Principia Logico-Metaphysica

(Draft/Excerpt)

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To my wife, Susanne Z. Riehemann

DRAFT EXCERPT

NOTE: This is an excerpt from an incomplete draft of the monograph *Principia Logica-Metaphysica*. The monograph currently has four parts:

Part I: Prophilosophy
Part II: Philosophy
Part III: Metaphilosophy
Part IV: Technical Appendices, Bibliography, Index

This excerpt was generated on October 28, 2016 and contains:

- Part II:

- Part IV:

Consequently, this excerpt omits the Preface, Acknowledgments, Part I (Chapters 1-6), Part II/Chapters 15–16 (which are being reworked), Part III (which is mostly unwritten), and some Appendices in Part IV. The excerpt contains references to some of this omitted content.

The work is ongoing and so the monograph changes constantly. Citations should explicitly reference this version of October 28, 2016, since page numbers, chapter numbers, section numbers, item (definition, theorem) numbers, etc., may all change in future versions.

Contents

Part I

Prophilosophy

Part II

Philosophy

Chapter 7

The Language

In this and subsequent chapters, our metalanguage makes use of informal notions and principles about numbers and sets so as to more precisely articulate and render certain definitions. These are *not* primitive notions or principles of our metaphysical system; ultimately, they are to be understood by using object theory to analyze the language of mathematics, as we shall do in several later chapters.

7.1 Metatheoretical Definitions

We begin with the definitions that create a particular second-order language in all of its details.

(**1**) **Definitions:** Simple Terms. A *simple term* of our language is any expression that is a *simple individual term* or a *simple n-place relation term* ($n \geq 0$), where these are listed as follows:

(.1) Simple Individual Terms: (Less Formal)
Individual Constants (Names):
$a_1, a_2, \ldots$ (a, b, c, d, e)
Individual Variables:
$x_1, x_2, \ldots$ (x, y, z, u, v, w)

(.2) Simple n-place Relation Terms ($n \geq 0$):
n-place Relation Constants (Names)
$P^n_1, P^n_2, \ldots$ $(P^n, Q^n, R^n, S^n, T^n)$
n-place Relation Variables:
$F^n_1, F^n_2, \ldots$ $(F^n, G^n, H^n, I^n, J^n)$

(.3) Distinguished 1-place Relation Constant:
$E!$ (read: 'being concrete' or 'concreteness')

A *constant* is any expression that is either an individual constant or an n-place relation constant ($n \geq 0$). A *variable* is any expression that is either an individual variable or an n-place relation variable ($n \geq 0$). The expressions listed in the column labeled 'Less Formal' are often used as replacements to facilitate readability. It is assumed that we never run out of primitive constants and variables, despite the fact that the less formal alternatives appear in a finite list.

(2) **Definitions:** Other Primitive Expressions of the Language. The simple terms of our language are primitive expressions. The other primitive expressions of our language are listed here. In what follows, we use α, β, γ as metavariables ranging over all variables and use μ, ν (Greek *mu*, *nu*), sometimes with a prime or an index, as metavariables ranging just over primitive individual variables. Thus, our language includes the following additional primitive expressions:

(.1) Unary Formula-Forming Operators:
$\neg$ ('it is not the case that' or 'it is false that')
$\mathcal{A}$ ('actually' or 'it is actually the case that')
$\Box$ ('necessarily' or 'it is necessary that')

(.2) Binary Formula-Forming Operator:
$\rightarrow$ ('if ..., then ...')

(.3) Variable-Binding Formula-Forming Operator:
$\forall\alpha$ ('every α is such that')
for every variable α

(.4) Variable-Binding Individual-Term-Forming Operator:
$\imath\nu$ ('the ν such that')
for every individual variable ν

(.5) Variable-Binding n-place Relation-Term-Forming Operators ($n \geq 0$):
$\lambda\nu_1 \ldots \nu_n$ ('being individuals $\nu_1 \ldots \nu_n$ such that')
for any distinct individual variables $\nu_1, \ldots, \nu_n$
When $n = 0$, the λ binds no variables and is read as 'that'

These primitive, syncategorematic expressions are referenced in the definition of the syntax of our language and become parts of the complex formulas and complex terms defined there.[80] In what follows, we sometimes call $\neg$ the *nega-*

[80] A syncategorematic expression is an expression that is neither a term, i.e., an expression that is assigned a denotation, nor a formula, i.e., an expression that is assigned truth conditions. Thus these are expressions that aren't assigned a semantic value in and of themselves.

tion operator, $\mathcal{A}$ the *actuality* operator, $\Box$ the *necessity* operator, $\rightarrow$ the *conditional*; $\forall\alpha$ a *universal quantifier*, and $\imath\nu$ a *definite description* operator. There is no special name for $\lambda\nu_1 \dots \nu_n$ operators.

(3) **Definitions:** Syntax of the Language. We present the syntax of our language by a simultaneous recursive definition of the following four kinds of expressions: *individual term*, *n-place relation term*, *formula*, and *propositional formula*. Before we give the definition, note that we shall be defining two kinds of (complex) formulas and we can intuitively describe the two kinds as ones that have encoding subformulas and ones that don't.[81] The latter are designated as *propositional formulas*. In what follows, we use the parenthetical expression '(propositional)' strategically so that we don't have to repeat clauses in the definition of *formula* to produce clauses in the definition of *propositional formula*. This technique is explained in more detail in Remark (4), which follows the definition.

(.1) Every simple individual term is an individual term and every simple n-place relation term is an n-place relation term ($n \geq 0$).

(.2) If Π^n is any n-place relation term and $\kappa_1, \dots, \kappa_n$ are any individual terms, then where $n \geq 1$,

(.a) $\Pi^n \kappa_1 \dots \kappa_n$ is a (propositional) formula ('$\kappa_1, \dots, \kappa_n$ exemplify Π^n')

(.b) $\kappa_1 \Pi^1$ is a formula ('κ_1 encodes Π^1')

and where Π^0 is any 0-place relation term,

(.c) Π^0 is a (propositional) formula ('Π^0 is true')

(.3) If φ, ψ are (propositional) formulas and α any variable, then
$(\neg\varphi)$, $(\varphi \rightarrow \psi)$, $\forall\alpha\varphi$, $(\Box\varphi)$, and $(\mathcal{A}\varphi)$ are (propositional) formulas.

(.4) If φ is any formula and ν any individual variable, then
$\imath\nu\varphi$ is an individual term

(.5) If φ is any propositional formula and $\nu_1, \dots, \nu_n$ are any distinct individual variables ($n \geq 0$), then

(.a) $[\lambda\nu_1 \dots \nu_n\, \varphi]$ is an n-place relation term, and

(.b) φ itself is a 0-place relation term[82]

[81] This description is intuitive because the notion of *subformula* hasn't yet been defined; it will, however, be defined below.

[82] One might wonder why we don't collapse clauses (3.2.a) and (3.2.c) by letting n in the former go to zero. There are several reasons for this. One is that we *read* the formulas of (3.2.a) differently from the way we read the formulas of (3.2.c). See the parenthetical annotation at the right margin

Note that according to clause (.5), when φ is propositional, $[\lambda\, \varphi]$ and φ are distinct 0-place relation terms. In general, the λ-expressions are *not* to be interpreted as denoting functions, but rather as denoting relations.

(**4**) **Remark:** On the Definition of Formula and Propositional Formula. We henceforth mark the distinction between formulas and propositional formulas by using $\varphi, \psi, \chi, \theta$ as metavariables ranging over formulas, and $\varphi^*, \psi^*, \chi^*, \theta^*$ as metavariables ranging over propositional formulas. To be absolutely clear about the distinction, note that some of the clauses in definition (3) include the parenthetical phrase '(propositional)'. This allows us to state the definitions of *formula* and *propositional formula* more efficiently. The clauses containing this parenthetical phrase, namely, (3.2.a), (3.2.c), and (3.3), should all be understood as conjunctions. In each of these clauses, remove the parenthetical phrase entirely to obtain a clause in the definition of *formula*; then remove only the parentheses from all the occurrences of '(propositional)' to obtain a clause in the definition of *propositional formula*. For example, clause (3.3) is short for:

> If φ, ψ are formulas and α any variable, then $(\neg\varphi)$, $(\varphi \to \psi)$, $\forall\alpha\varphi$, $(\Box\varphi)$, and $(\mathcal{A}\varphi)$ are formulas, and if φ^*, ψ^* are propositional formulas and α any variable, then $(\neg\varphi^*)$, $(\varphi^* \to \psi^*)$, $\forall\alpha\varphi^*$, $(\Box\varphi^*)$, and $(\mathcal{A}\varphi^*)$ are propositional formulas.

Thus, it should be clear from the definition that the propositional formulas are a subset of the formulas.

Note also that in clause (3.5), we require φ to be propositional, thereby restricting the subclauses introducing complex relation terms: only propositional formulas can appear in such terms. This restriction is achieved by using 'propositional' *without* parentheses in (3.5).

(**5**) **Definition:** Terms. We say that a *term* is any individual term or any n-place relation term ($n \geq 0$). We use τ to range over terms. The simple terms listed in item (1) are terms in virtue of clause (3.1). We say that a term τ is *complex* iff τ is not a simple term. Thus, clauses (3.4), (3.5.a), and (3.5.b) define types of complex terms, namely, *definite descriptions*, *λ-expressions*, and *propositional formulas*, respectively. We sometimes use ρ to range over complex terms. A consequence of our definitions is that every propositional formula is a term.

(**6**) **Definition:** A BNF Definition of the Syntax. We may succintly summarize the essential definitions of the context-free grammar of our language us-

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of both clauses. A second reason is that (3.2.a) intuitively defines *atomic* exemplification formulas. But (3.2.c) allows *any* 0-place relation term to be a formula, even the complex 0-place relation terms defined in clause (3.5.b). An example of the latter is $Pa\,\&\, \neg Qb$. So if we had collapsed (3.2.a) and (3.2.c), then one might be tempted to regard a complex 0-place relation term such as $Pa\,\&\, \neg Qb$ as a kind of atomic formula.

ing Backus-Naur Form (BNF). In the BNF definition, we repurpose our Greek metavariables as the names of grammatical categories, as follows:

δ	individual constants
ν	individual variables
Σ^n	n-place relation constants ($n \geq 0$)
Ω^n	n-place relation variables ($n \geq 0$)
α	variables
κ	individual terms
Π^n	n-place relation terms ($n \geq 0$)
φ^*	propositional formulas
φ	formulas
τ	terms

The BNF grammar for our language can now be stated as follows, where the Greek variables are now names of grammatical categories:[83]

$$
\begin{array}{lrcl}
 & \delta & ::= & a_1, a_2, \ldots \\
 & \nu & ::= & x_1, x_2, \ldots \\
(n \geq 0) & \Sigma^n & ::= & P^n_1, P^n_2, \ldots \\
(n \geq 0) & \Omega^n & ::= & F^n_1, F^n_2, \ldots \\
 & \alpha & ::= & \nu \mid \Omega^n \; (n \geq 0) \\
 & \kappa & ::= & \delta \mid \nu \mid \iota\nu\varphi \\
(n \geq 1) & \Pi^n & ::= & \Sigma^n \mid \Omega^n \mid [\lambda\nu_1 \ldots \nu_n \;\; \varphi^*] \\
 & \Pi^0 & ::= & \Sigma^0 \mid \Omega^0 \mid [\lambda \;\; \varphi^*] \mid \varphi^* \\
 & \varphi^* & ::= & \Pi^n\kappa_1 \ldots \kappa_n \; (n \geq 1) \mid \Pi^0 \mid (\neg\varphi^*) \mid (\varphi^* \rightarrow \varphi^*) \mid \forall\alpha\varphi^* \mid \\
 & & & (\Box\varphi^*) \mid (\mathcal{A}\varphi^*) \\
 & \varphi & ::= & \kappa_1\Pi^1 \mid \varphi^* \mid (\neg\varphi) \mid (\varphi \rightarrow \varphi) \mid \forall\alpha\varphi \mid (\Box\varphi) \mid (\mathcal{A}\varphi) \\
 & \tau & ::= & \kappa \mid \Pi^n \; (n \geq 0)
\end{array}
$$

Thus, if one defines a fragment of our language by giving a limiting value to n and listing a finite vocabulary of simple terms, the sentences of the resulting grammar can be parsed by any appropriately-configured off-the-shelf parsing engine.

(7) **Definitions:** Notational Conventions. We adopt the following conventions to facilitate readability:

(.1) We often substitute the less formal expressions listed in (1) for their more formal counterparts, and we often drop the superscript indicating the arity of a simple relation term when such terms appear in a formula,

[83] I'm indebted to Uri Nodelman for suggesting a simplification of the BNF grammar I had originally developed. That simplification is included in the definition.

since their arity can always be inferred from the number of individual terms in the formula. Thus, instead of writing $P^1_1 a_1$ and $P^1_2 a_2$, we write Pa and Qb, respectively; instead of $F^1_1 x_1$ and $F^1_2 x_2$, we write Fx and Gy, respectively; instead of $R^2_1 a_2 x_3$, we write Rbz; etc.

(.2) We substitute $p, q, r, \ldots$ for the 0-place relation variables, $F^0_1, F^0_2, \ldots$. If we need 0-place relation constants, we use $p_1, q_2, \ldots$ instead of $P^0_1, P^0_2, \ldots$.

(.3) We omit parentheses in formulas whenever we possibly can, i.e., whenever we can do so without ambiguity, such as in the following circumstances:

- We sometimes add parentheses and square brackets to assist in reading certain formulas and terms. Thus, for example, we write $\imath x(xQ)$ instead of $\imath xxQ$, and $F\imath x(xQ)$ instead of $F\imath xxQ$.
- We almost always drop outer parentheses and assume $\neg$, $\forall$, $\imath$, $\Box$, and $\mathcal{A}$ apply to as little as possible; $\rightarrow$ dominates $\neg$:
  - $\neg Pa \rightarrow Qb$ is short for $((\neg Pa) \rightarrow Qb)$, not $\neg(Pa \rightarrow Qb)$.
  - $\forall x Px \rightarrow Qx$ is to be read as $(\forall x Px) \rightarrow Qx$, not $\forall x(Px \rightarrow Qx)$.
  - $\neg\Box Pa$ is short for $(\neg(\Box Pa))$.
    $\Box Pa \rightarrow Qb$ is short for $(\Box Pa) \rightarrow Qb$, not $\Box(Pa \rightarrow Qb)$
  - $\neg\mathcal{A}Pa$ is short for $(\neg(\mathcal{A}Pa))$.
    $\mathcal{A}Pa \rightarrow Qb$ is short for $(\mathcal{A}Pa) \rightarrow Qb$, not $\mathcal{A}(Pa \rightarrow Qb)$.

(.4) We employ the usual definitions of (a) φ *and* ψ, (b) φ *or* ψ, (c) φ *if and only if* ψ, (d) *there exists an* α *such that* φ, and (e) *possibly* φ:

(a) $\varphi \,\&\, \psi =_{df} \neg(\varphi \rightarrow \neg\psi)$

(b) $\varphi \vee \psi =_{df} \neg\varphi \rightarrow \psi$

(c) $\varphi \equiv \psi =_{df} (\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)$

(d) $\exists\alpha\varphi =_{df} \neg\forall\alpha\neg\varphi$

(e) $\Diamond\varphi =_{df} \neg\Box\neg\varphi$

(.5) $\rightarrow$ and $\equiv$ dominate & and $\vee$:

$\neg\varphi \,\&\, \psi \rightarrow \chi$ is short for $(\neg\varphi \,\&\, \psi) \rightarrow \chi$, not $\neg\varphi \,\&\, (\psi \rightarrow \chi)$

$\varphi \vee \psi \equiv \psi \vee \varphi$ is short for $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$

Given the definitions of & and $\vee$ in (.3) and (.4), it should be clear that we are preserving a classical understanding of these connectives.

(8) **Definition:** Subformula. Where φ is any formula, we define a *subformula* of φ recursively as follows:

(.1) φ is a subformula of φ.

(.2) If $\neg\psi$, $\forall\alpha\psi$, $\Box\psi$, or $\mathcal{A}\psi$ is a subformula of φ, then ψ is a subformula of φ.

(.3) If $\psi \rightarrow \chi$ is a subformula of φ, then ψ is a subformula of φ and χ is a subformula of φ

(.4) If $[\lambda\, \psi^*]$ is a subformula of φ, then ψ^* is a subformula of φ.

(.5) Nothing else is a subformula of φ.

Given this definition, we may say that ψ is a *proper* subformula of φ just in case ψ is a subformula of φ and ψ is not φ.

One interesting feature of our definition of *subformula* is that it doesn't count the formula Gx as a subformula of the formulas $F\imath xGx$ or $[\lambda x\, \neg Gx]a$. Indeed, the definition generally doesn't count the formula φ in the term $\imath\nu\varphi$, or the formula φ^* in the term $[\lambda\nu_1 \ldots \nu_n\, \varphi^*]$ $(n \geq 1)$, as subformulas of any formula in which these terms occur. Consequently, if φ contains encoding subformulas, then the description $\imath\nu\varphi$ *may appear* in propositional formulas! For example, though the formula xQ contains an encoding subformula (namely, itself), the formula $F\imath x(xQ)$ qualifies as a propositional formula. This is an exemplification formula with the complex term $\imath x(xQ)$ appearing the argument to the relation term F. (8) doesn't define the notion *subformula of* for the term $\imath x(xQ)$ and so the formula xQ doesn't qualify as a subformula of $F\imath x(xQ)$. Consequently, the expressions $[\lambda F\imath x(xQ)]$ and $[\lambda y Ry\imath x(xQ)]$ are perfectly well-defined 0-place and 1-place λ-expressions, respectively. The reader might wish to consider how the semantics we developed in Chapter 5 provides truth conditions for (a) propositional formulas that contain descriptions with encoding formulas, such as $\imath x(xQ)$, and (b) propositional formulas such as $[\lambda y\, Ry\imath x(xQ)]z$, in which the λ-operator λy governs a propositional formula containing a term that includes encoding formulas.

Our definition of *subformula* of has some other important consequences, notably:

Metatheorem $\langle 7.1 \rangle$
If ψ is a subformula of χ and χ is a subformula of φ, then ψ is a subformula of φ.

Metatheorem $\langle 7.2 \rangle$
φ is a propositional formula if and only if no encoding formulas are subformulas of φ.

Metatheorem $\langle 7.3 \rangle$
φ and ψ are subformulas of $\varphi \,\&\, \psi$, $\varphi \vee \psi$, and $\varphi \equiv \psi$.

These metatheorems are proved or left as exercises in the Appendix to this chapter (see Part IV). Given Metatheorem ⟨7.2⟩ and clause (3.5), we can see that only formulas φ without encoding subformulas may appear in terms that denote relations.

(**9**) **Definitions:** Operator Scope and Free (Bound) Occurrences of Variables. We first define the *scope* of an occurrence of the formula- and term-building operators as follows:

(.1) The formulas $\neg\psi$, $\Box\psi$, and $\mathcal{A}\psi$ are the scope of the occurrence of the operators $\neg$, $\Box$ and $\mathcal{A}$ in their respective formulas. The formula $\psi \rightarrow \chi$ is the scope of the occurrence of the operator $\rightarrow$.

(.2) The formula $\forall\beta\psi$ is the scope of the left-most occurrence of the operator $\forall\beta$ in that formula. (We sometimes call ψ the *matrix* of the universal claim and the proper scope of $\forall\beta$ in $\forall\beta\psi$.)

(.3) The term $\imath\nu\psi$ is the scope of the left-most occurrence of the operator $\imath\nu$ in that term. (We sometimes call ψ the matrix of the description and the proper scope of $\imath\nu$ in $\imath\nu\psi$.)

(.4) The term $[\lambda\nu_1 \dots \nu_n\, \psi^*]$ ($n \geq 0$) is the scope of the left-most occurrence of the operator $\lambda\nu_1 \dots \nu_n$ in that term. (We sometimes call ψ^* the matrix of the λ-expression and proper scope of $\lambda\nu_1 \dots \nu_n$ in $[\lambda\nu_1 \dots \nu_n\, \psi^*]$.)

Now to define *free* and *bound* occurrences of a variable within a formula or term, we first say that $\forall\beta$ is a variable-binding operator *for* β, that $\imath\nu$ is a variable-binding operator *for* ν, and that $\lambda\nu_1 \dots \nu_n$ is a variable-binding operator *for* $\nu_1, \dots, \nu_n$. We then say, for any variable α and any formula φ (or any complex term τ):

(.5) An occurrence of α in φ (or τ) within the scope of an occurrence of a variable-binding operator for α is *bound*; otherwise, the occurrence is *free*.

Finally, we say:

(.6) Those occurrences of β that are free in ψ are *bound by* the left-most occurrence of $\forall\beta$ in $\forall\beta\psi$, as is the occurrence of β in that occurrence of $\forall\beta$; those occurrences of ν that are free in ψ are *bound by* the left-most occurrence of $\imath\nu$ in $\imath\nu\psi$, as is the occurrence of ν in that occurrence of $\imath\nu$; and those occurrences of ν_i ($1 \leq i \leq n$) that are free in ψ^* are *bound by* the left-most occurrence of $\lambda\nu_1 \dots \nu_n$ in $[\lambda\nu_1 \dots \nu_n\, \psi^*]$, as are the occurrences of $\nu_1, \dots, \nu_n$ in that occurrence of $\lambda\nu_1 \dots \nu_n$.

We henceforth say that the variable α *occurs free* or *is free* in formula φ or term τ if and only if at least one occurrence of α in φ or τ is free. Note that we don't need to define when an occurrence of a variable is free in the 0-place term φ^*, i.e., propositional formula φ^*. Since these expressions constitute a subset of the formulas, this case is covered by the general definition of free occurrence of a variable in a formula.

(**10**) **Remark:** Open and Closed Formulas and Terms. In the foregoing, we have rigorously distinguished between terms and formulas, though we've defined some expressions to be both terms and formulas. Intuitively, a *term* is an expression that may have a denotation, relative to some choice of values for the free variables it may contain. By contrast, a *formula* is, intuitively, an expression that is assertible and has truth conditions, relative to some choice of values for any free variables it may contain. A formula φ is *closed* if no variable occurs free in φ; otherwise φ is *open*. A formula φ is a *sentence* iff φ is a closed formula. We say further that a term τ is *closed* if no variable occurs free in τ; otherwise, τ is *open*. We may sometimes refer to open complex terms, such as $\imath xRxy$ and $[\lambda x\, Rxy]$, as *complex variables*. The former example is a complex individual variable, since for each choice for y, it may denote a different individual; the latter is a complex 1-place relation variable since, for each choice of y, it may denote a a different 1-place relation.

7.2 Definitions for Objects and Relations

(**11**) **Term Definitions:** Ordinary vs Abstract (i.e., Logical) Objects. We define *being ordinary* ('$O!$') as a new 1-place relation term:

(.1) $O! =_{df} [\lambda x \, \Diamond E!x]$

Strictly speaking, then, being ordinary is: being an x such that possibly, x exemplifies concreteness. In other words, being ordinary is being possibly concrete.

On the other hand, *being abstract* or *being logical*, written $A!$, is defined as:

(.2) $A! =_{df} [\lambda x \, \neg\Diamond E!x]$

So being abstract (or being logical) is being an x such that it is not possible that x exemplify concreteness, or more simply, being an x that couldn't possibly exemplify concreteness, or even more simply, not possibly being concrete.

(**12**) **Term Definition:** The Identity$_E$ Relation. We define the two-place relation *being identical*$_E$ ('$=_E$') as: being an individual x and an individual y such that x exemplifies being ordinary, y exemplifies being ordinary, and necessarily, x and y exemplify the same properties:

$$=_E \;=_{df}\; [\lambda xy\; O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]$$

Note that the definiens is a *bona fide* λ-expression (modulo the less formal variables) because there are no encoding subformulas in its matrix. Since the definiens is a well-defined 2-place relation term, so is the definiendum and thus, exemplification formulas like $=_E xy$ are well-formed.

(**13**) **Definition:** Identity$_E$ Infix Notation. We henceforth use the following notation for formulas containing the relation term $=_E$:

$$x =_E y \;=_{df}\; =_E xy$$

(**14**) **Remark:** Nested λ-expressions. It is of interest that the notion of a *haecceity* of individual a, namely, $[\lambda x\, x =_E a]$, which we discussed in earlier chapters, is now defined in a rather complex way, involving nested λ-expressions. For by the infix notation we've just defined,

$$[\lambda x\, x =_E a]$$

unpacks, by definition (13), to:

$$[\lambda x\; =_E xa]$$

which in turn expands, by definition (12) of $=_E$, to:

$$[\lambda x\, [\lambda xy\; O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]xa]$$

This, in turn, expands by definition (11.1) of $O!$, to:

$$[\lambda x\, [\lambda xy\, [\lambda x\, \Diamond E!x]x \,\&\, [\lambda x\, \Diamond E!x]y \,\&\, \Box\forall F(Fx \equiv Fy)]xa]$$

This expands, by definition (7.4.e) of $\Diamond$, to:

$$[\lambda x\, [\lambda xy\, [\lambda x\, \neg\Box\neg E!x]x \,\&\, [\lambda x\, \neg\Box\neg E!x]y \,\&\, \Box\forall F(Fx \equiv Fy)]xa]$$

Of course, this λ-expression is still not in primitive notation, since the definitions of & and $\equiv$ allow us to expand the expression even further. But enough has been said to alert the reader to the many layers of definition that are accumulating.

(**15**) **Definition:** Identity For Individuals. The symbol '=' for identity is not among the primitive expressions of our language. It can, however, be defined by cases, namely, for the two kinds of entities: individuals and n-place relations ($n \geq 0$). In turn, the definition of '=' for n-place relations will itself be given by cases (i.e., for properties, relations, and propositions) in (16). Note that though the following definitions allow us to show that identity is reflexive, the full power of identity as an equivalence condition is guaranteed by axiom (25).

But we first define identity for individuals. We say that $x = y$ iff either $x =_E y$ or x and y are both abstract and necessarily encode the same properties:

$$x = y \ =_{df} \ x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

Note that the definiens has encoding subformulas. Consequently, the expression $[\lambda xy\, x = y]$ is not a well-formed λ-expression; the matrix $x = y$, once expanded into primitive notation, contains encoding subformulas. (The defined formula '$x = y$' is therefore unlike, and is to be rigorously distinguished from, the defined term '$=_E$'.) Similarly, the expression $[\lambda x\, x = y]$ is not well-formed. This forestalls the McMichael/Boolos paradox discussed in Section 2.1. Moreover, the argument for the paradox fails to get any purchase on properties of the form $[\lambda x\, x =_E y]$.

(**16**) **Definitions:** Relation Identity. We define identity for properties, relations, and propositions as follows: (.1) properties F^1 and G^1 are identical iff necessarily, F^1 and G^1 are encoded by the same objects; (.2) relations F^n and G^n are identical iff for each way of plugging n–1 objects into the corresponding argument places of F^n and G^n, the resulting 1-place properties are identical; (.3) propositions F^0 and G^0 are identical iff *being such that* F^0 is identical to *being such that* G^0. Formally:

(.1) $F^1 = G^1 \ =_{df} \ \Box\forall x(xF^1 \equiv xG^1)$

(.2) $F^n = G^n \ =_{df}$ (where $n \geq 2$)

$$\begin{aligned}\forall x_1 \ldots \forall x_{n-1}(&[\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, G^n y x_1 \ldots x_{n-1}] \,\&\\ &[\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, G^n x_1 y x_2 \ldots x_{n-1}] \,\&\ldots\&\\ &[\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, G^n x_1 \ldots x_{n-1} y])\end{aligned}$$

(.3) $F^0 = G^0 \ =_{df} \ [\lambda y\, F^0] = [\lambda y\, G^0]$, i.e., using our conventions:
$p = q \ =_{df} \ [\lambda y\, p] = [\lambda y\, q]$

The last two definitions reduce the identity of relations and propositions, respectively, to the identity of properties. The simplest case of (.2) is where $n = 2$, which asserts:

$$F^2 = G^2 \ =_{df} \ \forall x_1([\lambda y\, F^2 y x_1] = [\lambda y\, G^2 y x_1] \,\&\, [\lambda y\, F^2 x_1 y] = [\lambda y\, G^2 x_1 y])$$

This asserts that F^2 and G^2 are identical if and only if for any object x_1, both (a) the property *bearing* F^2 *to* x_1 is identical to the property *bearing* G^2 *to* x_1 and (b) the property *being an object to which* x_1 *bears* F^2 is identical to the property *being an object to which* x_1 *bears* G^2. By (.1), the properties asserted to be identical in (a) and (b) have to be necessarily encoded by the same objects. Hence, F^2 and G^2 are identical if and only if, for every x_1, (a) there couldn't possibly be an object that encodes $[\lambda y\, F^2 y x_1]$ and fails to encode $[\lambda y\, G^2 y x_1]$ (or vice versa), and (b) there couldn't possibly be an object that encodes $[\lambda y\, F^2 x_1 y]$ and fails to encode $[\lambda y\, G^2 x_1 y]$ (or vice versa).

(**17**) **Remark:** Variables in Definitions. Note that we formulated definitions (11) – (16) using the object-language variables listed in the 'Less Formal' column of item (1.1) and (1.2). Specifically, we always try to use $x, y, z, \ldots$ instead of $x_1, x_2, x_3, \ldots$, and use $F, G, H, \ldots$ instead of $F^1_1, F^1_2, F^1_3, \ldots$, etc. The reason for this convention should be clear: the definiens of (11.1), $[\lambda x\, \Diamond E!x]$, is easier to read than $[\lambda x_1\, \Diamond E!x_1]$, and the definiens of (12),

$$[\lambda xy\; O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]$$

is easier to read than:

$$[\lambda x_1 x_2\; O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F^1_1(F^1_1 x_1 \equiv F^1_1 x_2)]$$

Nevertheless, we should remember that the definienda, '$O!$' and '$=_E$', are particular symbols being introduced and their definientia are particular λ-expressions in the object language. The 'less formal' variables $x, y, z \ldots$ are merely notational shorthand for the variables that are officially part of the language.

This observation applies to both definiendum and definiens in definitions such as (16.1). The definiendum $F^1 = G^1$ and its definiens $\Box\forall x(xF^1 \equiv xG^1)$ are easier to read, respectively, than $F^1_1 = F^1_2$ and $\Box\forall x_1(x_1 F^1_1 \equiv x_1 F^1_2)$. The improvement in readability is obvious and should justify the deployment of 'less formal' variables. Indeed, that is why we shall henceforth suppose that the definiendum can be written simply as $F = G$ and that the definiens can be written as $\Box\forall x(xF \equiv xG)$. But, officially, the definiendum and definiens use variables that are part of our specified language.

(**18**) **Definitions:** Infix Notation for Negated Identities. We henceforth use the following infix notation for the negation of formulas involving the identity symbol:

$\alpha \neq \beta \;=_{df}\; \neg(\alpha = \beta)$,
where α, β are, respectively, either both individual variables or both n-place relation variables (for some $n \geq 0$).

(**19**) **Remark:** A Note about Definitions. In this work, we regard definitions not as metalinguistic abbreviations of the object language but rather as conventions for extending the object language with new terms, formulas and axioms. Our main reason for understanding definitions in this way is so that theorems involving defined notions become genuine philosophical statements of the object language rather than metaphilosophical statements of the metalanguage. Though we shall postpone a full discussion of the theory of definitions to Section 9.12 of Chapter 9, it is important to make a few observations about the way definitions are constructed and understood.

We shall be using two kinds of definitions to extend our system: *term definitions*, to introduce new terms, and *formula definitions*, to introduce formulas containing new syncategorematic expressions. An discussion of term definitions is in order before we discuss formula definitions.

The definitions in items (11) and (12) are term definitions: $O!$, $A!$, and $=_E$ are new terms introduced into the language. In these particular examples, the new terms are *relation constants* since the definientia have no free variables. However, free variables may occur in definitions: we may use complex variables as definientia to introduce *functional terms*. For example, we might define the negation of property F as follows:

$$\overline{F} =_{df} [\lambda x \, \neg Fx]$$

In this definition, the variable F occurs free in both definiendum and definiens, and the definiens $[\lambda x \, \neg Fx]$ is a complex variable that serves to define the new functional term $\overline{F}$. For each property that F takes as value, $\overline{F}$ denotes the negation of that property.[84]

Indeed, one can uniformly substitute appropriate property terms for the free variable F to produce instances of the definition. Consider the the open property term $[\lambda y \, Gy \,\&\, Hy]$, in which G and H are free variables. We can substitute $[\lambda y \, Gy \,\&\, Hy]$ uniformly for F to produce the following instance of the definition:

$$\overline{[\lambda y \, Gy \,\&\, Hy]} =_{df} [\lambda x \, \neg[\lambda y \, Gy \,\&\, Hy]x]$$

The property term $[\lambda y \, Gy \,\&\, Hy]$ is an appropriate substitution for F because it contains no free occurrences of the variable x; if a property term ρ contains a free occurrence of x, the occurrence of x would become bound by the operator λx if ρ were substituted for F in $[\lambda x \neg Fx]$, thereby changing the meaning of the definition. The notion of appropriateness will become clear when, in item (24), we define the conditions under which a term τ is *substitutable for* the variable α in a term or formula.[85]

We shall also have occasion to formulate definitions of new individual terms. In the simplest case, we define a new individual term by employing a closed

[84]We have taken care to call $\overline{F}$ a *functional term* instead of a *function term*. The reason is that, strictly speaking, $\overline{F}$ doesn't denote a function. Ontologically speaking, a function is a relation R such that $Rxy \,\&\, Rxz \rightarrow y = z$. The present language doesn't have the enough expressive power to permit talk about relations among properties, such as a relation that holds between F and G whenever $\forall x(Fx \equiv \neg Gx)$, i.e., whenever F is a negation of G. In the type-theoretic version of our system, one may talk about such relations, but for now, we simply call terms like $\overline{F}$ functional terms since if we step outside the system and into the metalanguage, the term $\overline{F}$ is assigned a denotation based on the denotation assigned to the variable F.

