Seminar on Axiomatic Metaphysics
Lecture 11
Philosophy of Mathematics I

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Two Kinds of Mathematics

- Mathematics has an intuitive division: natural mathematics and theoretical mathematics.
- Natural mathematics: ordinary, pretheoretic claims we make about mathematical objects.
  - The Triangle has 3 sides.
  - The number of planets is eight.
  - There are more individuals in the class of insects than in the class of humans.
  - Lines a and b have the same direction.
  - Figures a and b have the same shape.
- Theoretical mathematics: claims that occur in the context of some explicit or implicit (informal) mathematical theory, e.g., theorems.
  - In ZF, the null set is an element of the unit set of the null set.
  - In Real Number Theory, 2 is less than or equal to π.

Natural Mathematical Objects

- We've already analyzed the objects of natural mathematics:
  - The Triangle.
    $\Phi_T =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv \Box\forall y(Ty \rightarrow Fy)))$
  - The number of Gs.
    $\#G =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv F \approx_D G))$
    Theorem: $\#F = \#G \equiv F \approx_D G$ (Hume's Principle)
  - The extension of G.
    $\epsilon G =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv \forall y(Gy \equiv Fy)))$
    Theorem: $\epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx)$ (Basic Law V)
  - The direction of line a.
    $\vec{a} =_{df} \epsilon[\lambda x\, x||a]$
    Theorem: $\vec{a} = \vec{b} \equiv a||b$ (Directions)
- In what follows, we distinguish the natural numbers from the theoretical numbers of Peano Arithmetic (PA), and the natural extensions from the theoretical sets of ZF, ZFC, NBG, NF, etc.

Goal: An Analysis of Theoretical Mathematics

- Our goal is a philosophical analysis of theoretical mathematics.
- To achieve the goal, distinguish (Shapiro 2004) three kinds of foundations for mathematics:
  - (logico-)metaphysical: identifies denotations and truth conditions.
  - epistemological: explain knowledge of mathematical claims
  - mathematical: a distinguished mathematical theory in which all other mathematical theories should be formulated.
- We're not attempting to give mathematical foundations. That is a mathematical question. Our analysis is consistent with whatever mathematical foundations, if any, that mathematicians agree on.
- Our goal: logico-metaphysical and epistemological analysis of mathematical theories, terms, predicates and statements, presupposing no mathematics.
- Mathematical theories, terms and predicates are identified (assigned denotations); mathematical statements are assigned truth conditions (in terms of the denotations).

A Second Goal: Unify Philosophy of Maths

- The main question in phil maths is which to adopt:
  - Platonism (Plato, Gödel)
  - Naturalism (Quine)
  - Fictionalism (Field, Balaguer) / Nominalism (Goodman & Quine)
  - Structuralism (Dedekind, Benacerraf, Shapiro, Resnik)
  - Inferentialism (Wittgenstein, Sellars, Brandom)
  - Formalism (Hilbert, Curry)
  - Carnapianism (Carnap)
  - Logicism (Frege, Whitehead & Russell)

  Goal: Unify these.
- Psychologism offers no answer to $\emptyset_{ZF} = \ldots$?
- Intuitionism, Constructivism, and Finitism urge a methodology (a separate issue). Philosophers shouldn't tell the mathematicians how to practice.
- If-Thenism/Deductivism/Modal Structuralism (Putnam, Hellman): This is mathematical eliminativism. No de re knowledge. Discussed later.

Mathematical Theories

- Let T range over mathematical theories. Collapse theories that are notational variants or that have redundant axioms. Assume λ-Conversion is part of the logic of mathematical theories.
- Replace the function terms in T and their axioms with predicates and the corresponding relational axioms.
- Treat theories as situations:
  - *The theory* $T = \iota x(A!x \,\&\, \forall F(xF \equiv \exists p(T \models p \,\&\, F = [\lambda y\, p])))$, i.e.,
    $= \iota s \forall p(s \models p \equiv T \models p)$
- For each *sentence* φ that is a theorem of T, let φ^* be the result of indexing T's primary, closed terms and predicates to T. Example:
  - If $T = \text{ZF}$ and $\varphi = \emptyset \in \{\emptyset\}$ (so $T \vdash \varphi$), then $\varphi^* = \emptyset_{\text{ZF}} \in_{\text{ZF}} \{\emptyset\}_{\text{ZF}}$
- **Importation**: If $T \vdash \varphi$, then the following analytic claims are taken as truths of object theory: $T \models \varphi^*$ (read: φ^* *is true in* T).
- Truth in a theory is closed: $\varphi \vdash_T \psi$ and $T \models \varphi^*$, then $T \models \psi^*$
- **Reduction Axiom**: $\tau_T = \iota x(A!x \,\&\, \forall F(xF \equiv T \models F\tau_T))$

