Seminar on Axiomatic Metaphysics
Lecture 8
Fregean Senses

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Stage Setting I

- Frege's work roughly divides into (a) his logic and theory of functions and extensions (which he used to reduce/derive number theory) and (b) his theory of language.
- But Frege developed his theory of language in the service of his theory of extensions and numbers. He had two concerns: (a) show that mathematical propositions are not trivial, and (b) provide an epistemological explanation of how we apprehend mathematical content.
- Many mathematical theorems and formulas are identity statements. Since Frege was concerned to show that the content of such mathematical theorems does not consist of trivialities, Frege developed two notions of content, i.e., sense and reference.
- "If there are logical objects at all—and the objects of arithmetic are such objects—then there must also be a means of apprehending, or recognizing them. This … is performed … by the fundamental law of logic that permits the transformation of an equality holding generally into an equation." (Gg II, §147)

Stage Setting II

- But Frege's theories of language and mathematics are developed using different entities, namely, senses and extensions.
- While it seems clear that senses are abstract objects, they are not described by Frege other than by way of extrinsic features and roles they play in his philosophy of language; they are primitive entities. There are no principles that characterize them intrinsically as entities (cf. sets and set theory).
- By contrast, Frege does attempt to axiomatize extensions. Clearly, they too are abstract objects. But Frege doesn't suggest that extensions and senses are related.
- As far as we know, there is no theoretical connection between the objects Frege utilizes in his theory of language and the objects Frege utilizes in his theory of mathematics.
- Over the next few lectures, we develop such a theoretical connection. (Connect Zalta 1999 and with Zalta 1988, 2001.)

Fregean Theses About Natural Language: I

- Singular terms denote objects; predicates denote concepts.
- *The denotations of predicates map the denotations of singular terms to truth values.
- *Sentences denote truth values.
- The sense of a singular term is an object; the sense of a predicate is a function (?) – see Heck & May 2011.
- The sense of a sentence is a thought.
- Senses of predicates map senses of singular terms to thoughts (?)
- Senses are modes of presentation.
- *The sense of an expression E determines E's denotation.
- The thought expressed by S is true iff S denotes The True.
- *Fictional names denote nothing.
- If a part of S fails to denote, S fails to denote.
- Inside certain linguistic contexts, terms denote their senses.

Fregean Theses About Natural Language: II

- Substitution of co-denoting terms preserves denotation. [Compositionality for denotation]
- Substitution of co-expressing terms preserves sense. [Compositionality for sense]
- The senses of the parts of the sentence can be recovered from the sense of the whole sentence (?) [Decompositionality for sense]
- A word has meaning only in the context of a sentence. [Context Principle]
- Pelletier 2001: there is a split in the community of Frege scholars as to whether the Compositionality or Context principle is a more important contribution to the philosophy of language.
- Frege uses the Context Principle in the analysis of mathematics (in Gl): instances introduce logical and mathematical objects. [cf. The St. Andrews School]

Roles Senses Play in Frege's Philosophy of Language

- Burge 1977 lists three roles senses are supposed to play (p. 356)
- **Sense$_1$**: The mode of representation to the thinker which is associated with an expression. Sense$_1$ accounts for the information value associated with an expression.
- **Sense$_2$**: That which determines the reference or denotation associated with an expression; for singular terms, senses serve as "routes" to singling out the unique object, if any, denoted by the term.
- **Sense$_3$**: The entity denoted by the term in oblique contexts.

The General Picture We Shall Motivate

- The denotation of an individual term is typically an ordinary individual.
- The denotation of a predicate is a property or relation.
- The denotation of a sentence is a state of affairs (= 0-place relation). (Frege doesn't accept this.)
- The sense of a singular term is an abstract individual.
- The sense of a predicate is an abstract property or relation.
- The sense of a sentence is a logical complex, identical in structure to the state of affairs the sentence denotes, but which has senses as constituents.
- The cognitive significance of identity statements of the form $x = y$ (for individuals) and of the form $F = G$ (for properties, relations, etc.), can be given an uniform explanation.