[85]That definition makes it clear that even the term $[\lambda x \, Gx \,\&\, Hx]$ can be substituted for F in the definition to produce the instance:

$$\overline{[\lambda x \, Gx \,\&\, Hx]} =_{df} [\lambda x \, \neg[\lambda x \, Gx \,\&\, Hx]x]$$

The occurrences of x in $[\lambda x Gx \& Hx]$ are all bound and so the none get captured when we substitute this expression for F in $[\lambda x \, \neg Fx]$.

definite description (i.e., no free variables) that provably has a denotation. For example, since it will be provable that the abstract object that encodes all and only self-identical properties exists, we might introduce the individual constant $\boldsymbol{a_V}$ as follows:

$$\boldsymbol{a_V} =_{df} \iota x(A!x \,\&\, \forall F(xF \equiv F=F))$$

Free variables are also allowed in definitions of individual terms: open definite descriptions (i.e., those having free variables) are complex variables that can be used to define a functional term. For example, since it will be provable that for every property G, the abstract object that encodes all and only the properties identical to G exists, we might introduce the functional term $\boldsymbol{a}_G$ as follows:

$$\boldsymbol{a}_G =_{df} \iota x(A!x \,\&\, \forall F(xF \equiv F = G))$$

In this definition, the variable G occurs free in both definiendum and definiens, and the definiens is a complex variable that serves to define the new functional term $\boldsymbol{a}_G$. For each property that G takes as value, the functional term $\boldsymbol{a}_G$ denotes the abstract object that encodes just the properties identical to G.

Again, one can uniformly substitute appropriate property terms for the free variable G to produce instances of the definition. For example, we can substitute the term $[\lambda y\, \neg Hy]$ uniformly for G to produce the following instance of the definition:

$$\boldsymbol{a}_{[\lambda y\, \neg Hy]} =_{df} \iota x(A!x \,\&\, \forall F(xF \equiv F = [\lambda y\, \neg Hy]))$$

Indeed, we can produce an instance of the definition by substituting a defined term for G:

$$\boldsymbol{a}_{\overline{H}} =_{df} \iota x(A!x \,\&\, \forall F(xF \equiv F = \overline{H}))$$

Although it is not appropriate to substitute the variable F for G in the definition of $\boldsymbol{a}_G$ (since F would become bound by the quantifier $\forall F$ in the definiens), our theory considers the definition to be unchanged if both the bound and free variables in the definiendum and definiens are appropriately changed, for example, as follows:

$$\boldsymbol{a}_F =_{df} \iota x(A!x \,\&\, \forall G(xG \equiv G = F))$$

The justification for understanding the definition in this way is given in item (209).

Although we intend a classical understanding of the inferential role of term definitions, the subtleties involved in our language make it incumbent upon us to provide a thorough discussion of this role, and this we do in item (207). For now, the main points to remember are that in any properly formed term definition of the form $\tau =_{df} \tau'$:

- the definiendum and definiens both have the same free variables, if any;
- the definition is independent of the choice of any free and bound variables; any other appropriate variables could have been used instead;
- it must be provable that $\exists\beta(\beta = \tau')$, i.e., a term τ' can be used to define a new term τ only if it provably denotes something;
- whenever appropriate terms of the right type are uniformly substituted for any free variables in the definition, the result is an instance of the definition;
- every well-formed instance of the definition becomes assertible as a simple identity of the form $\tau = \tau'$ (207.2); and
- for any instance of the definition, the defieniendum τ and definiens τ' may be substituted for one another in any context (207.3).

Although it won't become completely clear why these facts govern the inferential role of definitions until (207) and (209), they assure us that expressions introduced into our language via term definitions are logically well-behaved.

One last point about term definitions. Many new individual terms will be introduced by way of definite descriptions whose matrices have encoding subformulas; for example, $\boldsymbol{a}_{\mathbf{V}}$ and $\boldsymbol{a}_{G}$ previewed above are such terms. It should be noted that the encoding subformulas of the matrix of the definite description that serves as the definiens are neither subformulas of the description nor subformulas of the new term. Our notion of *subformula* is defined solely with respect to formulas; a term does not have subformulas. For example, consider again the definition of $\boldsymbol{a}_{\mathbf{V}}$ previewed above:

$$\boldsymbol{a}_{\boldsymbol{V}} =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv F = F))$$

The encoding formula xF is a subformula of the matrix of the description. Indeed, there are other encoding subformulas of the defined formula $F = F$. None of these encoding formulas are *subformulas* of the description or of the term $\boldsymbol{a}_{\mathbf{V}}$. Thus, the term $\boldsymbol{a}_{\mathbf{V}}$ may appear in a propositional formula without undermining the status of that formula as a propositional formula. We'll see that the analogous situation does not apply to formula definitions, to which we turn next.

There are two types of *formula definitions*: those that use metavariables and those that use object language variables. Examples of the first kind are items (7.4.a), (7.4.d), and (7.4.e):

$$\varphi \,\&\, \psi =_{df} \neg(\varphi \rightarrow \neg\psi) \tag{7.4.a}$$

$$\exists\alpha\varphi =_{df} \neg\forall\alpha\neg\varphi \tag{7.4.d}$$

$$\Diamond\varphi =_{df} \neg\Box\neg\varphi \tag{7.4.e}$$

In these cases, new syncategorematic expressions &, $\exists$, $\Diamond$ are being introduced into the language by way of formulas. These symbols are akin to $\neg$, $\rightarrow$, $\forall$, and $\Box$. Syncategorematic expressions, unlike terms, don't denote anything, though the formulas in which they occur have truth conditions. Any uniform substitution of object-language formulas for the Greek metavariables results in an instance of the above definitions.

The second kind of formula definition is similar to the first except that it uses object language variables instead of metavariables. Examples are items (15) and (16.1):

$$x = y =_{df} x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)) \tag{15}$$

$$F = G =_{df} \Box\forall x(xF \equiv xG) \tag{16.1}$$

Note that this second kind of formula definition also introduces new syncategorematic expressions. In (15) and (16.1), the identity symbol '=' introduced is not a new 2-place relation term. Rather, '=' is a syncategorematic expression that functions as a formula-forming binary operator on variables of the same type, though the definition of the formula that results depends on the type of variable on which '=' operates.

We'll see additional examples, such as the following, in later chapters:

$$\mathit{ExtensionOf}(x, G) =_{df} A!x \,\&\, \forall F(xF \equiv \forall z(Fz \equiv Gz)) \tag{234}$$

$$\mathit{ClassOf}(x, G) =_{df} \mathit{ExtensionOf}(x, G) \tag{246}$$

Thus both *ExtensionOf* and *ClassOf* become binary formula-forming operators that takes an individual variable as its first argument and a property variable as its second. We can produce instances of these definitions when we substitute appropriate individual terms for x and appropriate property terms for G.

A full discussion of the conventions for, and inferential role of, formula definitions is postponed until items (208) and (209), when we have proved enough theorems to justify our practice. The main points to remember now are that in a properly formed formula definition of the form $\varphi =_{df} \psi$:

- the definiendum and definiens both have the same free metavariables or the same free object-language variables (depending on the kind of formula definition);

- the definition is independent of the choice of metavariables and independent of the choice of bound and free object-language variables; any other appropriate (meta)variables could have been used instead;

- whenever formulas are uniformly substituted for the free metavariables in the definition, or appropriate terms are uniformly substituted for the free object language variables in the definition, the result is an instance of the definition;
- the definition subsequently becomes assertible as a necessary equivalence of the form $\Box\varphi \equiv \psi$;[86] and
- for any instance of the definition, the definiendum φ and definiens ψ can be substituted for one another within any formula or term.

These facts ensure that new expressions introduced into our language via formula definitions are logically well-behaved.

It is important to note that many new syncategorematic expressions will be introduced by way of formulas that aren't propositional. In the examples discussed above, the definientia of $x = y$, $F = G$, $ExtensionOf(x, G)$, and $ClassOf(x, G)$ all contain encoding subformulas. By convention, we regard the definiendum in each case as a non-propositional formula. Since these definitions extend the object language with formulas containing new (syncategorematic) expressions, it is important to remember that the formulas being introduced are defined in terms of encoding subformulas. And if the new formula occurs within a formula φ as a subformula, then again, by convention, we regard φ as non-propositional. Thus, the formulas $x = y \vee \neg x = y$, $F = G \rightarrow \forall(Fx \equiv Gx)$, etc., should all be regarded as non-propositional.

Although formula definitions introduce new syncategorematic expressions, the expressions introduced nevertheless typically provide an analysis of the meaning of a logically or philosophically significant word or phrase of English in terms of the (primitive) notions represented by the primitive expressions of our language. In the above cases: (7.4.a) analyzes 'and', (7.4.d) analyzes 'there exists', and (7.4.e) analyzes 'possibly'. Similarly, definition (15) of '=' offers an analysis of the English expression 'is identical to' as used in claims such as "Hesperus is identical to Phosphorus". Definition (16.1), which provides a separate case of the definition of '=', offers an analysis of the English phrase 'is the same as' as used in such claims as "*being a brother* is the same as *being a male sibling*" and "*being a circle* is the same as *being a closed plane figure every point of which lies equidistant from some given point*". And so on for the other examples of formula definitions.

Of course, the hope is that such definitions provide insightful analyses that demonstrate the power of the primitive notions of the language. For example,

[86] Indeed, something stronger is warranted. When we define *theoremhood* in Chapter 9 and distinguish a special class of *modally strict* theorems that are derivable without appealing to an axiom or premise that can't be necessitated, then we can say: a formula definition of the form $\varphi =_{df} \psi$ becomes assertible as a modally strict theorem of the form $\varphi \equiv \psi$. See the Simple Rule of Equivalence by Definition in item (208.2).

the definition of $F = G$ is part of our *theory* of identity: it (a) analyzes the identity of properties in terms of the primitives of our language, (b) provides a precise formulation of a notion (i.e., property identity) thought to be mysterious (e.g., by Quine), (c) allows that properties may be distinct even if necessarily exemplified by the same objects, and (d) shows that the identity conditions of properties are extensional (semantically, the definition requires that properties F and G have the same encoding extension at every semantically-primitive possible world). We need not pursue these facts, since they have been discussed elsewhere in this work.

It is interesting to observe that term definitions and formula definitions are not mutually exclusive: in the special case where the definiens is a propositional formula, the definition is both a term definition as well as a formula definition. Such definitions have features of both formula and term definitions. Consider the following two examples:

$$G \Rightarrow F \ =_{df} \ \Box\forall x(Gx \rightarrow Fx) \qquad (339.1)$$

$$p \,\&\, q \ =_{df} \ \neg(p \rightarrow \neg q) \qquad \text{instance of (7.4.a)}$$

$$Px \,\&\, Qx \ =_{df} \ \neg(Px \rightarrow \neg Qx) \qquad \text{instance of (7.4.a)}$$

At first glance, these definitions appear to introduce $\Rightarrow$ and & as syncategorematic expressions by way of the formulas $G \Rightarrow F$ and $Px \,\&\, Qx$. In this respect, these definitions may be considered formula definitions and so the definitions become assertible as equivalences.

However, in each case, the definiens is a propositional formula, and so by convention, the definiendum is as well. The definition of $G \Rightarrow F$ can be understood as introducing the special new binary functional term that denotes, relative to any properties G and F, the proposition (i.e., 0-place relation) that necessarily, everything that exemplifies G exemplifies F. (Indeed, we could make the definiendum look more like a functional term by writing it as $\Rightarrow_{G,F}$.) The definition of $p \,\&\, q$ shows that in the special case where & conjoins propositional formulas, it is not just a syncategorematic expression but also a kind of binary functional symbol: $p \,\&\, q$ denotes a proposition relative to the values of p and q.[87] (Again, we could make the definiendum look more like a functional term by writing it as $\&_{p,q}$.) Consequently, all of these definitions or instances of definitions may be considered term definitions. On the basis of such definitions, we may assert the identities:

[87]For the reasons noted in footnote 84, we've taken care not to suggest that & denotes a binary function or that it therefore constitutes a binary *function* symbol. The present language doesn't have the the expressive power to permit talk about a relation that holds between p, q, and r whenever r is the conjunction of p and q. In the type-theoretic version of our system, one may talk about such relations.

$$(G \Rightarrow F) = \Box\forall x(Gx \rightarrow Fx)$$

$$(p \,\&\, q) = \neg(p \rightarrow \neg q)$$

$$(Px \,\&\, Qx) = \neg(Px \rightarrow \neg Qx)$$

In general, when a definiens is a propositional formula ψ^*, both the definiendum φ^* and definiens are terms and the definition becomes assertible as a simple identity of the form $\varphi^* = \psi^*$.

In the deductive system described in Chapter 9, identities of the form $\varphi^* = \psi^*$ will logically imply equivalences of the form $\varphi^* \equiv \psi^*$, but not vice versa. Since definitions involving propositional formulas yield such identities, we'll suppose that that they are governed by both our conventions for term definitions and for formula definitions. We are free to draw inferences that conform to either conventions. However, we'll *label* such definitions as **Definitions** and reserve the label **Term Definition** when the expressions flanking the $=_{df}$ sign are terms that aren't formulas. So, we'll continue to use separate labels despite the fact the categories are not mutually exclusive.

Finally, the above definitions and remarks, together with the full theory of definitions described below in Section 9.12, provide a precise syntactic characterization of the formalism and thus forestall the concerns that Gödel (1944, 120) raised in this regard about Whitehead and Russell's *Principia Mathematica*:

> It is to be regretted that this first comprehensive and thorough-going presentation of a mathematical logic and the derivation of mathematics from it [is] so greatly lacking in formal precision in the foundations (contained in *1–*21 of *Principia*) that it presents in this respect a considerable step backwards as compared with Frege. What is missing, above all, is a precise statement of the syntax of the formalism. Syntactical considerations are omitted even in cases where they are necessary for the cogency of the proofs, in particular in connection with the "incomplete symbols".

It is important to have taken some pains in the above development to ensure that such a criticism doesn't apply to the present effort.

Chapter 8

Axioms

Now that we have a precisely-specified philosophical language that allows to express claims using primitive and defined notions, we next assert the fundamental axioms of our theory in terms of these notions. We may group these axioms as follows:

- Classical axioms governing the negation and the conditional operators.
- Classical axioms for identity formulas.
- Classical axioms of quantification theory, modified only to accommodate rigid definite descriptions.
- Classical axioms for the actuality operator, including an axiom schema whose instances are not asserted to be necessary truths.
- Classical axioms for the necessity operator, supplemented by an axiom about the possibility of contingently nonconcrete objects.
- Classical axioms for the interaction of the necessity operator and actuality operator.
- Classical axioms governing complex terms, modified only to accommodate for rigid definite descriptions.
- Axioms for encoding formulas.

The statement of some axiom groups require preliminary definitions.

(**20**) **Metadefinition:** Closures. We define:

- φ is a *universal closure* or *generalization* of ψ if and only if for some variables $\alpha_1, \ldots, \alpha_n$ ($n \geq 0$), φ is $\forall\alpha_1 \ldots \forall\alpha_n\psi$.

- φ is an *actualization* of ψ iff φ is the result of prefacing ψ by a string of zero or more occurrences of $\mathcal{A}$.
- φ is a *necessitation* of ψ iff φ is the result of prefacing ψ by a string of zero or more occurrences of $\square$.

Furthermore, we say:

- φ is a *closure* of ψ if and only if for some variables $\alpha_1,\ldots,\alpha_n$ $(n \geq 0)$, φ is the result of prefacing ψ by any string of zero or more occurrences of universal quantifiers, $\mathcal{A}$ operators, and $\square$ operators.

Since we're counting the empty string as a string, the definitions yield that every formula is a closure of itself. In what follows, when we say that we are taking the closures of a *schema* as axioms, we mean that the closures of every instance of the schema is an axiom.

8.1 Axioms for Negations and Conditionals

(**21**) **Axioms**: Negations and Conditionals. To ensure that negation and conditionalization behave classically, we take the closures of the following schemata as axioms:

(.1) $\varphi \to (\psi \to \varphi)$

(.2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(.3) $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

8.2 Axioms of Identity

(22) **Remark:** The Symbol '=' in Both Object Language and Metalanguage. In this section, we begin with two metalinguistic definitions, (23) and (24), in which we use the symbol '=' in the metalanguage to represent a primitive metalinguistic notion of identity. Metalinguistic identity helps us to define certain syntactic notions, such as the formula that results when a term τ is substituted for the occurrences of a variable α in some given formula. We state various axioms using such notions, such as the axiom of for the substitution of identicals (25). Of course, in these axioms, the formulas involving the symbol = are defined in items (15) and (16) of the previous chapter. But we need metalinguistic identity to formulate the notions needed to state these axioms.

Consequently, in what follows, we continue to use the = symbol for both object-theoretic and metatheoretic identity. In general, it should always be clear when = is being used to assert something in the object language and when it is being used to assert something in the metalanguage. Of course, if our philosophical project is correct, the theory of identity developed in the object language provides an analysis of the pretheoretical and unanalyzed metalinguistic notion of identity.

(23) **Metadefinitions:** Substitutions. One or more of the definitions in this item and the next are required to state the axioms of identity in item (25), the axioms of quantification in item (29), and the axioms for complex relation terms in item (36). In what follows, we use ρ as a metavariable ranging over any term whatsoever.

- Where τ is any term and α any variable, we use the notation φ^τ_α and ρ^τ_α, respectively, to stand for the result of substituting the term τ for every free occurrence of the variable α in formula φ and in term ρ.

This notion may be defined more precisely by recursion, based on the syntactic complexity of ρ and φ as follows, where the parentheses serve only to eliminate ambiguity and we suppress the obvious superscript indicating arity on the metalinguistic relation variable Π:

- If ρ is a constant or variable other than α, $\rho^\tau_\alpha = \rho$.
  If ρ is α, $\rho^\tau_\alpha = \tau$

- If φ is $\Pi\kappa_1 \dots \kappa_n$, then $\varphi^\tau_\alpha = \Pi^\tau_\alpha \kappa_{1\alpha}^{\ \tau} \dots \kappa_{n\alpha}^{\ \tau}$.
  If φ is $\kappa_1\Pi$, then $\varphi^\tau_\alpha = \kappa_{1\alpha}^{\ \tau}\Pi^\tau_\alpha$.

- If φ is $\neg\psi$, $\Box\psi$ or $\mathcal{A}\psi$, then $\varphi^\tau_\alpha = \neg(\psi^\tau_\alpha)$ or $\Box(\psi^\tau_\alpha)$, or $\mathcal{A}(\psi^\tau_\alpha)$, respectively.
  If φ is $\psi \to \chi$, then $\varphi^\tau_\alpha = \psi^\tau_\alpha \to \chi^\tau_\alpha$.

- If φ is $\forall\beta\psi$, then $\varphi^\tau_\alpha = \begin{cases} \forall\beta\psi, \text{ if } \alpha = \beta \\ \forall\beta(\psi^\tau_\alpha), \text{ if } \alpha \neq \beta \end{cases}$

- If ρ is $\imath\nu\psi$, then $\rho^\tau_\alpha = \begin{cases} \imath\nu\psi, \text{ if } \alpha = \nu \\ \imath\nu(\psi^\tau_\alpha), \text{ if } \alpha \neq \nu \end{cases}$

- If ρ is $[\lambda\nu_1 \dots \nu_n \psi^*]$, then $\rho^\tau_\alpha = \begin{cases} [\lambda\nu_1 \dots \nu_n\ \psi^*], \text{ if } \alpha \text{ is one of } \nu_1, \dots, \nu_n \\ [\lambda\nu_1 \dots \nu_n\ \psi^{*\tau}_{\ \alpha}], \text{ if } \alpha \text{ is none of } \nu_1, \dots, \nu_n \end{cases}$

We shall also want to define multiple simultaneous substitutions of terms for variables in φ and ρ, but since such a recursive definition would be extremely difficult to read, we simply rest with the following definition: where $\tau_1, \dots, \tau_m$ are any terms and $\alpha_1, \dots, \alpha_m$ are any distinct variables, we let $\varphi^{\tau_1,\dots,\tau_m}_{\alpha_1,\dots,\alpha_m}$ stand for the result of simultaneously substituting the term τ_i for each free occurrence of the corresponding variable α_i in φ, for each i such that $1 \le i \le m$. In other words, $\varphi^{\tau_1,\dots,\tau_m}_{\alpha_1,\dots,\alpha_m}$ is the result of making all of the following substitutions simultaneously: (a) substituting τ_1 for every free occurrence of α_1 in φ, (b) substituting τ_2 for every free occurrence of α_2 in φ, etc. Similarly, where $\tau_1, \ldots, \tau_m$ are any terms and $\alpha_1, \ldots, \alpha_m$ are any distinct variables, we let $\rho^{\tau_1,\ldots,\tau_m}_{\alpha_1,\ldots,\alpha_m}$ stand for the result of simultaneously substituting the term τ_i for each free occurrence of the corresponding variable α_i in ρ, for each i such that $1 \leq i \leq m$.

(24) **Metadefinitions:** Substitutable at an Occurrence and Substitutable For. We say:

- Term τ is *substitutable at an occurrence* of α in formula φ or term ρ just in case that occurrence of α does not appear within the scope of any operator binding a variable that has a free occurrence in τ.

In other words, τ is substitutable at an occurrence of α in φ or ρ just in case every occurrence of any variable β free in τ remains an occurrence that is free when τ is substituted for that occurrence of α in φ or ρ. Then we say:

- τ is *substitutable for* α in φ or ρ just in case τ is substitutable at every free occurrence of α in φ or ρ.

In other words, τ is substitutable for α in φ or ρ just in case every occurrence of any variable β free in τ remains an occurrence that is free when τ is substituted for every free occurrence of α in φ^{τ}_{α} or ρ^{τ}_{α}.

The following are consequences of this definition:

- Every term τ is trivially substitutable for α in φ if there are no free occurrences of α in φ.
- α is substitutable for α in φ or ρ.
- If τ contains no free variables, then τ is substitutable for any variable α in any formula φ or complex term ρ.
- If none of the free variables in τ occur bound in φ or ρ, then τ is substitutable for any α in φ or ρ.

(25) **Axioms:** The Substitution of Identicals. The identity symbol '=' is not a primitive expression of our object language. Instead, identity was defined in items (15) and (16) for both individuals and n-place relations ($n \geq 0$). The classical law of the reflexivity of identity, i.e., $\alpha = \alpha$, where α is any variable, will be derived in a subsequent chapter — see item (71.1) in Chapter 9. By contrast, we take the classical law of the substitution of identicals as an axiom. Therefore the closures of the following schema are axioms of our system:

> $\alpha = \beta \rightarrow (\varphi \rightarrow \varphi')$, whenever β is substitutable for α in φ, and φ' is the result of replacing zero or more free occurrences of α in φ with occurrences of β.

This is an unrestricted principle of substitution of identicals: (a) if x and y are identical individuals, then anything true of x is true of y, and (b) if F^n and G^n are identical n-place relations, then anything true of F^n is true of G^n.

(26) **Remark:** Observation about the Substitution of Identicals. Note that Substitution of Identicals will guarantee that if abstract objects x and y necessarily encode the same properties, then they necessarily exemplify the same properties. For the former implies (by definition) that $x = y$ and so Substitution of Identicals would let us infer $\Box\forall F(Fx \equiv Fy)$ from the provable fact that $\Box\forall F(Fx \equiv Fx)$. Similarly, this axiom will guarantee that if properties F and G are necessarily encoded by the same objects, then they are necessarily exemplified by the same objects. For the former implies (by definition) that $F = G$ and so Substitution of Identicals would let us infer $\Box\forall x(Fx \equiv Gx)$ from the provable fact that $\Box\forall x(Fx \equiv Fx)$.

Indeed, given such facts, one might wonder whether we could (a) replace the Substitution of Identicals with the following axioms:

- $(A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)) \rightarrow \Box\forall F(Fx \equiv Fy)$
- $\Box\forall x(xF \equiv xG) \rightarrow \Box\forall x(Fx \equiv Gx)$

and then (b) prove the Substitution of Identicals.

It would seem that these two axioms are *not* sufficient to prove the substitution of identicals. Here is an instance of the substitution of identicals that would apparently remain unprovable:

$$x = y \rightarrow (\exists F(xF \,\&\, \neg Fx) \rightarrow \exists F(yF \,\&\, \neg Fy))$$

The reasoning that suggests this is unprovable goes as follows.

> Suppose we assume both $x = y$ and $\exists F(xF \,\&\, \neg Fx)$, to prove $\exists F(yF \,\&\, \neg Fy)$. From $x = y$, we know that x and y are either (a) both ordinary and necessarily exemplify the same properties or (b) both abstract and necessarily encode the same properties, in which case, we know by the first of the alternative axioms above that they exemplify the same properties. Now from $\exists F(xF \,\&\, \neg Fx)$, then it follows by axiom (38) stated below, which asserts that ordinary objects don't encode any properties, that x is abstract. Hence y is abstract, given $x = y$, and so by definition, x and y encode the same properties. So x and y not only exemplify the same properties but also encode the same properties. But it isn't immediately clear how either fact will yield $\exists F(yF \,\&\, \neg Fy)$.
>
> However, if we could abstract out from $\exists F(xF \,\&\, \neg Fx)$ a property that x exemplifies by using the principle of β-Conversion described in (36.2) below, then the fact that x and y exemplify the same properties would help us to prove $\exists F(yF \,\&\, \neg Fy)$. But this we cannot do; we may not infer

> $[\lambda z \exists F(zF \,\&\, \neg Fz)]x$ from $\exists F(xF \,\&\, \neg Fx)$ because the λ-expression isn't well-formed. Hence $\exists F(xF \,\&\, \neg Fx)$ doesn't yield a property that x exemplifies. So we can't infer $[\lambda z \exists F(zF \,\&\, \neg Fz)]y$ (from the fact that x and y exemplify the same properties) and then use β-Conversion in the reverse direction to obtain $\exists F(yF \,\&\, \neg Fy)$.

If, notwithstanding this reasoning, someone were to find a proof of substitution of identicals from the proposed alternative axioms, we would have an interesting question to consider: is it more elegant to formulate the system by *deriving* Substitution of Identicals from the proposed alternative axioms?

8.3 Axioms of Quantification

We begin with some preliminary definitions.

(27) **Metadefinitions:** Terms of the Same Type. We say that τ and τ' are terms *of the same type* iff τ and τ' are both individual terms or are, for some $n \geq 0$, both n-place relation terms.

(28) **Remark:** Logically Proper Terms. The sentences $\exists x(x = a)$ and $\exists F(F = P)$ *explicitly* assert, respectively, that there is such a thing as a and there is such a thing as P and, as we shall soon see, these will both be axioms. By contrast, $\exists x(x = \imath y \varphi)$ may be false if there is no unique individual such that φ; semantically, in such a case, $\imath y \varphi$ fails to denote. We shall say that axioms and theorems of the form $\exists \beta(\beta = \tau)$ (for any variable β of the same type as τ) assert that term τ is *logically proper*, and we will appeal to this notion of logical propriety instead of saying semantically that τ has a denotation. Using this terminology, we may say, also informally, that the axioms for quantification listed below guarantee: (a) that every term τ other than a description is logically proper,[88] (b) that quantification with respect to logically proper terms is classical, and (c) that descriptions appearing in true exemplification and encoding formulas are logically proper.

(29) **Axioms:** Quantification. Let α and β be variables of the same type, and let τ be a term of the same type as α and β. Then we assert that the closures of the following are axioms:

(.1) $\forall \alpha \varphi \rightarrow (\exists \beta(\beta = \tau) \rightarrow \varphi^{\tau}_{\alpha})$, provided τ is substitutable for α in φ.

[88] Recall the discussion in a previous chapter, where we saw that complex n-place relation terms containing non-denoting descriptions nevertheless have denotations. The denotations of such terms are assigned on the basis of the truth conditions of the matrix formula in the relation term; since those truth conditions are always well-defined, the relation term will get assigned a denotation, even if its matrix formula has a non-denoting term.

Add restriction: and τ is not an improper λ-expression; see the definition of 'improper λ-expression' below.

(.2) $\exists\beta(\beta=\tau)$, provided τ is not a description and β doesn't occur free in τ.[89]

(.3) $\forall\alpha(\varphi\to\psi)\to(\forall\alpha\varphi\to\forall\alpha\psi)$

Note: No labeled theorems are lost by this restriction, but some proofs may require alteration.

(.4) $\varphi\to\forall\alpha\varphi$, provided α doesn't occur free in φ

(.5) $\psi_\mu^{\imath x\varphi}\to\exists\nu(\nu=\imath x\varphi)$, provided (a) ψ is either an exemplification formula $\Pi^n\kappa_1\ldots\kappa_n$ $(n\geq 1)$ or an encoding formula $\kappa_1\Pi^1$, (b) μ is an individual variable that occurs in ψ and only as one or more of the κ_i $(1\leq i\leq n)$, and (c) ν is any individual variable that doesn't occur free in φ.[90]

The above axioms employ metatheoretical definitions (24) – (27) and the theoretical definitions (15) – (16).

Revise (.5) as indicated in red: (.5) $\phi\to\exists\beta(\beta=\tau)$, where (1) ϕ is (a) any exemplification formula $\mathbf{\Pi}\varkappa1\ldots\varkappa n$ and τ is either $\mathbf{\Pi}$,$\varkappa 1$, … , or $\varkappa n$, or (b) any encoding formula $\varkappa\mathbf{\Pi}$ and τ is either $\varkappa$ or $\mathbf{\Pi}$, and (2) β doesn't occur free in τ

8.4 Axioms of Actuality

(**30**) ★**Axioms:** Necessitation-Averse Axioms of Actuality. We now introduce axioms whose necessitations are not valid and we call them *necessitation-averse* axioms because the Rule of Necessitation won't be applicable to them. We've used the label '★**Axiom**' to signpost this fact and, henceforth, we reference the item number for this axiom as (30)★. We take universal closures of the following axiom schema as axioms of the system, where φ is any formula:

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[89]We need the proviso that β doesn't occur free in τ to rule out instances like $\exists F(F=[\lambda y\,\neg Fy])$, in which β $(=F)$ is free in τ $(=[\lambda y\,\neg Fy])$ and which asserts that some F is identical to its negation. This would yield a contradiction given β-Conversion (36.2), for suppose P is such an F, so that we know $P=[\lambda y\,\neg Py]$. By β-Conversion (36.2), it is axiomatic that $[\lambda y\,\neg Py]x\equiv\neg Px$. But then substituting P for $[\lambda y\,\neg Py]$ in this latter formula, we would end up with $Px\equiv\neg Px$, which will provably be a contradiction.

Interestingly, however, we need not similarly restrict (29.1) to those cases where β isn't free in τ. If in fact $\exists\beta(\beta=\tau)$ when β is free in τ, then we should be able to substitute τ everywhere for α in $\forall\alpha\varphi$. For example, F is free in $[\lambda y\,Fy]$, and it is true that $\exists F(F=[\lambda y\,Fy])$. Indeed, as we shall see, every property is a witness to this claim, by η-Conversion (36.3). So we should be able to instantiate $[\lambda y\,Fy]$ into universal claims of the form $\forall G\varphi$. Moreover, the first example in this footnote isn't a problem for (29.1), since the following instances of (29.1) remain valid:

$$\forall G\varphi\to(\exists F(F=[\lambda y\,\neg Fy])\to\varphi_G^{[\lambda y\,\neg Fy]})$$

In this case, the antecedent of the consequent is false in every interpretation (as we just saw), and so the consequent,

$$\exists F(F=[\lambda y\,\neg Fy])\to\varphi_G^{[\lambda y\,\neg Fy]}),$$

is true in every interpretation. So the instance of (29.1) is true in every interpretation.

[90]The restrictions (a) – (c) can be explained as follows. (a) and (b) are required because the truth of 'atomic' (i.e., exemplification or encoding) formulas imply that $\imath x\varphi$ has a denotation when the description appears as one or more of the principal individual terms; a molecular formula having the form of a tautology, e.g., $F\imath x\varphi\to F\imath x\varphi$, is true even when the $\imath x\varphi$ fails to denote. (c) We don't want ν to have a free occurrence in φ for then the quantifier $\exists\nu$ in the right side of the conditional would capture it.

Research by Daniel Kirchner (Freie Univeritāt Berlin), under the direction of Christoph Benzmüller, has established that an additional restriction on (29.2) is necessary: τ may not be an improper λ-expression, where λ-expression [λυ1…υn ϕ] (1≤i≤n) is proper iff no occurrence of υ_i bound by the λ appears in a definite description somewhere in ϕ*. Thus, 1-place relation terms such as [λx FιyRyx] , [λx Fιy(x[λz Rzy])], [λx Fιy(y=x & ϕ)], etc., are improper.*

$\mathcal{A}\varphi \equiv \varphi$

We take *only* the universal closures of the above as axioms because the actualizations of instances of the above, as well as the actualizations of universal closures of instances of the above, will be derivable as theorems; see (90.4) and (91.4), respectively.

(**31**) **Axioms:** Actuality. By constrast we take the all of the closures of the following axiom schemata to be axioms of the system:

(.1) $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$

(.2) $\mathcal{A}(\varphi \to \psi) \equiv (\mathcal{A}\varphi \to \mathcal{A}\psi)$

(.3) $\mathcal{A}\forall\alpha\varphi \equiv \forall\alpha\mathcal{A}\varphi$

(.4) $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$

We leave the axioms governing the interaction between the actuality operator and the necessity operator for Section 8.6.

8.5 Axioms of Necessity

(**32**) **Axioms:** Necessity. We take the closures of the following principles as axioms:

(.1) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ K schema

(.2) $\Box\varphi \to \varphi$ T schema

(.3) $\Diamond\varphi \to \Box\Diamond\varphi$ 5 schema

Furthermore, let us read the formula $\exists x(E!x \;\&\; \Diamond\neg E!x)$ as asserting that there are contingently concrete individuals, i.e., individuals that exemplify being concrete but possibly don't. Then the following axiom asserts both that it is metaphysically possible that there are contingently concrete individuals and that it is metaphysically possible that there are not:

(.4) $\Diamond\exists x(E!x \;\&\; \Diamond\neg E!x) \;\&\; \Diamond\neg\exists x(E!x \;\&\; \Diamond\neg E!x)$

(32.1) – (32.3) are well known. Axiom (32.4) may not be familiar, but it will play a significant role in what follows.[91]

[91] One might wonder why we didn't assert the simpler claim $\Diamond\exists xE!x \;\&\; \Diamond\neg\exists xE!x$ instead of (.4). The reason has to do with the second conjunct of the simpler claim — it rules out *necessarily concrete* objects like Spinoza's God. If Spinoza is correct that God just is Nature and that God is a necessary being, then it would follow that God (g) is necessarily concrete, i.e., that $\Box E!g$. If so, it wouldn't be correct to assert that it is possible that there are no concrete objects; at least, we shouldn't assert this *a priori*. Our axiom in the text doesn't contradict Spinoza's thesis, since it allows for the possible existence and possible nonexistence of *contingently* concrete objects.

