Seminar on Axiomatic Metaphysics
Lecture 9
Frege Numbers (Part 1)

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Frege's Goals and Methods

- The Dedekind/Peano axioms for number theory:
  - 0 is a number.
  - 0 is not a successor of any number, i.e.,
    No number (immediately) precedes zero.
  - No two numbers have the same successor, i.e.,
    If numbers x, y precede a number z, then $x = y$. (one-one)
  - Every number has a successor, i.e.,
    Every number precedes some number.
  - Mathematical Induction: If (a) 0 has F and (b) n's successor has F whenever n has F, then every number has F
- Boolos (1995, 293; 1996, 275), and Heck (2011, 288) include:
  - A successor of a number is a number, i.e.,
    Anything a number precedes is a number.
  - No number has two successors, i.e.,
    If a number x precedes numbers y, z, then $y = z$. (functional)
- Frege's Theorem: derive these axioms as theorems in 2OL + Hume's Principle: https://plato.stanford.edu/entries/frege-theorem/

Frege's Strategy for Proving Frege's Theorem

- Frege (1884): Start with equinumerosity: $F \approx G =_{df} \exists R[\forall x(Fx \rightarrow \exists! y(Gy \,\&\, Rxy)) \,\&\, \forall y(Gy \rightarrow \exists! x(Fx \,\&\, Rxy))]$
  - Example
- Define (1884) $\#G$ ('the number of Gs') as a second-order concept under which all the first-order concepts equinumerous to G fall. In (1893): $\#G = \epsilon[\lambda x\, \exists F(x = \epsilon F \,\&\, F \approx G)]$
- $\epsilon f = \epsilon g \equiv \forall x(f(x) = g(x)) \vdash \#F = \#G \equiv F \approx G$ $\qquad$ $V \vdash$ Hume
- He then defined:
  - $Precedes(x, y) =_{df} \exists F \exists z(Fz \,\&\, y = \#F \,\&\, x = \#[\lambda w\, Fw \,\&\, w \neq z])$
  - $0 =_{df} \#[\lambda x\, x \neq x]$
  - $Precedes^*(x, y)$ (The ancestral of *Precedes*)
  - $Precedes^+(x, y)$ (The weak ancestral of *Precedes*)
  - $NaturalNumber(x) =_{df} Precedes^+(0, x)$
- Derive the Dedekind/Peano axioms from Hume's Principle, without essential appeal to Basic Law V (Heck 1993). Reconstruction: in second-order logic, replace Frege's ϵF and Basic Law V with $\#F$ and Hume's Principle.

Issues With This Reconstruction

- Caesar problem: '$\#F = x$' isn't defined for arbitrary x.
- Hume's Principle (and other Fregean biconditionals) collapse existence and identity conditions.
- Hume's Principle only gives you one kind of abstract object. What about all the others?
- The St. Andrews school (Wright & Hale 2001, Cook 2003, etc.): add Fregean biconditionals such as Hume's Principle to introduce each different group of abstract objects.
- Problems:
  - The resulting theory of abstract objects is piecemeal, and each type of abstract object has its own Caesar problem.
  - The bad-company objection: some biconditionals lead to contradiction (like Basic Law V), and some don't (Boolos 1990).
  - Embarassment of riches problem (Weir 2003): indefinitely many consistent, but pairwise inconsistent, biconditionals.

The Limits of Abstraction: Fine 2002

- A permutation $\pi(x, y)$ of the domain of individuals is a 1-1 correspondence between the universal concept $[\lambda x\, x = x]$ and itself. A permutation π induces a permutation of concepts: $\pi(F, G) \equiv \forall x, y(\pi(x, y) \rightarrow (Fx \equiv Gy))$. We write πF for the unique G such that $\pi(F, G)$. A first-level concept F is *invariant* iff F is equivalent (coextensive) with πF, for every permutation π.
- Call a second-level concept $\boldsymbol{P}$ *invariant* if $\boldsymbol{P}(F)$ and $\boldsymbol{P}(\pi F)$ are equivalent, for every permutation π. The following second-level concepts are invariant: *no* F, *some* F, *all* F, *at most one* F, *exactly one* F, *at least one* F, *finitely many* F, *infinitely many* F, *evenly many* F, etc.
- A second-level relation $\boldsymbol{R}$ is *invariant* if $\boldsymbol{R}(F, G)$ and $\boldsymbol{R}(\pi F, \pi G)$ are equivalent, for any concepts F, G and any permutation π. A second-level equivalence relation $\boldsymbol{R}$ is *non-inflationary* if there are no more equivalence classes of concepts under R than there are individuals in the domain.
- The basic idea behind Fine's theory: An abstraction principle of the form $\S F = \S G \equiv \boldsymbol{R}(F, G)$ is acceptable only when $\boldsymbol{R}$ is a second-level equivalence relation that is invariant and non-inflationary.
- This yields abstracts (i.e., numbers as objects) corresponding to the invariant concepts such as *exactly one*, *exactly two*, etc., and you get the sets of natural numbers that correspond to concepts applying to the numbers.

