s}'$, and (b) if $\neg p$ is true in $\mathfrak{s}$ and $\mathfrak{s}'$ is a refinement of $\mathfrak{s}$, then $\neg p$ is true in $\mathfrak{s}'$, i.e.,

  $$\vdash \forall p((\mathfrak{s} \models p \;\&\; \mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \mathfrak{s}' \models p) \;\&\; (\mathfrak{s} \models \neg p \;\&\; \mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \mathfrak{s}' \models \neg p))$$

  Humberstone 1981, 318; 2011, 900.
- This can be simplified, though, since $\neg p$ can be substituted into the universal claim $\forall p \phi$:

  $$\vdash \forall p(\mathfrak{s} \models p \;\&\; \mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \mathfrak{s}' \models p)$$

  Cf. van Benthem 1981, 3 ('Heredity'); 2016, 3; Holliday 2014, 315; forthcoming, 15; and Ding & Holliday 2020, 155. But see also Restall 2000, Definition 1.2 (Heredity Condition); Berto 2015, 767 (HC); Berto & Restall 2019, 1128 (HC).
- Cf. Barwise 1989a (265): p is *persistent* if and only if whenever p is true in s, p is true in every s' of which s is a part:

  $$Persistent(p) \;\equiv_{df}\; \forall s(s \models p \rightarrow \forall s'(s \trianglelefteq s' \rightarrow \; s' \models p))$$

  It is an immediate consequence that $\forall p\, Persistent(p)$. Thus, our theory implies Alternative 6.1 at Choice 6 in Barwise 1989a, 265.

Lemmas for the Refinability Principle

- The situation of *absolute necessity* (written $\boldsymbol{s}_\Box$) is, by definition, the situation in which all and only necessary truths are true:

  $\boldsymbol{s}_\Box =_{df} \imath s \forall p(s \models p \equiv \Box p)$
- If p is contingent, absolute necessity has a gap on p:

  $\vdash \mathit{Contingent}(p) \rightarrow \mathit{GapOn}(\boldsymbol{s}_\Box, p)$
- Absolute necessity is a possibility:

  $\vdash \mathit{Possibility}(\boldsymbol{s}_\Box)$
- Situations that are proper parts of absolute necessity are not possibilities:

  $\vdash \forall s((s \trianglelefteq \boldsymbol{s}_\Box \ \& \ s \neq \boldsymbol{s}_\Box) \rightarrow \neg \mathit{Possibility}(s))$
- Every possibility is a refinement of absolute necessity:

  $\vdash \forall \mathfrak{s}(\mathfrak{s} \trianglerighteq \boldsymbol{s}_\Box)$
- If a possibility has a gap on p, then p is contingent:

  $\vdash \mathit{GapOn}(\mathfrak{s}, p) \rightarrow \mathit{Contingent}(p)$
- If $\mathfrak{s}$ has a gap on p, then $\mathfrak{s}$ has a gap on $\neg p$:

  $\vdash \forall p(\mathit{GapOn}(\mathfrak{s}, p) \rightarrow \mathit{GapOn}(\mathfrak{s}, \neg p))$
- Possibilities are possible situations:

  $\vdash \mathit{Possible}(\mathfrak{s})$, i.e., $\mathit{Possibility}(s) \rightarrow \mathit{Possible}(s)$

The Refinability Principle

- If $\mathfrak{s}$ has a gap on p, then there is an $\mathfrak{s}'$ that refines $\mathfrak{s}$ in which p is true and an $\mathfrak{s}''$ that refines $\mathfrak{s}$ in which $\neg p$ is true:

  $$\vdash GapOn(\mathfrak{s}, p) \equiv \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models p) \ \& \ \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models \neg p)$$

  Cf. Humberstone 1981, 318; Holliday 2014, 315; forthcoming, 15; and Ding & Holliday 2020, 155.

- Proof Sketch: Let r be an arbitrary, but fixed, proposition.