Whereas many systematizations of the quantified S5 modal logic use a primitive Rule of Necessitation (RN), we derive RN in item (51) (Chapter 9). RN is derived in a form that implies that if there is a proof of a formula φ that doesn't depend on the necessitation-averse axiom of actuality $\mathcal{A}\varphi \equiv \varphi$ (30)★, then there is a proof of $\Box\varphi$. Thus, we can't apply RN to any formula derived from our necessitation-averse axiom. Once RN is derived, we shall be able to derive the Barcan Formula and Converse Barcan Formula; this occurs in item (123).

8.6 Axioms of Necessity and Actuality

(**33**) **Axioms:** Necessity and Actuality. We take the closures of the following principles as axioms:

(.1) $\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi$

(.2) $\Box\varphi \equiv \mathcal{A}\Box\varphi$

8.7 Axioms for Descriptions

(**34**) **Axioms:** Descriptions. We take the closures of the following axiom schema as axioms:

$x = \imath x\varphi \;\equiv\; \forall z(\mathcal{A}\varphi^z_x \equiv z = x)$, provided z is substitutable for x in φ and doesn't occur free in φ

We may read this as: x is the individual that is (in fact) such that φ just in case all and only those individuals that are actually such that φ are identical to x. The notation $\mathcal{A}\varphi^z_x$ used in the axiom is a harmless ambiguity; it should strictly be formulated as $(\mathcal{A}\varphi)^\tau_\nu$. By the third bullet point in definition (23), we know that $(\mathcal{A}\varphi)^\tau_\nu = \mathcal{A}(\varphi^\tau_\nu)$. Note also that one other axiom for descriptions has already been stated, namely, (29.5).

8.8 Axioms for Complex Relation Terms

Note: We have been using the prime symbol ′ to avoid overspecificity. When we attach a prime symbol to a metavariable, the resulting metavariable stands for an expression that is typically distinct from the expression signified by the metavariable without the prime. For example, when we defined *terms of the same type* in (27), we used τ and τ' to represent any two *terms* in the language. In the axiom for the Substitution of Identicals (25), we used φ' to indicate the result of replacing zero or more free occurrences of the variable α in φ with occurrences of the variable β. (So in the case where zero occurrences are replaced, φ' is φ.) Sometimes we shall place primes on expressions in the object language; for example, in a later chapter, c is introduced as a restricted variable in the object-language that ranges over classes, and we let $c', c'', \ldots$ be *distinct* restricted variables for classes (and so on for other restricted variables). In the next item, however, we shall be use φ' to stand for an alphabetical-variant of the formula φ, and use ρ' to stand for an alphabetical-variant of the term ρ. Later we shall use ρ' to denote an η-variant of the relation term ρ. The context should always make it clear how the prime symbol $'$ is being used.

(**35**) **Metadefinitions:** Alphabetic Variants. To state one of the axioms governing complex relation terms in item (36.1), we require a definition of *alphabetically-variant* relation terms. In general, *alphabetically-variant* formulas and terms are complex expressions that intuitively have the same meaning because they exhibit an insignificant syntactic difference in their bound variables. In the simplest cases:

- alphabetically-variant formulas $\forall x Fx$ and $\forall y Fy$ have the same truth conditions
- alphabetically-variant descriptions $\imath x Fx$ and $\imath y Fy$ either both denote the unique individual that in fact exemplifies F if there is one, or both denote nothing if there isn't
- alphabetically-variant relation terms $[\lambda x\, \neg Fx]$ and $[\lambda y\, \neg Fy]$ denote the same relation

However, we shall also need to define the notion of *alphabetic variant* for formulas and terms of arbitrary complexity, so that the following pairs of expressions count as alphabetic variants:[92]

- $\forall F(Fx \equiv Fy)$ / $\forall G(Gx \equiv Gy)$

[92] Traditionally, two formulas or complex terms are defined to be *alphabetic variants* just in case some sequence of uniform permutations of the bound variables (in which no variable is captured during a permutation) transforms one expression into the other. So, in the first example below, the permutation sequence $F \to G$ transforms the first formula into the second; in the second example, the permutation sequence $y \to z, x \to y$ transforms the first description into the second; in the third pair, the permutation sequence $z \to x, y \to z$ transforms the first λ-expression into the second. And so on. However, we shall *not* follow the traditional definition but rather develop a definition that identifies alphabetic variants by way of symmetries among bound variable occurrences.

- $\imath x \forall y Myx$ / $\imath y \forall z Mzy$
- $[\lambda z\, Rz\imath yQy]$ / $[\lambda x\, Rx\imath zQz]$
- $[\lambda\, Pa \to \forall F\, Fa]$ / $[\lambda\, Pa \to \forall G\, Ga]$
- $[\lambda x\, \neg Fx]a \to \forall y My$ / $[\lambda z\, \neg Fz]a \to \forall x Mx$

We shall eventually prove that alphabetically variant formulas are equivalent, and that alphabetically variant definite descriptions have the same denotation if they have one. But in the case of n-place relation terms, we have to stipulate, as axioms, that alphabetically variant n-place relation terms denote the same relation.

The last three examples displayed above can be reformulated as instances of an axiomatic *equation schema* asserted below. The axiom schema α-Conversion introduced in (36.1) will assert that alphabetically-variant relation terms can be put into an equation:

$$[\lambda z\, Rz\imath yQy] = [\lambda x\, Rx\imath zQz]$$

$$[\lambda\, Pa \to \forall F\, Fa] = [\lambda\, Pa \to \forall G\, Ga]$$

$$[\lambda x\, \neg Fx]a \to \forall y My = [\lambda z\, \neg Fz]a \to \forall x Mx$$

Note that the third example is an equation between two 0-place relation terms. The above examples shed light on the opening sentence of this numbered item, where we said that the notion of *alphabetic variant* is needed for the statement of α-Conversion.

To precisely define the general notion of *alphabetic variant*, i.e., for formulas and terms of arbitrary complexity, we first define *linked* and *independent* occurrences of a variable.

(.1) Let α_1 and α_2 be occurrences of the variable α in the formula φ or in term ρ. Then we say that α_1 is *linked to* α_2 in φ or ρ (or say that α_1 and α_2 are *linked* in φ or ρ) just in case:

(a) either both α_1 and α_2 are free, or

(b) both α_1 and α_2 are bound by the same occurrence of a variable-binding operator.

Otherwise, we say that α_1 and α_2 are *independent* in φ or ρ.

For example:

- In the formula $\forall F(Fa \equiv Fb)$, each occurrence of the variable F is linked to every other occurrence.

- In the formula $\forall F Fa \equiv \forall F Fb$, the first two occurrences of F are linked to each other and the last two occurrences of F are linked to each other, while each of the first two occurrences is independent of each of the last two occurrences and vice versa.

- In the formula $\forall xFx \rightarrow Fx$, the first two occurrences of x are linked, and both are independent of the third, given that this formula is shorthand for $(\forall xFx) \rightarrow Fx$.

- In the term $\imath x(Fx \rightarrow Gy)$, the two occurrences of x are linked.

- In the term $[\lambda x\, \forall yGy \rightarrow (Gy \,\&\, Gx)]$, the two occurrences of x are linked (both are bound by the λ), the first two occurrences of y are linked, and both of those occurrences of y are independent of the third occurrence of y. Also, all three occurrences of G are free and hence linked.

Thus, we have:

> **Metatheorem** $\langle 8.1 \rangle$
> *Linked* is an equivalence condition on variable occurrences in a formula (or term).

A proof is given in the Appendix to this chapter. Thus, the occurrences of each variable in a formula (or term) can be partitioned into linkage groups; in any linkage group for the variable α, each occurrence of α in the group is linked to every other occurrence, while occurrences of α in different linkage groups are independent of one another. In the first example above, there is one linkage group for F, and in the fourth example, there is one linkage group for x. In the second example, there are two linkage groups for F and in the third example, there are two linkage groups for x. In the fifth example, there is one linkage group for x, two linkage groups for y, and one linkage group for G.

Now let us introduce some notation ('BV-notation') that makes explicit the occurrences of *bound* variables in formulas and terms:

(.2) When $\alpha_1, \ldots, \alpha_n$ is the list of all the variable occurrences bound in formula φ or complex term ρ, in order of occurrence and including repetitions of the same variable, the BV-notation for φ is $\varphi[\alpha_1, \ldots, \alpha_n]$, and the BV-notation for ρ is $\rho[\alpha_1, \ldots, \alpha_n]$, respectively, .

Some examples should help:

- When $\varphi = \forall F(Fx \equiv Fy)$, then φ in BV-notation is $\varphi[F, F, F]$

- When $\rho = \imath x \forall yMyx$, then ρ in BV-notation is $\rho[x, y, y, x]$

- When $\rho = [\lambda z\, Rz\imath yQy]$, then ρ in BV-notation is $\rho[z, z, y, y]$

- When $\rho = [\lambda\, Pa \to \forall F\, Fa]$, then ρ in BV-notation is $\rho[F,F]$
- When $\rho = [\lambda x\, \neg Fx]a \to \forall y My$, then ρ in BV-notation is $\rho[x,x,y,y]$

Next we introduce notation for replacing bound variables:

(.3) We write $\varphi[\beta_1/\alpha_1,\ldots,\beta_n/\alpha_n]$ to refer to the result of replacing α_i by β_i in $\varphi[\alpha_1,\ldots,\alpha_n]$, for $1 \le i \le n$. Analogously, we write $\rho[\beta_1/\alpha_1,\ldots,\beta_n/\alpha_n]$ to refer to the result of replacing α_i by β_i in term $\rho[\alpha_1,\ldots,\alpha_n]$.

Finally, we may define:

(.4) φ' is an *alphabetic variant* of φ just in case, for some n:

- $\varphi' = \varphi[\beta_1/\alpha_1,\ldots,\beta_n/\alpha_n]$, and
- for $1 \le i,j \le n$, α_i and α_j are linked in $\varphi[\alpha_1,\ldots,\alpha_n]$ if and only if β_i and β_j are linked in $\varphi'[\beta_1,\ldots,\beta_n]$.

(.5) ρ' is an *alphabetic variant* of ρ just in case, for some n:

- $\rho' = \rho[\beta_1/\alpha_1,\ldots,\beta_n/\alpha_n]$, and
- for $1 \le i,j \le n$, α_i and α_j are linked in $\rho[\alpha_1,\ldots,\alpha_n]$ if and only if β_i and β_j are linked in $\rho'[\beta_1,\ldots,\beta_n]$.

Though we saw numerous examples of alphabetically-variant λ-expressions above, it may still be useful to offer the following additional examples:

- $\varphi = \forall F\, Fa \equiv \forall G\, Gb$
  $\varphi' = \forall F\, Fa \equiv \forall F\, Fb$
- $\varphi = \forall x Rxx \to \exists y Syz$
  $\varphi' = \forall y Ryy \to \exists x Sxz$
- $\rho = \imath z(Fz \to Gy)$
  $\rho' = \imath x(Fx \to Gy)$
- $\rho = [\lambda z\, \forall x Gx \to (Gy\, \&\, Gz)]$
  $\rho' = [\lambda x\, \forall z Gz \to (Gy\, \&\, Gx)]$

Note that our definitions require that if β is to replace α to produce an alphabetic variant, then β must not occur free within the scope of a variable-binding operator that binds an occurrence of α in φ. For example, in the formula $\forall x Rxy$ ($= \varphi$), y occurs free within the scope of $\forall x$. We don't obtain an alphabetic variant of φ by substituting occurrences of y for the occurrences of x in the linkage group of bound occurrences of x. The formula that results from such a replacement, $\forall y Ryy$ ($= \varphi'$), is very different in meaning from the original. In this example, the single occurrence of y in φ gets captured when occurrences of y

replace the bound occurrences of x. Thus, φ in BV-notation is $\varphi[x,x]$ and φ' in BV-notation is $\varphi'[y,y,y]$, and so the first condition in the definition of *alphabetic variant* fails because the number of bound variables, counting multiple occurrences, is different for φ and φ'.

Note that though all free occurrences of variable α in φ (or ρ) are linked, they cannot be changed in any alphabetic variant φ' (or ρ') since the free occurrences are not listed in BV-notation.

Note also that our definitions imply that:[93]

> If ρ is a term occurring in φ and ρ' is an alphabetic variant of ρ, then if φ' is the result of replacing one or more occurrences of ρ in φ by ρ', then φ' is an alphabetic variant of φ.
>
> If φ is a formula occurring in ρ and φ' is an alphabetic variant of φ, then if ρ' is the result of replacing one or more occurrences of φ in ρ by φ', then ρ' is an alphabetic variant of ρ.

Here are some example pairs of the preceding facts (the first two pairs are examples of the first fact and the second two pairs are examples of the second fact):

- Where $\varphi_1 = \forall F(F\imath xPx \equiv Fb)$, $\rho_1 = \imath xPx$, and $\rho_1{}' = \imath yPy$, then $\forall F(F\imath yPy \equiv Fb)$ is an alphabetic variant of φ_1
- Where $\varphi_2 = \forall x([\lambda y\, \neg Fy]x \equiv \neg Fx)$, $\rho_2 = [\lambda y\, \neg Fy]$, and $\rho_2{}' = [\lambda z\, \neg Fz]$, then $\forall x([\lambda z\, \neg Fz]x \equiv \neg Fx)$ is an alphabetic variant of φ_2
- Where $\rho_3 = [\lambda y\, \forall xGx \to Gy]$, $\varphi_3 = \forall xGx$, and $\varphi_3{}' = \forall zGz$, then $[\lambda y\, \forall zGz \to Gy]$ is an alphabetic variant of ρ_3
- Where $\rho_4 = \imath y\forall F(Fy \equiv Fa)$, $\varphi_4 = \forall F(Fy \equiv Fa)$, and $\varphi_4{}' = \forall G(Gy \equiv Ga)$, then $\imath y\forall G(Gy \equiv Ga)$ is an alphabetic variant of ρ_4

In the first case, $\imath yPy$ is an alphabetic variant of $\imath xPx$, and replacing the latter by the former in φ_1 yields $\varphi_1{}'$. In this case, φ_1 in BV-notation is $\varphi[F,F,x,x,F]$ and $\varphi_1{}'$ in BV-notation is $\varphi_1{}'[F,F,y,y,F]$. So (a) $\varphi_1{}' = \varphi_1[F/F,F/F,y/x,y/x,F/F]$, and (b) any two variables in the list of bound variables for φ_1 are linked in φ_1 if and only the corresponding variables in the list of bound variables for $\varphi_1{}'$ are linked in $\varphi_1{}'$. We leave the explanation of the remaining cases as exercises for the reader.

It follows from our definitions that:

[93] The first of the following two facts holds because all the bound variable occurrences in $\rho[\alpha_1,\ldots,\alpha_n]$ will appear in $\varphi[\gamma_1,\ldots,\gamma_m]$ and the replacements for the γ_is needed to replace ρ with $\rho'[\beta_1,\ldots,\beta_n]$ can not break any linkage groups. The only variable occurrences in ρ that could become linked to non-ρ variable occurrences in φ are the free variables of ρ and those cannot be changed in ρ' as noted above. The same reasoning explains why the second of the following two facts holds.

Metatheorem ⟨8.2⟩
Alphabetic variance is an equivalence condition on the complex formulas of our language, i.e., alphabetic variance is reflexive, symmetric, and transitive.

A proof can be found in the Appendix to this chapter.

Another important metatheorem can be approached by way of an example. Let φ be a formula of the form $\neg\psi$. Then any alphabetic variant of φ we pick will be a formula of the form $\neg(\psi')$, for some alphabetic variant ψ' of ψ. For example, if φ is $\neg\forall xFx$, so that ψ is $\forall xFx$. Then pick any alphabetic variant of φ, say, $\neg\forall yFy$. In this case, the formula $\forall yFy$ is the witness ψ' such that φ' is $\neg(\psi')$. This generalizes to formulas of arbitrary complexity, so that we have:

Metatheorem ⟨8.3⟩: Alphabetic Variants of Complex Formulas and Terms.

(a) If φ is a formula of the form $\neg\psi$ (or $\mathcal{A}\psi$, or $\Box\psi$), then each alphabetic variant φ' is a formula of the form $\neg(\psi')$ (or $\mathcal{A}(\psi')$ or $\Box(\psi')$, respectively), for some alphabetic variant ψ' of ψ.

(b) If φ is a formula of the form $\psi \to \chi$, then each alphabetic variant φ' is a formula of the form $\varphi' \to \psi'$, for some alphabetic variants φ' and ψ', respectively, of φ and ψ.

(c) If φ is a formula of the form $\forall\alpha\psi$, then each alphabetic variant φ' is a formula of the form $\forall\beta(\psi'^{\beta}_{\alpha})$, for some alphabetic variant ψ' of ψ and some variable β substitutable for α in ψ' and not free in ψ'.[94]

(d) If τ is a term of the form $\imath\nu\varphi$, then each alphabetic variant τ' is a term of the form $\imath\mu(\varphi'^{\mu}_{\nu})$, for some alphabetic variant φ' of φ and individual variable μ substitutable for ν in φ' and not free in φ'.

(e) If τ is a term of the form $[\lambda\nu_1 \ldots \nu_n\, \varphi^*]$, then each alphabetic variant τ' is a term of the form $[\lambda\mu_1 \ldots \mu_n\, (\varphi^{*\prime\mu_1,\ldots,\mu_n}_{\ \nu_1,\ldots,\nu_n})]$, for some alphabetic variant φ' of φ and individual variables μ_i ($1 \le i \le n$) substitutable, respectively, for the ν_i in $\varphi^{*\prime}$ and not free in $\varphi^{*\prime}$).[95]

Again, a proof can be found in the Appendix to this chapter.

Finally, given the semantics defined in previous chapters, it follows that alphabetically-variant formulas logically imply one another:

Metatheorem ⟨8.4⟩
$\varphi \dashv\models \varphi'$, where φ' is an alphabetic variant of φ

The proof of this Metatheorem is left as an exercise.

[94]Note that this allows for the case where β is α or the case where ψ' is ψ.
[95]Note that this allows for the case where $\mu_1,\ldots,\mu_n$ are, respectively, $\nu_1,\ldots,\nu_n$.

(**36**) **Axioms:** Complex n-Place Relation Terms. The classical axioms of the functional λ-calculus, namely α-, β-, and η-Conversion, systematize the λ-expressions of our relational λ-calculus. We add an axiom called *ı-Conversion*.

> The following axioms govern λ-expressions only; we need not assert versions that govern propositional formulas. That's because the identity claim $[\lambda\varphi^*] = \varphi^*$, which is a special case of η-Conversion, is derived as a theorem in (131.1). From this, we can *derive* versions of α-, β-, and $\imath$-Conversion that apply to propositional formulas. See (131.2) – (131.4).

(.1) α-Conversion. If φ^* is any propositional formula and $\nu_1,\ldots,\nu_n$ any distinct object variables, the closures of the following are axioms ($n \geq 0$):

$[\lambda\nu_1\ldots\nu_n\,\varphi^*] = [\lambda\nu_1\ldots\nu_n\,\varphi^*]'$,
where $[\lambda\nu_1\ldots\nu_n\,\varphi^*]'$ is any alphabetic variant of $[\lambda\nu_1\ldots\nu_n\,\varphi^*]$

and provided [λυ1...υn ϕ] is proper; see p. 191*

(.2) β-Conversion. Where φ^* is any propositional formula, the closures of the following are axioms ($n \geq 1$):

$[\lambda y_1\ldots y_n\,\varphi^*]x_1\ldots x_n \equiv \varphi^{*}{}^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$, provided the x_i are substitutable, respectively, for the y_i in φ^* ($1 \leq i \leq n$). *and provided [λy1...yn ϕ*] is proper; see p. 191*

(.3) η-Conversion: Where $n \geq 0$, the closures of the following are axioms:

$$[\lambda x_1\ldots x_n\,F^n x_1\ldots x_n] = F^n$$

(.4) $\imath$-Conversion:[96] Where χ^* is any propositional formula and $\chi^{*\prime}$ is the result of substituting $\imath x\psi$ for zero or more occurrences of $\imath x\varphi$ anywhere the latter occurs in χ^*:

$$\mathcal{A}(\varphi \equiv \psi) \rightarrow ([\lambda x_1\ldots x_n\,\chi^*] = [\lambda x_1\ldots x_n\,\chi^{*\prime}]) \qquad (n \geq 0)$$

and provided [λx1…xn X] is proper; see p. 191*

In (.1), (.3), and (.4), remember that the identity symbol '=' is a defined notion in our object language.

α-Conversion guarantees that alphabetically-variant λ-expressions denote the same relation. β-Conversion is well-known: the left-to-right direction is sometimes referred to as λ-Conversion, while the right-to-left direction is sometimes referred to as λ-Abstraction. Note that we don't assert the 0-place case (i.e., $[\lambda\,\varphi^*] \equiv \varphi^*$) of β-Conversion as an axiom because it is derivable as a

[96] The reader may call this *iota*-Conversion even though, strictly speaking, we're using the LaTeX character $\imath$ (imath) instead of the inverted Greek letter ι (iota) to represent the rigid definite description operator. Though the inverted Greek letter ι is often used for definite descriptions, we the LaTeX character $\imath$ to signal that our descriptions are rigid. With this warning, it does no harm to call the following axiom *iota*-Conversion.

theorem; see item (131.2). Moreover, a stronger form of β-Conversion is derived in item (128); it applies to λ-expressions in which the leftmost λ binds any distinct individual variables and permits the exemplification formula on the left-side of the biconditional to consist of any variables, not just $x_1,\dots,x_n$, that are substitutable for the variables bound by the λ.

Note also that the above statement of η-Conversion is not a schema. However, in item (164.2), we derive the schema $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n] = \Pi^n$ for $n \geq 0$, where $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n]$ is any elementary λ-expression, Π^n is any n-place relation term, and $\nu_1,\dots,\nu_n$ are any distinct object variables that don't occur free in Π^n. Furthermore, in item (131.1), we derive the case where $n = 0$, that is, $[\lambda\, \varphi^*] = \varphi^*$.

Finally, note that ι-Conversion intuitively tells us that if it is actually the case that φ and ψ are materially equivalent, then we may identify the relations denoted by λ-expressions that are ι-variants with respect to this fact, i.e., identify the relations denoted by λ-expressions whose matrices differ only in that one has occurrences of $\imath x\varphi$ where the other has occurrences of $\imath x\psi$. This conditional identification is valid because if $\mathcal{A}(\varphi \equiv \psi)$ is true in an interpretation, then $\imath x\varphi$ and $\imath x\psi$ have exactly the same denotation conditions and so make the same semantic contribution, *whether they denote or not*, to any λ-expression in which they occur.[97]

8.9 Axioms of Encoding

(**37**) **Axiom:** Rigidity of Encoding. If an object x encodes a property F, it does so necessarily. That is, the closures of the following are axioms of the system:

$$xF \to \Box xF$$

In other words, encoded properties are rigidly encoded. From this axiom, we will be able to prove that the properties an object possibly encodes are necessarily encoded.

(**38**) **Axiom:** Ordinary Objects Fail to Encode Properties. The closures of the following are axioms:

$$O!x \to \neg\exists F\, xF$$

We prove in the next chapter that if x is ordinary, then *necessarily* x fails to encode any properties, i.e., that $O!x \to \Box\neg\exists F\, xF$.

[97] The semantic proof that (.4) is valid rests on the fact that every interpretation $\mathcal{I}$ determines a class of formula pairs φ, ψ such that $\models_{\mathcal{I}} \mathcal{A}(\varphi \equiv \psi)$. This, in turn, yields a class of formula pairs χ and χ' that are ι-variants, i.e., formulas that syntactically differ only by the fact that χ' may have occurrences of $\imath x\psi$ where χ has occurrences of $\imath x\varphi$. A constraint on denotations then requires that when χ and χ' are such propositional formulas, χ^* and $\chi^{*\prime}$, then the denotations of $[\lambda x_1 \dots x_n\, \chi^*]$ and $[\lambda x_1 \dots x_n\, \chi^{*\prime}]$ are the same. See Constraint 3 in Section 5.6.3 of Chapter 5.

(**39**) **Axioms:** Comprehension Principle for Abstract Objects ('Object Comprehension'). The closures of the following schema are axioms:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)), \text{ provided } x \text{ doesn't occur free in } \varphi$$

When x doesn't occur free in φ, we may think of φ as presenting a condition on properties F, whether or not F is free in φ (the condition φ being a vacuous one when F doesn't occur free). So this axiom guarantees that for every condition φ on properties F expressible in the language, there exists an abstract object x that encodes just the properties F such that φ.

(**40**) **Remark:** The Restriction on Comprehension. In the formulation of the Comprehension Principle for Abstract Objects in (39), the formula φ used in comprehension may not contain free occurrences of x. This is a traditional constraint on comprehension schemata. Without such a restriction, a contradiction would be immediately derivable by using the formula '$\neg xF$' as φ, so as to produce the instance:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \neg xF))$$

Any such object, say a, would be such that $\forall F(aF \equiv \neg aF)$, and a contradiction of the form $\varphi \equiv \neg\varphi$ would follow once we instantiate the quantifier $\forall F$ to any property term. Instances such as the above are therefore ruled out by the restriction.

Chapter 9

The Deductive System PLM

In science, that which is provable shouldn't be believed without proof.[98]
— Dedekind 1888

In this chapter, we introduce the deductive system PLM by combining the axioms of the previous chapter with a primitive rule of inference and defining the notions of derivation, proof, and theorem. We then develop a series of theorems, the proofs of which are often facilitated by the introduction of metarules, i.e., special metatheorems that allow us to infer the existence of derivations and proofs. The actual derivations, proofs of the theorems, and the justifications of the metarules all appear in the main Appendix, while the proofs of other metatheorems continue to appear in chapter appendices. The Appendices are gathered together in Part IV.

9.1 Primitive Rule of PLM: Modus Ponens

(**41**) **Primitive Rule**: Modus Ponens. PLM employs just a single primitive rule of inference:

> **Modus Ponens** (**Rule MP**)
> $\varphi, \varphi \to \psi \ / \ \psi$

i.e., ψ follows from the formulas φ and $\varphi \to \psi$.

[98] The original German:

> Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweiß geglaubt werden.

(Translation mine.) This is the opening line from the Preface to Dedekind's classic work of 1888. In the second edition (1893), p. vii; in the seventh edition (1939), p. iii.

9.2 (Modally Strict) Proofs and Derivations

(42) **Metadefinitions:** Derivations, Proofs, and Theorems of PLM. In what follows, we say that φ is an *axiom* of PLM whenever φ is one of the axioms asserted in Chapter 8. The set of axioms of PLM is recursive and we introduce the following symbol to refer to it:

$$\mathbf{\Lambda} = \{\varphi | \varphi \textit{ is an axiom}\}$$

Then we define:

(.1) A *derivation* in PLM *of* φ *from* a set of formulas Γ is any sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that φ is φ_n and for each i $(1 \leq i \leq n)$, φ_i is either an element of $\mathbf{\Lambda} \cup \Gamma$ or follows from two of the preceding members of the sequence by Rule MP. A formula φ *is derivable from*, or *follows from*, the set Γ in PLM, written $\Gamma \vdash \varphi$, just in case there exists a derivation of φ from Γ.

We now adopt the following conventions. We call the members of Γ the *premises* or *assumptions* of the derivation. We often write $\varphi_1, \ldots, \varphi_n \vdash \psi$ when $\Gamma \vdash \psi$ and Γ is the set $\{\varphi_1, \ldots, \varphi_n\}$. We often write $\Gamma, \psi \vdash \varphi$ when $\Gamma \cup \{\psi\} \vdash \varphi$. We often write $\Gamma_1, \Gamma_2 \vdash \varphi$ when $\Gamma_1 \cup \Gamma_2 \vdash \varphi$. Whenever a sequence $\varphi_1, \ldots, \varphi_n$ is a derivation of φ from Γ, we say the sequence is a *witness to the claim* that $\Gamma \vdash \varphi$ (henceforth, *a witness to* $\Gamma \vdash \varphi$), and we call the members of the sequence the *lines* of the derivation.

In virtue of the above definition, $\vdash$ refers to *derivability*; this is a multigrade, metatheoretic relation between zero or more formulas and a formula designated as the *conclusion*.

Now using the definition of *derivation*, we may define the notions of *proof* in PLM and *theorem* of PLM:

(.2) A *proof* of φ in PLM is any derivation of φ from Γ in PLM in which Γ is the empty set $\varnothing$. A formula φ is a *theorem* of PLM, written $\vdash \varphi$, if and only if there exists a proof of φ in PLM.

Two simple consequences of our definitions are:

(.3) $\vdash \varphi$ if and only if there is a sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that φ is φ_n and for each i $(1 \leq i \leq n)$, φ_i is either an element of $\mathbf{\Lambda}$ or follows from two of the preceding members of the sequence by Rule MP.

(.4) If $\Gamma = \varnothing$, then $\Gamma \vdash \varphi$ if and only if $\vdash \varphi$.

(43) **Metadefinitions:** Modally Strict Proofs, Theorems, and Derivations. To precisely identify those derivations and proofs that make no appeal to the

necessitation-averse axiom (30)★, we first say that φ is a *necessary* axiom whenever φ is any axiom for which we've taken *all* the closures and not just the $\Box$-free closures. So far, (30)★ is the only axiom for which we've taken just the $\Box$-free closures, but one could extend the system with other such necessitation-averse axioms. We introduce the following symbol to refer to the set of necessary axioms:

$$\mathbf{\Lambda}_\Box = \{\varphi \mid \varphi \textit{ is a necessary axiom}\}$$

Then the following definitions of modally-strict derivations and proofs mirror (42.1) and (42.2), with the exception that the definientia refer to $\mathbf{\Lambda}_\Box$ instead of $\mathbf{\Lambda}$ and the ensuing definitions are all correspondingly modified:

(.1) A *modally-strict derivation* (or *$\Box$-derivation*) *of* φ *from* a set of formulas Γ in PLM is any sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi = \varphi_n$ and for each i ($1 \leq i \leq n$), φ_i is either an element of $\mathbf{\Lambda}_\Box \cup \Gamma$ or follows from two of the preceding members of the sequence by Rule MP. A formula φ *is strictly derivable* (or $\Box$-derivable) *from* the set Γ in PLM, written $\Gamma \vdash_\Box \varphi$, just in case there exists a modally-strict derivation of φ from Γ.

(.2) A *modally-strict proof* (or *$\Box$-proof*) of φ in PLM is any modally-strict derivation of φ from Γ when Γ is the empty set. A formula φ is a *modally-strict theorem* (or *$\Box$-theorem*) of PLM, written $\vdash_\Box \varphi$, if and only if there exists a modally-strict proof of φ in PLM.

These two definitions have simple consequences analogous to (42.3) and (42.4). We shall suppose that all of the conventions introduced in (42) concerning $\vdash$ also apply to $\vdash_\Box$.

(44) **Remark:** Metarules of Inference. In what follows, we often introduce and prove certain metatheorems about derivations. These facts all have the following form:

If conditions … hold, then there exists a derivation of φ from Γ.

We call facts having this form *metarules of inference* (as opposed to *rules of inference*), since instead of allowing us to infer φ from zero or more formulas, they allow us to infer the existence of a derivation of φ given certain conditions. Metarules often shorten the reasoning we use in the Appendix to establish that $\Gamma \vdash \varphi$ since, frequently, in the process of deriving φ from Γ, we reach a point in the reasoning where what we have established thus far meets the conditions of a metarule whose consequent asserts that there is a sequence of formulas constituting a derivation of φ from Γ.

Consequently, when we reason with metarules to establish the claim that $\Gamma \vdash \varphi$, we don't actually produce a witness to the claim. However, the proof of the metarule in the Appendix (they are, after all, metatheorems) shows how to construct such a witness. We call such proofs *justifications* of the metarule. The justification shows that an appeal to a metarule during the course of reasoning can always be converted into a *bona fide* derivation.

(**45**) **Metarules:** Modally Strict Derivations are Derivations. It immediately follows from our definitions that: (.1) if there is a modally-strict derivation of φ from Γ, then there is a derivation of φ from Γ, and (.2) if there is a modally-strict proof of φ, then there is a proof of φ:

(.1) If $\Gamma \vdash_{\Box} \varphi$, then $\Gamma \vdash \varphi$

(.2) If $\vdash_{\Box} \varphi$, then $\vdash \varphi$

Clearly, however, the converses are not true in general, since derivations and proofs in which (necessitation-averse) axiom (30)$\star$ is used are not modally-strict. Consequently, modally-strict derivations and proofs constitute a proper subset, respectively, of all derivations and proofs.

(**46**) **Metarules:** Fundamental Properties of $\vdash$ and $\vdash_{\Box}$. The following facts are particularly useful as we prove new theorems and justify new metarules of PLM. Note that these facts come in pairs, with one member of the pair governing $\vdash$ and the other member governing $\vdash_{\Box}$:

(.1) If $\varphi \in \mathbf{\Lambda}$, then $\vdash \varphi$. ("Axioms are theorems")
If $\varphi \in \mathbf{\Lambda}_{\Box}$, then $\vdash_{\Box} \varphi$. ("Necessary axioms are modally-strict theorems")

(.2) If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$. (Note the special case: $\varphi \vdash \varphi$)
If $\varphi \in \Gamma$, then $\Gamma \vdash_{\Box} \varphi$. (Note the special case: $\varphi \vdash_{\Box} \varphi$)

(.3) If $\vdash \varphi$, then $\Gamma \vdash \varphi$.
If $\vdash_{\Box} \varphi$, then $\Gamma \vdash_{\Box} \varphi$.

(.4) If $\varphi \in \mathbf{\Lambda} \cup \Gamma$, then $\Gamma \vdash \varphi$.
If $\varphi \in \mathbf{\Lambda}_{\Box} \cup \Gamma$, then $\Gamma \vdash_{\Box} \varphi$.

(.5) If $\Gamma \vdash \varphi$ and $\Gamma \vdash (\varphi \to \psi)$, then $\Gamma \vdash \psi$.
If $\Gamma \vdash_{\Box} \varphi$ and $\Gamma \vdash_{\Box} (\varphi \to \psi)$, then $\Gamma \vdash_{\Box} \psi$.

(.6) If $\vdash \varphi$ and $\vdash (\varphi \to \psi)$, then $\vdash \psi$.
If $\vdash_{\Box} \varphi$ and $\vdash_{\Box} (\varphi \to \psi)$, then $\vdash_{\Box} \psi$.

(.7) If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$.
If $\Gamma \vdash_{\Box} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\Box} \varphi$.