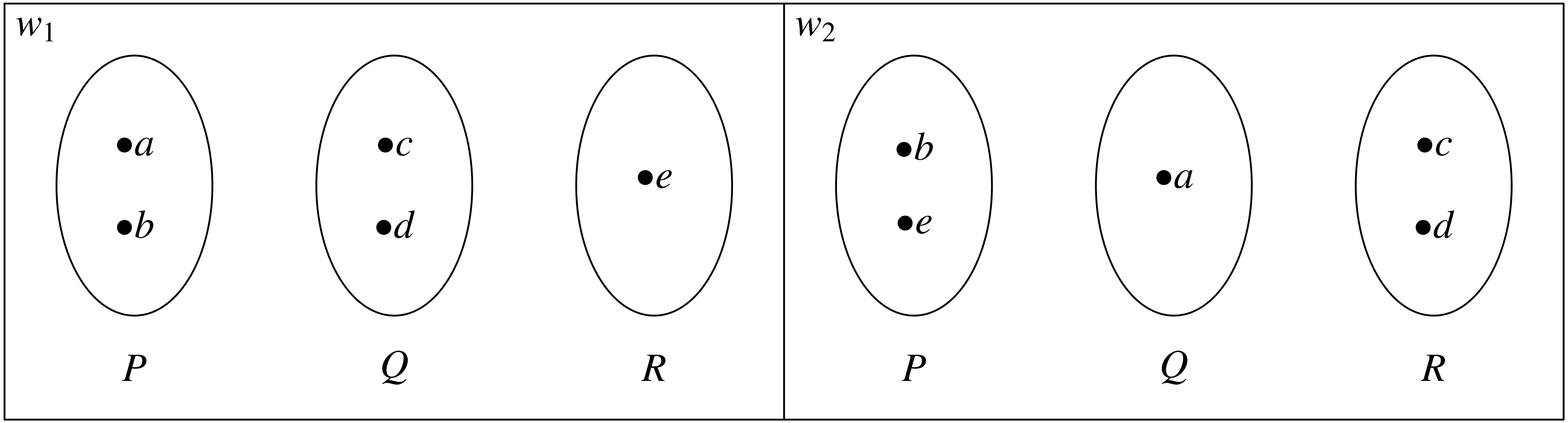
The Limits on Fine's Method of Abstraction

- Exactly which Fregean biconditionals are endorsed by this framework? This isn't made explicit.
- Significant parts of mathematics are not reducible using Fine's method of abstraction (Burgess 2003, Shapiro 2004).
- There is no solution to either the Julius Caesar problem or the epistemological question as to how we have knowledge of mathematics.
- There is no method for abstracting over equivalence relations on individuals: no *directions*, *shapes*, etc.
- The method requires that one assert the existence of at least two individuals in the domain of individuals.

A Problem for Fregean Reconstructions

- Frege's analysis of natural numbers gives rise to a problem in a modal setting: it yields different cardinal numbers in different modal contexts.
- At each possible world, the equivalence classes of equinumerous properties change; the numbers (as abstractions) that reify the equivalence classes at one world will be different from the numbers that reify the equivalence classes at each other possible world.
- The problem is that G might be exemplified by two objects in both w_1 and w_2, but the object that numbers G in w_1 is not identical to the object that numbers G in w_2.

Picture of the Problem



- The Fregean object 2 in w_1 includes P and Q since 2 is an abstraction of the equivalence class $\{P, Q\}$.
- The Fregean object 2 in w_2 includes P and R since 2 is an abstraction of the equivalence class $\{P, R\}$.
- So 2 in $w_1 \neq$ 2 in w_2: they're (abstracted from) different equivalence classes.

A Separate Problem For Object Theory

- We want to define $\#G$ as: $\iota x(A!x \ \& \ \forall F(xF \equiv F \approx G))$.
- But we have indiscernible abstract objects:

  $\exists x \exists y(A!x \ \& \ A!y \ \& \ x \neq y \ \& \ \forall F(Fx \equiv Fy))$
- A consequence: $\neg \exists G(A! \approx G)$
- *Proof*. Suppse $A!a$, $A!b$, $a \neq b$, and $\forall F(Fa \equiv Fb)$. Suppose, for reductio, that $\exists G(A! \approx G)$. Let Q be such a property, i.e., $A! \approx Q$. Then there is a witness R (one-one and onto) from $A!$ to Q. So Rac for some object c such that Qc. So $[\lambda z \, Rzc]a$. But, since a and b are indiscernible, $[\lambda z \, Rzc]b$, i.e., Rbc. But this contradicts the one-one character of R, for we have both Rac and Rbc and yet $a \neq b$.
- So $\forall G \neg(G \approx A!)$, and hence $\neg(A! \approx A!)$.
- A consequence: $\exists F(F \not\approx F)$, and since $\approx$ is not reflexive, $\approx$ is not an equivalence relation.
- So we can't define numbers as objects encoding up equivalence classes of equinumerous properties.

