Seminar on Axiomatic Metaphysics
Lecture 6
Impossible Worlds and Leibnizian Concepts

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Why Appeal to Impossible Worlds

- A valid reason: Some counterfactuals have impossible antecedents:
  - If Frege's system in *Grundgesetze* had been consistent, he would have established a form of logicism [or died a happy man, etc.]
  - If my parents had been different, my genes and heritage would have been different.
- Impossible worlds help us represent and interpret non-classical logics (e.g., paraconsistent logic) in which contradictions don't imply every proposition.
- We *don't* need impossible worlds to distinguish necessarily equivalent properties and propositions (Yagisawa 1988). OT is already hyperintensional.
- We *don't* need impossible worlds to model impossible objects (Priest 1995); use objects that encode inconsistent properties.
- Are some contradictions true? Priest (1998) argues: there are. But conditions that appear to imply a true contradiction can be analyzed in terms of objects that encode inconsistent properties.

Impossible Worlds I

- $Possible(s) \equiv_{df} \Diamond Actual(s)$, i.e., $\Diamond\forall p((s \models p) \rightarrow p)$
- $ImpossibleWorld(x) \equiv_{df}$
  $Situation(x) \mathbin{\&} Maximal(x) \mathbin{\&} \neg Possible(x)$
- Let i, i' be rigid restricted variables. Then $i \models p$ is defined and:
  $i = i' \equiv \forall p(i \models p \equiv i' \models p)$
- $\exists s ImpossibleWorld(s)$. Consider the universal situation $\boldsymbol{s_V}$:
  $\iota x(A!x \mathbin{\&} \forall F(xF \equiv \exists p(F = [\lambda y\, p])))$
- $TrivialSituation(x) \equiv_{df} Situation(x) \mathbin{\&} \forall p(x \models p)$
- $TrivialSituation(\boldsymbol{s_V})$
- $\bot =_{df} \iota x(A!x \mathbin{\&} \forall F(xF \equiv \exists p(\neg p \mathbin{\&} F = [\lambda y\, p])))$
- $ImpossibleWorld(\bot) \mathbin{\&} \neg TrivialSituation(\bot)$

Proof: **Situation**. (easy) **Maximal**: Reason from $\mathcal{A}p \vee \mathcal{A}\neg p$. If $\mathcal{A}q$, then $\mathcal{A}\neg\neg q$ and $\bot \models \neg q$. If $\mathcal{A}\neg q$, then $\bot \models q$. **Impossible**: Consider arbitrary q and the fact $\mathcal{A}\neg(q \mathbin{\&} \neg q)$. Then $\bot \models (q \mathbin{\&} \neg q)$. So, $\neg Possible(\bot)$. **Non-Trivial**: Where $p_0 = \forall x(E!x \rightarrow E!x)$, assume $\bot \models p_0$ (reductio). Then $\mathcal{A}\neg p_0$, i.e., $\neg\mathcal{A}p_0$. Contradiction.

Impossible Worlds II

- $\neg ModallyClosed(\bot)$
  Proof: $\bot$ encodes all and only (actual) falsehoods and so contradictions, which imply unencoded necessary truths.
- $s^{+p} =_{df} \iota s' \forall q(s' \models q \equiv (s \models q \vee q = p))$
- s^{+p} is 'strictly canonical': for any q, if $s \models q \vee q = p$, then this disjunction holds necessarily.
- Lemmas:
  - $s \models q \;\rightarrow\; s^{+p} \models q$
  - $s^{+p} \models p$
  - $\forall q(s^{+p} \models q \equiv (s \models q \vee q = p))$
- The Fundamental Theorem of Impossible World Theory:
  $\neg \Diamond p \rightarrow \exists s(ImpossibleWorld(s) \;\&\; s \neq s_{\boldsymbol{V}} \;\&\; s \models p)$
  - *Proof*: Suppose $\neg\Diamond p$. Consider $\boldsymbol{w}_{\alpha}^{+p}$

Cf. Zalta 1997, Nolan 1997.

