Seminar on Axiomatic Metaphysics
Lecture 10
Frege Numbers (Part 2)

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Predecessor

- We now add a new axiom that asserts the existence of an ordering relation, namely, being an x and y such that for some property F and discernible object u, (a) u exemplifies F, (b) y numbers F, and (c) x numbers: being an F-exemplifier other than u.
- Axiom: $[\lambda xy\ \exists F \exists u(Fu\ \&\ Numbers(y, F)\ \&\ Numbers(x, F^{-u}))]\downarrow$
- $\mathbb{P} =_{df} [\lambda xy\ \exists F \exists u(Fu\ \&\ Numbers(y, F)\ \&\ Numbers(x, F^{-u}))]$
- Note: No mathematical primitives are used to assert this axiom. The notion $Numbers(x, F)$ is defined in terms of the primitives of object theory.
- $\mathbb{P}xy \equiv \exists F \exists u(Fu\ \&\ Numbers(y, F)\ \&\ Numbers(x, F^{-u}))$
- $Rigid(\mathbb{P})$, i.e., $\Box \forall x \forall y(\mathbb{P}xy \rightarrow \Box \mathbb{P}xy)$

  Proof. The reasoning that shows $\mathbb{P}xy \rightarrow \Box \mathbb{P}xy$ is non-trivial – it requires an appeal to a rigidifying relation and so relies on the derivation of the Gallin axiom from the Kirchner Theorem. See Nodelman & Zalta chapter of Zalta m.s., *PLM*.

Predecessor is 1-1 and Functional

Thm: *1-1*($\mathbb{P}$) (Frege, Gg., Thm 89)

Proof. Assume $\mathbb{P}xz$ and $\mathbb{P}yz$. By definition of $\mathbb{P}$, these assumptions imply, respectively, that there are properties and discernible objects, say R, Q, a, b, such that:

(ϑ) $Qa \,\&\, Numbers(z, Q) \,\&\, Numbers(x, Q^{-a})$

(ξ) $Rb \,\&\, Numbers(z, R) \,\&\, Numbers(y, R^{-b})$

The second conjuncts of (ϑ) and (ξ) jointly yield $Q \approx_D R$. Since we also know Qa and Rb, it follows by a previous lemma that $Q^{-a} \approx_D R^{-b}$. But, separately, the 3rd conjuncts of (ϑ) and (ξ) jointly imply $x = y \equiv Q^{-a} \approx_D R^{-b}$, by the conditional underlying Hume's Principle. Hence $x = y$.

Thm: $\mathbb{P}xy \,\&\, \mathbb{P}xz \rightarrow y = z$ (Frege, Gg., Thm 71)

Proof. Assume both $\mathbb{P}xy$ and $\mathbb{P}xz$. By definition of $\mathbb{P}$, these assumptions imply, respectively, that there are properties and discernible objects, say Q, R, a, b, such that:

(ϑ) $Qa \,\&\, Numbers(y, Q) \,\&\, Numbers(x, Q^{-a})$

(ξ) $Rb \,\&\, Numbers(z, R) \,\&\, Numbers(x, R^{-b})$

Now the third conjuncts of (ϑ) and (ξ) jointly imply $Q^{-a} \approx_D R^{-b}$. Since we also know Qa and Rb, it follows by a previous lemma that $Q \approx_D R$. But independently, the second conjuncts of (ϑ) and (ξ) jointly imply $y = z \equiv Q \approx_D R$, by the conditional underlying Hume's Principle. Hence $y = z$.

Lemma: Non-Zero Cardinals Have Predecessors

- Thm: $NaturalCardinal(x)\ \&\ x \neq 0 \rightarrow \exists y \mathbb{P}yx$

  Proof. Assume $NaturalCardinal(x)$ and $x \neq 0$. By definition of $\mathbb{P}$, show:

  $$\exists y \exists F \exists u(Fu\ \&\ Numbers(x,F)\ \&\ Numbers(y,F^{-u})$$

  The first assumption implies, by definition $\exists G(x = \#G)$, and so by a previous equivalence it follows that $\exists G(Numbers(x,G))$. Suppose $Numbers(x,P)$. This and $x \neq 0$ imply $\exists uPu$. Suppose Pa. Then we know $[\lambda z\, Pz\ \&\ z \neq a]{\downarrow}$. Hence $P^{-a}{\downarrow}$. So $\exists y Numbers(y,P^{-a})$. Suppose $Numbers(b,P^{-a})$. Then, assembling what we know:

  $$Pa\ \&\ Number(x,P)\ \&\ Numbers(b,P^{-a})$$

  So $\exists y \exists F \exists u(Fu\ \&\ Number(x,F)\ \&\ Numbers(y,F^{-u}))$.