  ($\rightarrow$) Since $GapOn(\mathfrak{s}, r)$ implies $GapOn(\mathfrak{s}, \neg r)$, it suffices to show only:

  $$GapOn(\mathfrak{s}, r) \rightarrow \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models r)$$

  So assume $GapOn(\mathfrak{s}, r)$ and find a witness to $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models r)$. Consider $(\mathfrak{s}^{+r})^{\star}$; abbreviate this as $\mathfrak{s}^{+r\star}$. We have to show all of the following: (a) $\mathfrak{s}^{+r\star} \trianglerighteq \mathfrak{s}$, (b) $\mathfrak{s}^{+r\star} \models r$, and (c) $Possibility(\mathfrak{s}^{+r\star})$. And by definition, the last of the above requires us to show (d) $Consistent(\mathfrak{s}^{+r\star})$ and (e) $ModallyClosed(\mathfrak{s}^{+r\star})$. (Exercises)

  ($\leftarrow$) Assume: $\exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models r) \ \& \ \exists \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \ \& \ \mathfrak{s}' \models \neg r)$. Call this ($\vartheta$). For reductio, suppose $\neg GapOn(\mathfrak{s}, r)$. Then either $\mathfrak{s} \models r$ or $\mathfrak{s} \models \neg r$. Wlog, suppose $\mathfrak{s} \models r$. By Persistence Principle, every refinement of s makes r true. So there can't be a refinement that makes $\neg r$ true, contradicting the right conjunct of (ϑ).

The Cofinality, Negation, and Conjunction Principles

- Cofinality: If, for every possibility $\mathfrak{s}'$ that refines $\mathfrak{s}$, there is a possibility $\mathfrak{s}''$ that refines $\mathfrak{s}'$ in which p is true, then p is true in $\mathfrak{s}$:

  $$\vdash \forall \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \exists \mathfrak{s}''(\mathfrak{s}'' \trianglerighteq \mathfrak{s}' \ \& \ \mathfrak{s}'' \models p)) \rightarrow \mathfrak{s} \models p$$

  Cf. van Benthem 1981, 4; 2016, 3; and compare Humberstone's (2011, 900) new statement of the Refinement Principle.

- Negation: The negation of p is true in $\mathfrak{s}$ if and only if p fails to be true in every refinement of s:

  $$\vdash \mathfrak{s} \models \neg p \ \equiv \forall \mathfrak{s}'(\mathfrak{s}' \trianglerighteq \mathfrak{s} \rightarrow \neg \mathfrak{s}' \models p)$$

  Cf. Humberstone 1981, 320; 2011, 900.

- Conjunction: The conjunction *p and q* is true in $\mathfrak{s}$ if and only if both p and q are true in $\mathfrak{s}$:

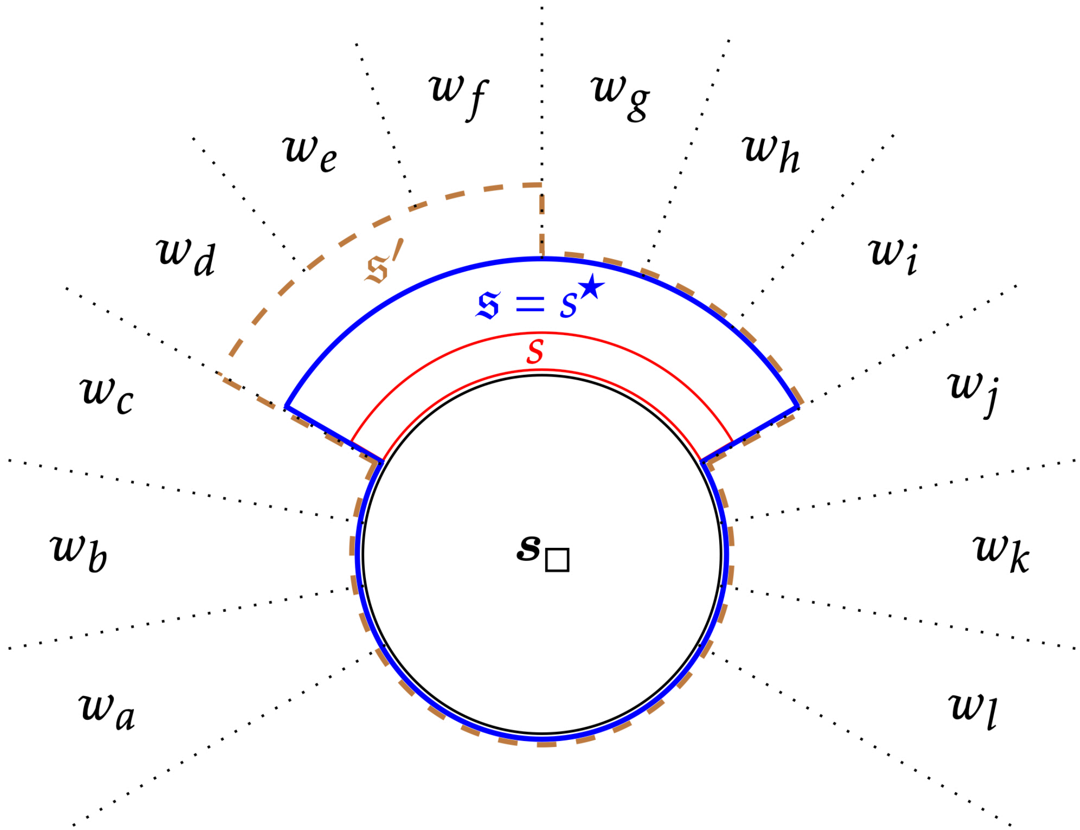
  $$\vdash \mathfrak{s} \models (p \ \& \ q) \equiv (\mathfrak{s} \models p \ \& \ \mathfrak{s} \models q)$$