Discernible Objects are Classical

- Recall the definitions of $D!$ and $=_D$ (Lecture 2):
  - $D! =_{df} [\lambda x\, \Box\forall y(y\neq x \rightarrow \exists F\neg(Fy \equiv Fx))]$
  - $=_D\ =_{df}\ [\lambda xy\, D!x\, \&\, D!y\, \&\, x=y]$
- We also proved a number of theorems about discernibles:
  - $O!x \rightarrow D!x$
  - $D!x \rightarrow \Box D!x$
  - $(D!x \vee D!z) \rightarrow (\forall F(Fx \equiv Fz) \rightarrow x=z)$
  - $=_D$ is reflexive (on discernibles), symmetric, and transitive
- $(D!x \vee D!y) \rightarrow \Box(x=y \equiv x=_D y)$

  Proof. First establish $D!x \rightarrow (x=y \equiv x=_D y)$. By RM, $\Box D!x \rightarrow \Box(x=y \equiv x=_D y)$. By cases: $D!x$; so $\Box D!x$; so …
- $D!y \rightarrow [\lambda x\, x=y]\downarrow$

  Proof. Assume $D!y$. We know $[\lambda x\, x=_D y]\downarrow$. Show $\Box\forall x(x=_D y \equiv x=y)$ and apply axiom.
- $(D!x\, \&\, D!y) \rightarrow (x\neq y \equiv [\lambda z\, z=x]\neq[\lambda z\, z=y])$

  Proof. Assume $D!x$, $D!y$, $x\neq y$, and for reductio, $[\lambda z\, z=x]=[\lambda z\, z=y]$. Since $D!x$, then $[\lambda z\, z=x]\downarrow$, so $x=x$ implies $[\lambda z\, z=x]x$. Hence $[\lambda z\, z=y]x$, i.e., $x=y$. Contradiction.

Equinumerosity$_D$ is Classical

- u, v range over discernibles; $\exists! u\varphi$ asserts unique existence.
- Df. *Correlates 1-1* w.r.t. discernibles and *equinumerosity$_D$* ($\approx_D$):
  - $R\,|\!:\, F \overset{1\text{-}1}{\longleftrightarrow}_D G \;\equiv_{df}\; R{\downarrow}\ \&\ F{\downarrow}\ \&\ G{\downarrow}\ \&$ $\forall u(Fu \to \exists! v(Gv\ \&\ Ruv))\ \&\ \forall v(Gv \to \exists! u(Fu\ \&\ Ruv))$
  - $F \approx_D G \;\equiv_{df}\; \exists R(R\,|\!:\, F \overset{1\text{-}1}{\longleftrightarrow}_D G)$
- $\approx_D$ is reflexive: $F \approx_D F$ ($=_D$ is witness)
- $\approx_D$ is symmetric: $F \approx_D G \to G \approx_D F$
  - If R is witness to $F \approx_D G$, consider $R^{-1} = [\lambda xy\, Ryx]$.
- $\approx_D$ is transitive: $F \approx_D G\ \&\ G \approx_D H \to F \approx_D H$
  - If R, S are witnesses to $F \approx_D G$ and $G \approx_D H$, define $R' = [\lambda xy\, \exists z(Gz\ \&\ Rxz\ \&\ Szy)]$.
- $(\neg\exists uFu\ \&\ \neg\exists vHv) \to F \approx_D H$
- Let F^{-u} designate $[\lambda z\, Fz\ \&\ z \neq u]$
  Let G^{-v} designate $[\lambda z\, Gz\ \&\ z \neq v]$
  - These exist by previous theorem: $[\lambda z\, D!z\ \&\ \varphi]{\downarrow}$, for any φ

Equinumerosity Facts

- $F \approx_D G \;\&\; Fu \;\&\; Gv \rightarrow F^{-u} \approx_D G^{-v}$ (Lemma)

  Proof: Assume $F \approx_D G$ (R = witness), Fu, Gv. Two cases: (1) If Ruv, then the witness to $F^{-u} \approx_D G^{-v}$ is R. (2) If $\neg Ruv$, then if b is the G-correlate of u and a the F-correlate of v, the witness to $F^{-u} \approx_D G^{-v}$ is (r, s restricted to discernibles):

  - $[\lambda rs\, (r \neq u \;\&\; s \neq v \;\&\; Rrs) \vee (r = a \;\&\; s = b) \vee (r = u \;\&\; s = v)]$

  [Note: This relation exists since $[\lambda xy\, D!x \;\&\; D!y \;\&\; \varphi]\downarrow$.]

- $F^{-u} \approx_D G^{-v} \;\&\; Fu \;\&\; Gv \rightarrow F \approx_D G$ (Lemma)

  Proof: Assume $F^{-u} \approx_D G^{-v}$ (R = witness), Fu, Gv. Then the witness to $F \approx_D G$ is: $[\lambda rs\, (F^{-u}r \;\&\; Q^{-v}s \;\&\; Rrs) \vee (r = u \;\&\; s = v)]$.

  [Note: This relation exists for the same reasons as in the previous theorem.]