Natural Cardinals are Discernible

Thm: $NaturalCardinal(x) \rightarrow D!x$

Proof. Assume $NaturalCardinal(x)$. Since $NaturalCardinal(0)$, we show $D!x$ by disjunctive syllogism from $x=0 \vee x\neq 0$. (a) $x=0$. We know $\exists x O!x$, say a. Then $[\lambda x\, x=a]\downarrow$, and so does $\#[\lambda x\, x=a]$ $(= b)$. Exercise: show $\mathbb{P}0b$ and, hence, $[\lambda z\, \mathbb{P}zb]0$. To show $D!0$, we show: $y\neq 0 \rightarrow \exists F\neg(Fy \equiv F0)$. So assume $y\neq 0$ and for reductio, $\neg\exists F\neg(Fy \equiv F0)$, i.e., $\forall F(Fy \equiv F0)$. Hence $[\lambda z\, \mathbb{P}zb]y$, and so $\mathbb{P}yb$. But $\mathbb{P}$ is a 1-1 relation and so by the definition of 1-1 and the fact that $\mathbb{P}\downarrow$ we may infer $0=y$ from $\mathbb{P}0b$ and $\mathbb{P}yb$. Contradiction.

(b) $x\neq 0$. Then since x is a natural cardinal, it follows by the previous theorem that $\exists y\mathbb{P}yx$. Suppose $\mathbb{P}cx$. Then $[\lambda z\, \mathbb{P}cz]x$. Again, to show $D!x$, we show: $y\neq x \rightarrow \exists F\neg(Fy \equiv Fx)$. So assume $y\neq x$ and, for reductio, $\neg\exists F\neg(Fy \equiv Fx)$, i.e., $\forall F(Fy \equiv Fx)$. Then $[\lambda z\, Pcz]y$, and hence $\mathbb{P}cy$. But $\mathbb{P}$ is functional and so from $\mathbb{P}cx$ and $\mathbb{P}cy$ it follows that $x=y$, which contradicts our assumption that $y\neq x$.

Some Corollaries

- Thm: $\mathbb{P}xy \rightarrow (NaturalCardinal(x)\ \&\ NaturalCardinal(y))$
- Thm: $\mathbb{P}xy \rightarrow (D!x\ \&\ D!y)$
- Thm: (immediate from the definition of $\mathbb{P}^*$)
  $\mathbb{P}^*xy \equiv \forall F((\forall z(\mathbb{P}xz \rightarrow Fz)\ \&\ \forall x'\forall y'(\mathbb{P}x'y' \rightarrow (Fx' \rightarrow Fy'))) \rightarrow Fy)$
- Thm: $\neg\exists x\mathbb{P}x0$ (Frege, Gg., Thm 108)

  Proof. Suppose not, e.g., $\mathbb{P}a0$. Then, for some property Q, and discernible b, $Qb\ \&\ Numbers(0,Q)\ \&\ Numbers(a,Q^{-b})$, by df $\mathbb{P}$. From Qb it follows that $\exists uQu$. But $Numbers(0,Q))$ implies $\neg\exists uQu$. Contradiction.
- Thm: $\neg\exists x\mathbb{P}^*x0$ (Frege, Gg., Thm 126)
- Thm: $\neg\mathbb{P}^*00$
- Thm: $\mathbb{P}^+xy \equiv \mathbb{P}^*xy \vee x =_D y$ (instance of the definition of $\mathbb{P}^+$)

Derivation of First 3 Dedekind/Peano Axioms

- $\mathbb{N} =_{df} [\lambda x\, \mathbb{P}^{+}0x]$ $\qquad$ $\mathbb{N}x \equiv \mathbb{P}^{+}0x$
- $NaturalNumber(0)$ $\qquad$ (D/P 1)

  Proof. $D!0$, since 0 is a natural cardinal and natural cardinals are discernible. So $0 =_D 0$, by reflexivity of $=_D$ on discernibles. So $\mathbb{P}^{*}00 \vee 0 =_D 0$ and hence $\mathbb{P}^{+}00$. Since $\mathbb{N}0 \equiv \mathbb{P}^{+}00$ (above), it follows that $\mathbb{N}0$.
- $\neg\exists n \mathbb{P}n0$ $\qquad$ (D/P 2) (Frege, Gg., Thm 126)

  (Zero doesn't succeed any natural number.)

  Proof. We've previously established $\neg\exists x Precedes(x, 0)$. *A fortiori*, no number precedes 0.
- $\forall n \forall m \forall k(\mathbb{P}nk \,\&\, \mathbb{P}mk \rightarrow m = n)$ $\qquad$ (D/P 3)

  No two numbers have the same successor.