  Cf. Humberstone 1981, 319; 2011, 900.

The Fundamental Theorems

- A proposition p is possible if and only if there is a possibility in which p is true:
  - $\Diamond p \equiv \exists \mathfrak{s}(\mathfrak{s} \models p)$
- A proposition p is necessary if and only if p is true in all possibilities:
  - $\Box p \equiv \forall \mathfrak{s}(\mathfrak{s} \models p)$

The Picture Again



$s_{\Box}$	=	the smallest possibility ('absolute necessity')
s	=	a *possible* situation
$s^{\star}$	=	the smallest possibility $\mathfrak{s}$ that contains s
$\mathfrak{s}'$	=	a refinement of $\mathfrak{s}$

Proof: $\forall x(\textit{PossibleWorld}(x) \rightarrow \textit{Maximal}(x))$

By GEN, we show $\textit{PossibleWorld}(x) \rightarrow \textit{Maximal}(x)$. So assume $\textit{PossibleWorld}(x)$. We have to show, for an arbitrary q, $x \models q \lor x \models \neg q$. We first appeal to an instance of a modal theorem, namely, $\Box(\phi \rightarrow \psi) \rightarrow (\Diamond\phi \rightarrow \Diamond\psi)$, where the instance is obtained by setting ϕ to $\forall p((x \models p) \equiv p)$ and ψ to $x \models q \lor x \models \neg q$. Then since $q \lor \neg q$, it follows that $\phi \rightarrow \psi$. Since we derived the conditional from no assumptions or contingent premises, it follows by RN that $\Box(\phi \rightarrow \psi)$. So by the instance of our modal theorem, $\Diamond\phi \rightarrow \Diamond\psi$. Since we know $\Diamond\phi$ (by the definition of possible world), we may infer $\Diamond\psi$, i.e., $\Diamond(x \models q \lor x \models \neg q)$. Then $\Diamond x \models q \lor \Diamond x \models \neg q$. But $\Diamond xF \rightarrow \Box xF$, and so $\Box x \models q \lor \Box x \models \neg q$. But by the T schema, $x \models q \lor x \models \neg q$.

A Lemma

- $\vdash \Diamond Situation(x) \rightarrow Situation(x)$
- Assume $\Diamond Situation(a)$, i.e., $\Diamond\forall F(aF \rightarrow \exists p(F{=}[\lambda y\, p]))$
- Show: $aG \rightarrow \exists p(G{=}[\lambda y\, p])$, where G is arbitrary.
- Assume aG, and so by rigidity, $\Box aG$
- By the Buridan schema: $\forall F\Diamond(aF \rightarrow \exists p(F{=}[\lambda y\, p]))$
- So in particular: $\Diamond(aG \rightarrow \exists p(G = [\lambda y\, p]))$.
- By modal logic and $\Box aG$: $\Diamond\exists p(G = [\lambda y\, p])$.
- By BF, $\exists p\Diamond(G = [\lambda y\, p])$.
- By the definition of =, $\exists p\Diamond\Box\forall x(xG \equiv x[\lambda y\, p])$.
- In S5, $\Diamond\Box\phi \rightarrow \Box\phi$, so reducing and applying the definition of =, it follows that $\exists p(G = [\lambda y\, p])$.
- By conditional proof, $aG \rightarrow \exists p(G = [\lambda y\, p])$.
- $\forall F(aF \rightarrow \exists p(F{=}[\lambda y\, p]))$, since G was arbitrary.