(.8) If $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$.
If $\Gamma \vdash_{\Box} \varphi$ and $\varphi \vdash_{\Box} \psi$, then $\Gamma \vdash_{\Box} \psi$.

(.9) If $\Gamma \vdash \varphi$, then $\Gamma \vdash (\psi \to \varphi)$, for any ψ.
If $\Gamma \vdash_{\Box} \varphi$, then $\Gamma \vdash_{\Box} (\psi \to \varphi)$, for any ψ.

(.10) If $\Gamma \vdash (\varphi \to \psi)$, then $\Gamma, \varphi \vdash \psi$.
If $\Gamma \vdash_{\Box} (\varphi \to \psi)$, then $\Gamma, \varphi \vdash_{\Box} \psi$.

Notice here that in the special case of (46.2), $\varphi \vdash_{\Box} \varphi$ holds even if φ isn't a necessary truth. In general, there can be modally-strict derivations in which neither the premises nor conclusion are necessary truths. The 3-element sequence $Pa, Pa \to Qb, Qb$ is a modally-strict derivation of Qb from the assumptions Pa and $Pa \to Qb$ whether or not the premises and the conclusion are necessary truths.

(47) **Remark:** Theorems That Aren't Modally Strict (⋆-Theorems). For the most part, we shall be interested in proofs generally, not just modally-strict ones, since our primary interest is what claims we can prove (*simpliciter*). But since significantly more modally-strict □-theorems are proved in what follows, it is useful the ones that are not. So the reader may assume that all of the items marked **Theorem** in what follows have modally-strict proofs, and that ⋆-**Theorems** do not. So the ⋆ makes it explicit that a proof of a theorem is *not* modally-strict. Moreover, we concatenate a theorem's item number with ⋆ when referencing a ⋆-theorem.[99] Similarly, in the case where the derivation of φ from Γ is not modally-strict, we may speak of ⋆-derivations and say that φ is ⋆-derivable from Γ.

It is important to recognize that ⋆-theorems are not defective in any way. Indeed, they are simply artifacts of a modal logic with an actuality operator and definite descriptions that are rigid, which produce contexts that are necessitation-averse.

(48) **Metadefinition:** Dependence. It is sometimes useful to indicate the difference between □-derivations and ⋆-derivations by saying that in the latter, the conclusion *depends* upon a necessitation-averse axiom, or depends upon a ⋆-theorem that in turn depends upon a necessitation-averse axiom, etc. To make this talk of dependence precise, we define the conditions under which one formula depends upon another within the context of a derivation:

> Let the sequence $\varphi_1, \ldots, \varphi_n$ be a derivation in PLM of φ ($= \varphi_n$) from the set of premises Γ and let ψ be a member of this sequence. Then we say that φ_i ($1 \leq i \leq n$) *depends upon* the formula ψ *in* this derivation iff either (a) $\varphi_i = \psi$, or (b) φ_i follows by Rule MP from two previous members of the sequence at least one of which depends upon ψ.

When $\Gamma = \varnothing$, this reduces to a definition of: φ depends on a formula ψ in a given proof of φ.

[99] The first such theorems are (94.1)⋆ – (94.2)⋆ and (96)⋆ – (101.3)⋆ below.

It follows from our definition that if a sequence S is a witness to $\Gamma \vdash \varphi$, then S is a witness to $\Gamma \vdash_{\Box} \varphi$ if and only if φ doesn't depend upon any instance of necessitation-averse axiom (30)★ in S. This holds even if, in S, φ depends upon a premise in Γ that isn't necessary. The sequence $S = Pa, Pa \to Qb, Qb$ is a witness to $Pa, Pa \to Qb \vdash Qb$ and even if Pa fails to be a necessary truth, it follows that $Pa, Pa \to Qb \vdash_{\Box} Qb$, since Qb doesn't depend on (30)★ in S. Thus, the ★-theorems below ultimately depend upon axiom (30)★, either directly or because they depend on other ★-theorems that depend upon (30)★, and so on.

9.3 Two Fundamental Metarules: GEN and RN

(**49**) **Metarule**: The Rule of Universal Generalization. The Rule of Universal Generalization (GEN) asserts that whenever there is a derivation of a claim (involving the variable α) of the form $\ldots\alpha\ldots$ from a set of premises Γ, and none of the premises in Γ is a special assumption about α, then there is a derivation from Γ of the claim $\forall\alpha(\ldots\alpha\ldots)$:

> **Rule of Universal Generalization** (**GEN**)
> If $\Gamma \vdash \varphi$ and α doesn't occur free in any formula in Γ, then $\Gamma \vdash \forall\alpha\varphi$.

When $\Gamma = \varnothing$, then GEN asserts that if a formula φ is a theorem, then so is $\forall\alpha\varphi$:

> If $\vdash \varphi$, then $\vdash \forall\alpha\varphi$

Note the proviso to GEN when φ is derived from Γ, namely, that the variable α doesn't occur free in any premise in Γ. This prohibits, for example, the meta-inference from $Rx \vdash Rx$ to $Rx \vdash \forall xRx$. (Henceforth we shall not distinguish between inferences and meta-inferences, since it is clear which ones are under discussion.) We know $Rx \vdash Rx$ by the special case of (46.2), but intuitively, we don't want $Rx \vdash \forall xRx$: from the premise that Rx is true (i.e., that some unspecified value of x exemplifies the property R), it doesn't follow that every individual exemplifies R. The proviso to GEN, of course, is unnecessary when φ is a theorem since φ is then derivable from the empty set of premises. Whenever any formula φ with free variable α is a theorem, we may invoke GEN to conclude that $\forall\alpha\varphi$ is also a theorem. For example, we shall soon prove that $\varphi \to \varphi$ is a theorem (53), so that the instance $Px \to Px$ is a theorem. From this latter it follows by GEN that $\forall x(Px \to Px)$.

Here is an example of GEN in action. The following reasoning sequence establishes that $\forall x(Qx \to Px)$ is derivable from the premise $\forall xPx$, even though strictly speaking, the sequence is not a witness to this derivability claim:

1.	$\forall x P x$	Premise
2.	$(\forall x P x) \to P x$	Instance, Axiom (29.1)
3.	$P x$	MP, 1,2
4.	$P x \to (Q x \to P x)$	Instance, Axiom (21.1)
5.	$Q x \to P x$	MP, 3,4
6.	$\forall x P x \vdash \forall x (Q x \to P x)$	GEN, 1–5

Lines 1–5 constitute a *bona fide* derivation of $Qx \to Px$ from $\forall x P x$: each of these lines is either an axiom, a member of Γ, or follows by MP from two previous lines. GEN, at this point, then tells us that given lines 1–5 and the fact that x doesn't appear free in the premise, there is a derivation of $\forall x(Qx \to Px)$ from the premise $\forall x P x$. This is what is asserted on line 6.

Though lines 1–6 above do not constitute a derivation of $\forall x(Qx \to Px)$ from $\forall x P x$, the justification (i.e., metatheoretic proof) of GEN given in the Appendix shows us how to convert the reasoning into a sequence of formulas that is a *bona fide* witness to the derivability claim. By studying the metatheoretic proof, it becomes clear that the above reasoning with GEN can be converted to the following derivation, in which no such appeal is made:[100]

Witness to $\forall x P x \vdash \forall x(Qx \to Px)$

1.	$\forall x P x$	Premise
2.	$\forall x(Px \to (Qx \to Px))$	Inst. Ax. (21.1)
3.	$\forall x(Px \to (Qx \to Px)) \to (\forall x P x \to \forall x(Qx \to Px))$	Inst. Ax. (29.3)
4.	$\forall x P x \to \forall x(Qx \to Px)$	MP, 2,3
5.	$\forall x(Qx \to Px)$	MP, 1,4

This sequence is a *bona fide* derivation of $\forall x(Qx \to Px)$ from $\forall x P x$, in the style of Frege and Hilbert. (In this particular example, the derivation is actually shorter by one step than the meta-derivation above that cited GEN. Most of the time, however, the meta-derivations that invoke GEN are shorter than *bona fide* derivations that don't. Of course, the reasoning with GEN already looks a bit more straightforward than the reasoning without it.) In any case, this example shows how we can use a metarule with our deductive calculus to show that universal claims are derivable.

In light of the above facts, we shall adopt the following convention for reasoning with GEN. In the Appendix, we henceforth establish a claim such as $\forall x P x \vdash \forall x(Qx \to Px)$ as follows:

> From the premise $\forall x P x$ and the instance $\forall x P x \to P x$ of axiom (29.1), it follows that Px, by MP. From this last conclusion and the instance $Px \to (Qx \to Px)$ of axiom (21.1), it follows by MP that $Qx \to Px$. Since x isn't free in our premise, it follows that $\forall x(Qx \to Px)$, by GEN. ⋈

[100]Lines 2 and 3 in the following derivation are indeed instances of the axiom schemata cited since we've take the closures of the instances of the schema as axioms.

Note that this takes the liberty of treating GEN as if it were a rule of inference instead of a metarule. However, the above discussion should have made it clear just what has and has not been accomplished in this piece of reasoning. We sometimes deploy other metarules in just this way.

(**50**) **Remark:** Conventions Regarding Metarules. Although GEN was formulated to apply to $\vdash$, it also applies to $\vdash_{\Box}$. The following version can be proved by a trivial reworking of the justification for (49):

- If $\Gamma \vdash_{\Box} \varphi$ and α doesn't occur free in any formula in Γ, then $\Gamma \vdash_{\Box} \forall\alpha\varphi$.

However, in what follows, we shall *not* formulate metarules twice, with one form for $\vdash$ and a second form for $\vdash_{\Box}$. Instead, we henceforth adopt the conventions:

(.1) Whenever a metarule of inference is formulated generally, so as to apply to $\vdash$, we omit the statement of the rule for the case of $\vdash_{\Box}$.

(.2) No metarule is to be adopted if the justification of the rule depends on a necessitation-averse axiom such as (30)$\star$.

Though these conventions will be discussed on other occasions below, the following brief remarks may be sufficient for now. As noted previously, the justifications of metarules provided in the Appendix show how to convert reasoning with the metarules into *bona fide* derivations that don't use them. A justification of a rule stated for $\vdash$ can be repurposed, with just a few obvious and trivial changes, to a justification of the analogous rule for $\vdash_{\Box}$, i.e., to a proof that any modally-strict reasoning using the metarule can be converted into a modally-strict derivation or proof that doesn't use the rule. As long as the justification doesn't depend on the necessitation-averse axiom (30)$\star$ or any theorem derived from it, then any metarule of inference that applies to derivations and proofs *generally* will be a metarule of inference that also applies to modally-strict derivations and proofs.

The next rule we consider, RN, contrasts with GEN because it is not a metarule that applies generally to all derivations and proofs. The antecedent of the Rule of Necessitation requires the existence of a modally-strict derivation or proof for the metarule to be applied.

(**51**) **Metarule:** Rule of Necessitation. The Rule of Necessitation (RN) is formulated in a way that prevents its application, in a derivation or proof, to any formula that depends on an a claim that isn't necessary. RN is based on a simple idea: if the derivation of a formula φ from the premise ψ is modally-strict (i.e., if φ doesn't depend on the necessitation-averse axiom (30)$\star$ in this derivation), then there is a derivation of $\Box\varphi$ from $\Box\psi$.

To formulate RN generally, however, let us introduce a definition:

- $\Box\Gamma =_{df} \{\Box\psi \mid \psi \in \Gamma\}$ (Γ any set of formulas)

So $\Box\Gamma$ is the result of prefixing a $\Box$ to every formula in Γ. Then the RN may be stated as follows:

> **Rule of Necessitation (RN)**
> If $\Gamma \vdash_{\Box} \varphi$, then $\Box\Gamma \vdash \Box\varphi$.

In other words, if there is a modally-strict derivation of φ from Γ, then there is a derivation of $\Box\varphi$ from $\Box\Gamma$. In the case where $\Gamma = \varnothing$, then the RN reduces to:

> If $\vdash_{\Box} \varphi$, then $\vdash \Box\varphi$.

That is, if there is a modally-strict proof of φ, then there is a proof of $\Box\varphi$.

As with GEN, the justification of RN in the Appendix shows us how to turn reasoning that appeals to RN into reasoning that does not. Here is an example application of the metarule in which $\Gamma = \{Pa, Pa \to Qb\}$ and $\varphi = Qb$:

Example 1

1.	Pa	Premise
2.	$Pa \to Qb$	Premise
3.	Qb	MP, 1,2
4.	$\Box Pa, \Box(Pa \to Qb) \vdash \Box Qb$	RN, 1–3

Lines 1–3 in this example constitute a witness to $Pa, Pa \to Qb \vdash_{\Box} Qb$ since (a) Qb follows by MP from two previous members of the sequence, both of which are in Γ, and (b) the derivation of Qb from the premises doesn't involve necessitation-averse axiom (30)$\bigstar$. RN then asserts that from lines 1–3 it follows that $\Box Pa, \Box(Pa \to Qb) \vdash \Box Qb$, i.e., there is a derivation of $\Box Qb$ from $\Box Pa$ and $\Box(Pa \to Qb)$.

Although the reasoning in the above example doesn't qualify as a witness to the derivability claim on line 4, the justification of RN in the Appendix shows us how to convert line 4 into the following 5-element annotated sequence that does so qualify:

Witness to $\Box Pa, \Box(Pa \to Qb) \vdash \Box Qb$

1.	$\Box Pa$	Premise in $\Box\Gamma$
2.	$\Box(Pa \to Qb)$	Premise in $\Box\Gamma$
3.	$\Box(Pa \to Qb) \to (\Box Pa \to \Box Qb)$	Instance, Axiom (32.1)
4.	$\Box Pa \to \Box Qb$	MP, 2,3
5.	$\Box Qb$	MP, 1,4

This conversion works generally for any formulas φ and ψ: since there is a modally strict derivation of ψ from φ and $\varphi \to \psi$, there is a derivation of $\Box\psi$ from $\Box\varphi$ and $\Box(\varphi \to \psi)$.

Given the above discussion, it should be straightforward to see why we shall adopt the following, less formal style of reasoning in the Appendix when presented with a case like Example 1:

> Assume $\Box Pa$ and $\Box(Pa \rightarrow Qb)$ as 'global' premises. Then note that by taking Pa and $Pa \rightarrow Qb$ as 'local' premises, it follows by MP that Qb. Since this is a modally-strict derivation of Qb from Pa and $Pa \rightarrow Qb$, it follows from our global premises that $\Box Qb$, by RN. ⋈

In effect, we have reasoned by producing a 'sub-derivation' showing $Pa, Pa \rightarrow Qb \vdash Qb$, within the larger derivation of $\Box Qb$ from $\Box Pa$ and $\Box(Pa \rightarrow Qb)$.

Here is an example that uses RN to conclude that $\Box\forall xPx \vdash \Box\forall x(Qx \rightarrow Px)$ given that $\forall xPx \vdash_{\Box} \forall x(Qx \rightarrow Px)$; it involves a slight variant of the example we used to illustrate GEN.

Example 2

1.	$\forall xPx$	Premise
2.	$(\forall xPx) \rightarrow Px$	Instance, Axiom (29.1)
3.	Px	MP, 1,2
4.	$Px \rightarrow (Qx \rightarrow Px)$	Instance, Axiom (21.1)
5.	$Qx \rightarrow Px$	MP, 3,4
6.	$\forall xPx \vdash_{\Box} \forall x(Qx \rightarrow Px)$	GEN, 1–5
7.	$\Box\forall xPx \vdash \Box\forall x(Qx \rightarrow Px)$	RN, 6

Note that since lines 1–5 constitute a modally-strict derivation of $Qx \rightarrow Px$ from $\forall xPx$, we apply, on line 6, the version of GEN that governs $\vdash_{\Box}$, which was discussed in (50). So line 6 satisfies the condition for the application of RN, which then implies the conclusion on line 7. The justification of RN itself shows us how to convert line 7 into a witness for $\Box\forall xPx \vdash \Box\forall x(Qx \rightarrow Px)$:

Witness to $\Box\forall xPx \vdash \Box\forall x(Qx \rightarrow Px)$

1.	$\Box\forall xPx$	Premise
2.	$\Box[\forall x(Px \rightarrow (Qx \rightarrow Px)) \rightarrow (\forall xPx \rightarrow \forall x(Qx \rightarrow Px))]$	Inst. Ax. (29.3)
3.	$\Box[\forall x(Px \rightarrow (Qx \rightarrow Px)) \rightarrow (\forall xPx \rightarrow \forall x(Qx \rightarrow Px))] \rightarrow$ $(\Box\forall x(Px \rightarrow (Qx \rightarrow Px)) \rightarrow \Box(\forall xPx \rightarrow \forall x(Qx \rightarrow Px)))$	Inst. Ax. (32.1)
4.	$\Box\forall x(Px \rightarrow (Qx \rightarrow Px)) \rightarrow \Box(\forall xPx \rightarrow \forall x(Qx \rightarrow Px))$	MP, 2,3
5.	$\Box\forall x(Px \rightarrow (Qx \rightarrow Px))$	Inst. Ax. (21.1)
6.	$\Box(\forall xPx \rightarrow \forall x(Qx \rightarrow Px))$	MP, 4,5
7.	$\Box(\forall xPx \rightarrow \forall x(Qx \rightarrow Px)) \rightarrow (\Box\forall xPx \rightarrow \Box\forall x(Qx \rightarrow Px))$	Inst. Ax. (32.1)
8.	$\Box\forall xPx \rightarrow \Box\forall x(Qx \rightarrow Px)$	MP, 6,7
9.	$\Box\forall x(Qx \rightarrow Px)$	MP, 1,8

This is a *bona fide* derivation of the conclusion from the premise since every line is either an axiom, a premise, or follows from previous lines by MP. It should now be clear how the reasoning using GEN and RN in Example 2 is far easier to develop, or even grasp, when compared to the above reasoning. Indeed, we may compress the reasoning in Example 2 even further. The reasoning used in the Appendix for examples like this goes as follows:

> Let $\Box\forall xPx$ be our global premise. Now from the local premise $\forall xPx$ and the instance $\forall xPx \to Px$ of axiom (29.1), it follows that Px, by MP. From this last conclusion and the instance $Px \to (Qx \to Px)$ of axiom (21.1), it follows by MP that $Qx \to Px$. Since x isn't free in our local premise, it follows that $\forall x(Qx \to Px)$, by GEN. Since this constitutes a modally-strict derivation of $\forall x(Qx \to Px)$ from the local premise $\forall xPx$, it follows from our global premise, by RN, that $\Box\forall x(Qx \to Px)$. ⋈

Given the preceding discussion, this reasoning should be transparent; though it is not an actual derivation, it shows us how to construct one. Consequently, we have a way to show derivability claims without producing actual derivations.

(**52**) **Remark:** Digresson on the Converse of RN. If we focus on proofs rather than derivations, then RN asserts that if $\vdash_\Box \varphi$, then $\vdash \Box\varphi$. The converse, however, doesn't hold: there are formulas φ such that $\vdash \Box\varphi$ but not $\vdash_\Box \varphi$. Though we aren't yet in a position to demonstrate this, it is not too hard to outline conditions under which the converse of RN fails. Suppose that φ is an encoding formula xF, indeed a theorem, and that the proofs of φ *depend on* some necessitation-averse axiom. Then $\Box\varphi$ is also a theorem, since axiom (37) asserts that encoding formulas are necessary if true. But since every proof of φ depends on a necessitation-averse axiom, there is no modally strict proof of φ.

An example of this scenario is discussed in some detail in (185), where we consider what happens if we apply our system by adding axioms that assert that some particular object, say b, exemplifies some property, say P, but possibly fails to exemplify P. That is, we consider what happens if we assert as axioms both that Pb and that $\Diamond\neg Pb$. Since Pb is a contingently true, it has to be asserted as a necessitation-averse axiom, on pain of contradiction. Now consider an abstract object, say a, that encodes exactly the properties that b in fact exemplifies. It will be provable that aP and, by (37), that $\Box aP$. But no modally strict proof of aP exists, since the only way to prove aP is by way of (i) a's description as an abstract object that encodes exactly the properties that b in fact exemplifies, and (ii) the contingent (and necessitation-averse) fact that Pb. Since any reasoning that concludes aP depends on the necessitation-averse axiom Pb, there is no modally strict proof of aP. So if we were to extend our system in the above way and let φ be aP, the converse of RN would fail.

Though this shows how the converse of RN fails if we extend our system with a specific contingent truth, item (185) concludes with a discussion of how we can show, without extending our system, that the converse of RN fails. Further discussion is therefore postponed until then.

9.4 The Theory of Negations and Conditionals

(**53**) **Theorems:** A Useful Fact. The following fact is derivable and is crucial to the proof of the Deduction Theorem:

$$\varphi \to \varphi$$

Although the notion of a *tautology* is a semantic notion and isn't officially defined in our formal system, we saw in Section 6.2 that the notion can be precisely defined if one takes on board the required semantic notions. It won't hurt, therefore, if we use the notion unofficially and label the above claim a tautology. Other tautologies will be derived below. As we will see, all tautologies are derivable, but it will be some time before we have assembled all the facts needed for the prove of this metatheoretic fact.

(**54**) **Metarule:** Deduction Theorem and Conditional Proof (CP). If there is a derivation of ψ from a set of premises Γ together with an additional premise φ, then there is a derivation of $\varphi \to \psi$ from Γ:

Rule CP
If $\Gamma, \varphi \vdash \psi$, then $\Gamma \vdash (\varphi \to \psi)$.

This rule is most-often used when $\Gamma = \varnothing$:

If $\varphi \vdash \psi$, then $\vdash \varphi \to \psi$.

When we cite this metarule in the proof of other metarules, we reference it as the *Deduction Theorem*. However, we shall adopt the following convention: during the course of reasoning, once we have produced a derivation of ψ from φ, we shall *infer* $\varphi \to \psi$ and cite *Conditional Proof* (CP), as opposed to concluding $\vdash \varphi \to \psi$ and citing the Deduction Theorem. The proof of the Deduction Theorem in the Appendix guarantees that we can indeed construct a proof of the conditional $\varphi \to \psi$ once we have derived ψ from φ.[101]

(**55**) **Metarules:** Corollaries to the Deduction Theorem. The following metarules are immediate consequences of the Deduction Theorem. They help us to prove the tautologies in (58) and (63). Recall that 'Γ_1, Γ_2' indicates '$\Gamma_1 \cup \Gamma_2$':

(.1) If $\Gamma_1 \vdash \varphi \to \psi$ and $\Gamma_2 \vdash \psi \to \chi$, then $\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$

(.2) If $\Gamma_1 \vdash \varphi \to (\psi \to \chi)$ and $\Gamma_2 \vdash \psi$, then $\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$

[101] Metatheorem ⟨6.7⟩, which is proved in the Appendix to Chapter 6, establishes that $\Gamma, \varphi \models \psi$ if and only if $\Gamma \models (\varphi \to \psi)$. Furthermore, Metatheorem ⟨6.8⟩, which is also proved in the Appendix to Chapter 6, establishes that $\varphi \models \psi$ if and only if $\models \varphi \to \psi$.

It is interesting that the above metarules have the following Variant forms, respectively:

(.3) $\varphi \to \psi,\ \psi \to \chi \vdash \varphi \to \chi$

(.4) $\varphi \to (\psi \to \chi),\ \psi \vdash \varphi \to \chi$

(55.3) is a Variant of (55.1) because we can derive each one from the other.[102] Similarly, (55.4) is a Variant of (55.2).

(**56**) **Derived Rules:** Hypothetical Syllogism. Note that the Variants (55.3) and (55.4) are somewhat different from the stated metarules (55.1) and (55.2): the Variants don't have the form of a conditional but instead simply assert the existence of a derivation. Of course they can be put into the traditional metarule form by conditionalizing them upon the triviality "If any condition holds" or "Under all conditions". But, given that these metarules hold without preconditions, we may transform them into *derived* rules of inference, i.e., rules of inference, like Modus Ponens, that can be used to infer formulas (as opposed to metarules, which only let us infer the existence of derivations):

(.1) $\varphi \to \psi,\ \psi \to \chi \,/\, \varphi \to \chi$ [**Hypothetical Syllogism**]

(.2) $\varphi \to (\psi \to \chi),\ \psi \,/\, \varphi \to \chi$

Thus, (55.3), for example, can be reconceived as a derived rule and not just a metarule. We may justifiably use this rule within derivations. The justification of (55.3) in the Appendix establishes that any derivation that yields a conclusion by an application of the above derived rule of Hypothetical Syllogism can be converted to a derivation in which this rule isn't used.

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[102] Here is a proof. ($\hookrightarrow$) Assume (55.1), i.e., if $\Gamma_1 \vdash \varphi \to \psi$ and $\Gamma_2 \vdash \psi \to \chi$, then $\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$. We want to show $\varphi \to \psi,\ \psi \to \chi \vdash \varphi \to \chi$. If we let Γ_1 be $\{\varphi \to \psi\}$, then since by (46.2) we know $\varphi \to \psi \vdash \varphi \to \psi$, we have $\Gamma_1 \vdash \varphi \to \psi$. By similar reasoning, if we let Γ_2 be $\{\psi \to \chi\}$, then we have $\Gamma_2 \vdash \psi \to \chi$. Hence, by (55.1), it follows that $\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$. But, this is just $\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$. ($\hookleftarrow$) Assume (55.3), i.e., $\varphi \to \psi,\ \psi \to \chi \vdash \varphi \to \chi$. Then by (46.7), it follows that:

$$\Gamma_1, \Gamma_2, \varphi \to \psi,\ \psi \to \chi \vdash \varphi \to \chi$$

From this, by two applications of the Deduction Theorem, we have:

(ϑ) $\Gamma_1, \Gamma_2 \vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

Now to show (55.1), assume $\Gamma_1 \vdash \varphi \to \psi$ and $\Gamma_2 \vdash \psi \to \chi$. So by (46.7), it follows, respectively, that:

(a) $\Gamma_1, \Gamma_2 \vdash \varphi \to \psi$

(b) $\Gamma_1, \Gamma_2 \vdash \psi \to \chi$

By (a) and (ϑ), it follows by (46.5) that:

(ξ) $\Gamma_1, \Gamma_2 \vdash (\psi \to \chi) \to (\varphi \to \chi)$

From (ξ) and (b), it follows by (46.5) that $\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$. ⋈

(**57**) **Remark:** Metarules vs. Derived Rules. In (55) we observed that some metarules have equivalent, variant versions that assert the existence of derivations without antecedent conditions. In (56), we observed that the variant versions could then be reconceived as *derived* rules instead of as metarules, and that one can count the justification of the metarule as a proof of the derived rule. This pattern will be repeated in this section; many of the metarules for reasoning with negation and conditionals have unconditional variants that will be regarded as derived rules. We shall formulate the metarule and its variants, and then leave the formulation of the derived rule as an obvious transformation of the variant. In the Appendix, however, we always reason with the *derived* form of the rule whenever it is available.

(**58**) **Theorems:** More Useful Tautologies. The tautologies listed below (and their proofs) follow the presentation in Mendelson 1997 (Lemma 1.11, pp. 38–40). We present them as a group because they are needed in the Appendix to this chapter to establish Lemma ⟨9.1⟩ and Metatheorem ⟨9.2⟩, i.e., that every tautology is derivable.

(.1) $\neg\neg\varphi \to \varphi$

(.2) $\varphi \to \neg\neg\varphi$

(.3) $\neg\varphi \to (\varphi \to \psi)$

(.4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

(.5) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

(.6) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$

(.7) $(\neg\varphi \to \psi) \to (\neg\psi \to \varphi)$

(.8) $\varphi \to (\neg\psi \to \neg(\varphi \to \psi))$

(.9) $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)$

(.10) $(\varphi \to \neg\psi) \to ((\varphi \to \psi) \to \neg\varphi)$

Note that (58.5) is used to prove Modus Tollens (59).

(**59**) **Metarules/Derived Rules:** Modus Tollens. We formulate Modus Tollens (MT) as two metarules:

Rules of Modus Tollen (MT)
(.1) If $\Gamma_1 \vdash (\varphi \to \psi)$ and $\Gamma_2 \vdash \neg\psi$, then $\Gamma_1, \Gamma_2 \vdash \neg\varphi$
(.2) If $\Gamma_1 \vdash (\varphi \to \neg\psi)$ and $\Gamma_2 \vdash \psi$, then $\Gamma_1, \Gamma_2 \vdash \neg\varphi$

By reasoning analogous to that in footnote 102, the following are equivalent, variant versions, respectively:

$$\varphi \to \psi, \neg\psi \vdash \neg\varphi$$
$$\varphi \to \neg\psi, \psi \vdash \neg\varphi$$

In light of Remark (57), these may be transformed into well-known derived rules.

(**60**) **Metarules/Derived Rules:** Contraposition. These metarules also come in two forms:

Rules of Contraposition
(.1) $\Gamma \vdash \varphi \to \psi$ if and only if $\Gamma \vdash \neg\psi \to \neg\varphi$
(.2) $\Gamma \vdash \varphi \to \neg\psi$ if and only if $\Gamma \vdash \psi \to \neg\varphi$

If we define $\chi \dashv\vdash \theta$ ('χ is interderivable with θ') to mean $\chi \vdash \theta$ and $\theta \vdash \chi$, then the equivalent, variant versions of (.1) and (.2) are, respectively:[103]

$$\varphi \to \psi \dashv\vdash \neg\psi \to \neg\varphi$$
$$\varphi \to \neg\psi \dashv\vdash \psi \to \neg\varphi$$

By Remark (57), we henceforth use the derived rules based on these variants.

(**61**) **Metarules/Derived Rules:** Reductio Ad Absurdum. Two classic forms of Reductio Ad Absurdum (RAA) are formulated as follows:

Rules of Reductio Ad Absurdum (RAA)
(.1) If $\Gamma_1, \neg\varphi \vdash \neg\psi$ and $\Gamma_2, \neg\varphi \vdash \psi$, then $\Gamma_1, \Gamma_2 \vdash \varphi$
(.2) If $\Gamma_1, \varphi \vdash \neg\psi$ and $\Gamma_2, \varphi \vdash \psi$, then $\Gamma_1, \Gamma_2 \vdash \neg\varphi$

The equivalent, variant versions are, respectively:[104]

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[103] Although the reasoning is again analogous to that in footnote 102, we show here the left-to-right direction of (.1) is equivalent to the variant $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$. ($\hookrightarrow$) Assume metarule (.1): if $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \neg\psi \to \neg\varphi$. Now let Γ be $\{\varphi \to \psi\}$. Then we have $\Gamma \vdash \varphi \to \psi$, by the special case of (46.2). But then it follows from our assumption that $\Gamma \vdash \neg\psi \to \neg\varphi$, i.e., $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$. ($\hookleftarrow$) Assume $\varphi \to \psi \vdash \neg\psi \to \neg\varphi$. Then by (46.7), it follows that $\Gamma, \varphi \to \psi \vdash \neg\psi \to \neg\varphi$. By the Deduction Theorem, it follows that $\Gamma \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$. But from this fact, we can derive $\Gamma \vdash \neg\psi \to \neg\varphi$ from the assumption that $\Gamma \vdash \varphi \to \psi$, by (46.5).

We leave the other direction, and the proof of the equivalence of (.2) and its variant, as exercises.

[104] We can show that (61.1) is equivalent to the variant as follows. ($\hookrightarrow$) Assume (61.1), i.e., if $\Gamma_1, \neg\varphi \vdash \neg\psi$ and $\Gamma_2, \neg\varphi \vdash \psi$, then $\Gamma_1, \Gamma_2 \vdash \varphi$. Now to derive the variant, note that if we let $\Gamma_1 = \{\neg\varphi \to \neg\psi\}$ and $\Gamma_2 = \{\neg\varphi \to \psi\}$, then we know by MP both that $\Gamma_1, \neg\varphi \vdash \neg\psi$ and $\Gamma_2, \neg\varphi \vdash \psi$. Hence by our assumption, $\Gamma_1, \Gamma_2 \vdash \varphi$. ($\hookleftarrow$) Assume the variant version, i.e., $\neg\varphi \to \neg\psi, \neg\varphi \to \psi \vdash \varphi$. By (46.7), it follows that:

$$\Gamma_1, \Gamma_2, \neg\varphi \to \neg\psi, \neg\varphi \to \psi \vdash \varphi$$

So by two applications of the Deduction Theorem, we know:

$$(\vartheta) \quad \Gamma_1, \Gamma_2 \vdash (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$$

Now to show (61.1), assume $\Gamma_1, \neg\varphi \vdash \neg\psi$ and $\Gamma_2, \neg\varphi \vdash \psi$. Then by (46.7), it follows, respectively, that $\Gamma_1, \Gamma_2, \neg\varphi \vdash \neg\psi$ and $\Gamma_1, \Gamma_2, \neg\varphi \vdash \psi$. By applying the Deduction Theorem to each of these, we obtain, respectively:

$$\neg\varphi \to \neg\psi,\ \neg\varphi \to \psi \vdash \varphi$$
$$\varphi \to \neg\psi,\ \varphi \to \psi \vdash \neg\varphi$$

We may therefore use the derived rules based on these variants, if needed.