  Proof. Since $\mathbb{P}$ is a 1-1 relation generally, it is a 1-1 relation on the numbers.

Lemma: Generalized Induction

Thm: $[Fz \,\&\, \forall x \forall y((\underline{G}^+zx \,\&\, \underline{G}^+zy) \rightarrow (\underline{G}xy \rightarrow (Fx \rightarrow Fy)))] \rightarrow \forall x(\underline{G}^+zx \rightarrow Fx)$

Proof. Assume the antecedent:

(ϑ) $Fz \,\&\, \forall x \forall y((\underline{G}^+zx \,\&\, \underline{G}^+zy) \rightarrow (\underline{G}xy \rightarrow (Fx \rightarrow Fy)))$

To show $\forall x(\underline{G}^+zx \rightarrow Fx)$, assume $\underline{G}^+zx$, to show Fx. We use the lemma:

$$(Fx \,\&\, \underline{G}^+xy \,\&\, Hereditary(F, \underline{G})) \rightarrow Fy$$

Instantiate F to $[\lambda y\, Fy \,\&\, \underline{G}^+zy]$, x to z, and y to x and simplify. Then we know:

(ξ) $[Fz \,\&\, \underline{G}^+zz \,\&\, \underline{G}^+zx \,\&\, Hereditary([\lambda y\, Fy \,\&\, \underline{G}^+zy], \underline{G})] \rightarrow (Fx \,\&\, \underline{G}^+zx)$

So to show Fx, we prove the antecedent of (ξ). Fz by assumption. $\underline{G}^+zz$ follows from the main fact about $\underline{G}^+$ and $z =_D z$ for discernible z. $\underline{G}^+zx$ also holds by assumption. So it remains to establish:

$$Hereditary([\lambda y\, Fy \,\&\, \underline{G}^+zy], \underline{G})$$

By definition and simplification, show:

$$\forall x, y[\underline{G}xy \rightarrow ((Fx \,\&\, \underline{G}^+(z, x)) \rightarrow (Fy \,\&\, \underline{G}^+(z, y)))].$$

Proof. Let a, b be arbitrary objects. Assume $\underline{G}ab$, Fa, and $\underline{G}^+za$, to show $Fb \,\&\, \underline{G}^+zb$. The second conjunct $\underline{G}^+zb$ follows easily: from the facts that $\underline{G}^+za$ and $\underline{G}ab$, it follows by a previous lemma that $\underline{G}^*zb$, which implies $\underline{G}^+zb$, by a previous theorem. So it remains to show Fb. Since we now have $\underline{G}^+za$, $\underline{G}^+zb$, $\underline{G}ab$, and Fa, it follows from the second conjunct of (ϑ) that Fb.

Derivation of D/P Axiom 4: Mathematical Induction

- Since $\mathbb{P}^+$ is a relation, we can instantiate Generalized Induction to $\mathbb{P}^+$ and 0 to get:

  $$F0 \mathbin{\&} \forall x \forall y[\mathbb{P}^+0x \mathbin{\&} \mathbb{P}^+0y \mathbin{\&} \mathbb{P}xy \rightarrow (Fx \rightarrow Fy)] \rightarrow \forall x(\mathbb{P}^+0x \rightarrow Fx)$$

- Now substitute $\mathbb{N}x$ for $\mathbb{P}^+0x$, and $\mathbb{N}y$ for $\mathbb{P}^+0y$, and the result is:

  $$F0 \mathbin{\&} \forall x \forall y[\mathbb{N}x \mathbin{\&} \mathbb{N}y \mathbin{\&} \mathbb{P}xy \rightarrow (Fx \rightarrow Fy)] \rightarrow \forall x(\mathbb{N}x \rightarrow Fx)$$

- Simplify with restricted variables:
  - $F0 \mathbin{\&} \forall n \forall m(\mathbb{P}nm \rightarrow (Fn \rightarrow Fm)) \rightarrow \forall n Fn$ (D/P 4)

Lemmas for Final D/P Axiom

- $\mathbb{N}x \to \mathit{NaturalCardinal}(x)$

  Proof. Assume $\mathbb{N}x$. Then $\mathbb{P}^{+}0x$. Reason by cases from $x=0 \vee x\neq 0$. If $x=0$, then $\mathit{NaturalCardinal}(x)$, by previous thm. If $x\neq 0$, then it follows that $\mathbb{P}^{*}0x$, definition of $\mathbb{P}^{+}$ and the fact that $x\neq 0 \to x\neq_D 0$. By a lemma about the weak ancestral, it follows *a fortiori* that $\exists z\mathbb{P}zx$. Let a be such an object, so that we know $\mathbb{P}ax$. Then by a previous fact, $\mathit{NaturalCardinal}(x)$.