Proof: There is a Unique Actual World: I

- To show $\Diamond\forall p(a \models p \equiv p)$, let q be an arbitrary proposition, and first show: $a \models q \equiv q$.
  - $(\rightarrow)$
  - $a \models q$, i.e., $a[\lambda y\, q]$ — assumption
  - $\exists p(p \;\&\; [\lambda y\, q] = [\lambda y\, p])$ — defn of a
  - $r \;\&\; [\lambda y\, q] = [\lambda y\, r]$ — r arbitrary
  - $q = r$ — defn of $q = r$
  - q — by =E
  - $(\leftarrow)$
  - q — assumption
  - $q \;\&\; [\lambda y\, q] = [\lambda y\, q]$ — =I
  - $\exists p(p \;\&\; [\lambda y\, q] = [\lambda y\, p])$ — ∃I
  - $a[\lambda y\, q]$, i.e., $a \models q$ — by (θ)
- So $\forall p(a \models p \equiv p)$, and *a fortiori*, $\Diamond\forall p(a \models p \equiv p)$
- Thus, $\mathit{PossibleWorld}(a)$.

Proof: There is a Unique Actual World: II

- Show: $Actual(a)$
- But we previously showed: $\forall p(a \models p \equiv p)$. *A fortiori*, $\forall p(a \models p \rightarrow p)$. So it remains to show uniqueness, i.e.,
- Show: $\forall x(PossibleWorld(x) \,\&\, Actual(x) \rightarrow x{=}a)$
  - Assume, for reductio, that b is an actual world distinct from a.
  - Then, since a, b are distinct abstract objects, they differ by at least one encoded property.
  - Without loss of generality, suppose aP and $\neg bP$.
  - Since a is a situation, there is a proposition, say q, such that $P = [\lambda y\, q]$.
  - So, by definition, $a \models q$ and $\neg b \models q$.
  - Then by maximality, $b \models \neg q$.
  - But both a and b are actual, so q (given that a is actual) and $\neg q$ (given that b is actual). Contradiction.

Proof of the Strengthened Lewis Principle

Theorem: $\forall p(\Diamond p \equiv \exists w(w \models p))$

Show: $(\rightarrow)$: $\Diamond q \rightarrow \exists w(w \models q)$, where q is arbitrary.

Proof strategy:

Stage A: Show: $\Diamond q \rightarrow \Diamond\exists w(w \models q)$

Stage B: Show $\Diamond\exists w(w \models q) \rightarrow \exists w(w \models q)$.

Stage A Strategy:

- Show: $\Box(q \rightarrow \exists w(w \models q))$:
  - Assume q
  - Derive: $\exists w(w \models q)$
  - Use Conditional Proof: $q \rightarrow \exists w(w \models q)$
  - Use RN: $\Box(q \rightarrow \exists w(w \models q))$
- Conclude: $\Diamond q \rightarrow \Diamond\exists w(w \models q)$, by modal theorem

Proof of the Strengthened Lewis Principle

- So assume q. We want to show: $\exists w(w \models q)$, i.e.,

  $\exists x(PossibleWorld(x) \;\&\; x \models q)$
- By Comprehension:

  $\exists x(A!x \;\&\; \forall F(xF \equiv \exists p(p \;\&\; F = [\lambda y\, p])))$
- Let a be such an object:

  $A!a \;\&\; \forall F(aF \equiv \exists p(p \;\&\; F = [\lambda y\, p]))$ (θ)
- Show: $PossibleWorld(a) \;\&\; a \models q$:
  - $PossibleWorld(a)$ — by previous reasoning
  - q — by assumption
  - $q \;\&\; [\lambda y\, q] = [\lambda y\, q]$ — $=$I
  - $\exists p(p \;\&\; [\lambda y\, q] = [\lambda y\, p])$ — $\exists$I
  - $a[\lambda y\, q]$ — from (θ)
  - $a \models q$ — by definition
- So, by CP, $q \rightarrow \exists w(w \models q)$. Since no contingent premises were used, it follows by RN: $\Box(q \rightarrow \exists w(w \models q))$. And thus, given modal logic, it follows that $\Diamond q \rightarrow \Diamond\exists w(w \models q)$ Stage A (✓).