Mathematical Individuals

- **Reduction Axiom**: Theoretically identify individual κ_T as follows:
  - $\kappa_T = \iota x(A!x \,\&\, \forall F(xF \equiv \; T \models F\kappa_T))$
  - $0_{\text{PA}} = \iota x(A!x \,\&\, \forall F(xF \equiv \; \text{PA} \models F0_{\text{PA}}))$
  - $\emptyset_{\text{ZF}} = \iota x(A!x \,\&\, \forall F(xF \equiv \; \text{ZF} \models F\emptyset_{\text{ZF}}))$
- **Consequence**: **Equivalence Theorem**:
  - $\kappa_T F \equiv T \models F\kappa_T$

Mathematical Properties and Relations

- The types and typed object theory sketched.
- Using typed object theory, assert comprehension for abstract objects at every type. Examples:
  - $\exists x(A!x \ \& \ \forall F(xF \equiv \varphi))$, φ has no free xs — x has type i
  - $\exists P(\boldsymbol{A!}P \ \& \ \forall \boldsymbol{F}(P\boldsymbol{F} \equiv \varphi))$, φ has no free Ps — P has type $\langle i \rangle$
  - $\exists R(\boldsymbol{A!}R \ \& \ \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \varphi))$, φ has no free Rs — R has type $\langle i, i \rangle$
- Recall **Importation Rule**: If $T \vdash \varphi$, then $T \models \varphi^*$.
- **Reduction Axiom**: Theoretically identify relation Π:
  - $\Pi_T = \imath R(\boldsymbol{A!}R \ \& \ \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \ T \models \boldsymbol{F}\Pi_T))$
  - $N_{\text{PA}} = \imath P(\boldsymbol{A!}P \ \& \ \forall \boldsymbol{F}(P\boldsymbol{F} \equiv \ \text{PA} \models \boldsymbol{F}N_{\text{PA}}))$
  - $\in_{\text{ZF}} = \imath R(\boldsymbol{A!}R \ \& \ \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \ \text{ZF} \models \boldsymbol{F}\in_{\text{ZF}}))$
- Consequence: **Equivalence Theorem**:
  - $\Pi_T\boldsymbol{F} \ \equiv \ T \models \boldsymbol{F}\Pi_T$

Two Examples of the Analysis

- By mathematical practice, both $\vdash_{ZF} \emptyset \in \{\emptyset\}$ and $\vdash_{\mathbb{R}} 2 \leq \pi$, and so:

  $\vdash_{ZF} [\lambda x\, x \in \{\emptyset\}]\emptyset$ $\quad \vdash_{ZF} [\lambda x\, \emptyset \in x]\{\emptyset\}$ $\quad \vdash_{ZF} [\lambda R\, \emptyset R\{\emptyset\}]\in_{ZF}$

  $\vdash_{\mathbb{R}} [\lambda x\, x \leq \pi]2$ $\quad \vdash_{\mathbb{R}} [\lambda x\, 2 \leq x]\pi$ $\quad \vdash_{\mathbb{R}} [\lambda R\, 2R\pi]\leq$

- By Importation : $\mathrm{ZF} \models \emptyset_{ZF} \in_{ZF} \{\emptyset\}_{ZF}$ and $\mathbb{R} \models 2_{\mathbb{R}} \leq_{\mathbb{R}} \pi_{\mathbb{R}}$, and:

  $\mathrm{ZF} \models [\lambda x\, x \in \{\emptyset\}]_{ZF}\emptyset_{ZF}$ $\quad \mathrm{ZF} \models [\lambda x\, \emptyset \in x]_{ZF}\{\emptyset\}_{ZF}$ $\quad \mathrm{ZF} \models [\lambda R\, \emptyset R\{\emptyset\}]_{ZF} \in_{ZF}$