(**62**) **Metarules/Derived Rules**. Alternative Forms of RAA. It is also useful to formulate Reductio Ad Absurdum in the following forms:

(.1) If $\Gamma, \varphi, \neg\psi \vdash \neg\varphi$, then $\Gamma, \varphi \vdash \psi$ [Variant: $\varphi, \neg\psi \to \neg\varphi \vdash \psi$]

(.2) If $\Gamma, \neg\varphi, \neg\psi \vdash \varphi$, then $\Gamma, \neg\varphi \vdash \psi$ [Variant: $\neg\varphi, \neg\psi \to \varphi \vdash \psi$]

(**63**) **Theorems:** Other Useful Tautologies. Since we not only have a standard axiomatization of negations and conditionals but also employ the standard definitions of the connectives &, $\vee$, and $\equiv$, many classical and other useful tautologies governing these connectives are derivable:

(.1) Principles of Noncontradiction:

(.a) $\neg(\varphi \,\&\, \neg\varphi)$

(.b) $\neg(\varphi \equiv \neg\varphi)$

(.2) Principle of Excluded Middle: $\varphi \vee \neg\varphi$

(.3) Idempotent, Commutative, and Associative Laws of &, $\vee$, and $\equiv$:

(.a) $(\varphi \,\&\, \varphi) \equiv \varphi$ (Idempotency of &)

(.b) $(\varphi \,\&\, \psi) \equiv (\psi \,\&\, \varphi)$ (Commutativity of &)

(.c) $(\varphi \,\&\, (\psi \,\&\, \chi)) \equiv ((\varphi \,\&\, \psi) \,\&\, \chi)$ (Associativity of &)

(.d) $(\varphi \vee \varphi) \equiv \varphi$ (Idempotency of $\vee$)

(.e) $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$ (Commutativity of $\vee$)

(.f) $(\varphi \vee (\psi \vee \chi)) \equiv ((\varphi \vee \psi) \vee \chi)$ (Associativity of $\vee$)

(.g) $(\varphi \equiv \psi) \equiv (\psi \equiv \varphi)$ (Commutativity of $\equiv$)

(.h) $(\varphi \equiv (\psi \equiv \chi)) \equiv ((\varphi \equiv \psi) \equiv \chi)$ (Associativity of $\equiv$)

(.4) Simple Biconditionals:

(.a) $\varphi \equiv \varphi$

(ξ) $\Gamma_1, \Gamma_2 \vdash \neg\varphi \to \neg\psi$

(ζ) $\Gamma_1, \Gamma_2 \vdash \neg\varphi \to \psi$

But from (ϑ) and (ξ), it follows by (46.5) that:

$$\Gamma_1, \Gamma_2 \vdash (\neg\varphi \to \psi) \to \varphi$$

And from this last conclusion and (ζ) it follows again by (46.5) that $\Gamma_1, \Gamma_2 \vdash \varphi$. ⋈ We leave the proof of the equivalence of (.2) and its variant as an exercise.

(.b) $\varphi \equiv \neg\neg\varphi$

(.5) Conditionals and Biconditionals:

(.a) $(\varphi \rightarrow \psi) \equiv \neg(\varphi \,\&\, \neg\psi)$

(.b) $\neg(\varphi \rightarrow \psi) \equiv (\varphi \,\&\, \neg\psi)$

(.c) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

(.d) $(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi)$

(.e) $(\varphi \equiv \psi) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \chi))$

(.f) $(\varphi \equiv \psi) \rightarrow ((\chi \rightarrow \varphi) \equiv (\chi \rightarrow \psi))$

(.g) $(\varphi \equiv \psi) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \chi))$

(.h) $(\varphi \equiv \psi) \rightarrow ((\chi \equiv \varphi) \equiv (\chi \equiv \psi))$

(.i) $(\varphi \equiv \psi) \equiv ((\varphi \,\&\, \psi) \vee (\neg\varphi \,\&\, \neg\psi))$

(.j) $\neg(\varphi \equiv \psi) \equiv ((\varphi \,\&\, \neg\psi) \vee (\neg\varphi \,\&\, \psi))$

(.k) $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$

(.6) De Morgan's Laws:

(.a) $(\varphi \,\&\, \psi) \equiv \neg(\neg\varphi \vee \neg\psi)$

(.b) $(\varphi \vee \psi) \equiv \neg(\neg\varphi \,\&\, \neg\psi)$

(.c) $\neg(\varphi \,\&\, \psi) \equiv (\neg\varphi \vee \neg\psi)$

(.d) $\neg(\varphi \vee \psi) \equiv (\neg\varphi \,\&\, \neg\psi)$

(.7) Distribution Laws:

(.a) $(\varphi \,\&\, (\psi \vee \chi)) \equiv ((\varphi \,\&\, \psi) \vee (\varphi \,\&\, \chi))$

(.b) $(\varphi \vee (\psi \,\&\, \chi)) \equiv ((\varphi \vee \psi) \,\&\, (\varphi \vee \chi))$

(.8) Exportation and Importation:

(.a) $((\varphi \,\&\, \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$ (Exportation)

(.b) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \,\&\, \psi) \rightarrow \chi)$ (Importation)

(.9) Conjunction Simplification:

(.a) $(\varphi \,\&\, \psi) \rightarrow \varphi$

(.b) $(\varphi \,\&\, \psi) \rightarrow \psi$

(.10) Other Miscellaneous Tautologies:

(.a) $\varphi \rightarrow (\psi \rightarrow (\varphi \,\&\, \psi))$ (Adjunction)

(.b) $(\varphi \rightarrow (\psi \rightarrow \chi)) \equiv (\psi \rightarrow (\varphi \rightarrow \chi))$ (Permutation)

(.c) $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \,\&\, \chi)))$ (Composition)

(.d) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$

(.e) $(\varphi \to \psi) \to ((\chi \to \theta) \to ((\varphi \,\&\, \chi) \to (\psi \,\&\, \theta)))$ (Double Composition)

(.f) $((\varphi \,\&\, \psi) \equiv (\varphi \,\&\, \chi)) \equiv (\varphi \to (\psi \equiv \chi))$

(.g) $((\varphi \,\&\, \psi) \equiv (\chi \,\&\, \psi)) \equiv (\psi \to (\varphi \equiv \chi))$

We leave the proof of these tautologies as exercises.

(**64**) **Metarules/Derived Rules:** The Classical Introduction and Elimination Rules. Our standard axiomatization of negation and conditionalization and standard definitions of the connectives &, ∨, and ≡ allow us to reason using all the *classical* introduction and elimination rules. However, we formulate them, in the first instance, as metarules.

Note that the metarules for the introduction and elimination of → and ¬ have already been presented. (46.5) is the metarule for →Elimination (→E) and the Deduction Theorem is the metarule for →Introduction (→I). Reductio Ad Absurdum, when formulated as in (61.1), is a metarule for ¬Elimination (¬E), and when formulated as in (61.2), is a metarule for ¬Introduction (¬I). So we formulate below the introduction and elimination metarules for &, ∨, ≡, and double negation. We also state the variant versions in each case, though we leave the proof that they are equivalent to the stated metarules for the reader. We also assume that the variant metarules can be transformed into derived rules (in which / has been substituted for ⊢), in the manner described in Remark (57).

(.1) &Introduction (&I):

If $\Gamma_1 \vdash \varphi$ and $\Gamma_2 \vdash \psi$, then $\Gamma_1, \Gamma_2 \vdash \varphi \,\&\, \psi$ [Variant: $\varphi, \psi \vdash \varphi \,\&\, \psi$]

(.2) &Elimination (&E):

(.a) If $\Gamma \vdash \varphi \,\&\psi$, then $\Gamma \vdash \varphi$ [Variant: $\varphi \,\&\, \psi \vdash \varphi$]

(.b) If $\Gamma \vdash \varphi \,\&\psi$, then $\Gamma \vdash \psi$ [Variant: $\varphi \,\&\, \psi \vdash \psi$]

(.3) ∨Introduction (∨I):

(.a) If $\Gamma \vdash \varphi$, then $\Gamma \vdash \varphi \vee \psi$ [Variant: $\varphi \vdash \varphi \vee \psi$]

(.b) If $\Gamma \vdash \psi$, then $\Gamma \vdash \varphi \vee \psi$ [Variant: $\psi \vdash \varphi \vee \psi$]

(.4) ∨Elimination (∨E):

(.a) Reasoning by Cases:
If $\Gamma_1 \vdash \varphi \vee \psi$, $\Gamma_2 \vdash \varphi \to \chi$, and $\Gamma_3 \vdash \psi \to \chi$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \chi$
[Variant: $\varphi \vee \psi,\ \varphi \to \chi,\ \psi \to \chi \vdash \chi$]

(.b) Disjunctive Syllogism:
If $\Gamma_1 \vdash \varphi \vee \psi$ and $\Gamma_2 \vdash \neg\varphi$, then $\Gamma_1, \Gamma_2 \vdash \psi$ [Variant: $\varphi \vee \psi$, $\neg\varphi \vdash \psi$]

(.c) Disjunctive Syllogism:
If $\Gamma_1 \vdash \varphi \vee \psi$ and $\Gamma_2 \vdash \neg\psi$, then $\Gamma_1, \Gamma_2 \vdash \varphi$ [Variant: $\varphi \vee \psi$, $\neg\psi \vdash \varphi$]

(.d) Disjunctive Syllogism:
If $\Gamma_1 \vdash \varphi \vee \psi$, $\Gamma_2 \vdash \varphi \rightarrow \chi$, and $\Gamma_3 \vdash \psi \rightarrow \theta$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \chi \vee \theta$
[Variant: $\varphi \vee \psi$, $\varphi \rightarrow \chi$, $\psi \rightarrow \theta \vdash \chi \vee \theta$]

(.e) Disjunctive Syllogism:
If $\Gamma_1 \vdash \varphi \vee \psi$, $\Gamma_2 \vdash \varphi \equiv \chi$, and $\Gamma_3 \vdash \psi \equiv \theta$, then $\Gamma_1, \Gamma_2, \Gamma_3 \vdash \chi \vee \theta$
[Variant: $\varphi \vee \psi$, $\varphi \equiv \chi$, $\psi \equiv \theta \vdash \chi \vee \theta$]

(.5) $\equiv$Introduction ($\equiv$I):

If $\Gamma_1 \vdash \varphi \rightarrow \psi$ and $\Gamma_2 \vdash \psi \rightarrow \varphi$, then $\Gamma_1, \Gamma_2 \vdash \varphi \equiv \psi$
[Variant: $\varphi \rightarrow \psi$, $\psi \rightarrow \varphi \vdash \varphi \equiv \psi$]

(.6) $\equiv$Elimination ($\equiv$E) (Biconditional Syllogisms):

(.a) If $\Gamma_1 \vdash \varphi \equiv \psi$ and $\Gamma_2 \vdash \varphi$, then $\Gamma_1, \Gamma_2 \vdash \psi$ [Variant: $\varphi \equiv \psi$, $\varphi \vdash \psi$]

(.b) If $\Gamma_1 \vdash \varphi \equiv \psi$ and $\Gamma_2 \vdash \psi$, then $\Gamma_1, \Gamma_2 \vdash \varphi$ [Variant: $\varphi \equiv \psi$, $\psi \vdash \varphi$]

(.c) If $\Gamma_1 \vdash \varphi \equiv \psi$ and $\Gamma_2 \vdash \neg\varphi$, then $\Gamma_1, \Gamma_2 \vdash \neg\psi$
[Variant: $\varphi \equiv \psi$, $\neg\varphi \vdash \neg\psi$]

(.d) If $\Gamma_1 \vdash \varphi \equiv \psi$ and $\Gamma_2 \vdash \neg\psi$, then $\Gamma_1, \Gamma_2 \vdash \neg\varphi$
[Variant: $\varphi \equiv \psi$, $\neg\psi \vdash \neg\varphi$]

(.e) If $\Gamma_1 \vdash \varphi \equiv \psi$ and $\Gamma_2 \vdash \psi \equiv \chi$, then $\Gamma_1, \Gamma_2 \vdash \varphi \equiv \chi$
[Variant: $\varphi \equiv \psi$, $\psi \equiv \chi \vdash \varphi \equiv \chi$]

(.f) If $\Gamma_1 \vdash \varphi \equiv \psi$ and $\Gamma_2 \vdash \varphi \equiv \chi$, then $\Gamma_1, \Gamma_2 \vdash \chi \equiv \psi$
[Variant: $\varphi \equiv \psi$, $\varphi \equiv \chi \vdash \chi \equiv \psi$]

(.7) Double Negation Introduction ($\neg\neg$I):

If $\Gamma \vdash \varphi$, then $\Gamma \vdash \neg\neg\varphi$ [Variant: $\varphi \vdash \neg\neg\varphi$]

(.8) Double Negation Elimination ($\neg\neg$E):

If $\Gamma \vdash \neg\neg\varphi$, then $\Gamma \vdash \varphi$ [Variant: $\neg\neg\varphi \vdash \varphi$]

We leave the justification of these metarules and their variants as exercises and henceforth use the corresponding derived rules within proofs and derivations.

(**65**) **Remark:** Not All Tautologies Are Yet Derivable. Rule MP and our axioms (21.1) – (21.3) for negations and conditionals are not yet sufficient for deriving all of the formulas that qualify as tautologies, as the latter notion was defined in Section 6.2. We discovered in that section that our system contains a new class of tautologies that arise in connection with 0-place relation terms of the form $[\lambda\, \varphi^*]$. Instances of the following schemata are members of this new class of tautologies: $[\lambda\, \varphi^*] \to \varphi^*$, $[\lambda\, \varphi^*] \equiv \varphi^*$, $[\lambda\, \varphi^*] \to \neg\neg\varphi^*$, etc. To derive these tautologies, we must first prove that $[\lambda\, \varphi^*] \equiv \varphi^*$ is a theorem (131.2), and to do that, we will need to show that $[\lambda\, \varphi^*] = \varphi^*$ is a theorem (131.1). The derivations of these latter theorems appeal to η-Conversion, GEN, Rule $\forall$E (a rule of quantification theory derived in item (77) below), and Rule SubId (i.e., the rule of substitution of identicals, derived in item (74.2) below). Once we've derived all of these key principles, and (131.2) in particular, we will be in a position to prove Metatheorem ⟨9.2⟩, i.e., that all tautologies are derivable. This Metatheorem is proved in the Appendix to this chapter. With such a result, we can derive Rule T, which is formulable using the semantic notions defined in Section 6.2, as a rule for our system:

> **Rule T**
> If $\Gamma \vdash \varphi_1$ and ... and $\Gamma \vdash \varphi_n$, then if $\{\varphi_1, \ldots, \varphi_n\}$ tautologically implies ψ, then $\Gamma \vdash \psi$.

Rule T asserts that ψ is derivable from Γ whenever the formulas of which it is a tautological consequence are all derivable from Γ. We won't use this rule in proving theorems, since it requires semantic notions. But it is a valid shortcut. Rule T is proved as Metatheorem ⟨9.4⟩ in the Appendix to this chapter.

9.5 The Theory of Identity

(**66**) **Theorems:** Necessarily, Every Individual and Relation Exists, and Necessarily Exists. Where α and β are both variables of the same type, it is a consequence of our axioms and rules that:

(.1) $\forall\alpha\exists\beta(\beta = \alpha)$

(.2) $\Box\exists\beta(\beta = \alpha)$

(.3) $\Box\forall\alpha\exists\beta(\beta = \alpha)$

(.4) $\forall\alpha\Box\exists\beta(\beta = \alpha)$

(.5) $\Box\forall\alpha\Box\exists\beta(\beta = \alpha)$

Note that when α and β are the individual variables x and y, respectively, then given one standard reading of the quantifiers, the above assert that: (.1) every individual exists; (.2) necessarily x exists; (.3) necessarily, every individual exists; (.4) every individual necessarily exists; and (.5) necessarily, every individual necessarily exists. We get corresponding readings when α and β are relation terms of the same arity. Clearly, on these readings, the symbol $\exists$ is being used to assert the logical existence, and not the physical existence, of the entities in question.

(**67**) **Theorems:** Identity for Properties, Relations, and Propositions is Classical. Our axioms and definitions imply that identity for properties, relations and propositions is reflexive, symmetric, and transitive.

(.1) $F^1 = F^1$

(.2) $F^1 = G^1 \rightarrow G^1 = F^1$

(.3) $F^1 = G^1 \;\&\; G^1 = H^1 \rightarrow F^1 = H^1$

(.4) $F^n = F^n$ $(n \geq 2)$

(.5) $F^n = G^n \rightarrow G^n = F^n$ $(n \geq 2)$

(.6) $F^n = G^n \;\&\; G^n = H^n \rightarrow F^n = H^n$ $(n \geq 2)$

(.7) $p = p$

(.8) $p = q \rightarrow q = p$

(.9) $p = q \;\&\; q = r \rightarrow p = r$

(**68**) **Metarule/Derived Rule:** Substitution of Alphabetically-Variant Relation Terms. Our principles of α-Conversion (36.1) and substitution of identicals (25) allow us to formulate a new metarule of inference: if we can derive a fact about a complex relation term τ from some premises Γ, then for any alphabetic variant τ' of τ, a corresponding fact about τ' can be derived from Γ:

This applies to logically proper relation terms τ, i.e., $\vdash \exists\beta(\beta = \tau)$

Substitution of Alphabetically-Variant Relation Terms[105]
Where (a) τ is any complex n-place relation term ($n \geq 0$), (b) τ' is an alphabetic variant of τ, (c) τ and τ' are both substitutable for the n-place relation variable α in φ, and (d) φ' is the result of substituting τ' for zero or more occurrences of τ in φ^τ_α, then if $\Gamma \vdash \varphi^\tau_\alpha$, then $\Gamma \vdash \varphi'$.

[Variant: $\varphi^\tau_\alpha \vdash \varphi'$]

As a simple example of the Variant version, consider:

$\Box[\lambda x\, \Diamond E!x]a \vdash \Box[\lambda y\, \Diamond E!y]a$

[105] ($\hookrightarrow$) To see that the stated version implies the Variant version, assume the stated version, i.e., that if $\Gamma \vdash \varphi^\tau_\alpha$, then $\Gamma \vdash \varphi'$. Then to see that the Variant version holds, note that we have as a special case of (46.4) that $\varphi^\tau_\alpha \vdash \varphi^\tau_\alpha$. But if we let $\Gamma = \{\varphi^\tau_\alpha\}$, then our stated version has the instance: if $\varphi^\tau_\alpha \vdash \varphi^\tau_\alpha$, then $\varphi^\tau_\alpha \vdash \varphi'$. Hence $\varphi^\tau_\alpha \vdash \varphi'$. ($\hookleftarrow$) Conversely, to see that the Variant version implies the stated version, assume the Variant version, i.e., $\varphi^\tau_\alpha \vdash \varphi'$. Now assume the antecedent of the stated version, i.e., $\Gamma \vdash \varphi^\tau_\alpha$. Then it follows by (46.8) that $\Gamma \vdash \varphi'$.

In this example:

$$\begin{array}{lcl} \varphi & = & \Box Fa \\ \alpha & = & F \\ \tau & = & [\lambda x\, \Diamond E!x] \\ \tau' & = & [\lambda y\, \Diamond E!y] \\ \varphi^{\tau}_{\alpha} & = & \Box[\lambda x\, \Diamond E!x]a \\ \varphi' & = & \Box[\lambda y\, \Diamond E!y]a \end{array}$$

In the manner of (57), we can transform the variant version into a derived rule and appeal to the latter in the course of proving theorems.

(**69**) **Theorems:** Useful Theorems About Identity$_E$ and Identity. The following are simple, but useful theorems: (.1) $x =_E y$ if and only if x exemplifies being ordinary, y exemplifies being ordinary, and x and y necessarily exemplify the same properties; (.2) whenever objects are identical$_E$, they are identical; and (.3) whenever objects are identical, then either they are both ordinary objects that necessarily exemplify the same properties or they are both abstract objects that encode the same properties:

(.1) $x =_E y \equiv (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$

(.2) $x =_E y \to x = y$

(.3) $x = y \equiv [(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))]$

The proof of (69.1) can be shortened once we establish the strengthened principle of β-Conversion (128). But at present, we haven't established the lemmas needed for its proof. So we have developed a proof of (69.1) in strict form, since this fact is needed for the theorem that follows (70.1).

(**70**) **Theorems:** Identity for Objects is an Equivalence Condition. Our axioms and definitions imply that identity for objects is reflexive, symmetric, and transitive:

(.1) $x = x$

(.2) $x = y \to y = x$

(.3) $x = y \,\&\, y = z \to x = z$

Together with the subsitutiton of identicals, these guarantee that identity is classical.

(**71**) **Theorems:** General Identity is an Equivalence Condition. We have now established that identity for relations (67) and for objects (70) is an equivalence condition. We may represent each group of facts in terms of single schemata, where α, β, γ are any three distinct variables of the same type:

(.1) $\alpha = \alpha$

(.2) $\alpha = \beta \rightarrow \beta = \alpha$

(.3) $\alpha = \beta \;\&\; \beta = \gamma \rightarrow \alpha = \gamma$

Though these claims would remain true even if the variables aren't all distinct, they wouldn't express the idea that identity is reflexive, symmetric and transitive, respectively.

(72) **Theorems:** Self-Identity and Necessity. It is a consequence of the foregoing that (.1) necessarily everything is self-identical, and that (.2) everything is necessarily self-identical. Where α is any variable:

(.1) $\Box\forall\alpha(\alpha = \alpha)$

(.2) $\forall\alpha\Box(\alpha = \alpha)$

These well-known principles of self-identity and necessity are thus provable.

(73) **Theorems:** Term Identities Imply Logical Propriety. For any terms τ and τ', if $\tau = \tau'$, then both τ and τ' are logically proper:

(.1) $\tau = \tau' \rightarrow \exists\beta(\beta = \tau)$, provided β isn't free in τ

(.2) $\tau = \tau' \rightarrow \exists\beta(\beta = \tau')$, provided β isn't free in τ'

where τ is an term other than an improper λ-expression; see p. 191

Note that this theorem holds even when τ or τ' is a definite description. If they appear in a true identity statement, then they are logically proper.

(74) **Metarules/Derived Rules:** Rules for Reasoning with Identity. We now formulate two metarules: one for the reflexivity of identity and one for the substitution of identicals:

(.1) **Rule for the Reflexivity of Identity** (**Rule ReflId**)
$\vdash \tau = \tau$, where τ is any term other than a description. *and other than an improper λ-expression; see p. 191*

(.2) **Rule of Substitution for Identicals** (**Rule SubId**)
If $\Gamma_1 \vdash \varphi^{\tau}_{\alpha}$ and $\Gamma_2 \vdash \tau = \tau'$, then $\Gamma_1, \Gamma_2 \vdash \varphi'$, whenever τ and τ' are any terms substitutable for α in φ, and φ' is the result of replacing zero or more occurrences of τ in φ^{τ}_{α} with occurrences of τ'. [Variant: $\varphi^{\tau}_{\alpha}, \tau = \tau' \vdash \varphi'$]

Since Rule ReflId requires no special conditions of application, we may regard it as a derived rule that asserts: $\tau = \tau$, for any term τ other than a description, is a theorem. The Variant of Rule SubId can also be converted to a derived rule: φ' follows from φ^{τ}_{α} and $\tau = \tau'$, where φ' is the result of replacing zero or more occurrences of τ in φ^{τ}_{α} with occurrences of τ'.

Note that in Rule SubId, τ and τ' might both be definite descriptions: as long as $\tau = \tau'$ is an assumption, one may substitute τ' for τ in φ even when one or both of τ, τ' is a description. Under the assumption that $\tau = \tau'$, both terms are logically proper, as theorems (73.1) and (73.2) establish, and so one may freely substitute τ and τ' for one another in that context.

Note also that when φ' is the result of replacing *all* of the occurrences of τ in φ^{τ}_{α} by τ', then φ' just is $\varphi^{\tau'}_{\alpha}$ and we have the following special case of Rule SubId:

(.3) **Rule SubId Special Case**
If $\Gamma_1 \vdash \varphi^{\tau}_{\alpha}$ and $\Gamma_2 \vdash \tau = \tau'$, then $\Gamma_1, \Gamma_2 \vdash \varphi^{\tau'}_{\alpha}$ [Variant: $\varphi^{\tau}_{\alpha}, \tau = \tau' \vdash \varphi^{\tau'}_{\alpha}$]

(**75**) **Theorems:** Identity and Necessity. Where α, β are both variables of the same type, we can establish that α and β are identical if and only if it is necessary that they are identical:

$$\alpha = \beta \equiv \Box \alpha = \beta$$

While the left-to-right direction requires reasoning by cases, the right-to-left direction of the above follows immediately by the T schema (32.2).

The left-to-right direction of this theorem is the famous *necessity of identity* principle (Kripke 1971); it goes further than Kripke's principle since it governs not only the identity of individuals but also the identity of relations. We've already seen that the *definitions* of object identity (15) and relation identity (16) ground the reflexivity of identity (71.1). The reflexivity of identity is one of the key facts used in the proof of the necessity of identity principle. Thus, our proof of this principle is derived within a theory in which identity is not a primitive; cf. Kripke 1971.

(**76**) **Theorems:** Identity, Necessity, and Descriptions. It is an interesting fact that the necessity of identity holds even when objects are described:

$$\imath x\varphi = \imath y\psi \equiv \Box \imath x\varphi = \imath y\psi$$

Notice that the theorem is not restricted to logically proper descriptions. To prove each direction of the biconditional, we need only consider the case where the antecedent of that direction is true. However, in the case where the descriptions fail to be logically proper, both sides of the biconditional are false. Thus, the necessity of identity principle and its converse, which are combined in (75), apply to every pair of terms of the same type.

9.6 The Theory of Quantification

(**77**) **Metarules/Derived Rules:** $\forall$Elimination ($\forall$E). The elimination rule for the universal quantifier has two forms (with the first being the primary form): (.1) legitimizes the instantiation of any term τ with the same type as the quantified variable provided τ is logically proper, while (.2) states that every term of the same type as the quantified variable, other than a description, is instantiable:

Rule $\forall$E
(.1) If $\Gamma_1 \vdash \forall\alpha\varphi$ and $\Gamma_2 \vdash \exists\beta(\beta = \tau)$, then $\Gamma_1, \Gamma_2 \vdash \varphi^\tau_\alpha$, provided τ is substitutable for α in φ [Variant: $\forall\alpha\varphi, \exists\beta(\beta = \tau) \vdash \varphi^\tau_\alpha$]

(.2) If $\Gamma \vdash \forall\alpha\varphi$, then $\Gamma \vdash \varphi^\tau_\alpha$, provided τ is substitutable for α in φ and τ is not a description [Variant: $\forall\alpha\varphi \vdash \varphi^\tau_\alpha$]

In the usual manner, we may convert the variants into the derived rules of $\forall$E and use them to produce genuine derivations.

Rule (77.2) and its Variant have special cases when τ is the variable α. Since α is not a description and is substitutable for itself in any formula φ with the result that $\varphi^\alpha_\alpha = \varphi$, the following special cases obtain:

- If $\Gamma \vdash \forall\alpha\varphi$, then $\Gamma \vdash \varphi$
- $\forall\alpha\varphi \vdash \varphi$

(**78**) **Remark:** A Misuse of Rule $\forall$E. Note that the following attempt to derive a contradiction involves a misuse of Rule $\forall$E. Suppose we let φ be the formula $\neg yF$ and formulate the following instance of Comprehension for Abstract Objects:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \neg yF))$$

So by GEN, we may derive as a theorem:

$$(\vartheta)\quad \forall y \exists x(A!x \,\&\, \forall F(xF \equiv \neg yF))$$

Since the variable x is not a description, one may now be tempted to instantiate the quantifier $\forall y$ in (ϑ) to x, by applying Rule $\forall$E Variant (77.2), to obtain:

$$(\xi)\quad \exists x(A!x \,\&\, \forall F(xF \equiv \neg xF))$$

This is easily shown to be a contradiction, for assume a is an arbitrary such object, so that we know $A!a \,\&\, \forall F(aF \equiv \neg aF)$. By &E, it follows that $\forall F(aF \equiv \neg aF)$. Now for any property you pick, say P, it follows that $aP \equiv \neg aP$, which by (63.1.b) is a contradiction.

What prevents such reasoning within our system is the fact that Rule $\forall$E Variant (77.2) has been incorrectly applied in the move from (ϑ) to (ξ). The rule states: $\forall\alpha\psi \vdash \psi^\tau_\alpha$, provided τ is substitutable for α in ψ and τ is not a description. In the present case, α is y, τ is x, (ϑ) has the form $\forall y\psi$, where $\psi = \exists x(A!x \,\&\, \forall F(xF \equiv \neg yF))$, and ($\xi$) has the form ψ^x_y. The move from (ϑ) to (ξ) obeys the condition that x not be a description, but it doesn't obey the condition that x be substitutable for y in ψ. The definition of *substitutable for* in (24) requires that for x to be substitutable for y in ψ, every variable that occurs free in the term x must remain free after we substitute x for y in ψ. But x is itself a variable that is free in the term x, yet it doesn't remain free when x is substituted for y in ψ (i.e., to produce ψ^x_y). Instead, x is captured by (i.e., falls within the scope of) the existential quantifier $\exists x$ in ψ^x_y. So the above reasoning fails to correctly apply (77.2) in the move from (ϑ) to (ξ).

(**79**) **Theorems:** Classical Quantifier Axioms as Theorems. Our principles yield two classical quantifier axioms as theorems:

(.1) $\forall\alpha\varphi \to \varphi^\tau_\alpha$, provided τ is substitutable for α in φ and is not a description

(.2) $\forall\alpha(\varphi \to \psi) \to (\varphi \to \forall\alpha\psi)$, provided α is not free in φ

These are used in Mendelson 1997, for example, as the two principal axioms of classical quantification theory (his formulation of predicate logic with function terms assumes that all terms have a denotation). Also, given what was said above about φ^α_α, formulas of the form:

(.3) $\forall\alpha\varphi \to \varphi$

are special cases of theorem (79.1).

(**80**) **Metarule:** Generalization or Universal Introduction ($\forall$I) on Constants. We introduce and explain the $\forall$Introduction Rule for Constants by way of analogy. One simple way to establish $\forall F\forall x(Fx \equiv Fx)$ is to argue that it follows by two applications of GEN from $Fx \equiv Fx$, where the latter is a theorem by being an instance of the tautology $\varphi \equiv \varphi$ (63.4.a). However, we often argue as follows: assume P is an arbitrary property and a an arbitrary object; then $Pa \equiv Pa$ is an instance of our tautology; since P and a are arbitrary, the claim holds for all properties and objects; hence, $\forall F\forall x(Fx \equiv Fx)$. In this reasoning, universal generalization is permissible in the last step because in the course of the reasoning, we haven't invoked any special assumptions about the arbitrarily chosen entities; no special facts about the property P or the individual a played a part in our conclusion that $Pa \equiv Pa$.

To formulate Rule $\forall$I generally, we first introduce some notation. Where τ is any constant and α any variable of the same type as τ:

- φ^α_τ is the result of replacing every occurrence of the constant τ in φ by an occurrence of α

We then have:

Rule $\forall$I
If $\Gamma \vdash \varphi$ and τ is a constant that does not occur in Γ or $\mathbf{\Lambda}$, then $\Gamma \vdash \forall\alpha\varphi^\alpha_\tau$, provided α is a variable that does not occur in φ.

Note that our axioms in $\mathbf{\Lambda}$ count as special facts about the named individuals or relations (such as $E!$) that might appear in them, and so the names of those individuals and relations are not acceptable instances of τ.

Here is an example of how we will use this rule. Consider the following reasoning that shows $\forall x(Px \to Qx), \forall y Py \vdash \forall x Qx$:

1.	$\forall x(Px \to Qx)$	Premise
2.	$\forall y Py$	Premise
3.	$Pa \to Qa$	$\forall$E, 1
4.	Pa	$\forall$E, 2
5.	Qa	MP, 3, 4
6.	$\forall x(Px \to Qx), \forall y Py \vdash \forall x Qx$	$\forall$I, 1–5

In this example, we set $\Gamma = \{\forall x(Px \to Qx), \forall y Py\}$, $\varphi = Qa$, and $\tau = a$. Given that $\forall$E is a derived rule and not just a metarule, lines 1–5 constitute a genuine derivation that is a witness to $\Gamma \vdash \varphi$. Since a doesn't occur in Γ and x doesn't occur in φ, we have an instance of the Rule $\forall$I in which α is the variable x, which we can then apply to lines 1–5 to infer the derivability claim on line 6.

Since $\forall$I is a metarule, we could have reached the conclusion on line 6 without it using the following reasoning, which doesn't involve the constant a:

1.	$\forall x(Px \to Qx)$	Premise
2.	$\forall y Py$	Premise
3.	$Px \to Qx$	$\forall$E, 1
4.	Px	$\forall$E, 2
5.	Qx	MP, 3, 4
6.	$\forall x(Px \to Qx),\ \forall y Py\ \vdash\ \forall x Qx$	GEN, 1–5

The application of GEN on line 6 is legitimate since we have legitimately derived Qx on line 5 from the premises $\forall x(Px \to Qx)$ and $\forall y Py$ and the variable x doesn't occur free in the premises. Of course, GEN itself is a metarule, but we already know how to eliminate it.

(**81**) **Lemmas:** Re-replacement Lemmas. In the following re-replacement lemmas, we assume that α, β, and τ are all of the same type:

(.1) If β is substitutable for α in φ and β doesn't occur free in φ, then α is substitutable for β in φ^{β}_{α} and $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta} = \varphi$.

(.2) If τ is a constant symbol that doesn't occur in φ, then $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau} = \varphi^{\beta}_{\alpha}$.

(.3) If β is substitutable for α in φ and doesn't occur free in φ, and τ is any term substitutable for α in φ, then $(\varphi^{\beta}_{\alpha})^{\tau}_{\beta} = \varphi^{\tau}_{\alpha}$.

It may help to read the following Remark before attempting to prove the above.

(**82**) **Remark:** Explanation of the Re-replacement Lemmas. By discussing (81.1) in some detail, (81.2) and (81.3) become more transparent and less in need of commentary. It is relatively easy to show that, in general, in the absence of any preconditions, $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta} \neq \varphi$. The variable α may occur in $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta}$ at a place where it does not occur in φ, and α may occur in φ at a place where it does not occur in $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta}$. Here is an example of each case:

- $\varphi = Ryx$. Then $\varphi^{y}_{x} = Ryy$ and though x is substitutable for y in φ^{y}_{x} (y doesn't fall under the scope of any variable-binding operator that binds x), $(\varphi^{y}_{x})^{x}_{y} = Rxx$. Hence $(\varphi^{y}_{x})^{x}_{y} \neq \varphi$. In this example, x occurs at a place in $(\varphi^{y}_{x})^{x}_{y}$ where it does not occur in φ.

- $\varphi = \forall yRxy$. Then $\varphi^{y}_{x} = \forall yRyy$. Since x is trivially substitutable for y in φ^{y}_{x} (there are no free occurrences of y in φ^{y}_{x}), $(\varphi^{y}_{x})^{x}_{y} = \varphi^{y}_{x} = \forall yRyy$. By inspection, then, $(\varphi^{y}_{x})^{x}_{y} \neq \varphi$. In this example, x occurs at a place in φ where it does not occur in $(\varphi^{y}_{x})^{x}_{y}$.

These two examples nicely demonstrate why the two antecedents of (81.1) are crucial. The first example fails the proviso that y not occur free in Ryx; the second example fails the proviso that y be substitutable for x in $\forall yRxy$. But here is an example of (81.1) in which the antecedents obtain:

- $\varphi = \forall yPy \rightarrow Qx$. In this example, the free occurrence of x is not within the scope of the quantifier $\forall y$. So y is substitutable for x in φ and y does not occur free in φ. Thus, $\varphi^{y}_{x} = \forall yPy \rightarrow Qy$, and since y has a free occurrence in φ^{y}_{x} not under the scope of a variable-binding operator binding x, x is substitutable for y in φ^{y}_{x}. Hence $(\varphi^{y}_{x})^{x}_{y} = \forall yPy \rightarrow Qx$, and so $(\varphi^{y}_{x})^{x}_{y} = \varphi$.