- $\mathbb{N}x \to D!x$ (Exercise)

- $\mathbb{P}nx \to \mathbb{N}x$ (Successors are numbers)

  Proof. Assume $\mathbb{P}nx$. Since $\mathbb{N}n$, by hypothesis, it follows from the definition of $\mathbb{N}$ that $\mathbb{P}^{+}0n$. Since $\mathbb{P}$ is a rigid relation on discernibles, a fact about the weak ancestral implies: $(\mathbb{P}^{+}0n \;\&\; \mathbb{P}nx) \to \mathbb{P}^{*}0x$. So $\mathbb{P}^{*}0x$. Hence, by definition of $\mathbb{P}^{+}$, it follows that $\mathbb{P}^{+}0x$. So $\mathbb{N}x$.

- $\mathbb{P}nm \;\&\; \mathbb{P}nk \to m=k$

  Proof. Predecessor is functional *tout court*, and so functional on the natural numbers.

Lemma

Thm: $\forall x(\mathbb{N}x \rightarrow \neg\mathbb{P}^*xx)$

Proof. Assume $\mathbb{N}b$; show $\neg\mathbb{P}^*bb$, use: $(Fx \,\&\, \underline{G}^+(x,y) \,\&\, Hereditary(F,\underline{G})) \rightarrow Fy$. Instantiate F to $[\lambda z\, \neg\mathbb{P}^*zz]$, x to 0, y to b, and since $\mathbb{P}$ is a rigid relation on discernibles, instantiate $\underline{G}$ to $\mathbb{P}$. Simplify the result to:

$$(\neg\mathbb{P}^*00 \,\&\, \mathbb{P}^+0b \;\&\, Hereditary([\lambda z\, \neg\mathbb{P}^*zz], \mathbb{P})) \rightarrow \neg\mathbb{P}^*bb$$

So show:

(ϑ) $\neg\mathbb{P}^*00$

(ξ) $\mathbb{P}^+0b$

(ζ) $Hereditary([\lambda z\, \neg\mathbb{P}^*zz], \mathbb{P})$

(ϑ): from theorem $\neg\exists x\mathbb{P}^*x0$.
(ξ): from $\mathbb{N}b$ (assumption) and the definition of $\mathbb{N}$.
(ζ): By definition, show:

$$\mathbb{P}{\downarrow} \;\&\, [\lambda z\, \neg\mathbb{P}^*zz]{\downarrow} \;\&\, \forall x\forall y(\mathbb{P}xy \rightarrow ([\lambda z\, \neg\mathbb{P}^*zz]x \rightarrow [\lambda z\, \neg\mathbb{P}^*zz]y))$$

$\mathbb{P}{\downarrow}$ and $[\lambda z\, \neg\mathbb{P}^*zz]{\downarrow}$ are easy. So simplify and show: $\mathbb{P}xy \rightarrow (\neg\mathbb{P}^*xx \rightarrow \neg\mathbb{P}^*yy)$. Assume $\mathbb{P}xy$ and $\neg\mathbb{P}^*xx$. Now since $\mathbb{P}$ is a 1-1 rigid relation on discernibles, we can apply a previous theorem about such relations, to infer:

$$(\mathbb{P}xy \,\&\, \neg\mathbb{P}^*xx) \rightarrow \neg\mathbb{P}^*yy$$

Hence $\neg\mathbb{P}^*yy$. ⋈

Lemma

Thm: $(\mathbb{N}x \,\&\, \mathbb{P}yx) \rightarrow (Numbers(z, [\lambda z\, \mathbb{P}^{+}zy]) \equiv Numbers(z, [\lambda z\, \mathbb{P}^{+}zx]^{-x}))$

Proof. Assume $\mathbb{N}x$ and $\mathbb{P}yx$. Since $G \equiv_D H \rightarrow (Numbers(x, G) \equiv Numbers(x, H))$ show, by definition $(\equiv_D)$, that $[\lambda z\, \mathbb{P}^{+}zy]u \equiv [\lambda z\, \mathbb{P}^{+}zx]^{-x}u$. Since $\mathbb{N}x \rightarrow D!x$, we can apply definition of $[\lambda z\, \mathbb{P}^{+}zx]^{-x}$ and simplify by λ-Conversion and Substitution. So show:

$$\mathbb{P}^{+}uy \equiv \mathbb{P}^{+}ux \,\&\, u \neq x$$

$(\rightarrow)$ Assume $\mathbb{P}^{+}uy$. From this, assumption $\mathbb{P}yx$, and $\mathbb{P}$ is a rigid relation on discernibles, it follows that $\mathbb{P}^{*}ux$. Hence $\mathbb{P}^{+}ux$. Suppose $u = x$, for reductio. Then from $\mathbb{P}^{*}ux$, it follows that $\mathbb{P}^{*}xx$, which contradicts a previous lemma given that $\mathbb{N}x$.