  $\mathbb{R} \models [\lambda x\, x \leq \pi]_{\mathbb{R}}2_{\mathbb{R}}$ $\quad \mathbb{R} \models [\lambda x\, 2 \leq x]_{\mathbb{R}}\pi_{\mathbb{R}}$ $\quad \mathbb{R} \models [\lambda R\, 2R\pi]_{\mathbb{R}} \leq_{\mathbb{R}}$

- Instances of Equivalence Theorem:

  $\emptyset_{ZF}F \equiv \mathrm{ZF} \models F\emptyset_{ZF}$ $\quad \{\emptyset\}_{ZF}F \equiv \mathrm{ZF} \models F\{\emptyset\}_{ZF}$ $\quad \in_{ZF}\mathcal{F} \equiv \mathrm{ZF} \models \mathcal{F}\in_{ZF}$

  $2_{\mathbb{R}}F \equiv \mathbb{R} \models F2_{\mathbb{R}}$ $\quad \pi_{\mathbb{R}}F \equiv \mathbb{R} \models F\pi_{\mathbb{R}}$ $\quad \leq_{\mathbb{R}}\mathcal{F} \equiv \mathbb{R} \models \mathcal{F}\leq_{\mathbb{R}}$

- Consequences:

  $\emptyset_{ZF}[\lambda x\, x \in \{\emptyset\}]_{ZF}$ $\quad \{\emptyset\}_{ZF}[\lambda x\, \emptyset \in x]_{ZF}$ $\quad \in_{ZF} [\lambda R\, \emptyset R\{\emptyset\}]_{ZF}$

  $2_{\mathbb{R}}[\lambda x\, x \leq \pi]_{\mathbb{R}}$ $\quad \pi_{\mathbb{R}}[\lambda x\, 2 \leq x]_{\mathbb{R}}$ $\quad \leq_{\mathbb{R}} [\lambda R\, 2R\pi]_{\mathbb{R}}$

Ontological (Analytic) Reduction of Mathematics

- We now know what is denoted by mathematical terms and predicates in theoretical contexts, and what the truth conditions are for truth in a theory T.
- To complete our reduction, we give readings of unadorned (theoretical) mathematical statements on which they are true.
- Simple Case: '0 is a number' (relative to Peano Number Theory): Two readings (suppressing subscripts):
  - $0\mathbb{N}$ (true)
  - $\mathbb{N}0$ (false)
- So the unadorned data is subject to an ambiguity in predication.
- The true reading, $0\mathbb{N}$, is derivable in object theory from the analytic truth $\text{PA} \models \mathbb{N}0$, by the Equivalence Theorem.

Ontological Reduction Generalized

- Consider any R and note that it is an axiom that:
  - $xyR \equiv x[\lambda z\, Rzy]\ \&\ y[\lambda z\, Rxz]$
- Base case: unadorned theoretical mathematical claims of the form 'a bears R to b' (relative to theory T) get two readings (suppressing subscripts): abR (true) and Rab (false).
- Complex case (ZF): No set is a member of the empty set. The standard translation is false: $\neg\exists x(Sx\ \&\ x \in \emptyset)$.
- The reading on which it is true:

$$\emptyset_{\text{ZF}}[\lambda y^{i}\ \neg\exists x(Sx\ \&\ x \in y)]_{\text{ZF}}\ \&$$
$$S_{\text{ZF}}[\lambda F^{\langle i\rangle}\ \neg\exists x(Fx\ \&\ x \in \emptyset)]_{\text{ZF}}\ \&$$
$$\in_{\text{ZF}}[\lambda F^{\langle i,i\rangle}\ \neg\exists x(Sx\ \&\ Fx\emptyset]_{\text{ZF}}$$

- General analysis: where φ^* is the representation of a theorem φ of theory τ and φ^- is the result of substituting new variables $y^{t_1}, \ldots, y^{t_n}$ for $\kappa^{t_1}, \ldots, \kappa^{t_n}$ in φ:
  - $\kappa_{\tau}^{t_1} \ldots \kappa_{\tau}^{t_n}[\lambda y^{t_1} \ldots y^{t_n}\ \varphi^-]_{\tau}$
- By the above axiom:

$$\kappa_{\tau}^{t_1}[\lambda y^{t_1}\ \varphi(y^{t_1}/\kappa^{t_1})]_{\tau}\ \&\ \ldots\ \&\ \kappa_{\tau}^{t_n}[\lambda y^{t_n}\ \varphi(y^{t_n}/\kappa^{t_n})]_{\tau}$$