Impossible Worlds III

- *Ex contradictione quodlibet* fails for impossible worlds:
  $\exists p \exists q \exists s(\mathit{ImpossibleWorld}(s) \;\&\; s \models (p \;\&\; \neg p) \;\&\; s \not\models q)$
  - *Proof*: Let p_1 be any proposition and formulate $[\lambda y\, p_1 \;\&\; \neg p_1]$. Now let q_1 be any false proposition other than $[\lambda\, p_1 \;\&\; \neg p_1]$. Then consider $\boldsymbol{w}_{\alpha}^{+(p_1 \;\&\; \neg p_1)}$.
- Disjunctive Syllogism fails for impossible worlds:
  $\exists s[\mathit{ImpossibleWorld}(s) \;\&\; s \models \neg p \;\&\; s \models (p \vee q) \;\&\; s \not\models q]$
  - *Proof*: Consider any two propositions p_1, q_1 where p_1 is true and q_1 is false. Consider: $\boldsymbol{w}_{\alpha}^{+\neg p_1}$. This encodes all the truths as well as one other proposition, namely, the falsehood $\neg p_1$. Since $\boldsymbol{w}_{\alpha}^{+\neg p_1}$ encodes all the truths, it encodes p_1 and therefore $p_1 \vee q_1$. But it also fails to encode q_1, since q_1 is false. So $\boldsymbol{w}_{\alpha}^{+\neg p_1}$ is an impossible world where $p_1 \vee q_1$ and $\neg p_1$ are true, but q_1 isn't.

3 Threads in Leibniz on Concepts

- Leibniz's work on concepts divides up into areas:
  - A non-modal 'calculus' of concepts
  - The concept containment theory of truth
  - The modal metaphysics of complete individual concepts.
- The non-modal calculus is more like an algebra of concepts.
  - Scan of Leibniz 1690 (LP, 131–144)
  - List of propositions in Leibniz 1690
- Leibniz takes a primitive operation $\oplus$ on concepts: every pair of concepts has a sum
  - Axiom 1: $\oplus$ is commutative
  - Axiom 2: $\oplus$ is idempotent
  - Leibniz omits associativity of $\oplus$, and some proofs fail.
- Leibniz 'defines' a primitive relation ($\preceq$): x is included in y:
  - Definition 3. That A 'is in' L, or, that L 'contains' A, is the same as that L is assumed to be coincident with several terms taken together, among which is A. $[A \preceq L \ =_{df} \ \exists B(L = A \oplus B)]$
- Computational studies using PROVER9.

Leibnizian Concept Theory I

- *x is a Leibnizian concept* ('$C!x$') $=_{df}$ $A!x$
- Let $c, d, e \ldots$ range over (Leibnizian) concepts ($C!$ is a rigid property: $\Box\forall x C!x \rightarrow \Box C!x$).
- $c = c \quad c = d \rightarrow d = c \quad c = d \;\&\; d = e \rightarrow c = e$
- $SumOf(c, d, e) \equiv_{df} \forall F(cF \equiv dF \vee eF)$
- Since $\exists! c SumOf(c, d, e)$, we define:
  The sum of d and e ('$d \oplus e$') $=_{df}$ $\iota c SumOf(c, d, e)$
- Sums are 'strictly canonical', since $dF \vee eF \rightarrow \Box(dF \vee eF)$. So $\forall F(d \oplus eF \equiv dF \vee eF)$ and $SumOf(d \oplus e, d, e)$.
- $\oplus$ is idempotent, commutative, and associative:
  - $c \oplus c = c$.
    - *Proof*. From $(\phi \vee \phi) \equiv \phi$
  - $c \oplus d = d \oplus c$.
    - *Proof*. From $\phi \vee \psi \equiv \psi \vee \phi$
  - $(c \oplus d) \oplus e = c \oplus (d \oplus e)$.
    - *Proof*. From $(\phi \vee \psi) \vee \chi \equiv \phi \vee (\psi \vee \chi)$