Proof of the Strengthened Lewis Principle

- Stage B: Show $\Diamond\exists w(w \models q) \rightarrow \exists w(w \models q)$.
- Assume $\Diamond\exists w(w \models q)$ Show: $\exists w(w \models q)$.
- Eliminating the restricted variable w:
  $\Diamond\exists x(PossibleWorld(x) \ \&\ x \models q)$.
- By BF:
  $\exists x\Diamond(PossibleWorld(x) \ \&\ x \models q)$.
- Let a be such an object; i.e.,
  $\Diamond(PossibleWorld(a) \ \&\ a \models q)$
- By modal logic: $\Diamond PossibleWorld(a) \ \&\ \Diamond a \models q$.
- Show each possibility is a non-modal fact!
  Show $PossibleWorld(a)$.
  Show $a \models q$.

Proof of the Strengthened Lewis Principle

- Show $PossibleWorld(a)$.
- By definition, from $\Diamond PossibleWorld(a)$, we know:
  $\Diamond(A!a \;\&\; \forall F(aF \rightarrow \exists p(F = [\lambda y\, p])) \;\&\; \Diamond\forall p(a \models p \equiv p))$.
- By modal logic:
  $\Diamond A!a \;\&\; \Diamond\forall F(aF \rightarrow \exists p(F = [\lambda y\, p])) \;\&\; \Diamond\Diamond\forall p(a \models p \equiv p) \quad (\theta)$
- We have to show:
  1. $A!a$
  2. $\forall F(aF \rightarrow \exists p(F = [\lambda y\, p]))$
  3. $\Diamond\forall p(a \models p \equiv p)$
- (1) follows from the 1st conjunct of (θ):
  $\Diamond A!a \rightarrow \Diamond\neg\Diamond E!a \rightarrow \Diamond\Box\neg E!a \rightarrow \Box\neg E!a \rightarrow \neg\Diamond E!a \rightarrow A!a$
- (2) follows from the second conjunct of (θ) by our Lemma.
- (3) follows from the third conjunct of (θ) by the S4 theorem.

Proof of the Strengthened Lewis Principle

- Show: $a \models q$.
- We already know $\Diamond a \models q$.
- By definition, $\Diamond a[\lambda y\ q]$
- By the Rigidity of Encoding, $\Box a[\lambda y\ q]$.
- By the T schema, $a[\lambda y\ q]$
- By definition, $a \models q$

- Thus, we've shown $\Diamond\exists w(w \models q) \rightarrow \exists w(w \models q)$.
- Stage B (✓)
- So from Stage A and Stage B: $\Diamond q \rightarrow \exists w(w \models q)$.
- Proof of ($\rightarrow$) direction of theorem is done.

Proof of the Strengthened Lewis Principle

- Theorem: $\forall p(\Diamond p \equiv \exists w(w \models p))$
- Show: $(\leftarrow)$: $\exists w(w \models q) \rightarrow \Diamond q$, where q is arbitrary.
- Assume $\exists w(w \models q)$, i.e., $\exists x(\mathit{PossibleWorld}(x) \;\&\; x \models q)$.
- Let a be such an object: $\mathit{PossibleWorld}(a) \;\&\; a \models q$ $\qquad(\theta)$
- By definition, the left conjunct yields: $\Diamond\forall p(a \models p \equiv p)$.
- By the Buridan formula: $\forall p\Diamond(a \models p \equiv p)$.
- So in particular: $\Diamond(a \models q \equiv q)$.
- By definition: $\Diamond(a[\lambda y\, q] \equiv q)$
- *A fortiori*, $\Diamond(a[\lambda y\, q] \rightarrow q)$, i.e., $\Diamond(\neg a[\lambda y\, q] \vee q)$
- By basic modal logic: $\Diamond\neg a[\lambda y\, q] \vee \Diamond q$
- By (θ), $a \models q$, and by rigidity, $\Box a \models q$, i.e., $\Box a[\lambda y\, q]$.
- By Disjunctive Syllogism, $\Diamond q$.
- Thus, $\exists w(w \models q) \rightarrow \Diamond q$
- The $(\leftarrow)$ direction of the theorem is done.
- Q.E.D.

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