Seminar on Axiomatic Metaphysics
Lecture 12
Philosophy of Mathematics II

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Review

- Third-order fragment of typed object theory:
  - $\exists!x(A!x \mathbin{\&} \forall F(xF \equiv \varphi))$, φ has no free xs — x has type i
  - $\exists!R(\boldsymbol{A!}R \mathbin{\&} \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \varphi))$, φ has no free Rs — R has type $\langle i, i\rangle$
- **Importation**: If $T \vdash \varphi$, then the following analytic claims are taken as truths of object theory: $T \models \varphi^*$ (read: *φ^* is true in T*), where this latter is defined as $T[\lambda y\, \varphi^*]$.
- $\kappa_T = \iota x(A!x \mathbin{\&} \forall F(xF \equiv T \models F\kappa_T))$ — F has type $\langle i\rangle$

  e.g., $\emptyset_{\mathrm{ZF}} = \iota x(A!x \mathbin{\&} \forall F(xF \equiv \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}))$
- $\Pi_T = \iota R(\boldsymbol{A!}R \mathbin{\&} \forall \boldsymbol{F}(R\boldsymbol{F} \equiv T \models \boldsymbol{F}\Pi_T))$ — $\boldsymbol{F}$ has type $\langle\langle i, i\rangle\rangle$

  e.g., $\in_{\mathrm{ZF}} = \iota R(\boldsymbol{A!}R \mathbin{\&} \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \mathrm{ZF} \models \boldsymbol{F}\in_{\mathrm{ZF}}))$
- **Consequence**: **Equivalence Theorem**:
  - $\kappa_T F \equiv T \models F\kappa_T$
  - $\Pi_T\boldsymbol{F} \equiv T \models \boldsymbol{F}\Pi_T$
- Examples: encoded properties derived from $\vdash_{\mathrm{ZF}} \emptyset \in \{\emptyset\}$:

  $\emptyset_{\mathrm{ZF}}[\lambda x\, x \in \{\emptyset\}]_{\mathrm{ZF}}$ and $\in_{\mathrm{ZF}}[\lambda R\, \emptyset R\{\emptyset\}]_{\mathrm{ZF}}$

Issues To Be Addressed by a Mathematical Structuralism

- In what precise sense is mathematics about abstract structure?
- In what precise sense are the elements of a structure incomplete?
- What are the essential properties of the elements of a structure?
- Can there be identity/distinctness between elements of different structures?
  - Is the natural number 2 the same as the number 2 of PA and are these identical to the number 2 of $\mathfrak{R}$?
  - Does the answer suffer from the Julius Caesar problem?
- Do the elements of a mathematical structure ontologically depend on the structure?
- Do the elements of a structure have haecceities?
- Is indiscernibility a problem for structuralism?

These issues addressed in Nodelman and Zalta 2014 (Preprint available online).

Issue: How Is Mathematics About Abstract Structure?

- When we import T, we may identify T as an incomplete abstract object that simply records just the truths of T:

  $$\textit{The structure } T = \iota x(A!x \;\&\; \forall F(xF \equiv \exists p(T \models p \;\&\; F = [\lambda y\, p])))$$
- So structures are abstract objects that encode just the T-facts.
- Moreover, we may define:

  x *is a structural element of* $T =_{df} T \models \forall y(y \neq_T x \rightarrow \exists F(Fx \;\&\; \neg Fy))$

  R *is a relation of* $T =_{df} T \models \forall S(S \neq_T R \rightarrow \exists \boldsymbol{F}(\boldsymbol{F}R \;\&\; \neg \boldsymbol{F}S))$
- Thms: $\emptyset_{\text{ZF}}$ is a structural element, and $\in_{\text{ZF}}$ a relation, of ZF.
- So the relations and elements of a structure are abstract.
- Physical system K has the structure T if the relations of K exemplify the properties encoded by the relations of T or if there is an isomorphism between the relations and objects of K and T.

Issue: Are Structural Elements and Relations Incomplete?