- Property Equivalence$_D$ ($\equiv_D$) and Equinumerosity$_D$:
  - $F \equiv_D G \;\equiv_{df}\; F\downarrow \;\&\; G\downarrow \;\&\; \forall u(Fu \equiv Gu)$
  - $F \equiv_D G \rightarrow F \approx_D G$
  - $F \approx_D G \;\&\; G \equiv_D H \rightarrow F \approx_D H$

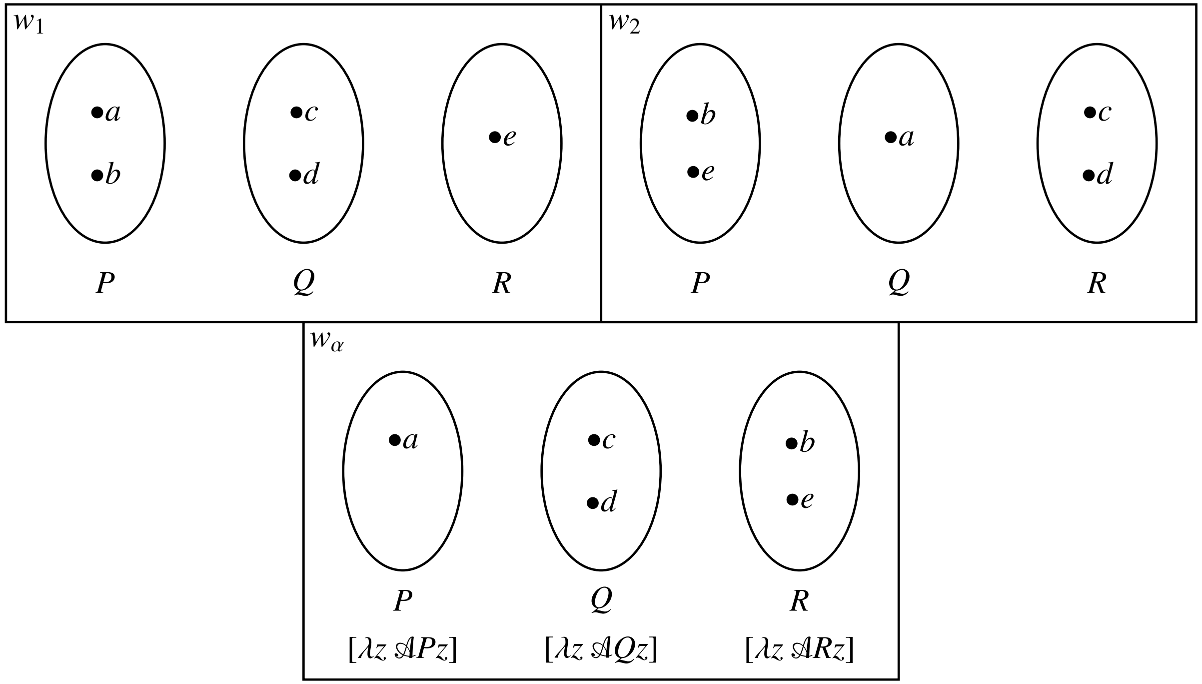
- Actuality, Rigidity, and Equinumerosity$_D$:

$Rigid([\lambda z\, \mathcal{A}Fz])$ $\qquad F \approx_D G \equiv \forall H([\lambda z\, \mathcal{A}Hz] \approx_D F \equiv [\lambda z\, \mathcal{A}Hz] \approx_D G)$

$Rigid(F) \rightarrow F \approx_D [\lambda z\, \mathcal{A}Fz]$ $\qquad (Rigid(F) \;\&\; Rigid(G)) \rightarrow \Box(F \approx_D G \rightarrow \Box F \approx_D G)$

Numbering a Property

$$Numbers(x, G) \equiv_{df} A!x \,\&\, G{\downarrow} \,\&\, \forall F(xF \equiv [\lambda z\, \mathcal{A}Fz] \approx_D G)$$



w_α: $Numbers(2, Q)$, $Numbers(2, R)$

w_1: $Numbers(2, P)$, $Numbers(2, Q)$

w_2: $Numbers(2, P)$, $Numbers(2, R)$

Numbering and Equinumerosity$_D$

- $\forall G \exists ! x Numbers(x, G)$
- $(Numbers(x, G) \ \& \ Numbers(x, H)) \rightarrow G \approx_D H$
- $Rigid(G) \rightarrow \Box \forall x (Numbers(x, G) \rightarrow \Box Numbers(x, G))$
- $\Box \forall x (Numbers(x, [\lambda z \, \mathcal{A} Gz]) \rightarrow \Box Numbers(x, [\lambda z \, \mathcal{A} Gz]))$
- $(Numbers(x, G) \ \& \ Numbers(y, H)) \rightarrow (x = y \equiv G \approx_D H)$ (pre-Hume)
- $\#G \ =_{df} \ \iota x Numbers(x, G)$

Aside: Hume's Principle is a $\star$-theorem:

- $\star \vdash [\lambda z \, \mathcal{A} Fz] \approx_D F$ (you need $\star$Axiom $\mathcal{A}\varphi \rightarrow \varphi$)
- $\star \vdash Numbers(\#G, G)$ (by $\star$-description theory)
- $\star \vdash \#F = \#G \equiv F \approx_D G$

  Proof. $(Numbers(\#F, F) \ \& \ Numbers(\#G, G)) \rightarrow (\#F = \#G \equiv F \approx_D G)$, by pre-Hume. But by the theorem just proved, $Numbers(\#F, F)$ and $Numbers(\#G, G)$.

A Necessary Version of Hume's Principle

- $Rigid(F) \rightarrow (Numbers(\#F, F))$

  Proof. If $Rigid(F)$, then by a previous theorem and the T schema, $\forall x(Numbers(x, F) \rightarrow \Box Numbers(x, F))$. When this holds, then by the theory of descriptions: $\exists! x Numbers(x, F) \rightarrow (\forall y(y = \iota x Numbers(x, F) \rightarrow Numbers(y, F)))$. The antecedent was recently established and so $\forall y(y = \iota x Numbers(x, F) \rightarrow Numbers(y, F))$. Instantiate to $\#F$ and apply the definition.