Leibnizian Concept Theory II

- *c is included in d* ('$c \leq d$') $=_{df}$ $\forall F(cF \rightarrow dF)$
  - Note the connection with the *part-of* relation in situation theory.
- *d contains c* ('$d \geq c$') $=_{df}$ $c \leq d$
- $\leq$ ($\geq$) is reflexive, anti-symmetric, and transitive:

  $c \leq c$ $\qquad$ $c \geq c$

  $c \leq d \rightarrow (c \neq d \rightarrow d \not\leq c)$ $\qquad$ $c \geq d \rightarrow (c \neq d \rightarrow d \not\geq c)$

  $c \leq d \;\&\; d \leq e \rightarrow c \leq e$ $\qquad$ $c \geq d \;\&\; d \geq e \rightarrow c \geq e$
- $\forall e(e \leq c \equiv e \leq d) \rightarrow c = d$ $\qquad$ $\forall e(c \geq e \equiv d \geq e) \rightarrow c = d$
  - *Proof*. Assume $\forall e(e \leq c \equiv e \leq d)$. For reductio, assume $c \neq d$. Without loss of generality, suppose cQ and $\neg dQ$, where Q is arbitrary. Consider the concept, say c', that encodes just Q. Then $\forall F(c'F \rightarrow cF)$. So $c' \leq c$. But then $c' \leq d$, i.e., $\forall F(c'F \rightarrow dF)$. So dQ. Contradiction.

Theorems Involving Concept Summation and Inclusion

- $\forall e(c \preceq e \equiv d \preceq e) \;\rightarrow\; c{=}d \qquad \forall e(e \succeq c \equiv e \succeq d) \rightarrow c{=}d$
  - *Proof*. Assume $\forall e(c \preceq e \equiv d \preceq e)$ and $c \neq d$. From the latter w.l.o.g., assume cP and $\neg dP$. So $c \npreceq d$. But instantiating our assumption to d, $c \preceq d \equiv d \preceq d$. But we know $d \preceq d$. So $c \preceq d$. Contradiction.
- $c \preceq c{\oplus}d \qquad c{\oplus}d \succeq c$
  $d \preceq c{\oplus}d \qquad c{\oplus}d \succeq d$
- $c \preceq d \;\rightarrow\; e{\oplus}c \preceq e{\oplus}d$ Leibniz 1690, Prop. 12
  $c \succeq d \;\rightarrow\; e{\oplus}c \succeq e{\oplus}d$
- Leibniz 1690, Corollary to Prop. 15, Prop. 18, Prop. 20:
  $c{\oplus}d \preceq e \;\rightarrow\; c \preceq e \;\&\; d \preceq e \qquad e \succeq c{\oplus}d \;\rightarrow\; e \succeq c \;\&\; e \succeq d$
  $c \preceq e \;\&\; d \preceq e \;\rightarrow c{\oplus}d \preceq e \qquad e \succeq c \;\&\; e \succeq d \;\rightarrow e \succeq c{\oplus}d$
  $c \preceq d \;\&\; e \preceq f \;\rightarrow c{\oplus}e \preceq d{\oplus}f \qquad c \succeq d \;\&\; e \succeq f \;\rightarrow c{\oplus}e \succeq d{\oplus}f$