$(\leftarrow)$ Assume $\mathbb{P}^{+}ux$ and $u \neq x$, and for reductio, $\neg\mathbb{P}^{+}uy$. From $u \neq x$, we know $u \neq_D x$. and from this and $\mathbb{P}^{+}ux$ it follows that $\mathbb{P}^{*}ux$. But since $\mathbb{P}$ is a 1-1 rigid relation on discernibles. we can instantiate a previous lemma to obtain $(\mathbb{P}yx \,\&\, \mathbb{P}^{*}ux) \rightarrow \mathbb{P}^{+}uy$, i.e., $(\mathbb{P}yx \,\&\, \neg\mathbb{P}^{+}uy) \rightarrow \neg\mathbb{P}^{*}ux)$. But from $\mathbb{P}yx$ (assumption) and $\neg\mathbb{P}^{+}uy$ (hypothesis), $\neg\mathbb{P}^{*}ux$. Contradiction. $\bowtie$

Main Lemma

Thm: $\forall n \exists y(Numbers(y, [\lambda z\, \mathbb{P}^{+}zn]) \;\&\; \mathbb{P}ny)$ Frege: $\forall n \mathbb{P}n\#[\lambda z\, \mathbb{P}^{+}zn]$

Proof. Consider:

$$[\lambda x\, \exists y(Numbers(y, [\lambda z\, \mathbb{P}^{+}zx]) \;\&\; \mathbb{P}xy)] \qquad (Q)$$

By λC, our theorem has the form $\forall nQn$. So, by induction, we show that $Q0$ and $\forall n \forall m(\mathbb{P}nm \to (Qn \to Qm))$.

Base Case: Show $Q0$, i.e., $\exists y(Numbers(y, [\lambda z\, \mathbb{P}^{+}z0]) \;\&\; \mathbb{P}0y)$. We know $\forall G \exists y Numbers(y, G)$. So let $Numbers(a, [\lambda z\, \mathbb{P}^{+}z0])$, and then show $\mathbb{P}0a$, i.e., show:

$$\exists F \exists u(Fu \;\&\; Numbers(a, F) \;\&\; Numbers(0, F^{-u}))$$

We pick our witness for F to be $[\lambda z\, \mathbb{P}^{+}z0]$ and pick our witness for u to be 0 (since $D!0$, given it is a natural cardinal and so discernible). So show:

(ϑ) $[\lambda z\, \mathbb{P}^{+}z0]0$

(ξ) $Numbers(a, [\lambda z\, \mathbb{P}^{+}z0])$

(ζ) $Numbers(0, [\lambda z\, \mathbb{P}^{+}z0]^{-0})$

(ϑ): Show $\mathbb{P}^{+}00$. But since $D!0$, $0 =_D 0$, and so $\mathbb{P}^{+}00$, by a fact about $\mathbb{P}^{+}$.

(ξ): holds by assumption.

Proof of Main Lemma (continued)

(ζ): $Numbers(0, [\lambda z\, \mathbb{P}^{+}z0]^{-0})$. By previous thm, it suffices to show:

$$\neg\exists u([\lambda z\, \mathbb{P}^{+}z0]^{-0}u)$$

Suppose not, and suppose $[\lambda z\, \mathbb{P}^{+}z0]^{-0}b$. Then by definition $[\lambda z\, [\lambda z\, \mathbb{P}^{+}z0]z \;\&\; z \neq 0]b$. Simplify to $[\lambda z\, \mathbb{P}^{+}z0]b \;\&\; b \neq 0$ and then to $\mathbb{P}^{+}b0 \;\&\; b \neq 0$. The 2nd conjunct implies $b \neq_D 0$, and so the first conjunct and the main fact about $\mathbb{P}^{+}$ imply $\mathbb{P}^{*}b0$, which contradicts $\neg\exists x \mathbb{P}^{*}x0$.

Inductive Case: Show $\mathbb{P}nm \rightarrow (Qn \rightarrow Qm)$, i.e.,

$\mathbb{P}nm \rightarrow (\exists y(Numbers(y, [\lambda z\, \mathbb{P}^{+}zn]) \;\&\; \mathbb{P}ny) \rightarrow \exists y(Numbers(y, [\lambda z\, \mathbb{P}^{+}zm]) \;\&\; \mathbb{P}my))$

So assume (IH):

(A) $\mathbb{P}nm$

(B) $\exists y(Numbers(y, [\lambda z\, \mathbb{P}^{+}zn]) \;\&\; \mathbb{P}ny)$

For (B), let $Numbers(b, [\lambda z\, \mathbb{P}^{+}zn])$ and $\mathbb{P}nb$. To find a witness for consequent, let c be such that $Numbers(c, [\lambda z\, \mathbb{P}^{+}zm])$ (every property is numbered!). To show $\mathbb{P}mc$, we have to show:

(C) $\exists F \exists u(Fu \;\&\; Numbers(c, F) \;\&\; Numbers(m, F^{-u}))$

Pick $[\lambda z\, \mathbb{P}^{+}zm]$ as witness for F, and m as witness for u (since $\mathbb{N}m \rightarrow D!m$).