How Abstract Objects Can Play the Role of Sense$_i$

- Sense$_1$: By encoding properties of ordinary individuals (relations), an abstract individual can present, or represent, an individual (relation). (Examples: Mark Twain, woodchuck)
- *Global* and *local* options: On the global option, the sense of a term does not vary from speaker to speaker. On the local option, the sense of a term does varies from speaker to speaker
- Sense$_2$: We can isolate a subclass of abstract objects x that *individuate* objects: necessarily, if there is an object that exemplifies all the properties x encodes, there is a unique object that exemplifies all the properties x encodes.

$$IndividualConcept(x) \ =_{df}$$
$$\Box[\exists y \forall F(xF \rightarrow Fy) \rightarrow \exists! y \forall F(xF \rightarrow Fy)]$$

- Sense$_3$: Since abstract individuals are individuals and abstract relations are relations, there exists 0-place relations which have both abstract individuals and abstract relations as constituents.

First Preliminary Consideration

- Recall global and local options: On the global option, the sense of a term does not vary from speaker to speaker. On the local option, the sense of a term does vary from speaker to speaker.
- May (2006) argues that Frege endorsed the global option. Let's suppose so. We will assume that for today. But if we put aside the question of Frege scholarship, there is a lot going for the local option, whether Frege endorsed it or not.
- We frequently learn, and competently use, proper names even though:
  - in a given context, different people associate different information with the name they learn, and
  - the information presented in the learning context doesn't determine or individuate anyone, or might even contain misinformation.

Digression: Example

- Consider this situation: You see a sign while walking along the road: “Dr. Gustav Lauben”, “GP”, “8am–5pm”, and fine print.
- Some people may not read the fine print, and so not take in the whole sign; the information someone comes away with will differ from person to person.
- Furthermore, the sign may present misinformation: Lauben may have lost his license 2 days before, sold his office, ...
- But competent speakers of the language can hold beliefs about Lauben after encountering the sign, and use the name “Dr. Gustav Lauben” in well-formed, meaningful sentences to assert propositions about Lauben.
- Here we might consider allowing different abstract objects to serve as senses of “Dr. Gustav Lauben” for different people.
- And we might consider using abstract objects which are not individual concepts, indeed, which encode misinformation.

Second Preliminary Consideration

- You might think that such models of the local option (senses may be non-individuating and may vary from person to person), would fail because everyone would speak a different language.
- But the idea that everyone would speak a different language rests on the idea that one and the same sentence would express different propositions for different people.
- Yet this wouldn't characterize our system, for the notion of 'proposition' is not univocal; it is replaced by thoughts and states of affairs. Our system can represent both the denotation of an English sentence as a logically complex entity, and its sense as a different logically complex entity.
- The fundamental metaphysical notion of truth resides at the level of states of affairs: a state of affairs may or may not obtain, and the thought expressed by a sentence is true or false depending on whether the state of affairs denoted by that sentence obtains.

The Data To Be Explained/Represented

- Identity Statements:
  - Cicero = Tully
  - Mark Twain = Samuel Clemens
  - The morning star is the evening star.

  - Being a woodchuck just is being a groundhog
  - Being a circle is being a plane figure every point of which lies equidistant from a given point.
  - Being a brother is being a male sibling.
- Failure of Substitution in Propositional Attitude Reports:
  - John believes Mark Twain wrote *Huckleberry Finn*.
  - John doesn't believe Samuel Clements wrote *Huckleberry Finn*.
  - MT = SC.

  - John believes my pet Woodie is a woodchuck.
  - John doesn't believe my pet Woodie is a groundhog.
  - being a woodchuck just is being a groundhog.

Type-Theoretic Object Theory

- Definition of the Types:
  - i (type for individuals)
    $\langle t_1, \ldots, t_n \rangle$ (type for relations, where $t_1, \ldots, t_n$ are any types)
    $\langle\,\rangle$ type for propositions
- Type the language and axioms of our system:
  - Atomic exemplification formulas: $F^{\langle t_1,\ldots,t_n\rangle}x^{t_1}\ldots x^{t_n}$
  - Atomic encoding formulas: $x^t F^{\langle t\rangle}$ (generally: $x^{t_1}\ldots x^{t_n}F^{\langle t_1,\ldots,t_n\rangle}$)
  - The usual complex formulas and complex terms.
  - Typed λ-Conversion:
    - $[\lambda x^{t_1}\ldots x^{t_n}\, \varphi]x^{t_1}\ldots x^{t_n} \equiv \varphi$
    - $[\lambda\, p^{\langle\,\rangle}] \equiv p$
  - Typed Object Comprehension:
    - $\exists x^t(A!^{\langle t\rangle}x \,\&\, \forall F^{\langle t\rangle}(xF \equiv \varphi))$, where φ has no free xs
- Specific Levels of Comprehension:
  - $\exists x^i(A!^{\langle i\rangle}x \,\&\, \forall F^{\langle i\rangle}(xF \equiv \varphi))$, where φ has no free xs
  - $\exists x^{\langle i\rangle}(A!^{\langle\langle i\rangle\rangle}x \,\&\, \forall F^{\langle\langle i\rangle\rangle}(xF \equiv \varphi))$, where φ has no free xs
  - $\exists x^{\langle i,i\rangle}(A!^{\langle\langle i,i\rangle\rangle}x \,\&\, \forall F^{\langle\langle i,i\rangle\rangle}(xF \equiv \varphi))$, where φ has no free xs