- Answer: Yes, with respect to encoding, but not exemplification.
- Let x^t range over abstract entities of type t, and $F^{\langle t \rangle}$ range over properties of entities of type t, and $\bar{F}^{\langle t \rangle}$ denote $[\lambda y^t\ \neg F^{\langle t \rangle} y]$:

$$x^t \textit{ is incomplete } =_{df} \exists F^{\langle t \rangle}(\neg xF \;\&\; \neg x\bar{F})$$

- Cf. Russell's 1903 discussion of Dedekind.
- Benacerraf's (1965) argument (from 'numbers have no properties other than structural ones' to 'the numbers aren't objects') fails for structural elements that both encode and exemplify properties.
- This undermines Shapiro (2006)/Linnebo (2008), who say: a mathematical object can't have only the mathematical properties defined by its structure—the natural number 3 has: numbering Shapiro's children, being Linnebo's favorite number, etc.

Issue: Which Properties Are Essential To Structural Elements?

- Answer: their encoded properties.
- Reason: Because these are the properties by which they are theoretically identified via the Reduction Axiom and by which they are individuated by the definition of identity for abstract objects; they make them the objects that they are.
- Encoded mathematical properties are even more important than properties necessarily exemplified (e.g., not being a building, being abstract, etc.) (cf. Shapiro 2006, 116; Hellman 2001, 193)
- Theory explains the asymmetry between Socrates and {Socrates}. Let M be modal set theory plus urelements. After importation:

  $$\mathrm{M} \models s\in\{s\}_{\mathrm{M}} \quad \mathrm{M} \models [\lambda z\, s\in z]_{\mathrm{M}}\{s\}_{\mathrm{M}} \quad \mathrm{M} \models [\lambda z\, z\in\{s\}]_{\mathrm{M}}s$$

- The middle claim implies $\{s\}_{\mathrm{M}}[\lambda z\, s\in z]_{\mathrm{M}}$, which in turn implies that {Socrates} essentially has Socrates as an element; but we can't abstract out any properties about Socrates from these claims: they involve encoding claims. (cf. Fine 1994)

Issue: Can There Be Cross-structural Identity/Distinctness?

- Resnik 1981, MacBride 2005 (no fact of the matter); Parsons 1990 (cautious); Shapiro (2006) ($S \neq S' \rightarrow x \neq x'$).
- Our theory yields: $2 \neq 2_{\mathrm{PNT}} \neq 2_{\mathfrak{R}}$, since they encode different properties. (cf. Frege, Shapiro 2006, 128)
- Our theory also yields: $2 \neq \{\{\emptyset\}\}_{\mathrm{ZF}} \neq \{\emptyset, \{\emptyset\}\}_{\mathrm{ZF}}$. (These encode different properties.)
- No Caesar problem: $2_T \neq$ Caesar, given that Caesar is ordinary.
- When T and T' have the same theorems (e.g., because T' has a redundant axiom), or are notational variants, we collapse them (and their objects) prior to importation.
- What can we say about the structure PNT and the ZF set ω? Answer: There is a 1–1 correspondence between the structural elements of PNT and the members of ω_{ZF}, just as there is between the elements of ω_{ZF} and ω_{ZFC}.

Issue: Do Elements Ontologically Depend on Their Structure?

- Shapiro 1997 (structure is prior); Linnebo 2008 (ZF elements don't depend on structure); Hellman 2001, MacBride 2006 (circularity threatens).
- Answer: A structure and its relations/elements ontologically depend on each other, just as offices and office-buildings ontologically depend on each other.
- Reason: The structure and its relations/elements all exist as abstractions grounded in facts of the form $T \models p$. No circularity!
- This applies to both algebraic and non-algebraic mathematical theories.
- Our answer therefore is in conflict with Linnebo 2008: "no set is strongly dependent on the structure of the entire universe of sets. For every set can be individuated without proceeding via this structure" (79). But this assumes some given $\in$ relation.

Issue: Do the Elements of a Structure Have Haecceities?