Proof of Main Lemma (continued)

Show:

(ϑ) $[\lambda z\, \mathbb{P}^{+}zm]m$

(ξ) $Numbers(c, [\lambda z\, \mathbb{P}^{+}zm])$

(ζ) $Numbers(m, [\lambda z\, \mathbb{P}^{+}zm]^{-m})$

(ϑ): Show $\mathbb{P}^{+}mm$. Since $\mathbb{N}m$, $D!m$, we know $m =_D m$. Hence $\mathbb{P}^{+}mm$, by fact about $\mathbb{P}^{+}$.

(ξ): holds by assumption.

(ζ): By $\mathbb{N}m$ (hypothesis), $\mathbb{P}nm$ (assumption), and a previous lemma, we know:

(D) $Numbers(m, [\lambda z\, \mathbb{P}^{+}zn]) \equiv Numbers(m, [\lambda z\, \mathbb{P}^{+}zm]^{-m})$

Note that $\mathbb{P}nm$, by (A), and $\mathbb{P}nb$, by hypothesis. So $m = b$, by the functionality of predecessor. Since we also know $Numbers(b, [\lambda z\, \mathbb{P}^{+}zn])$ by hypothesis, it follows that $Numbers(m, [\lambda z\, \mathbb{P}^{+}zn])$. So by (D), $Numbers(m, [\lambda z\, \mathbb{P}^{+}zm]^{-m})$. ⋈

Note: Frege's version of this Lemma:

- $\forall n \mathbb{P}n\#[\lambda z\, \mathbb{P}^{+}zn]$

is also provable. See Nodelman & Zalta chapter of Zalta m.s., *PLM*.

Derivation of D/P Axiom 5: Every Number Has a Successor

- $\forall n \exists ! m \mathbb{P}nm$. (D/P 5)
- *Proof*. By GEN, it suffices to show $\exists ! m \mathbb{P}nm$. But since $\mathbb{P}$ is a functional relation (by a previous theorem), it suffices to show that $\exists m \mathbb{P}nm$. Moreover, we know that if n immediately precedes anything, that thing is a natural number (by a previous theorem), and so it suffices to show that $\exists y \mathbb{P}ny$. But this follows, *a fortiori* from the main lemma which tells us $\exists y(Numbers(y, [\lambda z\ \mathbb{P}^{+}zn]) \ \&\ \mathbb{P}ny)$.
- *Proof Variant* (Frege-Style): By GEN, it suffices to show $\exists ! m \mathbb{P}nm$. But since $\mathbb{P}$ is a functional relation (by previous theorem), it suffices to show that $\exists m \mathbb{P}nm$. Moreover, we know that if n immediately precedes anything, that thing is a natural number (by previous theorem), and so it suffices to show that $\exists y \mathbb{P}ny$. But this follows *a fortiori* from Frege's version of the main lemma, which tells us that $\mathbb{P}n\#[\lambda z\ \mathbb{P}^{+}zn]$.

Arithmetic

- Define notation for Successors
  - $n' =_{df} \imath m \mathbb{P}nm$

  n' is well defined, by D/P 5.
- Define numerals:
  - $1 =_{df} 0'$
  - $2 =_{df} 1'$
  - $3 =_{df} 2'$
  - $\vdots$
- Restrictions: when G is a 2-place relation and F a property:

  $$G_{\upharpoonright F} =_{df} [\lambda xy\, Fx \,\&\, Gxy]$$
- Define $<$, $\leq$, $>$, $\geq$:
  - $< =_{df} \mathbb{P}^*_{\upharpoonright \mathbb{N}}$
  - $\leq =_{df} \mathbb{P}^+_{\upharpoonright \mathbb{N}}$
  - $> =_{df} [\lambda xy\, y < x]$
  - $\geq =_{df} [\lambda xy\, y \leq x]$
- Prove theorems about $<$, $\leq$, $>$, $\geq$. E.g., $<$ is asymmetric and transitive, $\leq$ is reflexive, anti-symmetric and transitive, etc.