Notation for (Global) Senses: I

- The sense of a term of type i is an abstract object (individual concept) of type i.
  - Let '$\underline{\tau}$' denote the sense of the proper name τ
- The sense of a term of type $\langle i \rangle$ is an abstract object (individual concept) of type $\langle i \rangle$.
  - Let '$\underline{\Pi}$' denote the sense of the simple predicate Π
- Let 'B' denote the belief relation, of type $\langle i, p \rangle$.
- Case:

  Johnny believes that Woodie is a woodchuck.
  Johnny doesn't believe that Chuckie is a woodchuck.
  Woodie is identical to Chuckie.
  Johnny doesn't believe that Woodie is a groundhog.
  Being a woodchuck is identical to being a groundhog.

Application of Notation for (Global) Senses: I

- Johnny believes that Woodie is a woodchuck.
  $B(j, [\lambda\, Ww])$ (de re)
  $B(j, [\lambda\, \underline{W}\, \underline{w}])$ (de dicto)
- Johnny doesn't believe that Chuckie is a woodchuck.
  $\neg B(j, [\lambda\, Wc])$ (de re)
  $\neg B(j, [\lambda\, \underline{W}\, \underline{c}])$ (de dicto)
- Woodie just is Chuckie.
  $w = c$
- Johnny doesn't believe that Woodie is a groundhog.
  $\neg B(j, [\lambda\, Gw])$ (de re)
  $\neg B(j, [\lambda\, \underline{G}\, \underline{w}])$ (de dicto)
- Being a woodchuck just is being a groundhog.
  $W = G$

Application of Notation for (Global) Senses: II

- Consider the de dicto reading of ‘Johnny believes that Woodie is a woodchuck’:
  $B(j, [\lambda\ \underline{W}\ \underline{w}])$ (de dicto)
- Question: Under what conditions is the thought $[\lambda\ \underline{W}\ \underline{w}]$ true?
- Answer: $[\lambda\ \underline{W}\ \underline{w}]$ is true iff Ww.
- General Answer: The thought represented by $[\lambda\ \varphi]$ is true iff φ^*, where φ^* is the result of removing all the underlines from φ.
- So, in the case of de dicto readings, *x truly believes that* φ is a relation between a person and a true thought, where a true thought is a logical complex that represents a state of affairs that obtains.
- x truly believes that $\varphi\ =_{df} B(x, [\lambda\ \varphi]) \ \&\ \varphi^*$

Some Subtleties: I

- We can revise the logic to implement the local option:
  - Johnny believes that Woodie is a woodchuck.
    $B(j, [\lambda\, Ww])$ (de re)
    $B(j, [\lambda\, \underline{W}_j\, \underline{w}_j])$ (de dicto)
- '*x truly believes that* φ' is defined as above.
- One can introduce a representation relation:
  $[\lambda\, \underline{W}_j\, \underline{w}_j]$ represents $[\lambda\, Ww]$ to j

Some Subtleties II: Descriptions

- The sense of ‘the man wearing a hat’ = the abstract x that encodes exactly the properties implied by being a unique man wearing a hat.
- ‘man wearing a hat’ $[\lambda y\, My \;\&\; Wy]$
- ‘the man wearing a hat’ $\iota x([\lambda y\, My \;\&\; Wy]x)$
- The sense of ‘the man wearing a hat’
  $\iota x^{i}(A!x \;\&\; \forall F(xF \equiv [\lambda y\, My \;\&\; Wy \;\&\; \forall z(Mz \;\&\; Wz \rightarrow z =_{E} y)] \Rightarrow F))$
- Where ‘the …’ is translated as $\iota x\varphi$, then the sense of ‘the …’ is:
  $\underline{\iota x\varphi} =_{df} \iota x^{i}(A!x \;\&\; \forall F(xF \equiv [\lambda y\, \forall z(\varphi^{z}_{x} \equiv z =_{E} y)] \Rightarrow F))$
- Observations: The data is in ordinary English, so the translations require only formulas φ that can be used in λ-expressions. We can increase fine-grainedness by appealing to senses of names in the senses of descriptions. We can sidestep the debate between direct reference theorists and descriptivists.