These remarks and the proof of (81.1) should suffice to clarify the remaining two replacement lemmas. (81.1) is used to prove (83.12), (86.10), and the Rule of Alphabetic Variants (115). Lemma (81.3) is used in the proof of (86.8).

(**83**) **Theorems:** Basic Theorems of Quantification Theory. The following are all basic consequences of our quantifier axioms and (derived) rules:

(.1) $\forall\alpha\forall\beta\varphi \equiv \forall\beta\forall\alpha\varphi$

(.2) $\forall\alpha(\varphi \equiv \psi) \equiv (\forall\alpha(\varphi \rightarrow \psi)\ \&\ \forall\alpha(\psi \rightarrow \varphi))$

(.3) $\forall\alpha(\varphi \equiv \psi) \rightarrow (\forall\alpha\varphi \equiv \forall\alpha\psi)$

(.4) $\forall\alpha(\varphi\ \&\ \psi) \equiv (\forall\alpha\varphi\ \&\ \forall\alpha\psi)$

(.5) $\forall\alpha_1 \ldots \forall\alpha_n\varphi \rightarrow \varphi$

(.6) $\forall\alpha\forall\alpha\varphi \equiv \forall\alpha\varphi$

(.7) $(\varphi \rightarrow \forall\alpha\psi) \equiv \forall\alpha(\varphi \rightarrow \psi)$, provided α is not free in φ

(.8) $(\forall\alpha\varphi \vee \forall\alpha\psi) \rightarrow \forall\alpha(\varphi \vee \psi)$

(.9) $(\forall\alpha(\varphi \rightarrow \psi) \,\&\, \forall\alpha(\psi \rightarrow \chi)) \rightarrow \forall\alpha(\varphi \rightarrow \chi)$

(.10) $(\forall\alpha(\varphi \equiv \psi) \,\&\, \forall\alpha(\psi \equiv \chi)) \rightarrow \forall\alpha(\varphi \equiv \chi)$

(.11) $\forall\alpha(\varphi \equiv \psi) \equiv \forall\alpha(\psi \equiv \varphi)$

(.12) $\forall\alpha\varphi \equiv \forall\beta\varphi^{\beta}_{\alpha}$, provided β is substitutable for α in φ and doesn't occur free in φ

The two provisos on (83.12) can be explained by referencing and adapting the examples used in Remark (82) that helped us to understand the antecedent of the Re-replacement Lemma (81.1):

- In the formula $\varphi = Rxy$, y is substitutable for x but also occurs free. Without the second proviso in (83.12), we could set α to x and β to y and obtain the instance: $\forall xRxy \equiv \forall yRyy$. Clearly, this is not valid: the left side asserts that everything bears R to y while the right asserts that everything bears R to itself.

- In the formula $\varphi = \forall yRxy$, y is not substitutable for x even though it does not occur free. Without the first proviso in (83.12), we could set α to x and β to y and obtain the instance: $\forall x\forall yRxy \equiv \forall y\forall yRyy$. Again, clearly, this is not valid: the left side is true when everything bears R to everything while the right side, which by (83.6) is equivalent to $\forall yRyy$, is true only when everything bears R to itself.

(83.12) is a special case of the interderivability of alphabetic variants; indeed, it is a special case of a special case. The interderivability of alphabetically-variant universal generalizations is a special case of the interderivability of alphabetically-variant formulas of arbitrary complexity. But within that special case, there are two basic ways in which a universal generalization of the form $\forall\alpha\varphi$ can have an alphabetic variant. (83.12) concerns one of those ways, namely, alphabetic variants of the form $\forall\beta\varphi^{\beta}_{\alpha}$. But it follows from Metatheorem $\langle 8.3\rangle$(e) (in Chapter 8) that $\forall\alpha\varphi$ can also have alphabetic variants of the form $\forall\alpha\varphi'$, where φ' is an alphabetic variant of φ. We aren't yet in a position to prove the interderivability of the latter, much less prove the interderivability of alphabetically-variant formulas of *arbitrary* complexity. The case proved above tells us only that whenever we have established a theorem of the form $\forall\alpha\varphi$, we may infer any formula with the same exact form but which differs only by the choice of the variable bound by the leftmost universal quantifier, provided the choice of new variable β is a safe one, i.e., one that will preserve the meaning of the original formula when the substitution is carried out.

(**84**) **Metarules/Derived Rules:** ∃Introduction (∃I). The metarules of ∃Introduction allow us to infer the existence of derivations of existential generalizations, though their variant forms yield derived rules that let us existentially generalize, within a derivation, on any term τ that is logically proper. In that regard, it is perfectly standard. But ∃I rules have to be formulated carefully. There is a form that applies to any term whatsoever and a restricted form that applies to any term other than a description:

Rule ∃I

(.1) If $\Gamma_1 \vdash \varphi$ and $\Gamma_2 \vdash \exists\beta(\beta = \tau)$, then $\Gamma_1, \Gamma_2 \vdash \exists\alpha\varphi'$, whenever α is a variable of the same type as τ and φ' is obtained from φ by substituting α for zero or more occurrences of τ, provided both (1) when τ is a variable, all of the replaced occurrences of τ in φ are free occurrences, and (2) all of the substituted occurrences of α are free in φ'. [Variant: $\varphi, \exists\beta(\beta = \tau) \vdash \exists\alpha\varphi'$]

(.2) If $\Gamma \vdash \varphi$, then $\Gamma \vdash \exists\alpha\varphi'$, whenever φ' is obtained from φ by substituting the variable α for zero or more occurrences of some term τ of the same type as α, provided (1) τ is not a description, (2) when τ is a variable, all of the replaced occurrences of τ are free in φ, and (3) all of the substituted occurrences of α are free in φ'. [Variant: $\varphi \vdash \exists\alpha\varphi'$]

The simplest two examples of the variant version of (.2) are $Gy \vdash \exists xGx$ and $Gy \vdash \exists F(Fy)$. In the first case, φ is Gy, φ' is Gx, τ is y, and α is x. Condition (1) in the rule is met because y is not a description; condition (2) is met because all of the replaced occurrences of y are free in Gy; and condition (3) is met because all of the substituted occurrences of x are free in Gx.

Note also that the inference from $P\imath xQx$ and $\exists y(y = \imath xQx)$ to $\exists xPx$ is justified by the variant version of (84.1), whereas the inference from $Pa \,\&\, Pb$ to $\exists F(Fa \,\&\, Pb)$ is justified by (84.2). Since this rule is covered in detail in basic courses on predicate logic, we omit both the formulation of more complex examples and further explanation of the conditions that must be satisfied for the rule to be applied.

(**85**) **Metarule:** ∃Elimination (∃E) on Constants. If we have asserted $\exists\alpha\varphi$ as a theorem or premise, we often continue reasoning by saying "Assume τ is an arbitrary such φ, so that we know φ^{τ}_{α}," where τ is a 'fresh' constant that hasn't previously appeared in the context of reasoning or even in our axioms. If we then validly reason our way to ψ from (some premises and) φ^{τ}_{α} without making any special assumptions about τ other than φ^{τ}_{α}, then ∃E allows us to discharge our assumption about τ and validly conclude that we can derive ψ from (the premises we used and) $\exists\alpha\varphi$:

Rule $\exists$**E**
If $\Gamma, \varphi^\tau_\alpha \vdash \psi$, then $\Gamma, \exists\alpha\varphi \vdash \psi$, provided τ is a constant that does not occur in φ, ψ, Γ, or $\mathbf{\Lambda}$.

In other words, if there is a derivation of ψ from $\Gamma \cup \{\varphi^\tau_\alpha\}$, where Γ, φ and ψ make no special assumptions about τ (i.e., τ is arbitrary with respect to Γ, $\mathbf{\Lambda}$, φ and ψ), then there is a derivation of ψ from $\Gamma \cup \{\exists\alpha\varphi\}$.

(**86**) **Theorems:** Further Theorems of Quantification Theory. The foregoing series of rules for quantification theory facilitate the derivation of many of the following theorems:

(.1) $\forall\alpha\varphi \to \exists\alpha\varphi$

(.2) $\neg\forall\alpha\varphi \equiv \exists\alpha\neg\varphi$

(.3) $\forall\alpha\varphi \equiv \neg\exists\alpha\neg\varphi$

(.4) $\neg\exists\alpha\varphi \equiv \forall\alpha\neg\varphi$

(.5) $\exists\alpha(\varphi \,\&\, \psi) \to (\exists\alpha\varphi \,\&\, \exists\alpha\psi)$

(.6) $\exists\alpha(\varphi \lor \psi) \equiv (\exists\alpha\varphi \lor \exists\alpha\psi)$

(.7) $\exists\alpha\varphi \equiv \exists\beta\varphi^\beta_\alpha$, provided β is substitutable for α in φ and doesn't occur free in φ

(.8) $\varphi \equiv \exists\beta(\beta = \alpha \,\&\, \varphi^\beta_\alpha)$, provided β is substitutable for α in φ and doesn't occur free in φ.

(.9) $\varphi^\tau_\alpha \equiv \exists\alpha(\alpha = \tau \,\&\, \varphi)$, provided τ is any term other than a description and is substitutable for α in φ.

(.10) $(\varphi \,\&\, \forall\beta(\varphi^\beta_\alpha \to \beta = \alpha)) \equiv \forall\beta(\varphi^\beta_\alpha \equiv \beta = \alpha)$,
provided α, β are distinct variables of the same type, and β is substitutable for α in φ and doesn't occur free in φ.

(.11) $(\forall\alpha\varphi \,\&\, \forall\alpha\psi) \to \forall\alpha(\varphi \equiv \psi)$

(.12) $(\neg\exists\alpha\varphi \,\&\, \neg\exists\alpha\psi) \to \forall\alpha(\varphi \equiv \psi)$

(.13) $(\exists\alpha\varphi \,\&\, \neg\exists\alpha\psi) \to \neg\forall\alpha(\varphi \equiv \psi)$

(.14) $\exists\alpha\exists\beta\varphi \equiv \exists\beta\exists\alpha\varphi$

A simple example of (86.8) is $Px \equiv \exists y(y = x \,\&\, Py)$, and a simple example (86.9) is $Qa \equiv \exists x(x = a \,\&\, Qx)$. But these theorems also apply to relation terms; as simple examples we have $Fa \equiv \exists G(G = F \,\&\, Ga)$ and $Pa \equiv \exists F(F = P \,\&\, Fa)$, respectively. The reader should produce examples in which φ has greater complexity.

Note that (86.9) is restricted to terms τ other than descriptions. An example that illustrates why this is necessary comes readily to hand. Let φ be $Px \rightarrow Px$, and let τ be $\imath yQy$, so that $\varphi_x^{\imath yQy}$ is $P\imath yQy \rightarrow P\imath yQy$. Then $\varphi_x^{\imath yQy}$ is a tautology and so true no matter whether $\imath yQy$ is logically proper or not. But in the case where $\imath yQy$ is not logically proper, the following would be an invalid instance of (86.9): $(P\imath yQy \rightarrow P\imath yQy) \equiv \exists x(x = \imath yQy \; \& \; (Px \rightarrow Px))$. When nothing is the unique Q object, then it is not the case that something x is both the unique Q object and such that if Px then Px.

Theorem (86.10) is noteworthy because the two sides of the main biconditional are equivalent ways of asserting an important claim. When α is an individual variable, both sides of the biconditional assert the claim that α *is a unique individual such that* φ, and when α is a relation variable, they both represent the claim that α *is a unique relation such that* φ. (In the formal mode, we would say, in both cases, that α uniquely satisfies φ.) The left condition comes to us from Russell's (1905) classic analysis of uniqueness, whereas the right condition is a slightly more efficient way of expressing the uniqueness claim.

(**87**) **Definitions:** Unique Existence Quantifier. As a consequence of the preceding observation, we introduce, in the usual way, a special quantifier $\exists!$ to conveniently assert that there exists a unique α such that φ. Where β is substitutable for α in φ and doesn't occur free in φ, we stipulate that:

(.1) $\exists!\alpha\varphi \;=_{df}\; \exists\alpha(\varphi \;\&\; \forall\beta(\varphi_\alpha^\beta \rightarrow \beta = \alpha))$

It is important not to confuse the defined unique-existence quantifier '$\exists!$' with the simple predicate '$E!$' in what follows. Moreover, in light of (86.10), the above definition is equivalent to:

(.2) $\exists!\alpha\varphi \;=_{df}\; \exists\alpha\forall\beta(\varphi_\alpha^\beta \equiv \beta = \alpha)$

That is, there exists a unique α such that φ if and only if there exists an α such that all and only the entities which are such that φ are identical to α.

(**88**) **Theorems:** Uniqueness and Necessity. It is now provable that if every individual such that φ is necessarily such that φ, then if there is a unique object such that φ, there is a unique object necessarily such that φ:

$$\forall x(\varphi \rightarrow \Box\varphi) \rightarrow (\exists!x\varphi \rightarrow \exists!x\Box\varphi)$$

In other words, if φ is a formula that *necessarily holds* of x whenever it holds of x, then if there is exactly one object such that φ, there is exactly one object necessarily such that φ.

9.7 The Theory of Actuality and Descriptions

Although the theorems in this section sometimes involve the necessity operator, no special principles for necessity other than the axioms and rules introduced thus far are required to prove the basic theorems and metarules governing actuality. We turn first to three groups of modally strict theorems that are needed to prove the Rule of Actualization.

(**89**) **Theorems:** Necessity Implies Actuality. It is straightforward to show that if necessarily φ, then actually φ:

$$\Box\varphi \to \mathcal{A}\varphi$$

Note that in the Appendix, we do *not* give the following proof of this theorem: assume $\Box\varphi$, infer φ by the T schema, infer $\mathcal{A}\varphi$ by the necessitation-averse axiom for actuality (30)$\star$, and conclude $\Box\varphi \to \mathcal{A}\varphi$ by conditional proof. Though this is a perfectly good proof, it is not modally strict. By contrast, the proof in the Appendix is modally strict and so one may apply RN to our theorem to obtain $\Box(\Box\varphi \to \mathcal{A}\varphi)$.

(**90**) **Theorems:** Actuality, Conjunctions, and Biconditionals. The following theorems also have modally strict proofs. (.1) it is actually the case that if actually φ then φ; (.2) it is actually the case that if φ then actually φ; (.3) if it is actually the case that φ and actually the case that ψ, then it is actually the case that both φ and ψ; and (.4) it is actually the case that, actually φ if and only if φ:

(.1) $\mathcal{A}(\mathcal{A}\varphi \to \varphi)$

(.2) $\mathcal{A}(\varphi \to \mathcal{A}\varphi)$

(.3) $(\mathcal{A}\varphi \,\&\, \mathcal{A}\psi) \to \mathcal{A}(\varphi \,\&\, \psi)$

(.4) $\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$

Note that (.4) tells us that the actualizations of instances of the necessitation-averse axiom (30)$\star$ are modally strict theorems.

(**91**) **Theorems:** Actualizations and Universal Closures of the Previous Theorem. Note that the left-to-right direction of axiom (31.4) tells us that if actually φ, then actually, actually φ. By applying this principle to theorem (90.4) and repeating the process, then (.1) every actualization of the necessitation-averse axiom (30)$\star$ is derivable:

(.1) $\mathcal{A}\ldots\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$, for any finite string of actuality operators $\mathcal{A}\ldots\mathcal{A}$

Moreover, by applying GEN to (31.4), we know (.2) every α is such that it is actually the case that, actually φ if and only if φ:

(.2) $\forall\alpha\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$

Now it follows from (.2) by axiom (31.3) that (.3) actually, every α is such that: actually φ if and only if φ, i.e.,

(.3) $\mathcal{A}\forall\alpha(\mathcal{A}\varphi \equiv \varphi)$

Hence the actualization of a universal closure of an instance of axiom (30)★ is derivable. Moreover, by repeating steps (.2) and (.3) enough times, it follows that:

(.4) $\mathcal{A}\forall\alpha_1 \ldots \forall\alpha_n(\mathcal{A}\varphi \equiv \varphi)$

(.4) tells us that the actualization of any universal closure of any instance of $\mathcal{A}\varphi \equiv \varphi$ is a modally strict theorem. Thus, given (.4) and (90.4), the actualizations of all the axioms asserted in (30)★ are modally strict theorems. This fact is needed in the proof of the Rule of Actualization, to which we now turn.

(**92**) **Metarule:** Rule of Actualization (RA). We first define:

- $\mathcal{A}\Gamma = \{\mathcal{A}\psi \mid \psi \in \Gamma\}$ (Γ any set of formulas)

Thus, $\mathcal{A}\Gamma$ is the result of adding the actuality operator to the front of every formula in Γ. We then have the following metarule:

Rule of Actualization (**RA**)
If $\Gamma \vdash \varphi$, then $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$

We most often use this rule in the form in which Γ is empty:

- If $\vdash \varphi$, then $\vdash \mathcal{A}\varphi$

In other words, whenever φ is a theorem, so is $\mathcal{A}\varphi$.

(**93**) **Remark:** Observations about RA. Several important observations about RA are in order. First, note that given this rule, whenever φ is a theorem, so is $\Box\mathcal{A}\varphi$, for RA yields that $\mathcal{A}\varphi$ is a theorem if φ is and so axiom (33.1), i.e., $\mathcal{A}\varphi \to \Box\mathcal{A}\varphi$, then implies $\Box\mathcal{A}\varphi$. Indeed, this holds even if our initial theorem φ is a ★-theorem. Here is a simple example. Axiom (30)★ is $\mathcal{A}\varphi \equiv \varphi$. So by RA, $\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$. Hence by (33.1), it follows that $\Box\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$. While this reasoning establishes $\vdash \Box\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$, it doesn't show $\vdash_\Box \Box\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$! But we know there is such a modally strict proof: apply RN to the modally strict theorem (90.4).

Second, it is important to recognize why RA is formulated as in (92) above, as opposed to the following alternative:

If $\Gamma \vdash \varphi$, then $\Gamma \vdash \mathcal{A}\varphi$

Here, the consequent of the rule doesn't require $\mathcal{A}\Gamma$ but only Γ. One can prove that this version of the rule is semantically valid.[106] However, the justification of this rule depends on the necessitation-averse axiom of actuality (30)★,

[106] It is provable that if $\Gamma \models \varphi$, then $\Gamma \models \mathcal{A}\varphi$. Intuitively, if $\Gamma \models \varphi$, i.e., if φ is true at the distinguished world in every interpretation in which all the formulas in Γ are true at the distinguished world, then it follows that in every interpretation in which all the formulas in Γ are true at the distinguished world, $\mathcal{A}\varphi$ is true at the distinguished world, i.e., it follows that $\Gamma \models \mathcal{A}\varphi$.

i.e., (30)★ is *used* in the justification.[107] Clearly, the application of this alternative rule would undermine modally-strict derivations. For example, given that $\varphi \vdash \varphi$ is an instance of derived rule (46.4), this alternative version of RA would allow us to conclude $\varphi \vdash \mathcal{A}\varphi$, which by the Deduction Theorem (54) yields $\varphi \rightarrow \mathcal{A}\varphi$ as a theorem. But we certainly don't want this to be a modally-strict theorem—we know that its necessitation, $\Box(\varphi \rightarrow \mathcal{A}\varphi)$, fails to be valid. By formulating the consequent of RA with $\mathcal{A}\Gamma$, we forestall such a derivation. All that follows from $\varphi \vdash \varphi$ via RA, as officially formulated in (92), is that $\mathcal{A}\varphi \vdash \mathcal{A}\varphi$, which by the Deduction Theorem, yields only $\vdash \mathcal{A}\varphi \rightarrow \mathcal{A}\varphi$. Moreover, this derivation of $\mathcal{A}\varphi \rightarrow \mathcal{A}\varphi$ is modally-strict and we may happily apply RN to derive a valid necessary truth.

Call those rules (like the alternative to RA considered above) whose justification depends on the necessitation-averse axiom (30)★ *non-strict rules*. If we were to use non-strict rules, we would have to tag any theorem proved by means of such a rule a ★-theorem and, indeed, tag the rule itself as a ★-rule. This explains why we adopted convention (50.2) described in Remark (50), namely, that we avoid metarules whose justification depends on necessitation-averse axioms. With such a convention in place, we don't have to worry about redefining *modal strictness* to ensure that derivations that depend on a necessitation-averse axiom *or* on a non-strict rule fail to be modally-strict.

Third, and finally, note that there are other valid but non-strict metarules that we shall eschew because they violate our convention. Consider, for example, the following rule:

If $\Gamma \vdash \mathcal{A}\varphi$, then $\Gamma \vdash \varphi$.

This rule can be justified from the basis we now have.[108] Again, however, the justification depends on the necessitation-averse axiom (30)★. Since this reasoning shows us how to turn a proof using the metarule into a proof that doesn't use that rule, it becomes apparent that any proof that uses the above rule implicitly involves an appeal to the necessitation-averse axiom (30)★. Unless we take further precautions, this rule could permit us to derive invalidi-

[107] To see this, assume the antecedent, i.e., $\Gamma \vdash \varphi$. Now since $\varphi \rightarrow \mathcal{A}\varphi$ is a simple consequence of the necessitation-averse axiom of actuality (30)★, $\mathcal{A}\varphi \equiv \varphi$, we know $\vdash \varphi \rightarrow \mathcal{A}\varphi$. So by (46.10), we have $\varphi \vdash \mathcal{A}\varphi$. But from $\Gamma \vdash \varphi$ and $\varphi \vdash \mathcal{A}\varphi$, it follows by (46.8) that $\Gamma \vdash \mathcal{A}\varphi$.

Note here how the justification of RA (92) in the Appendix doesn't similarly *use* (30)★. In the base case of the justification, we essentially showed that if φ is any axiom, then $\vdash \mathcal{A}\varphi$, even when φ is axiom (30)★, i.e., even in the case where φ is $\mathcal{A}\psi \equiv \psi$. But in showing that $\vdash \mathcal{A}\varphi$ in this case, we didn't *use* (30)★ in the reasoning. Indeed, we showed $\vdash \mathcal{A}\varphi$ *without* using it. See the justification of (92) in the Appendix.

[108] To see this, assume the antecedent, i.e., $\Gamma \vdash \mathcal{A}\varphi$. Note that since the necessitation-averse axiom of actuality (30)★ asserts that $\mathcal{A}\varphi \equiv \varphi$, it follows by (46.1) that $\vdash \mathcal{A}\varphi \equiv \varphi$. So by (46.3), it follows that $\Gamma \vdash \mathcal{A}\varphi \equiv \varphi$. Then from $\Gamma \vdash \mathcal{A}\varphi$ and $\Gamma \vdash \mathcal{A}\varphi \equiv \varphi$, it follows by biconditional syllogism (64.6.a) that $\Gamma \vdash \varphi$.

ties.[109] So instead of taking such precautions as tagging the rule with a ⋆ (to mark it as non-strict) and tagging any derivations involving the rule as non-strict, we simply avoid non-strict rules altogether.

(**94**) ⋆**Theorems:** Actuality and Negation. The following are simple consequences of (30)⋆:

(.1) $\neg\mathcal{A}\varphi \equiv \neg\varphi$

(.2) $\neg\mathcal{A}\neg\varphi \equiv \varphi$

Given that the proofs of these theorems depend on (30)⋆, we may not apply RN to either theorem.

(**95**) **Theorems**. Modally Strict Theorems of Actuality.

(.1) $\mathcal{A}\varphi \vee \mathcal{A}\neg\varphi$

(.2) $\mathcal{A}(\varphi \,\&\, \psi) \equiv (\mathcal{A}\varphi \,\&\, \mathcal{A}\psi)$

(.3) $\mathcal{A}(\varphi \equiv \psi) \equiv (\mathcal{A}(\varphi \to \psi) \,\&\, \mathcal{A}(\psi \to \varphi))$

(.4) $(\mathcal{A}(\varphi \to \psi) \,\&\, \mathcal{A}(\psi \to \varphi)) \equiv (\mathcal{A}\varphi \equiv \mathcal{A}\psi)$

(.5) $\mathcal{A}(\varphi \equiv \psi) \equiv (\mathcal{A}\varphi \equiv \mathcal{A}\psi)$

(.6) $\Diamond\varphi \equiv \mathcal{A}\Diamond\varphi$

(.7) $\mathcal{A}\varphi \equiv \Box\mathcal{A}\varphi$

(.8) $\mathcal{A}\Box\varphi \to \Box\mathcal{A}\varphi$

(.9) $\Box\varphi \to \Box\mathcal{A}\varphi$

(.10) $\mathcal{A}(\varphi \vee \psi) \equiv (\mathcal{A}\varphi \vee \mathcal{A}\psi)$

(.11) $\mathcal{A}\exists\alpha\varphi \equiv \exists\alpha\mathcal{A}\varphi$

Note that one can develop far simpler proofs of some of the above theorems than the ones given in the Appendix by using the necessitation-averse axiom (30)⋆. But such proofs would fail to be modally strict. By contrast, the proofs we give in the Appendix are modally-strict. Note also that (.2) is used in the proof of (.3), and (.3) and (.4) are used in the proof of (.5). The latter is used to prove (112.1), which is the key lemma in the proof of the Rule of Substitution

[109] Here is a simple example. As an instance of $\varphi \vdash \varphi$ (46.4), we know: $\mathcal{A}\varphi \vdash \mathcal{A}\varphi$. So the proposed rule of actuality elimination would allow one to infer $\mathcal{A}\varphi \vdash \varphi$. But by the Deduction Theorem (54), it would follow that $\mathcal{A}\varphi \to \varphi$ is a modally-strict theorem. We know that the necessitation of this claim is invalid, but without further constraints on RN, it would follow that $\Box(\mathcal{A}\varphi \to \varphi)$ is a theorem.

(113). This is an important rule that can and should be derived in a way that complies with our convention of eschewing non-strict rules.

(**96**) ⋆**Lemmas:** A Consequence of the Necessitation-Averse Equivalence of φ and $\mathcal{A}\varphi$. It is a straightforward consequence of the necessitation-averse axiom for actuality (30)⋆ that an object x is uniquely such that $\mathcal{A}\varphi$ if and only if x is uniquely such that φ:

> $\forall z(\mathcal{A}\varphi^z_x \equiv z = x) \equiv \forall z(\varphi^z_x \equiv z = x)$, provided z is substitutable for x in φ and doesn't occur free in φ

(**97**) ⋆**Theorems:** Fundamental Theorems Governing Descriptions. It follows from the previous lemma that x is the individual that is (in fact) such that φ just in case x is uniquely such that φ:

> $x = \imath x\varphi \equiv \forall z(\varphi^z_x \equiv z = x)$, provided z is substitutable for x in φ and doesn't occur free in φ

The proof of this theorem goes by way of the axiom for descriptions (34) and lemma (96)⋆, and hence depends on the necessitation-averse axiom for actuality (30)⋆. So we may not apply RN to derive its necessitation. But from this ⋆-theorem, we may prove other important and famous principles involving descriptions which, in the present context, are also ⋆-theorems. Examples are Hintikka's schema (98)⋆ and Russell's analysis of descriptions (99)⋆. Though the classical statement of these famous principles are derived in a way that is not modally-strict, the principles can be slightly modified so as to be derivable by modally-strict proofs. This will become apparent below.

(**98**) ⋆**Theorems:** Hintikka's Schema. We may derive the instances of Hintikka's schema for definite descriptions (1959, 83), namely, x is identical to the x (in fact) such that φ if and only if φ and everything such that φ is identical to x, i.e.,[110]

> $x = \imath x\varphi \equiv (\varphi \,\&\, \forall z(\varphi^z_x \rightarrow z = x))$, provided z is substitutable for x in φ and doesn't occur free in φ

Note that in contrast to Hintikka's original schema, the one above involves a defined rather than primitive identity sign.

The proof of Hintikka's schema appeals to the ⋆-theorem (97)⋆ and so the schema fails to be a modally-strict theorem. When φ is within the scope of a rigidifying operator like $\imath x$ or $\mathcal{A}$ on one side of a true conditional (or biconditional) but not within the scope of such an operator on the other side, the necessitation of the conditional (or biconditional) is invalid. For the discussion

[110] We've changed one of the variables in Hintikka's original schema to simplify the statement of the theorem.

of an example involving a description, see Section 5.5.2. The example discussed in detail there, $Q\imath xPx \rightarrow \exists yPy$, should help one to see why Hintikka's schema, though valid, doesn't have a valid necessitation.

(**99**) $\star$**Theorems:** Russell's Analysis of Descriptions. Our derived quantifier rules also help us to more easily prove, as a *theorem*, a version of Russell's famous (1905) analysis of definite descriptions. For any exemplification or an encoding formula ψ, if the x (in fact) such that φ is such that ψ, then there exists an x such that both φ and anything z such that φ just is x, i.e.,

> $\psi_x^{\imath x\varphi} \equiv \exists x(\varphi \;\&\; \forall z(\varphi_x^z \rightarrow z = x) \;\&\; \psi)$, provided (a) ψ is either an exemplification formula $\Pi^n\kappa_1 \dots \kappa_n$ $(n \geq 1)$ or an encoding formula $\kappa_1\Pi^1$, (b) x occurs in ψ and only as one or more of the κ_i $(1 \leq i \leq n)$, and (c) z is substitutable for x in φ and doesn't appear free in φ

This asserts: the individual (in fact) such that φ is ψ if and only if something x is such that φ, everything such that φ just is x, and x is such that ψ. It is also important to note that with rigid definite descriptions, Russell's analysis is a $\star$-theorem; the proof relies on Hintikka's schema, which in turn depends on the $\star$-theorem (97)$\star$.

(**100**) $\star$**Theorems:** Logically Proper Descriptions and Uniqueness. There exists something that is identical to the individual (in fact) such that φ if and only if there exists a unique x such that φ:

> $\exists y(y = \imath x\varphi) \equiv \exists! x\varphi$, provided y doesn't occur free in φ

The left-to-right direction captures definition $*14\cdot02$ in *Principia Mathematica*, since we know, from the discussion on p. 31 (Introduction, Chapter I) of that text, that the intent of Whitehead and Russell's predicate $E!$ in the definiendum $E!\imath x\varphi$ is to assert that the x such that φ exists. However, note that since the proof of this theorem appeals to Hintikka's schema, this $\star$-theorem is not subject to RN. Both directions of the biconditional fail to be necessary, and there are counterexamples to their necessitations. We can find formulas φ and interpretations for which the following is a counterexample to the necessitation of the left-to-right direction: *possibly*, something is the (actual) φ and nothing is uniquely φ. And we can find a φ and an interpretation for which the following is a counterexample to the necessitation of the right-to-left direction: *possibly*, something is uniquely φ and nothing is the (actual) φ.

(**101**) $\star$**Theorems:** Logically Proper Descriptions Apply To Themselves.

(.1) $x = \imath x\varphi \rightarrow \varphi$

(.2) $z = \imath x\varphi \rightarrow \varphi_x^z$, provided z is substitutable for x in φ and doesn't occur free in φ

(.3) $\exists y(y = \imath x\varphi) \rightarrow \varphi_x^{\imath x\varphi}$, provided y doesn't occur free in φ

These theorems license substitutions into the matrix of a description under certain conditions. The last one tells us we can substitute a description into its own matrix if we know that the description is logically proper. Note that since the proofs appeal to Hintikka's schema, they are all ★-theorems.

(**102**) **Lemmas:** Consequence of the Necessary Equivalence of $\mathcal{A}\varphi$ and $\mathcal{A}\mathcal{A}\varphi$. One of the necessary axioms for actuality is (31.4), namely, $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$. It is a straightforward, modally-strict consequence of this axiom that an object x is uniquely such that $\mathcal{A}\varphi$ if and only if x is uniquely such that $\mathcal{A}\mathcal{A}\varphi$:

> $\forall z(\mathcal{A}\varphi_x^z \equiv z = x) \equiv \forall z(\mathcal{A}\mathcal{A}\varphi_x^z \equiv z = x)$, provided z is substitutable for x in φ and doesn't occur free in φ

The necessitation-averse axiom for actuality (30)★ is not needed to prove this lemma.

(**103**) **Theorems:** More Fundamental Theorems of Descriptions and Actuality. It is provable that (.1) x is identical to the individual (in fact) such that φ if and only if x is identical to the individual (in fact) such that actually φ, and (.2) If there is something which is the individual (in fact) such that φ, then it is identical to the individual (in fact) such that actually φ:

(.1) $x = \imath x\varphi \;\equiv\; x = \imath x\mathcal{A}\varphi$

(.2) $\exists y(y = \imath x\varphi) \rightarrow \imath x\varphi = \imath x\mathcal{A}\varphi$, provided y doesn't occur free in φ

These are modally-strict theorems. **Exercise:** Use these two theorems to find modally strict proofs of:

(.3) $\psi_x^{\imath x\varphi} \rightarrow \exists\nu(\nu = \imath x\mathcal{A}\varphi)$, provided (a) ψ is either an exemplification formula $\Pi^n\kappa_1\ldots\kappa_n$ ($n \geq 1$) or an encoding formula $\kappa_1\Pi^1$, (b) x occurs in ψ and only as one or more of the κ_i ($1 \leq i \leq n$), and (c) ν is any individual variable that doesn't occur free in φ.

(.4) $\psi_x^{\imath x\varphi} \rightarrow \imath x\varphi = \imath x\mathcal{A}\varphi$, provided (a) ψ is either an exemplification formula $\Pi^n\kappa_1\ldots\kappa_n$ ($n \geq 1$) or an encoding formula $\kappa_1\Pi^1$ and (b) x occurs in ψ and only as one or more of the κ_i ($1 \leq i \leq n$).

Note how the first of these exercises is a variant of axiom (29.1).

(**104**) **Theorems:** Modally Strict Version of Hintikka's Schema. By a judicious placement of the actuality operator, we obtain the following version of Hintikka's schema, namely, x is the individual (in fact) such that φ if and only if it is actually the case that φ and everything for which it is actually the case that φ is identical to x, i.e.,

$x = \imath x\varphi \equiv \mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = x)$, provided z is substitutable for x in φ and doesn't occur free in φ.

Since the proof of this theorem appeals to the necessary axiom (34) instead of to theorem (97)★, it is modally-strict.