Basic Recursive Functions are Relations

- Operations: Rigid functional relations on numbers, e.g., successor (s) is $\mathbb{P}$ and numerical identity ($\doteq$) is $=_{D\restriction\mathbb{N}}$.
- Constant Operations (where $n' = \imath m\mathbb{P}nm$):
  - $C^{n'}_m =_{df} [\lambda x_1 \ldots x_n y\, \mathbb{N}x_1 \,\&\, \ldots \,\&\, \mathbb{N}x_n \,\&\, y \doteq m]$ $\qquad (n, m \geq 0)$

  It now follows that $C^{n'}_m$ is an n-ary operation:
  - $Op^n(C^{n'}_m)$
- Projection Operations: $\pi^{i'}_k$ takes i arguments, returns the kth ($1 \leq k \leq i$). (The arity of the relation is i', which includes the value of the function.), i.e., $\pi^{i'}_k =_{df} [\lambda n_1 \ldots n_i m\; m \doteq n_k]$
  - $Op^i(\pi^{i'}_k)$ $\qquad (1 \leq k \leq i)$
- Composition Operations:
  - $G \circ H =_{df} [\lambda xy\, \exists z(Hxz \,\&\, Gzy)]$
  - $Op^1(H) \,\&\, Op^1(G) \rightarrow (Op^1(G \circ H) \,\&\, \forall x([G \circ H](x) = G(H(x))))$
  - Generalizes to n-ary composition: $G \circ (H_1, \ldots, H_m)$, where G is any m'-ary relation ($m \geq 1$) and $H_1, \ldots, H_m$ are any n'-ary relations ($n \geq 0$)

Recursion Theorem

- Our approach has the following parts:
  - Start with given operations $H(n)$ and $G(n,m,j)$.
  - Inductively define (suppressing the index to H and G):

    $$\boldsymbol{F}_0 =_{df} H$$
    $$\boldsymbol{F}_{m'} =_{df} G \circ (\pi^2_1, C^2_m, \boldsymbol{F}_m)$$

    Lemma: each $\boldsymbol{F}_m$ exists and is an unary operation.
  - Then define $\boldsymbol{F}_{H,G} =_{df} [\lambda nmj\, F_m(n) \doteq j]$
- Lemma: $\boldsymbol{F}_{H,G}$ exists and is a binary operation.
- Recursion Thm: where H is a unary op and G is a ternary op: $Op^2(\boldsymbol{F}_{H,G})$ & $\boldsymbol{F}_{H,G}(n,0) = H(n)$ & $\boldsymbol{F}_{H,G}(n,m') = G(n,m,\boldsymbol{F}_{H,G}(n,m)))$
- Example: standard recursive definition of Addition ($\boldsymbol{A}$).
  - $\boldsymbol{A} =_{df} \boldsymbol{F}_{\pi^2_1, \boldsymbol{s}\circ\pi^4_3}$

  It follows that:
  - $\boldsymbol{A}(n,0) = n$

    $\boldsymbol{A}(n,m') = (\boldsymbol{A}(n,m))'$

  Or, in infix notation:
  - $n + 0 = n$

    $n + m' = (n+m)'$

Derivation of Second-Order Peano Arithmetic

- The Dedekind-Peano axioms (including mathematical induction) are theorems.
- By the Recursion Theorem, the axioms for recursive addition and multiplication become theorems once addition $\boldsymbol{A}$ and multiplication $\boldsymbol{M}$ are defined:
  - $n + 0 =_{df} n$
    $n + m' =_{df} (n + m)'$
  - $n \times 0 =_{df} 0$
    $n \times m' =_{df} n + (n \times m)$

  Note: You have to introduce the multiplication in a manner similar to that of addition:
  - $\boldsymbol{M} =_{df} \boldsymbol{F}_{C_0^2, \boldsymbol{A} \circ (\pi_1^4, \pi_3^4)}$

  From which it follows that:
  - $\boldsymbol{M}(n, 0) = 0$
    $\boldsymbol{M}(n, m') = n + \boldsymbol{M}(n, m)$
- Comprehension Principle for Properties is already a theorem.
- So second-order Peano Arithmetic has been derived.

An Infinite Cardinal Exists

- For κ NatCard: $Finite(\kappa) \equiv_{df} \mathbb{N}\kappa$ $\quad Infinite(\kappa) \equiv_{df} \neg Finite(\kappa)$
- Lemma 1: $\forall u([\lambda z\, \mathbb{P}^{+}zm]u \rightarrow \mathbb{N}u)$ *Proof*. By induction on m.
- Lemma 2: $\neg\exists n Numbers(n, \mathbb{N})$