- $(Rigid(F) \& Rigid(G)) \rightarrow (\#F = \#G \equiv F \approx_D G)$

  Proof. Assume $Rigid(F)$ and $Rigid(G)$. Independently, instantiate pre-Hume to F, G, $\#F$ and $\#G$, to obtain $(Numbers(\#F, F) \& Numbers(\#G, G)) \rightarrow (\#F = \#G \equiv F \approx_D G)$. But our previous theorem and our assumptions imply $Numbers(\#F, F)$ and $Numbers(\#G, G)$. So $\#F = \#G \equiv F \approx_D G$.

- $\mathcal{A}Numbers(x, G) \equiv Numbers(x, [\lambda z\, \mathcal{A}Gz])$
- $Numbers(x, [\lambda z\, \mathcal{A}Gz]) \equiv x = \#G$
- $\forall F(\#GF \equiv [\lambda z\, \mathcal{A}Fz] \approx_D [\lambda z\, \mathcal{A}Gz])$
- See Nodelman & Zalta forthcoming, and Chapter 14 (Nodelman & Zalta) in *Principia Logico-Metaphysica*

Natural Cardinals

- $NaturalCardinal(x) \equiv_{df} \exists G(x = \#G)$
- $Numbers(x, G) \to NaturalCardinal(x)$ *Proof*. Assume $Numbers(x, G)$.

  Independently, Gallin's axiom (which we derived previously) implies: $\exists F Rigidifies(F, G)$. Suppose $Rigidifies(P, G)$. Then, by definition: $Rigid(P)$ & $\forall x(Px \equiv Gx)$. $\forall x(Px \equiv Gx)$ implies $P \equiv_D G$, which in turn implies $Numbers(x, P)$: $Rigid(P)$ implies $P \approx_D [\lambda z\, \mathcal{A}Pz]$, by a previous theorem. So $Numbers(x, [\lambda z\, \mathcal{A}Pz])$ Hence $x = \#P$, by a previous theorem. Trivially, $\exists F(x = \#F)$, and so $NaturalCardinal(x)$.
- $\exists G(x = \#G) \equiv \exists G(Numbers(x, G))$
- $NaturalCardinal(x) \to (xF \equiv x = \#F)$

  Proof. Assume $NaturalCardinal(x)$. Then $\exists G(x = \#G)$. Suppose $x = \#P$. Then $Numbers(x, [\lambda z\, \mathcal{A}Pz])$ $(= \vartheta)$. We show: $xF \equiv x = \#F$. By a previous theorem: we know: $\#PF \equiv [\lambda z\, \mathcal{A}Fz] \approx_D [\lambda z\, \mathcal{A}Pz]$. Hence: $xF \equiv [\lambda z\, \mathcal{A}Fz] \approx_D [\lambda z\, \mathcal{A}Pz]$ $(= \xi)$. By propositional logic, we know: $\vartheta \to ((xF \equiv (\psi \equiv \vartheta)) \equiv (xF \equiv \psi))$ $(= \zeta)$. Reason as follows:

xF	$\equiv$	$[\lambda z\, \mathcal{A}Fz] \approx_D [\lambda z\, \mathcal{A}Pz]$	by (ξ)
	$\equiv$	$Numbers(x, [\lambda z\, \mathcal{A}Fz]) \equiv Numbers(x, [\lambda z\, \mathcal{A}Pz])$	by previous thm
	$\equiv$	$Numbers(x, [\lambda z\, \mathcal{A}Fz])$	via (ϑ) and (ζ)
	$\equiv$	$x = \#F$	by previous theorem

Zero

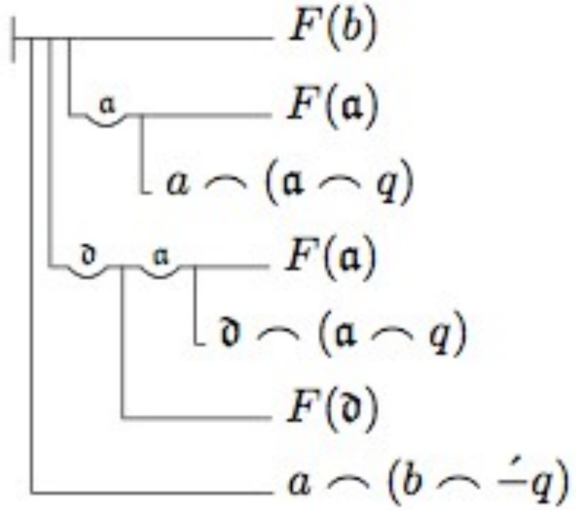
- $0 =_{df} \#[\lambda x\, D!x \,\&\, x \neq x] =_{df} \#[\lambda u\, u \neq u]$
- $NaturalCardinal(0)$
- $\neg\exists uFu \equiv Numbers(0, F)$