(**105**) **Theorems:** Descriptions Based on Actual Equivalences. The modally strict version of Hintikka's scheme allows us to formulate and prove a nice theorem in connection with descriptions $\imath x\varphi$ and $\imath x\psi$ whose matrices are actually equivalent. If it is actually the case that φ if and only if ψ, then x is identical to the individual (in fact) such that φ if and only if x is identical to the individual (in fact) such that ψ:

$$\mathcal{A}(\varphi \equiv \psi) \rightarrow \forall x(x = \imath x\varphi \equiv x = \imath x\psi)$$

Note that we can't prove $\mathcal{A}(\varphi \equiv \psi) \rightarrow \imath x\varphi = \imath x\psi$: the consequent implies that the descriptions are logically proper, but this is something that is not guaranteed by the antecedent. Consider a situation in which both $\neg\exists x\mathcal{A}\varphi$ and $\neg\exists x\mathcal{A}\psi$. Then by (86.12), it would follow that $\forall x(\mathcal{A}\varphi \equiv \mathcal{A}\psi)$, and hence $\mathcal{A}\varphi \equiv \mathcal{A}\psi$. It would then follow that $\mathcal{A}(\varphi \equiv \psi)$, by (95.5). But in this situation, the descriptions $\imath x\varphi$ and $\imath x\psi$ both fail to be logically proper, and $\imath x\varphi = \imath x\psi$ would be false.

(**106**) **Theorems:** Modally Strict Version of Russell's Analysis of Descriptions. By a another judicious placement of the actuality operator, we can prove a modally strict version of Russell's analysis of descriptions. For any exemplification or encoding formula ψ, if the x (in fact) such that φ is such that ψ, then there exists an x such that (i) it is actually the case that φ, and (ii) anything z for which it is actually the case that φ just is x, i.e.,

$\psi^{\imath x\varphi}_x \equiv \exists x(\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = x) \,\&\, \psi)$, provided (a) ψ is either an exemplification formula $\Pi^n\kappa_1 \ldots \kappa_n$ $(n \geq 1)$ or an encoding formula $\kappa_1\Pi^1$, (b) x occurs in ψ and only as one or more of the κ_i $(1 \leq i \leq n)$, and (c) z is substitutable for x in φ and doesn't occur free in φ

No appeal to (30)★ or (99)★ is necessary.

(**107**) **Theorems:** Theorems for Logically Proper Descriptions and Actuality. Where y doesn't occur free in φ, we have:

(.1) $\exists y(y = \imath x\varphi) \equiv \exists! x\mathcal{A}\varphi$

(.2) $x = \imath x\varphi \rightarrow \mathcal{A}\varphi$

(.3) $z = \imath x\varphi \rightarrow \mathcal{A}\varphi^z_x$, provided z is substitutable for x in φ and doesn't occur free in φ

(.4) $\exists y(y = \imath x\varphi) \to \mathcal{A}\varphi_x^{\imath x\varphi}$, provided y doesn't occur free in φ

These are modally-strict theorems. Compare (.1) with (100)★, and (.2) – (.4) with (101.1)★ – (101.3)★, respectively.

(**108**) **Theorems:** Modally Strict Conditions for Logically Proper Descriptions to Apply to Themselves. Here are some interesting facts about descriptions. (.1) If there exists a unique individual that is necessarily such that φ, then anything identical to the x such that φ is such that φ:

(.1) $\exists!x\Box\varphi \to \forall y(y = \imath x\varphi \to \varphi_x^y)$, provided y is substitutable for x in φ and doesn't occur free in φ

This is modally strict; by comparing this theorem with (101.2)★, we see that $\exists!x\Box\varphi$ provide modally strict conditions under which a logically proper descriptions applies to themselves.

Furthermore (.1) helps us to prove another useful theorem, namely, that (.2) if everything such that φ is necessarily such that φ, then if there is a unique thing such that φ, then anything identical to the x such that φ is such that φ:

(.2) $\forall x(\varphi \to \Box\varphi) \to (\exists!x\varphi \to \forall y(y = \imath x\varphi \to \varphi_x^y))$

Later in this chapter, we'll see how this theorem helps us to prove facts about a distinguished group of *canonical* abstract objects.

9.8 The Theory of Necessity

(**109**) **Theorems:** Tautologies Proved Thus Far Are Necessary.

- The tautologies proved in Section 9.4, in items (53), (58), and (63), are all necessary.

In each case, the necessitation follows by an application of RN. This applies to any other tautology derivable from our axioms for negations and conditionals.

But as noted in Remark (65), we haven't yet shown that *all* tautologies are derivable, for there are some tautologies involving 0-place λ-expressions that are not derivable from the axioms for negations and conditionals alone. The remaining principles needed for the proof that every tautology is derivable are established in the present section. In particular, once we have the proof of (131.2) in hand (i.e., the 0-place case of β-Conversion, namely, $[\lambda\, \varphi^*] \equiv \varphi^*$), Metatheorems $\langle 9.2 \rangle$ and $\langle 9.3 \rangle$ (in the Appendix to this chapter) provide proofs, respectively, of the facts that *every* tautology is derivable and that every tautology is a necessary truth.

(**110**) **Metarules:** Rules RM and RM$\Diamond$. The classical rule RM of the logic of necessity asserts that if $\vdash \varphi \to \psi$, then $\vdash \Box\varphi \to \Box\psi$. However, in our system, rule

RM has to be adjusted slightly to accommodate necessitative-averse axioms and contingent premises.

To see why, take a simple case in which one extends our theory by asserting both ψ and $\Diamond\neg\psi$ as axioms. Consider the 3-element sequence of formulas $\psi, \psi \to (\varphi \to \psi), \varphi \to \psi$. This sequence is a proof of $\varphi \to \psi$, since the first member is an axiom by hypothesis, the second member is an instance of axiom schema (21.1) for conditionals, and the third member follows from the first two by MP. So this establishes $\vdash \varphi \to \psi$. By the classical rule RM, it would follow that $\vdash \Box\varphi \to \Box\psi$. But if φ were, say, an instance of a necessary axiom, so that $\Box\varphi$ is also an axiom, then by (46.1), it would follow that $\vdash \Box\varphi$ and, by (46.6), that $\vdash \Box\psi$. But, by hypothesis, $\Diamond\neg\psi$. So the unmodified rule RM would allow us to derive the necessitation of a formula (ψ) that is possibly false.

The problem in this case, of course, is that in the initial proof of $\varphi \to \psi$, the conclusion depends on ψ, which is known to be true but possibly false. If we formulate RM so that it applies only to conditional theorems that have modally-strict proofs or derivations, we can forestall the potential derivation of a falsehood (from a truth). But we first formulate the rule for derivations generally:

(.1) **Rule RM**:
If $\Gamma \vdash_{\Box} \varphi \to \psi$, then $\Box\Gamma \vdash \Box\varphi \to \Box\psi$.

In other words, if there is a modally-strict derivation of $\varphi \to \psi$ from Γ, then there is a derivation of $\Box\varphi \to \Box\psi$ from the necessitations of the formulas in Γ. When $\Gamma = \varnothing$, then RM reduces to the principle:

If $\vdash_{\Box} \varphi \to \psi$, then $\vdash \Box\varphi \to \Box\psi$

i.e., if $\varphi \to \psi$ is a modally-strict theorem, then $\Box\varphi \to \Box\psi$ is a theorem.

RM$\Diamond$ is a corresponding rule:

(.2) **Rule RM$\Diamond$**:
If $\Gamma \vdash_{\Box} \varphi \to \psi$, then $\Box\Gamma \vdash \Diamond\varphi \to \Diamond\psi$.

In other words, if there is a modally-strict derivation of $\varphi \to \psi$ from Γ, then there is a derivation of $\Diamond\varphi \to \Diamond\psi$ from the necessitations of the formulas in Γ. When $\Gamma = \varnothing$, then RM$\Diamond$ reduces to the principle:

If $\vdash_{\Box} \varphi \to \psi$, then $\vdash \Diamond\varphi \to \Diamond\psi$

i.e., if $\varphi \to \psi$ is a modally-strict theorem, then $\Diamond\varphi \to \Diamond\psi$ is a theorem.

(**111**) **Theorems:** Basic K Theorems. The presentation and proof of some of the following basic theorems that depend upon the K schema has been informed by Hughes and Cresswell 1968 and 1996:

(.1) $\Box\varphi \to \Box(\psi \to \varphi)$

(.2) $\Box\neg\varphi \to \Box(\varphi \to \psi)$

(.3) $\Box(\varphi \,\&\, \psi) \equiv (\Box\varphi \,\&\, \Box\psi)$

(.4) $\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \to \psi) \,\&\, \Box(\psi \to \varphi))$

(.5) $(\Box(\varphi \to \psi) \,\&\, \Box(\psi \to \varphi)) \to (\Box\varphi \equiv \Box\psi)$

(.6) $\Box(\varphi \equiv \psi) \to (\Box\varphi \equiv \Box\psi)$

(.7) $(\Box\varphi \,\&\, \Box\psi) \to \Box(\varphi \equiv \psi)$

(.8) $\Box(\varphi \,\&\, \psi) \to \Box(\varphi \equiv \psi)$

(.9) $\Box(\neg\varphi \,\&\, \neg\psi) \to \Box(\varphi \equiv \psi)$

If we define φ *necessarily implies* ψ as $\Box(\varphi \to \psi)$, then theorem (.1) guarantees that if φ is necessarily true, then every claim formulable in our language necessarily implies φ. Similarly, (.2) guarantees that if φ is necessarily false, then φ necessarily implies any claim whatsoever. Where the classic notion of *strict implication* is understood in terms of the above definition of *necessary implication*, these results are among the classical "paradoxes of strict implication" (Lewis and Langford 1932 [1959], 511): a truth that is necessarily true is strictly implied by everything and a falsehood that is necessarily false strictly implies everything. However, given the meaning of the conditional, they are harmless.[111] (.3) establishes that the necessity operator distributes over the conjuncts of a conjunction (and vice versa!), while (.4) and (.5) are lemmas needed for the proof of (.6), which asserts that the necessity operator distributes over a biconditional.

Note that the converse of (.6), namely $(\Box\varphi \equiv \Box\psi) \to \Box(\varphi \equiv \psi)$, is not a theorem: the material equivalence of $\Box\varphi$ and $\Box\psi$ doesn't imply that the biconditional $\varphi \equiv \psi$ is necessary. To see this, consider an interpretation in which there are two worlds, $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$, such that (a) φ is true at $\boldsymbol{w}_0$ and false at $\boldsymbol{w}_1$, and (b) ψ is false at $\boldsymbol{w}_0$ and true at $\boldsymbol{w}_1$. Then clearly, both $\Box\varphi$ and $\Box\psi$ are false at $\boldsymbol{w}_0$ and so $\Box\varphi \equiv \Box\psi$ is true at $\boldsymbol{w}_0$ (since $\Box\varphi$ and $\Box\psi$ have the same truth value at $\boldsymbol{w}_0$). But the claim $\Box(\varphi \equiv \psi)$ is false at $\boldsymbol{w}_0$: the conditional $\varphi \equiv \psi$ fails at both worlds given that φ and ψ have different truth values at each world. (It is important to be familiar with this counterexample since as we shall see, there are special conditions under which the converse of (.6) holds, namely, if both φ and ψ necessarily hold whenever they hold. See (120) and (126.5) below.)

[111] (.1) is provably equivalent to the claim: $\Box\varphi \to \neg\Diamond(\psi \,\&\, \neg\varphi)$. This just follows *a fortiori* from the fact that $\Box\varphi \to \neg\Diamond\neg\varphi$, which is established below as item (.3). Similarly, (.2) is provably equivalent to the claim: $\Box\neg\varphi \to \neg\Diamond(\varphi \,\&\, \neg\psi)$. This just follows *a fortiori* from the fact that $\Box\neg\varphi \to \neg\Diamond\varphi$, which is established below as item (.4).

Finally, (.7) asserts that the biconditional $\varphi \equiv \psi$ is necessary if both φ and ψ are necessary. (.8) asserts that if a conjunction is necessary, then it is necessary that the conjuncts are materially equivalent. (.9) asserts that if the conjunction of $\neg\varphi$ and $\neg\psi$ is necessary, then it is necessary that $\varphi \equiv \psi$.

(**112**) **Metarules:** Rules of Necessary Equivalence. Theorems (63.5.d), (63.5.e), (63.5.f), (83.3), (95.5), and (111.6) each play a crucial role in establishing one of the cases of the following rule:

(.1) If $\vdash \Box(\psi \equiv \chi)$, then:

  (.a) $\vdash \neg\psi \equiv \neg\chi$

  (.b) $\vdash (\psi \to \theta) \equiv (\chi \to \theta)$

  (.c) $\vdash (\theta \to \psi) \equiv (\theta \to \chi)$

  (.d) $\vdash \forall\alpha\psi \equiv \forall\alpha\chi$

  (.e) $\vdash \mathcal{A}\psi \equiv \mathcal{A}\chi$

  (.f) $\vdash \Box\psi \equiv \Box\chi$

The above rule can be used to give an informal proof of the following rule, though we give both the informal and strict proof in the Appendix:

(.2) If $\vdash \Box(\psi \equiv \chi)$, then if φ' is the result of substituting the formula χ for zero or more occurrences ψ where the latter is a subformula of φ, then $\vdash \varphi \equiv \varphi'$.

Here are some examples of (.2):

Example.
If $\vdash \Box(A!x \equiv \neg\Diamond E!x)$, then $\vdash \exists x A!x \equiv \exists x \neg\Diamond E!x$

Example.
If $\vdash \Box(Rxy \equiv (Rxy \,\&\, (Qa \vee \neg Qa)))$, then
$\vdash (Pa \,\&\, \Box Rxy) \equiv (Pa \,\&\, \Box(Rxy \,\&\, (Qa \vee \neg Qa)))$

In the second example, we've conjoined a tautology $Qa \vee \neg Qa$ with the formula Rxy and the result is necessarily equivalent to Rxy. Hence we can take the formula, $Pa \,\&\, \Box Rxy$ ($= \varphi$), replace Rxy in this formula with its necessary equivalent, and the result φ' is materially equivalent to φ.
(.2) is the central theorem that helps to simplify the proofs of many of the theorems that follow. In many ways, it is the rule that deserves the title Rule of Substitution. However, in traditional modal systems in which all of the axioms and theorems are necessary truth, the Rule of Substitution asserts that the material equivalence of φ and ψ is sufficient for them to be substitutable for one another as subformulas of any formula. In our system, we formulate

this Rule of Substitution in the next item, but with the proviso that the material equivalence of φ and ψ must be derivable by a modally strict proof.

(**113**) **Metarule:** Rule of Substitution.

> **Rule of Substitution**
> If $\vdash_\Box \psi \equiv \chi$ and φ' is the result of substituting the formula χ for zero or more occurrences of ψ where the latter is a subformula of φ, then if $\Gamma \vdash \varphi$, then $\Gamma \vdash \varphi'$. [Variant: If $\vdash_\Box \psi \equiv \chi$, then $\varphi \vdash \varphi'$]

This rule tells us that if we can derive φ from the set Γ, then if there is a modally-strict proof of $\psi \equiv \chi$, we can derive φ' from Γ, where φ' has χ substituted (though not necessarily uniformly) for some subformula ψ in φ. Cf. Hughes and Cresswell 1996, 242, **Eq**.

Note that when $\vdash_\Box \psi \equiv \chi$, the Rule of Substitution does *not* allow us to substitute χ for ψ in any context whatsoever, but rather only when they occur as subformulas of some given formula. The Rule does *not*, for example, permit the substitution of χ for ψ within the formula $[\lambda y\, \psi \,\&\, \theta]x$ to obtain $[\lambda y\, \chi \,\&\, \theta]x$ (since ψ is not a subformula of $[\lambda y\, \psi \,\&\, \theta]x$). In this particular case, it just so happens that the substitution is valid: we may use β-Conversion on $[\lambda y\, \psi \,\&\, \theta]x$ to obtain $(\psi \,\&\, \theta)^x_y$, i.e., $\psi^x_y \,\&\, \theta^x_y$; then use the Rule of Substitution to obtain $\chi^x_y \,\&\, \theta^x_y$, which again by β-Conversion, yields $[\lambda y\, \chi \,\&\, \theta]x$. But such substitutions are not generally valid; just consider the similar case but with encoding formulas. From $\vdash_\Box \psi \equiv \chi$, we may not use the Rule of Substitution to infer $x[\lambda y\, \chi \,\&\, \theta]$ from $x[\lambda y\, \psi \,\&\, \theta]$, nor could such an inference be validly derived by other means. Of course, if ψ is a formula *defined* by χ, then rules for substituting definiens for definiendum would permit us to substituted χ for ψ in both $[\lambda y\, \psi \,\&\, \theta]x$ and $x[\lambda y\, \psi \,\&\, \theta]$. See the discussion in (208) and, in particular, the discussion of Rule SubDefForm in (208.5).

(**114**) **Remark:** Legitimate and Illegitimate Uses of the Rule of Substitution. Here are some legitimate examples of (113):

> **Example 1**.
> If $\Gamma \vdash \neg A!x$ and $\vdash_\Box A!x \equiv \neg\Diamond E!x$, then $\Gamma \vdash \neg\neg\Diamond E!x$.

> **Example 2**.
> If $\Gamma \vdash p \to Rxy$ and $\vdash_\Box Rxy \equiv (Rxy \,\&\, (Qa \vee \neg Qa))$, then
> $\Gamma \vdash p \to (Rxy \,\&\, (Qa \vee \neg Qa))$.

> **Example 3**.
> If $\vdash \exists x A!x$ and $\vdash_\Box A!x \equiv \neg\Diamond E!x$, then $\vdash \exists x \neg\Diamond E!x$.

> **Example 4**.
> If $\vdash \mathcal{A}\neg\neg Px$ and $\vdash_\Box \neg\neg Px \equiv Px$, then $\vdash \mathcal{A}Px$.

Example 5.
If $\vdash \Box(\varphi \to \psi)$ and $\vdash_{\Box} (\varphi \to \psi) \equiv (\neg\psi \to \neg\varphi)$, then $\vdash \Box(\neg\psi \to \neg\varphi)$.

Example 6.
If $\vdash \Box(\varphi \to \psi)$ and $\vdash_{\Box} \psi \equiv \chi$, then $\vdash \Box(\varphi \to \chi)$.

Example 7.
If $\vdash \Box(\varphi \to \varphi)$ and $\vdash_{\Box} \varphi \equiv \neg\neg\varphi$, then $\vdash \Box(\neg\neg\varphi \to \varphi)$.

Though the following is also a legitimate instance of the Rule of Substitution, it can only be invoked in certain circumstances:

- If $\vdash \Box\mathcal{A}\varphi$ and $\vdash_{\Box} \mathcal{A}\varphi \equiv \varphi$, then $\vdash \Box\varphi$

For arbitrary φ, we can't establish that $\vdash_{\Box} \mathcal{A}\varphi \equiv \varphi$, and so we can't generally substitute φ for $\mathcal{A}\varphi$ in $\Box\mathcal{A}\varphi$ to obtain $\vdash \Box\varphi$ from $\vdash \Box\mathcal{A}\varphi$. One exception is the case where φ is some necessary truth, say $\Box\psi$, for then axiom (33.2) is that $\Box\psi \equiv \mathcal{A}\Box\psi$, which by contraposition is $\mathcal{A}\Box\psi \equiv \Box\psi$. Since there is, by definition, a modally strict proof of this axiom, we can substitute $\Box\psi$ for $\mathcal{A}\Box\psi$ in $\Box\mathcal{A}\Box\psi$ to obtain $\Box\Box\psi$.[112]

Note also the following *illegitimate* instance of the Rule of Substitution:

- If $\vdash Pa = Pa$ and $\vdash_{\Box} Pa \equiv (Pa \,\&\, (Qb \vee \neg Qb))$, then
  $\vdash Pa = (Pa \,\&\, (Qb \vee \neg Qb))$

Though we can establish both $\vdash Pa = Pa$ and $\vdash_{\Box} Pa \equiv (Pa \,\&\, (Qb \vee \neg Qb))$, we may not conclude $\vdash Pa = (Pa \,\&\, (Qb \vee \neg Qb))$ because Pa is *not* a subformula of $Pa = Pa$! When the defined notation $Pa = Pa$ is expanded, the formula Pa will appear only inside complex terms and so the definition of *subformula* won't count Pa as a subformula of the formula $Pa = Pa$.

Before we explain this in detail, note that $Pa = Pa$ is a defined identity formula of the form $\Pi^0 = \Pi^0$, where Π^0 is a 0-place relation term. It is easy to show $Pa = Pa$ is a theorem: we first prove $p = p$ (67.1), then apply GEN to obtain $\forall p(p = p)$, and then use Variant Rule $\forall$E to instantiate Pa into the universal claim to obtain $Pa = Pa$ (we can do this because Pa is a 0-place relation term substitutable for p in $p = p$). Moreover, we can show $\vdash_{\Box} Pa \equiv (Pa \,\&\, (Qb \vee \neg Qb))$, for it is an instance of a tautology $\varphi \equiv (\varphi \,\&\, (\psi \vee \neg\psi))$, which is easily provable as a modally-strict theorem.

Notwithstanding these derivations, the fact is that Pa is not a subformula of the theorem $Pa = Pa$. To see why, expand the latter by applying definition (16.3). For then we obtain: $[\lambda y\, Pa] = [\lambda y\, Pa]$. Though this, too, is a defined formula, at this point we should suspect that already Pa is not a subformula

[112] Another exception will become clear once we reach item (126.10), where we prove, via a modally strict proof, that $\mathcal{A}xF \equiv xF$. Thus, we can substitute xF for $\mathcal{A}xF$ whenever $\mathcal{A}xF$ appears somewhere as a subformula, and vice versa.

of $[\lambda y\, Pa] = [\lambda y\, Pa]$. However, to complete our proof that it is not, we expand $[\lambda y\, Pa] = [\lambda y\, Pa]$ by definition (16.1), to obtain:

$$\Box\forall x(x[\lambda y\, Pa] \equiv x[\lambda y\, Pa])$$

The above is the result of expanding $Pa = Pa$ into primitive notation, and by applying the definition of *subformula* (8) to the above formula, we can see that Pa doesn't count as one of its subformulas.[113] Hence, we may not use the Rule of Substitution to substitute $Pa\,\&\,(Qb \vee \neg Qb)$ for Pa in $Pa = Pa$ to obtain $Pa = (Pa\,\&\,(Qb \vee \neg Qb))$. The moral here is this: one may not use the Rule of Substitution to identify necessarily equivalent propositions.

By the same token, one may not use the Rule of Substitution to identify necessarily equivalent properties. Consider the following illegitimate application of the rule, related to the one above:

- If $\vdash Px = Px$ and $\vdash_\Box Px \equiv [\lambda y\, Py\,\&\,(Qb \vee \neg Qb)]x$, then $\vdash Px = [\lambda y\, Py\,\&\,(Qb \vee \neg Qb)]x$

One can establish $\vdash Px = Px$ in the same manner that we established $\vdash Pa = Pa$. Moreover, one can establish $\vdash_\Box Px \equiv [\lambda y\, Py\,\&\,(Qb \vee \neg Qb)]x$ as follows. First, appeal to the following instance of β-Conversion (36.2):

$$[\lambda y\, Py\,\&\,(Qb \vee \neg Qb)]x \equiv Px\,\&\,(Qb \vee \neg Qb)$$

Now $Px\&(Qb\vee\neg Qb)$ is provably equivalent Px; it is an instance of the tautology $(\varphi\,\&\,(\psi \vee \neg\psi)) \equiv \varphi$. Thus, it follows by a biconditional syllogism and commutativity of $\equiv$ that $Px \equiv [\lambda y\, Py\,\&\,(Qb \vee \neg Qb)]x$. Since this proof is modally-strict, we've established that:

(ϑ) $\vdash_\Box Px \equiv [\lambda y\, Py\,\&\,(Qb \vee \neg Qb)]x$

But though we have established both $\vdash Px = Px$ and (ϑ), we may not apply the Rule of Substitution to infer $\vdash Px = [\lambda y\, Py\,\&\,(Qb \vee \neg Qb)]x$. Px is not a subformula of $Px = Px$ and so we don't have a legitimate instance of the rule.

(**115**) **Metarule/Derived Rule:** Rule for Alphabetic Variants. The Rules of Necessary Equivalence not only help us to derive the Rule of Substitution, but also help us derive an important rule about alphabetically-variant formulas and terms, as these are defined in item (35):

Rule of Alphabetic Variants
$\Gamma \vdash \varphi$ if and only if $\Gamma \vdash \varphi'$, where φ' is any alphabetic variant of φ
[Variant $\varphi \dashv\vdash \varphi'$]

[113] By (8.1), $\Box\forall x(x[\lambda y\, Pa] \equiv x[\lambda y\, Pa])$ is a subformula of itself; so by (8.2), $\forall x(x[\lambda y\, Pa] \equiv x[\lambda y\, Pa])$ is also a subformula; by (8.2) again, $x[\lambda y\, Pa] \equiv x[\lambda y\, Pa]$ is also a subformula; and by Metatheorem $\langle 7.3\rangle$, $x[\lambda y\, Pa]$ is also a subformula. These are the only subformulas of $\Box\forall x(x[\lambda y\, Pa] \equiv x[\lambda y\, Pa])$.

As a special case, when $\Gamma = \varnothing$, our rule asserts that a formula is a theorem if and only if any of its alphabetic variants is a theorem. We henceforth use the Variant form within derivations as a derived rule.

(**116**) **Theorems:** Theorems About Alphabetic Variants. It follows from the preceding rule: (.1) that alphabetically-variant formulas are equivalent, and (.2) that alphabetic-variants of logically proper descriptions can be identified:

(.1) $\varphi \equiv \varphi'$, where φ' is an alphabetic variant of φ

(.2) $\exists y(y = \iota\nu\varphi) \rightarrow \iota\nu\varphi = (\iota\nu\varphi)'$, where y doesn't occur free in φ and $(\iota\nu\varphi)'$ is any alphabetic variant of $\iota\nu\varphi$

Note that (.2) is a consequence, given (115.1), of the facts that logically proper descriptions can be instantiated into the universal claim $\forall x(x = x)$ and that formulas of the form $\iota\nu\varphi = \iota\nu\varphi$ and $\iota\nu\varphi = (\iota\nu\varphi)'$ are alphabetic variants by virtue of the fact that they differ only with respect to alphabetically-variant terms.

(**117**) **Theorems:** Additional K Theorems. The Rules of Necessary Equivalence and Substitution often simplify the proof of the following theorems:

(.1) $\Box\varphi \equiv \Box\neg\neg\varphi$

(.2) $\neg\Box\varphi \equiv \Diamond\neg\varphi$

(.3) $\Box\varphi \equiv \neg\Diamond\neg\varphi$ (Df$\Box$)

(.4) $\Box\neg\varphi \equiv \neg\Diamond\varphi$

(.5) $\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$ (K$\Diamond$)

(.6) $\Diamond(\varphi \vee \psi) \equiv (\Diamond\varphi \vee \Diamond\psi)$

(.7) $(\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$

(.8) $\Diamond(\varphi \,\&\, \psi) \rightarrow (\Diamond\varphi \,\&\, \Diamond\psi)$

(.9) $\Diamond(\varphi \rightarrow \psi) \equiv (\Box\varphi \rightarrow \Diamond\psi)$

(.10) $\Diamond\Box\varphi \equiv \neg\Box\Diamond\neg\varphi$

(.11) $\Diamond\Diamond\varphi \equiv \neg\Box\Box\neg\varphi$

(.12) $\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Diamond\psi)$

(**118**) **Theorems:** A Theorem of T. The following theorem depends on the T schema (32.2):

$$\varphi \to \Diamond\varphi \qquad (\mathrm{T}\Diamond)$$

(**119**) **Theorems:** Basic S5 Theorems. The system S5 is based on the K, T, and 5 schemata, i.e., (32.1) – (32.3). Chellas 1980 (16–18) informed the development of the following list of theorems of that system:

(.1) $\Diamond\Box\varphi \to \Box\varphi$ (5$\Diamond$)

(.2) $\Box\varphi \equiv \Diamond\Box\varphi$

(.3) $\Diamond\varphi \equiv \Box\Diamond\varphi$

(.4) $\varphi \to \Box\Diamond\varphi$ (B)

(.5) $\Diamond\Box\varphi \to \varphi$ (B$\Diamond$)

(.6) $\Box\varphi \to \Box\Box\varphi$ (4)

(.7) $\Box\varphi \equiv \Box\Box\varphi$

(.8) $\Diamond\Diamond\varphi \to \Diamond\varphi$ (4$\Diamond$)

(.9) $\Diamond\Diamond\varphi \equiv \Diamond\varphi$

(.10) $\Box(\varphi \vee \Box\psi) \equiv (\Box\varphi \vee \Box\psi)$

(.11) $\Box(\varphi \vee \Diamond\psi) \equiv (\Box\varphi \vee \Diamond\psi)$

(.12) $\Diamond(\varphi \,\&\, \Diamond\psi) \equiv (\Diamond\varphi \,\&\, \Diamond\psi)$

(.13) $\Diamond(\varphi \,\&\, \Box\psi) \equiv (\Diamond\varphi \,\&\, \Box\psi)$

(.14) $\Box(\varphi \to \Box\psi) \equiv \Box(\Diamond\varphi \to \psi)$

(**120**) **Theorems:** Conditions for, and Consequences of, Modal Collapse. We now investigate a special case of the last theorem (119.14), in which we take ψ to be φ itself. It is a theorem that (.1) if necessarily, φ implies $\Box\varphi$, then φ is possible if and only if φ is necessary:

(.1) $\Box(\varphi \to \Box\varphi) \to (\Diamond\varphi \equiv \Box\varphi)$

So when the condition $\Box(\varphi \to \Box\varphi)$ holds, φ suffers from *modal collapse* — possibly φ becomes equivalent to necessarily φ and so φ is not subject to modal distinctions. Most modal systems try to avoid modal collapse, and as long as we're considering exemplification formulas, modal collapse *should* be avoided. But, as we shall see, modal collapse for encoding formulas is something that our system embraces, to good effect.

It also follows that (.2) if necessarily, φ implies $\Box\varphi$, then φ fails to be necessary if and only if necessarily φ is false:

(.2) $\Box(\varphi \to \Box\varphi) \to (\neg\Box\varphi \equiv \Box\neg\varphi)$

This theorem helps us to establish an important other fact about the conditions producing modal collapse, namely, (.3) if necessarily, φ implies $\Box\varphi$, and necessarily, ψ implies $\Box\psi$, then the material equivalence of $\Box\varphi$ and $\Box\psi$ implies that $\varphi \equiv \psi$ is necessary:

(.3) $(\Box(\varphi \to \Box\varphi) \,\&\, \Box(\psi \to \Box\psi)) \to ((\Box\varphi \equiv \Box\psi) \to \Box(\varphi \equiv \psi))$

In other words, (.3) tells us that when both φ and ψ are subject to conditions producing modal collapse, the converse of (111.6) holds. (.3) plays an important role in the proofs of (126.5) and (189.1), which are key theorems.

(**121**) **Metarules:** Consequences of the B and B$\Diamond$ Schemata. The following rules are derivable with the help of the B (119.4) and B$\Diamond$ (119.5) schemata:

(.1) If $\Gamma \vdash_{\Box} \Diamond\varphi \to \psi$, then $\Box\Gamma \vdash \varphi \to \Box\psi$

(.2) If $\Gamma \vdash_{\Box} \varphi \to \Box\psi$, then $\Box\Gamma \vdash \Diamond\varphi \to \psi$

When Γ is empty and there are no premises or assumptions involved, the above reduce to:

If $\vdash_{\Box} \Diamond\varphi \to \psi$, then $\vdash \varphi \to \Box\psi$

If $\vdash_{\Box} \varphi \to \Box\psi$, then $\vdash \Diamond\varphi \to \psi$

(**122**) **Theorems:** Barcan Formulas.

(.1) $\forall\alpha\Box\varphi \to \Box\forall\alpha\varphi$ — Barcan Formula (= BF)

(.2) $\Box\forall\alpha\varphi \to \forall\alpha\Box\varphi$ — (Converse Barcan Formula = CBF)

(.3) $\Diamond\exists\alpha\varphi \to \exists\alpha\Diamond\varphi$ — (BF$\Diamond$)

(.4) $\exists\alpha\Diamond\varphi \to \Diamond\exists\alpha\varphi$ — (CBF$\Diamond$)

(**123**) **Theorems:** Other Theorems of Modal Quantification.

(.1) $\exists\alpha\Box\varphi \to \Box\exists\alpha\varphi$ — (Buridan)

(.2) $\Diamond\forall\alpha\varphi \to \forall\alpha\Diamond\varphi$ — (Buridan$\Diamond$)

(.3) $\Diamond\exists\alpha(\varphi \,\&\, \psi) \to \Diamond(\exists\alpha\varphi \,\&\, \exists\alpha\psi)$

(.4) $(\Box\forall\alpha(\varphi \to \psi) \,\&\, \Box\forall\alpha(\psi \to \chi)) \to \Box\forall\alpha(\varphi \to \chi)$

(.5) $(\Box\forall\alpha(\varphi \equiv \psi) \,\&\, \Box\forall\alpha(\psi \equiv \chi)) \to \Box\forall\alpha(\varphi \equiv \chi)$

(**124**) **Theorems:** Identity, Necessity, and Possibility. The foregoing theorems and rules now allow us to prove: (.1) α and β are possibly identical if and only if they are identical; (.2) α and β are distinct if and only if they are necessarily distinct; and (.3) α and β are possibly distinct if and only if they are distinct:

(.1) $\Diamond\alpha = \beta \equiv \alpha = \beta$

(.2) $\alpha \neq \beta \equiv \Box\alpha \neq \beta$

(.3) $\Diamond\alpha \neq \beta \equiv \alpha \neq \beta$

As usual, these theorems assume that α and β are both variables of the same type, i.e., both individual variables, or for some n, both n-place relation variables.

(**125**) **Lemmas:** Logically Proper (Rigid) Descriptions and Necessity.

(.1) $\exists y(y = \imath x\varphi) \to \exists y\Box(y = \imath x\varphi)$

(.2) $\exists y(y = \imath x\varphi) \to \Box\exists y(y = \imath x\varphi)$

These theoorems have modally-strict proofs. Note that (.1) does not say that: if there exists *the* x that happens to be φ then there exists something that necessarily is the x that happens to be φ. Since the description is rigid, it should be read as: if something is the x in fact such that φ then something necessarily is the x in fact such that φ. This understanding is confirmed by the theorems in item (103).