  Proof. For reductio, suppose $\mathbb{N}a$ and $Numbers(a, \mathbb{N})$. Then by the main lemma for D/P 5, $\exists y(Numbers(y, [\lambda z\, \mathbb{P}^{+}za]) \,\&\, \mathbb{P}ay)$. Suppose $Numbers(b, [\lambda z\, \mathbb{P}^{+}za]) \,\&\, \mathbb{P}ab$. From $\mathbb{P}ab$ and $\mathbb{N}a$, we have $\mathbb{N}b$. From $\mathbb{P}ab$, it then follows that $a < b$. Now a fact we haven't proved is:

  $$(Numbers(n, F) \,\&\, Numbers(m, G) \,\&\, \forall u(Fu \rightarrow Gu)) \rightarrow n \leq m$$

  Instantiate $[\lambda z\, \mathbb{P}^{+}za]$ for F, $\mathbb{N}$ for G, a for m, and b for n:

  $$(Numbers(b, [\lambda z\, \mathbb{P}^{+}za]) \,\&\, Numbers(a, \mathbb{N}) \,\&\, \forall u([\lambda z\, \mathbb{P}^{+}za]u \rightarrow \mathbb{N}u)) \rightarrow b \leq a$$

  We already know the first two conjuncts. The third conjunct follows from the Lemma 1, with a instantiated for m. Hence $b \leq a$. But we previously established $a < b$. So by a simple fact $(n < m \,\&\, m \leq k \rightarrow n < k)$ (exercise), $a < a$, which contradicts $\neg(n < n)$ (exercise).
- $Infinite(\#\mathbb{N})$. *Proof*. It is provable that $Rigid(\mathbb{N})$. Then by a fact about numbering, $Numbers(\#\mathbb{N}, \mathbb{N})$. If, for reductio, $\mathbb{N}\#\mathbb{N}$, $\exists n Numbers(n, \mathbb{N})$, contradicting Lemma 2.
- Thus, $\exists x Infinite(x)$ has been derived from no math primitives! If $\aleph_0 =_{df} \#\mathbb{N}$, then the existence of $\aleph_0$ doesn't require mathematics.

Consistency of the Theory of Natural Numbers

- Use Aczel models in which the Urelements consist of: 1 ordinary object and let the set of special elements **S** contain a copy of the natural numbers $0^*, 1^*, 2^*, \ldots$.
- Let $\mathbf{R}_2$ contain $\mathbb{P}$, whose extension at $\boldsymbol{w}_0$ is:
  - $\{\langle 0^*, 0^*\rangle, \langle 2^*, 3^*\rangle, \langle 3^*, 4^*\rangle, \langle 4^*, 5^*\rangle, \ldots\}$

  $\langle 0^*, 0^*\rangle$ will represent $\mathbb{P}\aleph_0\aleph_0$.
- Let the domain of abstract objects **A** contains $0, 1, 2, \ldots$, and $\aleph_0$, where each is a set of properties whose extensions are equinumerous$_D$ at $\boldsymbol{w}_0$:
  - 0 is the set of properties whose extensions equinumerous$_D$ to the property denoted by $[\lambda x\, D!x \,\&\, x \neq x]$ at $\boldsymbol{w}_0$
  - n' is the set of properties whose extensions are equinumerous$_D$ to the property denoted by $[\lambda m\, \mathbb{P}^{+}mn]$ at $\boldsymbol{w}_0$

  Let n range over these objects. Set the proxy function so that $|\aleph_0| = 0^*$, $|i| = 1^*$ (where i is any indiscernible), and $|n| = (n+2)^*$, so that $2^*, 3^*, 4^*, \ldots$ are the proxies of the natural numbers for exemplification purposes.

Observations I

- Natural numbers and an infinite cardinal are definable and their principles are derivable in extended object theory.
- No mathematical primitives are used, and no mathematical axioms are asserted.
- The fundamental question of the philosophy of arithmetic (Heck 2011, 152): What is the basis of our knowledge of the infinity of the series of natural numbers? Answer: We can derive it as a theorem from principles that govern abstract objects generally.
- Frege's question: *Wie soll uns denn eine Zahl gegeben sein, wenn wir keine Vorstellung oder Anschauung von ihr haben können?* (1884, §62). Answer: By descriptions guaranteed to be well-defined by principles that govern abstract objects generally.
- Everything depends on logico-metaphysical principles that demonstrate how logic and metaphysics are entangled.

Observations: II

- We haven't asserted the existence of any concrete objects, but only that concrete objects might exist.
- There is no Julius Caesar problem. $\#F = x$ is defined for any value of x.
- We aren't postulating objects piecemeal, though we have had to extend object theory with 1 axiom and prove it is consistent.
- There is no 'bad company' objection, 'embarassment of riches' objection, etc.
- We've united the Fregean philosophy of mathematics (by deriving extensions and natural numbers) and Fregean philosophy of language (by identifying senses).
- We turn next to the analysis of theoretical mathematics.

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