The Strong Ancestral of R: I

- $Hereditary(F,G) \ =_{df} \ \forall x,y(Gxy \rightarrow (Fx \rightarrow Fy))$
- Definition of G^* ('*being a G-ancestor of*'): being an x and y such that y exemplifies every property F such that (a) F is exemplified by everything x bears G to and (b) F is hereditary w.r.t. G.
- $G^* \ =_{df} [\lambda xy\, \forall F(\forall z(Gxz \rightarrow Fz) \,\&\, Hereditary(F,G) \rightarrow Fy)]$
  $G^*xy \equiv \forall F[\forall z(Gxz \rightarrow Fz) \,\&\, Hereditary(F,G) \rightarrow Fy]$
- Properties of the Strong Ancestral.
- $Gxy \rightarrow G^*xy$
- *Proof*. Assume Gab. Pick an arbitrary property, say P, and assume $\forall z(Gaz \rightarrow Pz)$ and $Hereditary(P,G)$. Then Pb, by the first two of our three assumptions.
- $[G^*xy \,\&\, \forall z(Gxz \rightarrow Fz) \,\&\, Hereditary(F,G)] \rightarrow Fy$
  (Frege 1893, Theorem 123)
- *Proof*. Immediate from the definition.

Interlude: Sanity Check

We show how to translate/transform Frege's Theorem 123 in 1893 (p. 138), which is in Frege Notation:



into our representation of Theorem 123, which is in modern notation:

$$[G^{*}xy \mathbin{\&} \forall z(Gxz \rightarrow Fz) \mathbin{\&} Hereditary(F,G)] \rightarrow Fy$$

The Transformation

The Strong Ancestral of G: II

- $Fx\ \&\ G^*xy\ \&\ Hereditary(F,G) \rightarrow Fy$ (Gg., Thm. 128)
- *Proof*. Assume Pa, $G^*(a,b)$, and that $Hereditary(P,G)$. Then by the previous theorem (123), to show Pb we simply need to show $\forall z(Gaz \rightarrow Pz)$. So assume Gac, where c is arbitrary (to show Pc). Since P is hereditary w.r.t. G and Pa, it follows that Pc.
- $Gxy\ \&\ G^*(y,z) \rightarrow G^*(x,z)$ (Gg., Thm. 129)
- *Proof*. Assume Gab and $G^*(b,c)$. To prove $G^*(a,c)$, further assume $\forall z(Gaz \rightarrow Pz)$ and $Hereditary(P,G)$ (to show Pc). So Pb. But from Pb, $G^*(b,c)$, and $Hereditary(P,G)$, it follows that Pc, by (128).
- $G^*xy \rightarrow \exists zGzy$ (Gg., Thm. 124)
- *Proof*. Assume $G^*(a,b)$. If we instantiate a,b into (123) and instantiate F to $[\lambda w\ \exists zGzw]$. Then, after λ-conversion,

  $$[G^*(a,b)\ \&\ \forall x(Gax \rightarrow \exists zGzx)\ \&\ \forall x,y(Gxy \rightarrow (\exists zGzx \rightarrow \exists zGzy))] \rightarrow \exists zGzb$$

  We assumed $G^*(a,b)$, and the second and third conjuncts of the antecedent are immediate, reasoning with arbitrary objects: If Gac, then, $\exists zGzc$. If Gcd and $\exists zGzc$, then $\exists zGzd$. So $\exists zGzb$.

The Weak Ancestral of $\underline{G}$: I

- F is a rigid (binary) relation on discernibles (‘$Rigid_D(F)$’) iff $Rigid(F)\ \&\ \Box\forall x\forall y(Fxy \rightarrow (D!x\ \&\ D!y))$
- $Rigid_D(F) \rightarrow \Box Rigid_D(F)$
- $\underline{G}$ ranges over rigid (binary) relations on discernibles!
- $\underline{G}^+ \ =_{df}\ [\lambda xy\ \underline{G}^*xy\ \vee\ x{=}_D y]$
  $\underline{G}^+xy \equiv \underline{G}^*xy \vee x{=}_D y$
- Facts About the Weak Ancestral of $\underline{G}$
- $Gxy \rightarrow \underline{G}^+xy$
- *Proof*. Assume Gab. By the 1st property of $\underline{G}^*$, $\underline{G}^*ab$. So $\underline{G}^*ab\ \vee\ a{=}b$. So G^+ab.
- $Fx\ \&\ \underline{G}^+xy\ \&\ Hereditary(F,G) \rightarrow Fy$ (Gg., Thm. 144)
- *Proof*. Assume Pa, $\underline{G}^+ab$, and $Hereditary(P,\underline{G})$. So by definition, $\underline{G}^*ab$ or $a{=}b$. If $\underline{G}^*ab$, then Pb, by (128). If $a{=}b$, then Pb, from the assumption that Pa.