Fine's Puzzle: I

- *F is essential to x* $=_{df} \Box(E!x \rightarrow Fx)$ **(E)**
- Counterexample (Fine 1994a, 4):
  - Let $x = s =$ Socrates.
    Let $F = K = [\lambda y\, y \in \{s\}]$.
- In modal set theory, from the fact that singleton Socrates essentially has Socrates as an element, it follows:
  - Necessarily, if Socrates exists, he is an element of singleton Socrates $\Box(E!s \rightarrow Ks)$
- But, intuitively, being an element of singleton Socrates (i.e., K) is not essential to Socrates.

Fine's Puzzle: II

- The Problem: One can prove the counterintuitive claim that being an element of singleton Socrates, $[\lambda y\, y \in \{s\}]$, is essential to Socrates ('s'). It follows from the assumption that having Socrates as an element, $[\lambda y\, s \in y]$, is essential to singleton Socrates ('$\{s\}$'):
- *Proof*. Suppose $[\lambda y\, s \in y]$ is essential to $\{s\}$. Then, by (**E**) above, $\Box(E!\{s\} \rightarrow [\lambda y\, s \in y]\{s\})$, and by λ-conversion, it follows that $\Box(E!\{s\} \rightarrow s \in \{s\})$. But, it is a principle of modal set theory that necessarily, singleton Socrates exists iff Socrates exists, i.e., $\Box(E!\{s\} \leftrightarrow E!s)$. So, $\Box(E!s \rightarrow s \in \{s\})$ (by the S5 inference rule: from $\Box(\varphi \rightarrow \psi)$ and $\Box(\varphi \leftrightarrow \chi)$, we may infer $\Box(\chi \rightarrow \psi)$). And by λ-conversion, $\Box(E!s \rightarrow [\lambda y\, y \in \{s\}]s)$. Thus, by (**E**) again, $[\lambda y\, y \in \{s\}]$ is essential to Socrates.

Essence, Modality, and Abstract Objects

- What properties do abstract objects 'have' necessarily? (Restrict $x, y, \ldots$ to abstract objects.)
- Distinguish: $\Box Fx$ vs. $\Box xF$
- Definition: $Essential(F, x) \;=_{df}\; xF$
- Now we work towards proof that mathematical objects have their mathematical properties essentially. We do this for two arbitrarily selected mathematical objects and one of their properties.
- Show:
  - $Essential([\lambda x\, x \in \{\emptyset\}]_{\text{ZF}}, \emptyset_{\text{ZF}})$, i.e., $\emptyset_{\text{ZF}}[\lambda x\, x \in \{\emptyset\}]_{\text{ZF}}$

Remember Our Previous Example of the Analysis

- By mathematical practice, $\vdash_{ZF} \emptyset \in \{\emptyset\}$ and so:

  $\vdash_{ZF} [\lambda x\, x \in \{\emptyset\}]\emptyset \quad \vdash_{ZF} [\lambda x\, \emptyset \in x]\{\emptyset\} \quad \vdash_{ZF} [\lambda R\, \emptyset R\{\emptyset\}]\in$

- By Importation (suppressing indices): $\text{ZF} \models \emptyset \in \{\emptyset\}$ and further:

  $\text{ZF} \models [\lambda x\, x \in \{\emptyset\}]_{ZF}\emptyset_{ZF} \quad \text{ZF} \models [\lambda x\, \emptyset \in x]_{ZF}\{\emptyset\}_{ZF} \quad \text{ZF} \models [\lambda R\, \emptyset R\{\emptyset\}]_{ZF} \in_{ZF}$

- Instances of Equivalence Theorem:

  $\emptyset_{ZF}F \equiv \text{ZF} \models F\emptyset_{ZF} \quad \{\emptyset\}_{ZF}F \equiv \text{ZF} \models F\{\emptyset\}_{ZF} \quad \in_{ZF}\mathcal{F} \equiv \text{ZF} \models \mathcal{F}\in_{ZF}$

- Consequences:

  $\emptyset_{ZF}[\lambda x\, x \in \{\emptyset\}]_{ZF} \quad \{\emptyset\}_{ZF}[\lambda x\, \emptyset \in x]_{ZF} \quad \in_{ZF}[\lambda R\, \emptyset R\{\emptyset\}]_{ZF}$

- The first of these is, by definition: $\mathit{Essential}([\lambda x\, x \in \{\emptyset\}]_{ZF}, \emptyset_{ZF})$.