- Our answer: No. (cf. Shapiro 2006, Keränen 2006)
- Reason: Identity (simpliciter) is defined in terms of encoding formulas. Neither '$[\lambda xy\, x = y]$' nor '$[\lambda x\, x = a]$' are well-defined.
- Background: abstract objects can be modeled by sets of properties. But you can't, for each distinct set b of properties, have a distinct property $[\lambda x\, x = b]$ (violation of Cantor's Theorem).
- In this model, an abstract object x *exemplifies* a property F just in case a proxy of x (down at the level of individuals) is an element of F.
- So we get the theorem: $\exists a, b(a \neq b \;\&\; \forall F(Fa \equiv Fb))$
- So there are distinct abstract objects (i.e., they encode different properties) that are indiscernible from the point of view of exemplification. (Picture of Aczel models.)

Issue: Does Indiscernibility Pose a Problem for Structuralism?

- Answer: No. Consider dense, linear orderings, no endpoints:

Irreflexive:	$\forall x(x \not< x)$
Transitive:	$\forall x, y, z(x < y \;\&\; y < z \rightarrow x < z)$
Connected:	$\forall x, y(x \neq y \rightarrow (x < y \;\vee\; y < x))$
Dense:	$\forall x, y \exists z(x < z < y)$
No Endpoints:	$\forall x \exists y \exists z(z < x < y)$

- Aren't all the elements of this structure ('*D*') indiscernible?
- Answer: No. Reason: There are no elements of *D*. *D* is defined solely by general properties of the ordering relation $<_D$, which encodes such properties of relations as: $[\lambda R\; \forall x \neg xRx]$, $[\lambda R\; \forall x, y, z(xRy \;\&\; yRz \rightarrow xRz)]$, etc.
- Analogy: A novel asserts, "General X advanced upon Moscow with an army of 100,000 men". There aren't 100,002 characters, but only 3 (General X, Moscow, and the army of 100,000 men).

Another Example: The Case of i and $-i$

- To get $\mathbb{C}$, we take the axioms for $\mathfrak{R}$ and add the following:
  $i^2 = -1$
  (Strictly speaking, $i^2 =_{\mathbb{C}} -1$.)
- Objection: The structural elements i and $-i$ are collapsed in (our) structuralism.
- Reason: Any formula $\varphi(x)$ with only x free in the language of complex analysis that holds of i also holds of $-i$, and vice versa. Thus, i and $-i$ are indiscernible and after importing $\mathbb{C}$ we have $\mathbb{C}\models Fi \equiv \mathbb{C}\models F{-i}$. One might try to argue, by the Equivalence Theorem, that $iF \equiv \mathbb{C}\models Fi$ and $-iF \equiv \mathbb{C}\models F{-i}$. It would then follow that $iF \equiv -iF$. Therefore, $i = -i$, by the definition of identity for abstract objects.
- Response: This argument is blocked because i and $-i$ are not elements of the structure $\mathbb{C}$.

Formal Solution

- Our procedure: import φ of T into object theory by adding $T \models \varphi^*$, where φ^* is the result of replacing all the well-defined singular terms κ in φ by κ_T.
- *x is an element of* (structure) $\mathbb{C} =_{df} \mathbb{C} \models \forall y(y \neq_{\mathbb{C}} x \rightarrow \exists F(Fx \,\&\, \neg Fy))$
- By this definition, i and $-i$ aren't elements of $\mathbb{C}$.
- Our procedure for interpreting the language of $\mathbb{C}$: *before importation*, replace every theorem of the form $\varphi(\ldots i \ldots)$, by a theorem of the form: $\exists x(x^2 + 1 = 0 \,\&\, \varphi(\ldots x \ldots))$, and then import the result.
- Under this analysis, i and $-i$ disappear and we are left with structural properties of complex addition and complex multiplication. E.g., for complex addition $+_{\mathbb{C}}$, for each theorem $\exists x(x^2 + 1 = 0 \,\&\, \varphi(\ldots x \ldots))$, we can abstract out properties encoded by $+_{\mathbb{C}}$ of the form $[\lambda R\, \exists x(x^2 R 1 = 0 \,\&\, \varphi(\ldots x \ldots))]$. Similar techniques can be used for complex multiplication $\times_{\mathbb{C}}$.