The Weak Ancestral of R: II

- $\underline{G}^{+}xy \,\&\, Gyz \rightarrow \underline{G}^{*}xz$ (Gg., Thm. 134)
- *Proof*. Assume $\underline{G}^{+}ab$ and $\underline{G}bc$. Then from the disjunctive definition of $\underline{G}^{+}$, either (1) $\underline{G}^{*}ab$ and $\underline{G}bc$ or (2) $a\!=\!b$ and $\underline{G}bc$. Show $\underline{G}^{*}ac$ in both cases:
  - (1) $\underline{G}^{*}ab$ and $\underline{G}bc$. To show $\underline{G}^{*}ac$, pick an arbitrary property, P, and assume that $\forall z(\underline{G}az \rightarrow Pz)$ and $Hereditary(P, \underline{G})$, to show: Pc. From these assumptions and $\underline{G}^{*}ab$, it then follows that Pb, by the definition of $\underline{G}^{*}$. But from the facts that $Hereditary(P, \underline{G})$, $\underline{G}bc$, and Pb, it follows that Pc.
  - (2) $a\!=\!b$ and $\underline{G}bc$. Then $\underline{G}ac$, and so by the 1st property of $\underline{G}^{*}$, it follows that $\underline{G}^{*}ac$.
- $\underline{G}^{*}xy \,\&\, \underline{G}yz \rightarrow \underline{G}^{+}xz$
- *Proof*. Assume $\underline{G}^{*}ab$ and $\underline{G}bc$ (to show $\underline{G}^{+}ac$). From $\underline{G}^{*}ab$, it follows that $\underline{G}^{+}ab$ by definition of $\underline{G}^{+}$. So by (134), it follows that $\underline{G}^{*}ac$. So $\underline{G}^{+}ac$, by the definition of $\underline{G}^{+}$.

The Weak Ancestral of $\underline{G}$: III

- $\underline{G}xy \;\&\; \underline{G}^+yz \rightarrow \underline{G}^*xz$ (Gg., Thm. 132)
- *Proof*. Assume $\underline{G}ab$ and $\underline{G}^+bc$ (to show: $\underline{G}^*ac$). By definition of $\underline{G}^+$, either $\underline{G}^*bc$ or $b=c$. If $\underline{G}^*bc$, then given $\underline{G}ab$, we have $\underline{G}^*ac$, by (129). If $b=c$, then $\underline{G}ac$, in which case, $\underline{G}^*ac$, by the 1st property of $\underline{G}^*$.
- $\underline{G}^*xy \rightarrow \exists z(\underline{G}^+xz \;\&\; \underline{G}zy)$ (Gg., Thm. 141)
- *Proof*. Assume $\underline{G}^*ab$ (to show: $\exists z(\underline{G}^+az \;\&\; \underline{G}zb)$). The following is an instance of (123):

$$\underline{G}^*ab \;\&\; \forall x(\underline{G}ax \rightarrow Fx) \;\&\; \mathit{Hereditary}(F,\underline{G}) \rightarrow Fb$$

Instantiate this to: $[\lambda w\, \exists z(\underline{G}^+az \;\&\; \underline{G}zw)]$. Expand definitions and use λ-conversion:

$$\underline{G}^*ab \;\&\; \forall x(\underline{G}ax \rightarrow \exists z(\underline{G}^+az \;\&\; \underline{G}zx)) \;\&\; \forall x\forall y[\underline{G}xy \rightarrow (\exists z(\underline{G}^+az \;\&\; \underline{G}zx) \rightarrow \exists z(\underline{G}^+az \;\&\; \underline{G}zy))] \rightarrow \exists z(\underline{G}^+az \;\&\; \underline{G}zb)$$

Establish the antecedent. We have $\underline{G}^*ab$. For the 2nd conjunct, assume $\underline{G}ac$. By definition of $\underline{G}^+$, $\underline{G}^+aa$. From $\underline{G}^+aa \;\&\; \underline{G}ac$, it follows that $\exists z(\underline{G}^+az \;\&\; \underline{G}zc)$. For the 3rd conjunct, assume $\underline{G}cd$ and $\exists z(\underline{G}^+az \;\&\; \underline{G}zc)$. We have to show $\underline{G}^+ac$ since we have $\underline{G}cd$. So for some object, say e, $\underline{G}^+ae \;\&\; \underline{G}ec$. So by (134), it follows that $\underline{G}^*ac$. But, then $\underline{G}^+a,c$, by definition of $\underline{G}^+$.

One-to-One Rigid Relations on Discernibles

- $\mathit{1\text{-}1}(G) \equiv_{df} G{\downarrow} \;\&\; \forall x\forall y\forall z(Gxz \;\&\; Gyz \rightarrow x{=}y)$
- $\mathit{1\text{-}1}(\underline{G}) \rightarrow ((\underline{G}xy \;\&\; \underline{G}^{*}zy) \rightarrow \underline{G}^{+}zx)$

  Proof. Assume $\mathit{1\text{-}1}(\underline{G})$, $\underline{G}xy$, and $\underline{G}^{*}zy$. The latter implies, by a fact about $\underline{G}^{+}$, that some object, say a, is such that $\underline{G}^{+}za$ and $\underline{G}ay$. Since $\mathit{1\text{-}1}(\underline{G})$, $x{=}a$. So $\underline{G}^{+}zx$.
- $\mathit{1\text{-}1}(\underline{G}) \rightarrow ((\underline{G}xy \;\&\; \neg\underline{G}^{*}xx) \rightarrow \neg\underline{G}^{*}yy)$