Back to Fine: 'Impure' Abstracta

- M = Modal Set Theory + Urelements.
- Theorems of object theory which take the following form:
  - $\mathrm{M} \models F\{s\}_{\mathrm{M}}$
- Instance of the Theoretical Identification Principle:

$$\{s\}_{\mathrm{M}} = \iota x(A!x \;\&\; \forall F(xF \equiv \mathrm{M} \models F\{s\}_{\mathrm{M}}))$$

- Consequence: The properties essential to singleton Socrates are the properties it exemplifies according to M, since these are its encoded properties.

Essence, Modality, and 'Impure' Abstracta

- The data = theorems of M:
  - $\vdash_M s \in \{s\}$
  - $\vdash_M [\lambda z\, s \in z]\{s\}$
  - $\vdash_M [\lambda z\, z \in \{s\}]s$
- Under our analysis, we have the following theorems in object theory. NOTE: We don't index *s* to M.
  - $\mathrm{M} \models s \in_{\mathrm{M}} \{s\}_{\mathrm{M}}$ (ξ_1)
  - $\mathrm{M} \models [\lambda z\, s \in z]_{\mathrm{M}}\{s\}_{\mathrm{M}}$ (ξ_2)
  - $\mathrm{M} \models [\lambda z\, z \in \{s\}]_{\mathrm{M}}s$ (ξ_3)
- It follows from (ξ_2), given Equivalence:
  - $\{s\}_{\mathrm{M}}[\lambda z\, s \in z]_{\mathrm{M}}$ (ρ)

Essence, Modality, and ‘Impure’ Abstracta

- It follows from (ρ) and the definition of essential properties: for abstract objects:
  - $Essential([\lambda z\, s \in z]_{\mathrm{M}}, \{s\}_{\mathrm{M}})$

  This proves a premise of Fine’s counterexample.
- Socrates, as an ordinary object, doesn’t encode properties:
  - $\neg s[\lambda z\, z \in \{s\}]_{\mathrm{M}}$.
- Nothing about Socrates follows by either the Theoretical Identification Principle or the Equivalence Theorem from (ξ_1)–(ξ_3), since those principles don’t apply to Socrates. Nor can we abstract from them any properties of Socrates in virtue of the properties exemplified by singleton Socrates according to M (they are all encoding claims)
- The asymmetry between Socrates and singleton Socrates is established on theoretical grounds.

Some Traditional Philosophies of Mathematics

- Platonism: The terms and predicates of mathematical language denote abstract objects and abstract relations. Gödel 1944, 1947
- Naturalism: Accept only the mathematics needed for our best scientific theory.
- Fictionalism: Mathematical objects don't exist; mathematical statements are prefixed by a story operator. Field 1980, 1989
- Structuralism: Mathematical language is about pure structures or patterns.
  Resnik 1997, Shapiro 1997
- Inferentialism: The content of the terms of mathematical language is their inferential role in the discourse.
  Wittgenstein 1956, Sellars 1980, Dummett 1973, Brandom 2000
- Formalism: Mathematics consists of formal theories that manipulate formal symbols within uninterpreted formal systems.
  von Neumann 1931, Curry 1951.
- Carnapianism: Every mathematical theory is about (and true of) the objects in its own framework. External existence is just a matter of expedience.
- Logicism: Mathematics is reducible to logic & analytic truths.
  Frege 1893/1902, Russell & Whitehead 1910–1913

Observations:

- Traditional Platonism is ‘naive’ or piecemeal. No prior, rigorous theory of abstracta is offered. Epistemological problems as well.
- Fictionalism and If-Thenism don’t treat simple mathematical statements as predications, but the appearances are that they predicate properties of objects.
- Inferentialism needs systematicity. Can we formalize ‘roles’?
- Formalism requires a type/token distinction: the formalisms and rules are stated in terms of types.
- Structuralism offers no mathematics-free theory of structures or theory of patterns.
- Logicism seems to be a non-starter: mathematics has strong existence assumptions, but logic has very weak existence assumptions.