Theorems Involving Concept Summation and Inclusion

- $c \preceq d \;\equiv\; \exists e(c \oplus e = d)$ Leibniz, 1690, Definition 3
- *Proof*. ($\rightarrow$) Assume $c \preceq d$. (a) Suppose $c = d$. By the idempotency of $\oplus$, $c \oplus c = c$, in which case, $c \oplus c = d$. So, $\exists e(c \oplus e = d)$. (b) Suppose $c \neq d$. Then since $c \preceq d$, we know $\exists F(dF \,\&\, \neg cF)$. Consider, then, the concept that encodes any every property: $\imath c' \forall F(c'F \equiv dF \,\&\, \neg cF))$. Call this object '$e_1$'. We need only establish that $c \oplus e_1 = d$, i.e., that $c \oplus e_1$ and d encode the same properties. (($\rightarrow$)) Assume $c \oplus e_1 P$ (to show: dP). Then $cP \vee e_1 P$, by definition of $\oplus$. If cP, then by the fact that $c \preceq d$, it follows that dP. On the other hand, if $e_1 P$, then by definition of e_1, it follows that $dP \,\&\, \neg cP$. So in either case, we have dP. (($\leftarrow$)) Assume dP (to show $c \oplus e_1 P$). The alternatives are cP or $\neg cP$. If cP, then $cP \vee e_1 P$, then $c \oplus e_1 P$, by definition of $\oplus$. Alternatively, if $\neg cP$, then we have $dP \,\&\, \neg cP$. So $e_1 P$, by definition of e_1, and by familiar reasoning, it follows that $c \oplus e_1 P$. Combining both directions of our biconditional, we have established that $c \oplus e_1 P \equiv dP$, for an arbitrary P. So $c \oplus e_1 = d$, and we therefore have $\exists e(c \oplus e = d)$.

  ($\leftarrow$) Assume $\exists e(c \oplus e = d)$. Let '$e_2$' be such, so that we know $c \oplus e_2 = d$. To show $c \preceq d$, assume cP (to show dP). Then, $cP \vee e_2 P$, which by the definition of $\oplus$, entails that $c \oplus e_2 P$. But by hypothesis, $c \oplus e_2 = d$. So dP. $\bowtie$
- $c \preceq d \;\equiv\; c \oplus d = d$ (Leibniz 1690, Proposition 13)

A Complete Boolean Algebra of Concepts

- $ProductOf(c, d, e) \ \equiv_{df} \ \forall F(cF \equiv dF \ \& \ eF)$
- $d \otimes e \ =_{df} \ \imath c ProductOf(c, d, e)$
- $\otimes$ is idempotent, commutative, and associative
- Absorption Laws are theorems:

  $c \oplus (c \otimes d) = c \qquad c \otimes (c \oplus d) = c$
- Concepts form a bounded lattice ($\boldsymbol{a}_{\emptyset}$ = null; $\boldsymbol{a}_{V}$ = universal):

  $c \oplus \boldsymbol{a}_{\emptyset} = c$

  $c \otimes \boldsymbol{a}_{V} = c$

  $c \oplus \boldsymbol{a}_{V} = \boldsymbol{a}_{V}$

  $c \otimes \boldsymbol{a}_{\emptyset} = \boldsymbol{a}_{\emptyset}$
- Distribution Laws are theorems:

  $c \oplus (d \otimes e) = (c \oplus d) \otimes (c \oplus e)$

  $c \otimes (d \oplus e) = (c \otimes d) \oplus (c \otimes e)$
- Complements and complementation laws:

  $ComplementOf(c, d) \ \equiv_{df} \ \forall F(cF \equiv \neg dF)$

  $-d \ =_{df} \ \imath c ComplementOf(c, d)$

  $c \oplus -c = \boldsymbol{a}_{V}$

  $c \otimes -c = \boldsymbol{a}_{\emptyset}$

A Mereology of Concepts

Varzi (2019, §1) writes:

> … it is worth stressing that mereology assumes no ontological restriction on the field of 'part'. In principle, the relata can be as different as material bodies, events, geometric entities, or spatio-temporal regions, … as well as abstract entities such as properties, propositions, types, or kinds, … ….As a formal theory … mereology is simply an attempt to lay down the general principles underlying the relationships between an entity and its constituent parts, whatever the nature of the entity, just as set theory is an attempt to lay down the principles underlying the relationships between a set and its members. Unlike set theory, mereology is not committed to the existence of abstracta: the whole can be as concrete as the parts. But mereology carries no nominalistic commitment to concreta either: the parts can be as abstract as the whole.

Cf. *Principia Logico-Metaphysica*, Section 13.1.5.