Some Subtleties III: Complex Predicates

- The sense of 'is a brother' ('$\underline{B}$')
- The sense of 'is a male sibling' ('$\underline{[\lambda x\, Mx \,\&\, Sx]}$')

$$\iota F^{\langle i\rangle}(A!F \,\&\, \forall R(FR \equiv R[\lambda x\ \underline{M}x \,\&\, \underline{S}x]))$$

- The sense of 'is such that ...' (translated $[\lambda x\, \varphi]$) ('$\underline{[\lambda x\, \varphi]}$')

  $$= \iota F^{\langle i\rangle}(A!F \,\&\, \forall R(FR \equiv R[\lambda x\, \hat{\varphi}])),$$

  where $\hat{\varphi}$ results by replacing all the simple terms τ in φ by $\underline{\tau}$
- Observation: Now we have a Fregean explanation of the paradox of analysis: the identity statement '$B = [\lambda x\, Mx \,\&\, Sx]$' is true because the predicates denote the same property, but informative because the senses differ. (Cf. the Carnapian and Montagovian method: both predicates have the same extension and intension and so there is no explanation as to why their analysis explains informativeness.)

Some Subtleties IV: Senses of Sentences

- What is the thought that S?

  Consider atomic case: where Pa translates S.
- Previous slide suggests: $[\lambda\, \underline{P}\, \underline{a}]$
- But this doesn't describe/specify the thought as an abstract object (it is an ordinary propositions with abstract constituents).
- But we can identify the thought with an abstract object:

  $$\underline{Pa} = \iota p(A!p\ \&\ \forall R(pR \equiv R[\lambda\, \underline{P}\, \underline{a}]))$$
- The thought that S (where φ is translation of S) =

  $$\underline{\varphi} = \iota p(A!p\ \&\ \forall R(pR \equiv R[\lambda\, \hat{\varphi}]))$$
- This assures that the sense of the whole sentence is identified in the abstract domain.

Identity Statements

- '$a = a$' differs in cognitive significance from '$a = b$', and '$P = P$' differs in cognitive significance from '$P = Q$'.
- 'Mark Twain = Mark Twain' differs in cognitive significance from 'Mark Twain = Samuel Clemens', and '*being a woodchuck* just is *being a woodchuck*' differs in cognitive significance from '*being a woodchuck* just is *being a groundhog*'.
- There is a unified, but Fregean, explanation: the thought (= abstract proposition) $\underline{a = a}$ is self-identical but different from the thought $\underline{a = b}$ and the thought (= abstract proposition) $\underline{P = P}$ is self-identical but different from the thought $\underline{P = Q}$.

Our Fregean Philosophy of Natural Language

- Singular terms denote objects; predicates denote properties.
- Sentences denote states of affairs that have truth values.
- The sense of a singular term is an individual, the sense of a predicate is an property, and the sense of a sentence is a thought.
- Senses of predicates and singular terms are mapped to thoughts.
- Senses are modes of presentation.
- Sense determines denotation (on the global conception).
- The thought expressed by S is true iff S denotes a true state of affairs (i.e., has as a model-theoretic extension The True or 1).
- If a primary term in S fails to denote, S still has truth conditions – but these conditions don't obtain.
- In intensional contexts, terms (sentences) may denote their senses.
- Substitution of co-denoting terms preserves denotation and substitution of co-expressing terms preserves sense.

Observations

- By encoding properties, abstract objects and abstract relations can represent ordinary objects and ordinary relations that exemplify the encoded properties, and thereby serve as senses and modes of presentation.
- So object theory unifies Frege's theory of language with his theory of extensions. In the next lecture, we complete the unification of Frege's theory of language and mathematics by identifying the Frege numbers in the domain of abstract objects.

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