Elements and Symmetries

- What gives a mathematical structure its *structure*? Answer: the relations of the theory. Without relations, there's no structure.
- An *element* of a structure must be uniquely characterizable in terms of the relations of the structure—it must be discernible.
- Indiscernibles arise from symmetries (non-trivial automorphisms) of the structure. Mathematicians working *with* a structure find it useful to give separate names to indiscernibles. But these names don't denote elements of the structure. After all, the names are arbitrary and there is nothing (i.e., no property) within the theory that distinguishes the indiscernibles from each other.
- The mathematician's use of 'i' and '$-i$' in $\mathbb{C}$ is different from their use of '$1_{\mathbb{C}}$' and '$-1_{\mathbb{C}}$'. The naming of 1 and -1 is not arbitrary — you can't permute $1_{\mathbb{C}}$ and $-1_{\mathbb{C}}$ and retain the same structure. So it makes sense to say that 'i' and '$-i$' do not denote objects the way that '$1_{\mathbb{C}}$' and '$-1_{\mathbb{C}}$' do.

Inferentialism

- Recall that we import the theorems of mathematical theories into object theory but prefaced by their theory operator. Because we read $T \models p$ as "p is true in T", all the deductive consequences of propositions true in T are true in T.
- Now we can say: the content of mathematical terms is their inferential role in the theories in which they are axiomatized:
  $\emptyset_{\mathrm{ZF}} = \iota x(A!x \,\&\, \forall F(xF \equiv \mathrm{ZF} \models F\emptyset_{\mathrm{ZF}}))$
- Given our Importation rule and the fact that all the theorems of ZF become imported into object theory (prefaced by the theory operator), this abstracts out, and objectifies, the inferential role of '$\emptyset_{\mathrm{ZF}}$'
- This constitutes a new kind of proof-theoretic semantics: we abstract the semantic denotation of the term from its proof-theoretic role.

Inferentialism: II

- Note: Warren 2020 uses the 'Peano rules' an an example of unrestricted inferentialism. These are the meaning-constituting rules that give the meaning of the terms:

$$(P1)\ \frac{}{N0} \qquad (P2)\ \frac{N\alpha}{NS\alpha} \qquad (P3)\ \frac{N\alpha}{0 \neq S\alpha} \qquad (P4)\ \frac{N\alpha \quad N\beta \quad S\alpha{=}S\beta}{\alpha{=}\beta}$$

$$(P5)\ \frac{\varphi(0) \quad \forall\xi(\varphi(\xi) \rightarrow \varphi(S\xi)) \quad N\beta}{\varphi(\beta)}$$

- You can't specify the inferential role of 0 on this basis. You can give a theoretical description of the meaning of 0 as:
  - The inferential role of 0 $=_{df}$ ...
- You'd have to add some set theory and some way of abstracting out the role of 0 as a single entity.
- Without that, all you can do is use the inference rules to reason about 0.

Formalism

- Formalism: Mathematics consists of theories that manipulate formal symbols within a formal system. But a distinction between symbol *types* and tokens is needed.
- Option (1). Local type model: (a) identify symbol types as abstract objects that encode the distinguishing properties of the tokens, and (b) regard the object-theoretic individuals and relations used in the analysis of mathematics as symbol types.
- Option (2). Global type model: interpret our entire formalism: $A!x$ means x is an abstract symbol type, interpret the Fs as the distinguishing properties of the types, and interpret xF as identifying the formal properties that constitute the symbol type.
- Both options: '0_{PNT}', '$\pi_{\mathfrak{R}}$', '$\in_{\text{ZF}}$', denote abstract symbol types.
- Using Reduction Axioms, the terms and predicates encode all of the allowable symbol transformations built into each term or predicate.