Platonism

- The terms of mathematical language and theories denote abstract objects and abstract relations.
- There are true (encoding) readings of ordinary mathematical statements (i.e., those with no 'theory-operator' prefixed):
  '2 is prime' is ambiguous between '2*P*' (true) and '*P*2' (false)
- We've achieved one element of Gödel's program for solving the problem of the ontological status of mathematical objects and concepts (i.e., answering the question of their 'objective validity'): an axiomatization of metaphysics. (H. Wang 1996)
- Each mathematical theory is about its own domain of abstract objects.
- Epistemological problems addressed in Linsky & Zalta 1995.

Naturalism

- Linsky and Zalta 1995:
  - Reject traditional view of mind-independence and objectivity: abstracta aren't subject to appearance/reality, sparse, or complete.
  - They are a plenitude, and non-arbitrary. Can't have just ZF-sets and not NF, NBG, nonwellfounded sets, etc.
  - Parsimony: accept as few objects as possible in a non-arbitrary way. But with abstract objects this means: accept them all.
  - Knowledge by acquaintance and by description collapses
- Further thoughts: reconceptualize abstract objects as things naturalists already believe in. Use Aristotelian conception of immanent rather than transcendent objects; they arise as patterns in the natural world. How?
- The comprehension principle can't be instantiated until mathematicians put forward a theory. Once we have a theory, we can instantiate comprehension to determine the objects and relations required by the theory.

Fictionalism I

- Reinterpret the quantifier using the distinction between 'there is' and 'there exists' in natural language. (Contra Quine, don't rehabilitate language, but rather regiment it.)
- Interpret $E!$ as existence predicate. Distinguish 'there is an x such that φ' ($\exists x\varphi$) and 'there exists an x such that φ' ($\exists x(E!x \,\&\, \varphi)$).
- Our definitions become:
  - $A!x \;=\; [\lambda x\, \neg\Diamond E!x]$, i.e., necessarily nonexistent!
- So comprehension now asserts that there *are* (necessarily) nonexistent objects.
- On this interpretation, mathematical objects, e.g., 2, $\emptyset$, don't exist. Here, we speak with the learned, since this is what the fictionalist and Field claims.
- We preserve another element of Field's philosophy, his view that mathematical claims are false. On our view, ordinary mathematical statements do have a false reading. (cf. Field 1980)

Fictionalism II

- We now have an explanation as to why realists and anti-realists can't even agree on the data (i.e., the truth of mathematics). This is explained by an ambiguity in language. No other philosophy explains this.
- Balaguer's 1998 conclusion:
  - On every point, the arguments for and against (full-blooded) platonism or fictionalism evenly cancel out: none is conclusive and we could never know whether one is true. So there is no fact of the matter whether mathematical entities exist.
- Explanation: platonism and fictionalism are two incompatible interpretations of the same formalism and the regimentation of natural language in platonistic or fictionalistic terms is equally good.
- Our analysis isn't subject to the problem of Balaguer's full-blooded platonism: he doesn't have incomplete objects, and so the denotations of the terms of our theories (which are incomplete) can't uniquely specified.

Nominalism

- New forms of nominalism: Azzouni 2004, Priest 2005.
- Azzouni: Quantifier commitment vs. ontological commitment
- Priest: Interpret $\exists x\varphi$ as 'some x is s.t.φ', not as 'there is an x s.t. φ' or 'there exists an x s.t. φ'. So $\exists x\varphi$ is existentially neutral.
- Use these ably-defended suggestions to interpret OT.
- The result is Azzouni-Priest-Routley nominalism.
- This also makes sense of Rayo forthcoming ('ultrathin' objects) and Linnebo 2018 ("objects whose existence makes no substantial demand upon the world").
- Abstract objects are 'ultrathin' in a couple of senses: (a) a theoretical description is sufficient for acquaintance and reference – no information pathway needed; (b) they encode only the properties attributed in their respective theories.

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