Concepts of Properties and Individuals

- $ConceptOf(c, G) \equiv_{df} G\!\downarrow \;\&\; \forall F(cF \equiv G \Rightarrow F)$
- The concept G ('$\boldsymbol{c}_G$') $=_{df} \; \iota c ConceptOf(c, G)$
- Note the connection to Plato's Forms: $\boldsymbol{c}_G = \Phi_G$
- $\boldsymbol{c}_G F \equiv G \Rightarrow F$ (Lemma 1)
- Let u, v range over discernible individuals.
- $ConceptOf(c, u) \equiv_{df} \forall F(cF \equiv Fu)$
- The concept of u ('$\boldsymbol{c}_u$') $=_{df} \iota c ConceptOf(c, u)$
- $\star \vdash \boldsymbol{c}_u G \equiv Gu$ ($\star$Lemma 2)
  Proof: $ConceptOf(\boldsymbol{c}_u, u)$ follows from $\boldsymbol{c}_u = \iota c ConceptOf(c, u)$ by a $\star$-theorem; then apply the definition and instantiate to G.
- The biconditional in $\star$Lemma 2 is not subject to Rule RN; the reasoning to establish the biconditional depends upon contingencies. We'll see that this will help to explain something Leibniz says. Contrast: $\boldsymbol{c}_u G \equiv \mathcal{A} Gu$

The Containment Theory of Truth

- d *contains* c ('$d \geq c$') $=_{df}$ $\forall F(cF \rightarrow dF)$
- Leibnizian analysis of 'Alexander is king':
  The concept Alexander contains the concept king: $\boldsymbol{c}_a \geq \boldsymbol{c}_K$
- $\star \vdash Gu \equiv \boldsymbol{c}_u \geq \boldsymbol{c}_G$
- *Proof*: ($\rightarrow$) Assume Gu. To show $\boldsymbol{c}_u \geq \boldsymbol{c}_G$, show $\forall F(\boldsymbol{c}_G F \rightarrow \boldsymbol{c}_u F)$. Assume $\boldsymbol{c}_G F$. Then by Lemma 1, $G \Rightarrow F$. So Fu. Then $\boldsymbol{c}_u F$, by $\star$Lemma 2. ($\leftarrow$) Assume $\boldsymbol{c}_u \geq \boldsymbol{c}_G$ (to show Gu). So $\forall F(\boldsymbol{c}_G F \rightarrow \boldsymbol{c}_u F)$. Clearly, $\boldsymbol{c}_G G$. So $\boldsymbol{c}_u G$. And by definition of $\boldsymbol{c}_u$, it follows that Ga. $\bowtie$
- Fact: (1) we can't apply RN to this biconditional theorem since it is a $\star$-theorem. (2) We can't add the premise Fu, infer $\boldsymbol{c}_u \geq \boldsymbol{c}_F$ and apply RN, since any such reasoning fails to be modally strict – the reasoning depends on contingencies.

Observations

- Leibniz appealed to 'hypothetical necessity' to answer Arnauld's objection that truth as containment turns contingent propositions into necessities. If Leibniz's hypothetical necessities are necessary truths that depend upon a hypothesis, then we have a way to understand him. The proof of $\boldsymbol{c}_a \succeq \boldsymbol{c}_K$ depends on a contingent premise.
- Leibniz's analysis extends to generalized quantifiers:
  - $\boldsymbol{c}_{\forall G} =_{df} \imath c \forall F(cF \equiv \forall x(Gx \rightarrow Fx))$
  - $\boldsymbol{c}_{\exists G} =_{df} \imath c \forall F(cF \equiv \exists y(Gy \,\&\, Fy))$
- Leibnizian analysis of "Every person is rational":
  - The concept *every person* contains the concept *being rational*
  - $\boldsymbol{c}_{\forall P} \succeq \boldsymbol{c}_R$
- This is equivalence to the modern analysis, $\forall x(Px \rightarrow Rx)$, by the $\star$-theorem:
  - $\star \vdash \forall x(Gx \rightarrow Fx) \equiv \boldsymbol{c}_{\forall G} \succeq \boldsymbol{c}_F$
- Generalize to "Some person is rational" (exercise).
- Containment theory of truth anticipates generalized quantifiers.

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