  Proof. Assume $\mathit{1\text{-}1}(\underline{G})$, $\underline{G}xy$ and $\neg\underline{G}^{*}xx$, and $\underline{G}^{*}yy$ for reductio. By previous theorem (setting z to y): $(\underline{G}xy \;\&\; \underline{G}^{*}yy) \rightarrow \underline{G}^{+}yx$. So, $\underline{G}^{+}yx$. We also know the following instance of a fact about $\underline{G}^{+}$ (setting z to x): $(\underline{G}xy \;\&\; \underline{G}^{+}yx) \rightarrow \underline{G}^{*}xx$. Hence $\underline{G}^{*}xx$. Contradiction.
- $\mathit{1\text{-}1}(\underline{G}) \rightarrow ((\neg\underline{G}^{*}xx \;\&\; \underline{G}^{+}xy) \rightarrow \neg\underline{G}^{*}yy)$

  Proof. Assume $\mathit{1\text{-}1}(\underline{G})$, $\neg\underline{G}^{*}xx$, and $\underline{G}^{+}xy$. If we instantiate $[\lambda z\, \neg\underline{G}^{*}zz]$ into a previous fact about $\underline{G}^{+}$ to get $(\neg\underline{G}^{*}xx \;\&\; \underline{G}^{+}xy \;\&\; \mathit{Hereditary}([\lambda z\, \neg\underline{G}^{*}zz], \underline{G})) \rightarrow \neg\underline{G}^{*}yy$. The first two conjuncts are assumptions. Expanding and simplifying the third, we need to show, by GEN: $\underline{G}x'y' \rightarrow (\neg\underline{G}^{*}x'x' \rightarrow \neg\underline{G}^{*}y'y')$. But instantiating $\underline{G}$, x', and y' into the previous theorem implies (since $\mathit{1\text{-}1}(\underline{G})$): $(\underline{G}x'y' \;\&\; \neg\underline{G}^{*}x'x') \rightarrow \neg\underline{G}^{*}y'y'$. But this is equivalent to what we had to show.

Bibliography

- Boolos, G., 1990, "The Stanford of Equality of Numbers," in G. Boolos (ed.), *Meaning and Method: Essays in Honor of Hilary Putnam*, Cambridge: Cambridge University Press; reprinted in G. Boolos, *Logic, Logic, and Logic*, R. Jeffrey (ed.), Cambridge, MA: Harvard University Press, 1998, 202–219.
- Boolos, G., 1996, "On the Proof of Frege's Theorem," in A. Morton and S. Stich (eds.), *Benacerraf and His Critics*, Cambridge, MA: Blackwell, pp. 143–159; reprinted in G. Boolos, *Logic, Logic, and Logic*, R. Jeffrey (ed.), Cambridge, MA: Harvard University Press, 1998, 275–290. [Page reference to the reprint.]
- Boolos, G., 1995, "Frege's Theorem and the Peano Postulates," *Bulletin of Symbolic Logic*, 1: 317–326; reprinted in G. Boolos, *Logic, Logic, and Logic*, R. Jeffrey (ed.), Cambridge, MA: Harvard University Press, 1998, pp. 291–300. [Page reference to the reprint.]
- Burgess, J., 2003, "Review of Kit Fine, *The Limits of Abstraction*," *Notre Dame Journal of Formal Logic*, 44 (4): 227–251.
- Cook, R., 2003, "Iteration One More Time," *Notre Dame Journal of Formal Logic*, 44 (2): 63–92.

Bibliography

- Fine, K., 2002, *The Limits of Abstraction*, Oxford: Clarendon Press.
- Frege, G., 1884, *The Foundations of Arithmetic*, translated by J. L. Austin, Oxford: Blackwell, second revised edition, 1974.
- Frege, G., 1893 [1903], *Grundgesetze der Arithmetik*, Band I [II], Jena: Verlag Hermann Pohle.
- Hale, B., and C. Wright, 2001, *The Reason's Proper Study*, Oxford: Clarendon.
- Heck, R., 2011, *Frege's Theorem*, Oxford: Clarendon.
- Heck, R., 1993, "The Development of Arithmetic in Frege's *Grundgesetze Der Arithmetik*," *Journal of Symbolic Logic*, 58 (2): 579–601
- Nodelman, U., and E. Zalta, forthcoming, "Number Theory and Infinity Without Mathematics", *Journal of Philosophical Logic*. Preprint available online
- Shapiro, S., 2004, "Critical Study: The Nature and Limits of Abstracts," *Philosophical Quarterly*, 54 (214): 166–174.

Bibliography

- Weir, A., 2003, "Neo-Fregeanism: An Embarrassment of Riches," *Notre Dame Journal of Formal Logic*, 44 (1): 13–48.
- Wright, C., 1983, *Frege's Conception of Numbers as Objects*, Aberdeen: University of Aberdeen Press.
- Zalta, E., 1999, "Natural Numbers and Natural Cardinals as Abstract Objects: A Partial Reconstruction of Frege's *Grundgesetze* in Object Theory," *Journal of Philosophical Logic*, 28(6): 619–660.
- Zalta, E., 2018, "Frege's Theorem and Foundations for Arithmetic", *The Stanford Encyclopedia of Philosophy* (Spring 2018 Edition), Edward N. Zalta (ed.), URL = https://plato.stanford.edu/archives/spr2018/entries/frege-theorem/.
- Zalta, E., m.s., *Principia Logico-Metaphysica*, https://mally.stanford.edu/principia.pdf