Carnapianism

- Carnap took each linguistic framework to be about the objects and relations represented by its primitive notions (1950 [1956]).
- He refused to draw any conclusions about the 'external' existence of the objects and relations of *any* framework – such external existence questions were seen as questions about the expediency of adopting one framework rather than another.
- But given Carnap's interest in semantics, one might expect an explicit statement of the principle that guarantees the 'internal existence' of the appropriate objects for each logical framework.
- Our work provides such a principle; without it, we lack a semantic interpretation of the terms for *arbitrary* frameworks and can't therefore say why the 'internal' questions of existence for arbitrary frameworks are always answerable in the affirmative.

The Nature of Logicism

- Logicism: mathematics is reducible to logic & analytic truths.
  - Logicism about Concepts: Every mathematical concept is analyzable in terms of logical concepts.
  - Logicism about Propositions: For every mathematical theorem φ, if each mathematical concept in φ is replaced by the logical analysis, then the resulting proposition is true just in virtue of logical concepts.
- The goal of Leitgeb, Nodelman, and Zalta (m.s.) ("A Defense of Logicism") is to show that the object-theoretic analysis of maths is logicist in both senses.
- This would unify logic and mathematics (two *a priori* sciences) and produce the epistemological benefits of logicism.

The Epistemological Benefits of Logicism

- (Benacerraf 1981, 42–43):
  "But in reply to Kant, logicists claimed that these [mathematical] propositions are a priori because they are analytic—because they are true (false) merely 'in virtue of' the meanings of the terms in which they are cast. Thus to know their meanings is to know all that is required for a knowledge of their truth. No empirical investigation is needed. The philosophical point of establishing the view was nakedly epistemological: logicism, if it could be established, would show that our knowledge of mathematics could be accounted for by whatever would account for our knowledge of language. And, of course, it was assumed that knowledge of language could *itself* be accounted for in ways consistent with empiricist principles, that language was itself entirely learned. Thus, following Hume, all our knowledge could once more be seen as concerning either 'relations of ideas' (analytic and a priori) or 'matters of fact'."

Logicism About Concepts

- Logicism about Concepts: Every mathematical concept is analyzable in terms of logical and analytically-defined concepts.
- Claim: This is made true by our analyses of 0_{PNT}, $\emptyset_{\text{ZF}}$, $\in_{\text{ZF}}$, etc.
- These mathematical concepts are relativized to their theories and then identified as:
  - $0_{\text{PNT}} = \iota x(A!x \mathbin{\&} \forall F(xF \equiv \text{PNT} \models F0_{\text{PNT}}))$
  - $\emptyset_{\text{ZF}} = \iota x(A!x \mathbin{\&} \forall F(xF \equiv \text{ZF} \models F\emptyset_{\text{ZF}}))$
  - $N_{\text{PNT}} = \iota F(\boldsymbol{A!}F \mathbin{\&} \forall \boldsymbol{F}(F\boldsymbol{F} \equiv \text{PNT} \models \boldsymbol{F}N_{\text{PNT}}))$
  - $\in_{\text{ZF}} = \iota R(\boldsymbol{A!}R \mathbin{\&} \forall \boldsymbol{F}(R\boldsymbol{F} \equiv \text{ZF} \models \boldsymbol{F}\in_{\text{ZF}}))$
- Each mathematical concept is identified with its logically definable role within its governing mathematical theory, and is, therefore, a logical concept.

Logicism About Propositions

- Logicism about Propositions: For every mathematical theorem φ, if each mathematical concept in φ is replaced by the logical analysis, then the resulting proposition is true just in virtue of logical concepts.
- We've seen a general example: No set is a member of the empty set (derived in the context of ZF).
- The readings:

$$\text{ZF} \models \neg\exists x(S_{\text{ZF}}x \;\&\; x \in_{\text{ZF}} \emptyset_{\text{ZF}}) \qquad \text{(true)}$$
$$\emptyset_{\text{ZF}} S_{\text{ZF}} \in_{\text{ZF}} [\lambda y^i F^{\langle i \rangle} R^{\langle i,i \rangle}\; \neg\exists x(Fx \;\&\; xRy)]_{\text{ZF}} \qquad \text{(true)}$$
$$\neg\exists x(S_{\text{ZF}}x \;\&\; x \in_{\text{ZF}} \emptyset_{\text{ZF}}) \qquad \text{(false)}$$

- The first true reading is true in virtue of the concept denoted by 'ZF', and the second true reading is true (indeed *derivable*) in virtue of (the analysis of) the concepts $\emptyset_{\text{ZF}}$, S_{ZF}, and $\in_{\text{ZF}}$.
- This establishes Logicism about Propositions.

Object Theory Fragment For Mathematics is a Logic

- The smallest models of object theory analogous to 3OL: 1 ordinary object, 2 properties, and 4 abstract objects.
- Leitgeb, Nodelman, and Zalta (m.s.) argue: using a natural conception of *logic*, object theory is one.
- A sentence is logically true iff true in every interpretation in which the domain contains the entities required for the possibility of (complex) thoughts.
- Cf. Comprehension and an instance of λ-Conversion, say: $[\lambda y\, \neg Gy]x \equiv \neg Gx$). The latter abstracts out an existing logical pattern *not exemplifying* G. This is required if we are to understand and reason about negations.
- Similarly, Comprehension for Abstracta justifies reasoning about a relation (say $<_D$: dense, linear, orderings without endpoints) once mathematicians state the axioms.

Interlude

- The recent paper Zalta 2024 ("Mathematical Pluralism") summarizes the foregoing metaphilosophical form of mathematical pluralism – each philosophy of mathematics has a valid contribution..
- In the final section, the paper argues Modal Structuralism is not pluralism but reductionism.

From If-Thenism to Modal Structuralism

- OT and if-thenism (deductivism) both take the fundamental truths of T to be under the scope of an operator: "In T, ..." or "If the conjunction of the axioms of T hold, then ...".
- Modal Structuralism (MS) goes a step further. The constants and predicates of mathematical theories don't have a semantic content at our world and there is no reading of categorical predications on which they are true.
- MS method: replace constants and predicates in T by variables, so that each axiom φ of T becomes an open formula of the form $\varphi(\vec{x}, \vec{F})$, where $\vec{x}$ and $\vec{F}$ represent the sequence of individual and relation variables introduced.
- Using $\wedge T(\vec{x}, \vec{F})$ as the conjunction of the resulting axioms, MS recasts the categorical theorems φ of T as modal claims: (A) necessary logical theorems, and (B) a possibility claim:
  - (A) $\Box\forall\vec{x}\,\forall\vec{F}(\wedge T(\vec{x}, \vec{F}) \rightarrow \varphi(\vec{x}, \vec{F}))$
  - (B) $\Diamond\exists\vec{x}\,\exists\vec{F}(\wedge T(\vec{x}, \vec{F}))$

Why Modal Structuralism Can't Be Preserved

- Mathematical theories are not *about* structures or indeed about anything! CBF is invalid.
- This is mathematical eliminativism.
- To interpret T, math has to be conceived as modal. Indeed, they sometimes customize (B) to T (Hellman 1989, 27-30; 45; 1996). Is this general?
- There is no *de re* mathematical knowledge: we can't explain "π is more well-known that e", "At one time, mathematicians didn't believe that $\sqrt{-1}$ exists; "Fraenkel wondered whether $\omega + \omega$ could be proved in Z"; "0 wasn't always used for counting".
- Main question: If $\Diamond \exists \vec{x}\, \exists \vec{F}(\wedge T(\vec{x}, \vec{F}))$, then semantically, there are possible worlds where $\exists \vec{x}\, \exists \vec{F}(\wedge T(\vec{x}, \vec{F}))$ is true. So if mathematical objects exist at other possible worlds, on what conception of 'abstract object' do they exist at other possible worlds but not ours?

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