(**126**) **Theorem:** Encoding, Modality and Actuality. We now prove a host of important *modally-strict* theorems about the status of encoding predications: (.1) possibly x encodes F iff necessarily x encodes F; (.2) x encodes F iff necessarily x encodes F; (.3) possibly x encodes F iff x encodes F; (.4) xF and yG are materially equivalent iff $\Box xF$ and $\Box yG$ are materially equivalent; (.5) xF and yG are necessarily equivalent iff $\Box xF$ and $\Box yG$ are materially equivalent; (.6) xF and yG are materially equivalent iff it is necessary that they are materially equivalent; (.7) x fails to encode F iff necessarily x fails to encode F; (.8) possibly x fails to encode F iff x fails to encode F; (.9) possibly x fails to encode F iff necessarily x fails to encode F; and (.10) actually x encodes F iff x encodes F:

(.1) $\Diamond xF \equiv \Box xF$

(.2) $xF \equiv \Box xF$

(.3) $\Diamond xF \equiv xF$

(.4) $(xF \equiv yG) \equiv (\Box xF \equiv \Box yG)$

(.5) $\Box(xF \equiv yG) \equiv (\Box xF \equiv \Box yG)$

(.6) $(xF \equiv yG) \equiv \Box(xF \equiv yG)$

(.7) $\neg xF \equiv \Box\neg xF$

(.8) $\Diamond\neg xF \equiv \neg xF$

(.9) $\Diamond\neg xF \equiv \Box\neg xF$

(.10) $\mathcal{A}xF \equiv xF$

(.1) is especially significant. (.1) has a dual significance. Logically, (.1) is a fact about encoding predication, namely, that they are subject to modal collapse; the possible truth of xF is *equivalent* to its necessary truth. Metaphysically, (.1) is a fact about abstract objects, namely, that what they encode is not relative to any circumstance.

(.4) – (.6) are also interesting. (.4) tells us that if two encoding formulas are equivalent, then their necessitations are equivalent. (.5) is significant *not* because of the left-to-right direction, which is just an instance (111.6), but because of its right-to-left direction. In general, $\Box\varphi \equiv \Box\psi$ doesn't materially imply the claim $\Box(\varphi \equiv \psi)$, as we saw in the brief discussion following (111.6). But when φ and ψ are two encoding claims, such as xF and yG, the implication holds because of both encoding claims are subject to modal collapse, as we would expect from the discussion of theorem (120.3). (.6) is an simple but interesting consequence of (.4) and (.5). **Exercise:** Use (.6) to prove that $\Diamond(xF \equiv yG) \equiv (xF \equiv yG)$ and that $\Diamond(xF \equiv yG) \equiv \Box(xF \equiv yG)$.

Finally, (.10) is significant because it is a modally-strict theorem and has the form $\mathcal{A}\varphi \equiv \varphi$, but is provable without an appeal the necessitation-averse schema having this form (30)$\bigstar$.

9.9 The Theory of Relations

In this subsection, we describe some important theorems that govern properties, relations, and propositions.

(**127**) **Theorems:** Fact about Necessary Equivalence and Complex Relations. If φ^* and ψ^* are equivalent propositional formulas, then objects $x_1,\ldots,x_n$ stand in the relation *being* $y_1,\ldots,y_n$ *such that* φ^* if and only if they stand in the relation *being* $y_1,\ldots,y_n$ *such that* ψ^*:

(.1) $(\varphi^* \equiv \psi^*) \to [\lambda y_1 \ldots y_n\, \varphi^{*}{}^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n \equiv [\lambda y_1 \ldots y_n\, \psi^{*}{}^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n$, provided $y_1,\ldots,y_n$ are substitutable, respectively, for $x_1,\ldots,x_n$ in φ^* and ψ^*, and don't occur free in φ^* and ψ^*

This theorem generalizes:

(.1) and (.2) apply only to proper λ-expressions; see p. 191

(.2) $\Box\forall x_1 \dots \forall x_n(\varphi^* \equiv \psi^*) \rightarrow$
$\Box\forall x_1 \dots \forall x_n([\lambda y_1 \dots y_n\, \varphi^{*y_1,\dots,y_n}_{x_1,\dots,x_n}]x_1 \dots x_n \equiv [\lambda y_1 \dots y_n\, \psi^{*y_1,\dots,y_n}_{x_1,\dots,x_n}]x_1 \dots x_n)$,
provided $y_1, \dots, y_n$ are substitutable, respectively, for $x_1, \dots, x_n$ in φ^* and don't occur free in φ

As an example of (.2), we have:

$$\Box\forall x(\neg\Diamond E!x \equiv \Box\neg E!x) \rightarrow \Box\forall x([\lambda y\, \neg\Diamond E!y]x \equiv [\lambda y\, \Box\neg E!y]x)$$

Note that this does *not* imply that one can substitute $[\lambda y\, \Box\neg E!y]$ for $[\lambda y\, \neg\Diamond E!y]$ in any context; this theorem doesn't assert that $[\lambda y\, \neg\Diamond E!y] = [\lambda y\, \Box\neg E!y]$ whenever $\Box\forall x(\neg\Diamond E!x \equiv \Box\neg E!x)$.

(**128**) **Theorems:** Strengthened β-Conversion. Though we stated our axiom of β-Conversion (36.2) governing λ-expressions using (a) specific object-language variables in the expression $[\lambda y_1 \dots y_n\, \varphi^*]$ and (b) exemplification predications of the form $[\lambda y_1 \dots y_n\, \varphi^*]x_1 \dots x_n$, we can derive β-Conversion for: (i) any λ-expressions, (ii) exemplifications involving any variables other that $y_1, \dots, y_n$ that are bound by the λ and (iii) any variables other than $x_1, \dots, x_n$ in the exemplification predication.

Let $\mu_1, \dots, \mu_n$ be any distinct individual variables, and let $\nu_1, \dots, \nu_n$ be any individual variables. Strengthened β-Conversion then states:

$[\lambda \mu_1 \dots \mu_n\, \varphi^*]\nu_1 \dots \nu_n \equiv \varphi^{*\nu_1,\dots,\nu_n}_{\mu_1,\dots,\mu_n}$, provided the ν_i are substitutable, respectively, for the μ_i in φ^* $(1 \leq i \leq n)$ *and provided [λμ1…μn ϕ*] is proper; see p. 191*

Henceforth, when we appeal to instances of this theorem schema, we'll just cite β-Conversion instead of Strengthened β-Conversion.

(**129**) **Theorems:** Comprehension Principle for Properties and Relations. The following is a theorem schema derivable from β-Conversion (36.2):

(.1) $\exists F^n \Box\forall x_1 \dots \forall x_n(F^n x_1 \dots x_n \equiv \varphi^*)$, where $n \geq 1$ and φ^* is any propositional formula in which F^n is not free. *and provided no x_i bound by ∀ occurs in a description in ϕ*.*

When $n = 1$, we have a comprehension condition on properties:

(.2) $\exists F \Box\forall x(Fx \equiv \varphi^*)$, where φ^* is any propositional formula in which F is not free.

Since (129.1) and (129.2) provide existence conditions for relations and properties, respectively, and the definitions of $F^n = G^n$ $(n \geq 2)$ and $F^1 = G^1$ in (16.2) and (16.1) provide respective identity conditions, these principles constitute a rigorous theory of relations and properties. As noted previously, not only are there two senses in which properties can be *materially equivalent*, namely $\forall x(Fx \equiv Gx)$ and $\forall x(xF \equiv xG)$, but also there are two senses in which they can be *necessarily equivalent*, namely, $\Box\forall x(Fx \equiv Gx)$ and $\Box\forall x(xF \equiv xG)$. Necessary equivalence in the former sense does not guarantee property identity, whereas necessary equivalence in the latter sense does, by (16.1).

(**130**) **Theorem:** Equivalence and Identity of Properties. The following theorem establishes that our theory of properties constitutes an *extensional* theory of intensional entities, since it shows that all we need to do to prove the identity of two properties is establish that they are materially equivalent in the encoding sense:

$$\forall x(xF \equiv xG) \rightarrow F = G$$

Thus, to establish $F = G$, it suffices to establish that F and G are encoded by the same objects.

(**131**) **Theorems:** η-, β, α, and $\imath$-Conversion for Propositional Formulas. We have as *theorem* schemata the following conversion principles governing 0-place λ-expressions and propositional formulas:

(.1) $[\lambda\, \varphi^*] = \varphi^*$ (η-Conversion)

(.2) $[\lambda\, \varphi^*] \equiv \varphi^*$ (β-Conversion)

(.3) $\varphi^* = \varphi^{*\prime}$, where $\varphi^{*\prime}$ is any alphabetic variant of φ^* (α-Conversion)

(.4) $\mathcal{A}(\varphi \equiv \psi) \rightarrow (\chi^* = \chi^{*\prime})$, where φ and ψ are any formulas, χ^* is any propositional formula, and $\chi^{*\prime}$ is the result of substituting $\imath x\psi$ for zero or more occurrences of $\imath x\varphi$ anywhere the latter occurs in χ^* ($\imath$-Conversion)

In (.1), $[\lambda\, \varphi^*]$ and φ^* are being used as terms and so we read (.1) as: *that*-φ^* is identical to φ^*. In (.2), however, $[\lambda\, \varphi^*]$ and φ^* are being used as formulas and so we we (.2) as: *that*-φ^* is true if and only if φ^*. With (.2), we have established that the propositional version of the Tarski T-Schema is a *theorem* (Zalta 2014)! (.3) and (.4) are derivable from α-Conversion (36.1) and $\imath$-Conversion (36.4), respectively, with the help of (.1).

(**132**) **Remark:** On the Derivability of Tautologies. The derivation of (131.1) and (131.2) allows us to finish the proof of Metatheorem ⟨9.1⟩. This metatheorem is the key lemma needed for the proof of Metatheorem ⟨9.2⟩, i.e., that all tautologies are derivable. See the proofs of these metatheorems in the Appendix to this chapter. Note that from these metatheorems, Metatheorem ⟨9.3⟩ also follows, namely, that every tautology is necessary.

(**133**) **Theorems:** Comprehension Principle for Propositions. The following comprehension principle for propositions is a consequence of (131.2), by RN and $\exists$I:

$\exists p \Box (p \equiv \varphi^*)$,
where φ^* is any propositional formula with no free occurrences of p.

This comprehension principle and definition (16.3) jointly offer a precise theory of propositions or, if you prefer, *states of affairs*.[114] The claim that propositions are necessarily equivalent, i.e., $\Box(p \equiv q)$, does not entail that p and q are identical. For some propositions p, q, one may consistently assert, that $\Box(p \equiv q) \,\&\, p \neq q$.

(**134**) **Remark:** Identity of Relations. The theory of identity for n-place relations ($n \geq 0$) asserted in (16) takes on new significance in light of the Comprehension Principles for Relations, Properties, and Propositions. Given the present context, one can look back on the definitions in (16) and see that they are designed to answer one question in a precise way: under what conditions are relations identical? An answer to this questions tells us what we have to prove to establish that $F^n = G^n$, for any n. The theory does not take a stand on the question 'Is R identical to S?' for arbitrary relation expressions 'R' and 'S' of natural language. In some cases, one can prove that R and S are distinct, e.g., when it is provable that possibly, R and S aren't equivalent (see the next theorem). But in cases of necessarily equivalent relations, the matter is often open. For example, if one has a good reason to think that *being red and round* is identical to *being round and red*, one may, when the theory is applied, assert this identity.[115] By the same token, if one has good reason to think these properties are distinct, one may assert their distinctness, in the defined sense that there is an (abstract) object that encodes the one without encoding the other (and vice versa). Thus, we are free to answer the question, 'How fine-grained are relations?' in ways that match our intuitions. But once one adds the assertion that

[114] We may, for present purposes, treat propositions and states of affairs as the same entities. Ontologically speaking, it is probably better to conceive of 0-place relations as *states of affairs*, since 'proposition' sometimes has a connotation on which it signifies a linguistic or mental entity. I've used the term 'proposition' mostly as a matter of economy.

In a type-theoretic version of our system, which allows one to represent beliefs as higher-order relations between individuals and 0-place relations, there is more of a reason to distinguish between propositions and states of affairs. In the type theory developed in Chapter 15 (Section 15.2), there are entities of type $\langle i, \langle\,\rangle\rangle$ that relate entities of type i (the type for individuals) to entities of type $\langle\,\rangle$ (the type for 0-place relations). One may suppose that the *de re* objects of belief are *states of affairs*. These are entities of type $\langle\,\rangle$ whose constituents are the denotations of the terms inside the belief context. By contrast, whenever the objects of belief are entities of type $\langle\,\rangle$ whose constituents are the senses of the terms inside the belief context, we may suppose that these objects of *de dicto* beliefs are propositions.

Thus, when it comes time to represent beliefs as higher-order relations, a proposition becomes an entity of type $\langle\,\rangle$ having abtract constituents (e.g., abstract individuals or abstract relations) that *represent* other relations and individuals. These abstract individuals and abstract relations serve as the senses of individual terms and relation terms, respectively. In the present 'second-order' setting, however, where higher-order relations between individuals and 0-place relations are not expressible, we may forego the distinction between propositions and states of affairs and leave it for another occasion.

[115] Indeed, if one has a strong intuition about this kind of case, one could assert: $\forall F \forall G([\lambda x\, Fx \,\&\, Gx] = [\lambda x\, Gx \,\&\, Fx])$.

particular properties F and G are identical, then our theory guarantees that there are no abstract objects that encode the one without encoding the other.

(**135**) **Theorems:** Fact about Property Non-identity. Properties that might fail to be materially equivalent are distinct:

$$\Diamond\neg\forall x(Fx \equiv Gx) \rightarrow F \neq G$$

(**136**) **Term Definitions:** Definition of Relation Negation. We introduce $\overline{F^n}$ ('being non-F^n') to abbreviate *being* $y_1,\ldots,y_n$ *such that it is not the case that* $F^n y_1 \ldots y_n$ (for $n \geq 1$), and introduce $\overline{p}$ to abbreviate *that it is not the case that* p, as follows:

(.1) $\overline{F^n} \ =_{df} \ [\lambda y_1 \ldots y_n \, \neg F^n y_1 \ldots y_n]$ $\qquad (n \geq 1)$

(.2) $\overline{p} \ =_{df} \ [\lambda \, \neg p]$

So, when F is a property, $\overline{F} = [\lambda y \, \neg Fy]$, and when R is a 2-place relation, $\overline{R} = [\lambda yz \, \neg Ryz]$.

(**137**) **Theorems:** Relations and their Negations.

(.1) $\overline{F^n} x_1 \ldots x_n \equiv \neg F^n x_1 \ldots x_n$ $\qquad (n \geq 1)$

(.2) $\neg\overline{F^n} x_1 \ldots x_n \equiv F^n x_1 \ldots x_n$ $\qquad (n \geq 1)$

(.3) $\overline{p} \equiv \neg p$

(.4) $\neg\overline{p} \equiv p$

(.5) $F^n \neq \overline{F^n}$ $\qquad (n \geq 1)$

(.6) $p \neq \overline{p}$

Note that we also have:

(.7) $\overline{p} = \neg p$

(.8) $p \neq \neg p$

(.9) $p = q \rightarrow \neg p = \neg q$

(.10) $p = q \rightarrow \overline{p} = \overline{q}$

(**138**) **Definitions:** Noncontingent and Contingent Relations. Let us say: (.1) a relation F^n ($n \geq 0$) is *necessary* just in case necessarily, all objects $x_1,\ldots,x_n$ are such that $x_1,\ldots,x_n$ exemplify F^n; (.2) F^n is *impossible* just in case necessarily, all objects $x_1,\ldots,x_n$ are such that $x_1,\ldots,x_n$ fail to exemplify F^n; (.3) F^n is *noncontingent* whenever it is necessary or impossible; and (.4) F^n is *contingent* whenever it is neither necessary nor impossible:

(.1) $Necessary(F^n) =_{df} \Box\forall x_1 \ldots \forall x_n F^n x_1 \ldots x_n$ $(n \geq 0)$

(.2) $Impossible(F^n) =_{df} \Box\forall x_1 \ldots \forall x_n \neg F^n x_1 \ldots x_n$ $(n \geq 0)$

(.3) $NonContingent(F^n) =_{df} Necessary(F^n) \vee Impossible(F^n)$ $(n \geq 0)$

(.4) $Contingent(F^n) =_{df} \neg(Necessary(F^n) \vee Impossible(F^n))$ $(n \geq 0)$

(**139**) **Theorems:** Facts about Noncontingent and Contingent Properties. If we focus just on properties as opposed to n-place relations generally, the following facts can be established:

(.1) $NonContingent(F^1) \equiv NonContingent(\overline{F^1})$

(.2) $Contingent(F) \equiv \Diamond\exists xFx \;\&\; \Diamond\exists x\neg Fx$

(.3) $Contingent(F^1) \equiv Contingent(\overline{F^1})$

(**140**) **Theorems:** Some Noncontingent Properties. Let $L = [\lambda x\, E!x \rightarrow E!x]$ ('being concrete if concrete'). Then we have:

(.1) $Necessary(L)$

(.2) $Impossible(\overline{L})$

(.3) $NonContingent(L)$

(.4) $NonContingent(\overline{L})$

(.5) $\exists F\exists G(F \neq G \;\&\; NonContingent(F) \;\&\; NonContingent(G))$,
i.e., there are at least two noncontingent properties.

(**141**) **Lemmas:** A Symmetry. It is possible that something that exemplifies F might not have if and only if it is possible that something that doesn't exemplify F might have:

$$\Diamond\exists x(Fx \;\&\; \Diamond\neg Fx) \equiv \Diamond\exists x(\neg Fx \;\&\; \Diamond Fx)$$

If we think semantically for the moment and take possible worlds as primitive entities, then this lemma tells us:

> There is a world where something both exemplifies F and, at some (other) world, fails to exemplify F
> if and only if
> there is a world where something both fails to exemplify F and, at some (other) world, exemplifies F.

So, semantically, the symmetry involves possible worlds. Algebraically, however, the symmetry is between F and its negation $\overline{F}$, since the above is equivalent to:

$$\Diamond\exists x(Fx \,\&\, \Diamond\neg Fx) \equiv \Diamond\exists x(\overline{F}x \,\&\, \Diamond\neg\overline{F}x)$$

once we apply (137.1), (137.2) and the Rule of Substitution to the right-hand condition.

(**142**) **Theorems:** $E!$ and $\overline{E!}$ are Contingent Properties.

(.1) $\Diamond\exists x(\neg E!x \,\&\, \Diamond E!x)$

(.2) $Contingent(E!)$

(.3) $Contingent(\overline{E!})$

(.4) $\exists F\exists G(F \neq G \,\&\, Contingent(F) \,\&\, Contingent(G))$,
i.e., there are at least two contingent properties.

(**143**) **Theorems:** Facts about Property Existence. Where we continue to use L to abbreviate $[\lambda x\, E!x \rightarrow E!x]$, we have the following general and specific facts about the existence of properties:

(.1) $NonContingent(F) \rightarrow \neg\exists G(Contingent(G) \,\&\, G = F)$

(.2) $Contingent(F) \rightarrow \neg\exists G(NonContingent(G) \,\&\, G = F)$

(.3) $L \neq \overline{L} \,\&\, L \neq E! \,\&\, L \neq \overline{E!} \,\&\, \overline{L} \neq E! \,\&\, \overline{L} \neq \overline{E!} \,\&\, E! \neq \overline{E!}$,
i.e., L, $\overline{L}$, $E!$, and $\overline{E!}$ are pairwise distinct

(.4) There are at least four properties.

(**144**) **Theorems:** Facts about Noncontingent and Contingent Propositions. If we focus now just on propositions, the following facts can be established:

(.1) $NonContingent(p) \equiv NonContingent(\overline{p})$

(.2) $Contingent(p) \equiv \Diamond p \,\&\, \Diamond\neg p$

(.3) $Contingent(p) \equiv Contingent(\overline{p})$

(**145**) **Theorems:** Some Noncontingent Propositions. Let $p_0 = \forall x(E!x \rightarrow E!x)$. Then we have:

(.1) $Necessary(p_0)$

(.2) $Impossible(\overline{p_0})$

(.3) $NonContingent(p_0)$

(.4) $NonContingent(\overline{p_0})$

(.5) $\exists p \exists q(p \neq q \,\&\, NonContingent(p) \,\&\, NonContingent(q))$,
i.e., there are at least two noncontingent propositions.

(**146**) **Theorems:** Some Contingent Propositions. Let q_0 be the proposition $\exists x(E!x \,\&\, \Diamond\neg E!x)$. Intuitively, q_0 asserts the existence of contingently concrete objects (i.e., objects that exemplify being concrete but possibly do not). Given this definition of q_0, axiom (32.4) becomes $\Diamond q_0 \,\&\, \Diamond\neg q_0$. The following claims regarding propositions thus become derivable:

(.1) $\exists p(\Diamond p \,\&\, \Diamond\neg p)$

(.2) $Contingent(q_0)$

(.3) $Contingent(\overline{q_0})$

(.4) $\exists p \exists q(p \neq q \,\&\, Contingent(p) \,\&\, Contingent(q))$,
i.e., there are at least two contingent propositions.

(**147**) **Theorems:** Facts about Proposition Existence. Where we continue to use p_0 to abbreviate $\forall x(E!x \to E!x)$ and q_0 to abbreviate $\exists x(E!x \,\&\, \Diamond\neg E!x)$, we have the following general and specific facts about the existence of propositions:

(.1) $NonContingent(p) \to \neg\exists q(Contingent(q) \,\&\, q = p)$

(.2) $Contingent(p) \to \neg\exists q(NonContingent(q) \,\&\, q = p)$

(.3) $p_0 \neq \overline{p_0} \,\&\, p_0 \neq q_0 \,\&\, p_0 \neq \overline{q_0} \,\&\, \overline{p_0} \neq q_0 \,\&\, \overline{p_0} \neq \overline{q_0} \,\&\, q_0 \neq \overline{q_0}$,
i.e., p_0, $\overline{p_0}$, q_0, and $\overline{q_0}$ are pairwise distinct

(.4) There are at least four propositions.

(**148**) **Definitions:** Contingently True and Contingently False Propositions. Let us say that: (.1) a proposition p is *contingently true* just in case p is true and possibly false, and (.2) p is *contingently false* just in case p is false but possible true:

(.1) $ContingentlyTrue(p) =_{df} p \,\&\, \Diamond\neg p$

(.2) $ContingentlyFalse(p) =_{df} \neg p \,\&\, \Diamond p$

Note that if one were add, as an axiom, either the claim that some particular proposition is contingently true, or the claim that some particular proposition is contingently false, then these would have to be marked as necessitation-averse axioms.

(**149**) **Theorems:** Contingently True (False) and Contingent. As to be expected from the previous definitions and the definition of *contingent* in item (138.4), it follows that (.1) if p is contingently true, it is contingent; and (.2) if p is contingently false, it is contingent:

(.1) $ContingentlyTrue(p) \rightarrow Contingent(p)$

(.2) $ContingentlyFalse(p) \rightarrow Contingent(p)$

(.3) $ContingentlyTrue(p) \equiv ContingentlyFalse(\overline{p})$

(.4) $ContingentlyFalse(p) \equiv ContingentlyTrue(\overline{p})$

(**150**) **Theorems:** Facts about Contingently True and Contingently False Propositions. Recall that in (146) we let q_0 be the proposition $\exists x(E!x \,\&\, \Diamond\neg E!x)$. Clearly, given (32.4), it follows that (.1) q_0 is contingently true or contingently false; and (.2) either q_0 is contingently false or its negation is contingently true:

(.1) $ContingentlyTrue(q_0) \vee ContingentlyFalse(q_0)$

(.2) $ContingentlyFalse(q_0) \vee ContingentlyFalse(\overline{q}_0)$

The proof of (.1) goes by way of a disjunctive syllogism from the tautology $q_0 \vee \neg q_0$ and axiom (32.4) plays an important role, again revealing its importance. We've formulated (.2) because it plays a key role in the proof of the Fundamental Theorem governing impossible worlds (468).

These theorems allow us to conclude something stronger, namely, that (.3) some proposition is contingently true; and (.4) some proposition is contingently false:

(.3) $\exists p ContingentlyTrue(p)$

(.4) $\exists p ContingentlyFalse(p)$

Finally, two other theorems are worthy of mention. (.5) if p is contingently true and q is necessary, then p is not identical to q; and (.6) if p is contingently false and q is impossible, then p is not identical to q:

(.5) $(ContingentlyTrue(p) \,\&\, Necesary(q)) \rightarrow p \neq q$

(.6) $(ContingentlyFalse(p) \,\&\, Impossible(q)) \rightarrow p \neq q$

Recall that *Necessary*(q) *Impossible*(q) were defined as the 0-place cases of (138.1) and (138.2), respectively, as $\Box q$ and $\Box\neg q$.

(**151**) **Remark:** On the Existence of Contingent Truths and Falsehoods. Though many philosophers would surely grant that there are contingently true and contingently false propositions, the problem just solved by the previous theorems is: How are we to establish (150.3) and (150.4) at the present stage of development of the theory without either (a) asserting, for some particular individual x and property F, either $Fx\,\&\,\Diamond\neg Fx$ or $\neg Fx\,\&\,\Diamond Fx$, or (b) asserting as an axiom that both a contingent truth and a contingent falsehood exist? The solution involves axiom (32.4), which allows us to prove that there are contingent truths and contingent falsehoods. The proof isn't constructive, though; it doesn't tell us which proposition is contingently true and which is contingently false.

Since our deductive system is prepared for the addition of contingently true axioms, one might wonder why haven't we extended our system by asserting q_0, i.e., $\exists x(E!x\,\&\,\Diamond\neg E!x)$, as such an axiom. The reason is that we cannot justifiably assert q_0 *a priori*. Were q_0 justifiably assertible *a priori*, Berkeley's idealism would be convincingly refutable *a priori*. It isn't. *A posteriori* evidence derived from our senses may lead us to assign q_0 a very high probability. But all that our theory can and should assert *a priori* is embodied by axiom (32.4), i.e., that there might be contingently concrete objects and there might not be.

We've already justified axiom (32.4) on the grounds that it offers a way of making explicit what logicians and metaphysicians assume to be true, namely, that the domain of individuals might either contain, or be empty of, contingently concrete objects. But it also has the virtue of grounding a presupposition of Leibniz's famous question, "Why is there something rather than nothing?" (Article 7, Principles of Nature and Grace, 1714, PW 199, G.vi 602). Leibniz poses this question in the context of considering the natural world, and it is not unreasonable to suppose that he is asking, "Why is there something contingently concrete rather than nothing contingently concrete?" This presupposes that there might be no contingently concrete objects, which is precisely what the second conjunct of (32.4) asserts *a priori*. But we can't also justifiably assert q_0 *a priori*, only its possible truth and possible falsity. Fortunately, this is enough to imply *a priori* that there is a contingently true proposition and a contingently false proposition.

Exercise. Say what is wrong with the reasoning in (A) and (B), which both attempt to derive a contradiction in our system.

(A) By (150.3), we know $\exists p\mathit{ContingentlyTrue}(p)$. Suppose p_1 is an arbitrary such proposition, so that we know *ContingentlyTrue*(p_1). Then, by definition (148.1), $p_1\,\&\,\Diamond\neg p_1$. So by &E, p_1. Since this reasoning is modally strict, we may use RN to infer $\Box p_1$, i.e., $\neg\Diamond\neg p_1$, by (117.3). But this contradicts $\Diamond\neg p_1$.

(B) By (150.3), we know $\exists p\mathit{ContingentlyTrue}(p)$. Suppose p_1 is an arbitrary such proposition, so that we know $\mathit{ContingentlyTrue}(p_1)$. Then by RN, know $\Box\mathit{ContingentlyTrue}(p_1)$. But by definition (148.1), it follows that $\Box(p_1 \,\&\, \Diamond\neg p_1)$. By (111.3) and &E, it follows that $\Box p_1$. But now we have a contradiction, since it follows from the fact that p_1 is contingently true that $\Diamond\neg p_1$, by definition (148.1). But this is equivalent to $\neg\Box p_1$.

These examples are interesting because (150.3) and (150.4) are modally strict theorems, but their witnesses are necessitation-averse.

(**152**) **Theorems:** $O!$ and $A!$ Are Contingent. The properties *being ordinary* and *being abstract* are distinct, contradictory contingent properties:

(.1) $O! \neq A!$

(.2) $O!x \equiv \neg A!x$

(.3) $A!x \equiv \neg O!x$

(.4) $\mathit{Contingent}(O!)$

(.5) $\mathit{Contingent}(A!)$

Moreover, the negations of *being ordinary* and *being abstract* are distinct and contradictory contingent properties:

(.6) $\overline{O!} \neq \overline{A!}$

(.7) $\overline{O!}x \equiv \neg\overline{A!}x$

(.8) $\mathit{Contingent}(\overline{O!})$

(.9) $\mathit{Contingent}(\overline{A!})$

(**153**) **Theorems:** Further Facts about Being Ordinary and Being Abstract.

(.1) $O!x \rightarrow \Box O!x$

(.2) $A!x \rightarrow \Box A!x$

(.3) $\Diamond O!x \rightarrow O!x$

(.4) $\Diamond A!x \rightarrow A!x$

(.5) $\Diamond O!x \equiv \Box O!x$

(.6) $\Diamond A!x \equiv \Box A!x$

(.7) $O!x \equiv \mathcal{A}O!x$

(.8) $A!x \equiv \mathcal{A}A!x$

The last two theorems are especially interesting. By the commutativity of the biconditional (63.3.g), they both imply claims of the form $\mathcal{A}\varphi \equiv \varphi$ and so have a form similar to the necessitation-averse axiom (30)★. But we can prove them by modally-strict means, without an appeal to this axiom.

(**154**) **Definition:** Weakly Contingent Properties. We say that a property F is *weakly contingent* just in case F is contingent and anything that possibly exemplifies F necessarily exemplifies F:

$$\mathit{WeaklyContingent}(F) \ =_{df} \ \mathit{Contingent}(F) \,\&\, \forall x(\Diamond Fx \rightarrow \Box Fx)$$

(**155**) **Theorems:** Facts about Weakly Contingent Properties. (.1) F is weakly contingent iff $\overline{F}$ is weakly contingent; (.2) if F is weakly contingent and G is not, then F is not G:

(.1) $\mathit{WeaklyContingent}(F) \equiv \mathit{WeaklyContingent}(\overline{F})$

(.2) $(\mathit{WeaklyContingent}(F) \,\&\, \neg\mathit{WeaklyContingent}(G)) \rightarrow F \neq G$

(**156**) **Theorems:** Facts about $O!$, $A!$, $E!$, and L. If we continue to use L to abbreviate $[\lambda x\, E!x \rightarrow E!x]$ (i.e., *being concrete if concrete*), we have the following facts: (.1) *being ordinary* is weakly contingent; (.2) *being abstract* is weakly contingent; (.3) *being concrete* is not weakly contingent; (.4) *being concrete if concrete* is not weakly contingent; (.5) *being ordinary* is distinct from: $E!$, $\overline{E!}$, L, and $\overline{L}$; (.6) *being abstract* is distinct from: $E!$, $\overline{E!}$, L, and $\overline{L}$; (.7) There are at least six distinct properties:

(.1) $\mathit{WeaklyContingent}(O!)$

(.2) $\mathit{WeaklyContingent}(A!)$

(.3) $\neg\mathit{WeaklyContingent}(E!)$

(.4) $\neg\mathit{WeaklyContingent}(L)$

(.5) $O! \neq E! \,\&\, O! \neq \overline{E!} \,\&\, O! \neq L \,\&\, O! \neq \overline{L}$

(.6) $A! \neq E! \,\&\, A! \neq \overline{E!} \,\&\, A! \neq L \,\&\, A! \neq \overline{L}$

(.7) There are at least six properties, namely, $E!$, $\overline{E!}$, $O!$, $A!$, L, and $\overline{L}$.

Note, however, that our axioms and definitions don't appear to imply $O! \neq \overline{A!}$ or imply $A! \neq \overline{O!}$. Hence, the facts (exercise) that (.5) holds for $\overline{O!}$ (i.e., $\overline{O!}$ is distinct from $E!$, $\overline{E!}$, L, and $\overline{L}$) and that (.6) holds for $\overline{A!}$, don't imply that there are eight distinct properties.

(**157**) **Theorems:** Identity$_E$, Necessity, and Possibility. (.1) objects x and y are identical$_E$ if and only if they are necessarily identical$_E$; and (.2) objects x and y are possibly identical$_E$ if and only if they are identical$_E$:

(.1) $x =_E y \equiv \Box x =_E y$

(.2) $\Diamond x =_E y \equiv x =_E y$

(**158**) **Term and Formula Definitions:** Distinctness$_E$. We now introduce a new 2-place relation term *being distinct*$_E$:

(.1) $\neq_E \;=_{df}\; \overline{=_E}$

Here the definiens combines defined notation: it combines the defined negation overbar '$\overline{}$' (136.1) with the defined term $=_E$ (12). So $\neq_E$ is defined as: *being an x and y such that it is not the case that x bears $=_E$ to y*. To complement our term definition, we also introduce the following formula definition for infix notation:

(.2) $x \neq_E y \;=_{df}\; \neq_E xy$

These definitions play an important role when we develop the theory of Fregean numbers in Chapter 14: not only do λ-expressions such as $[\lambda x\, x \neq_E a]$ play a role in the definition of the *Predecessor* relation but also the modal properties of the $\neq_E$ relation are crucial to the derivation of the Dedekind-Peano axioms for number theory as theorems of the present theory.

It might come as a surprise just how many definitions must be unpacked in order to expand $[\lambda x\, x \neq_E a]$ into primitive notation and the defined & symbol:

$[\lambda x\;\, x \neq_E a]$
$= [\lambda x\;\, \neq_E xa]$ By (158.2)
$= [\lambda x\, \overline{=_E} xa]$ By (158.1)
$= [\lambda x\, [\lambda y_1 y_2\, \neg(=_E y_1 y_2)]xa]$ By (136.1)
$= [\lambda x\, [\lambda y_1 y_2\, \neg[\lambda xy\; O!x \;\&\; O!y \;\&\; \Box\forall F(Fx \equiv Fy)]y_1 y_2]xa]$ By (12)
$= [\lambda x\, [\lambda y_1 y_2\, \neg[\lambda xy\; [\lambda x\, \Diamond E!x]x \;\&\; [\lambda x\, \Diamond E!x]y \;\&\; \Box\forall F(Fx \equiv Fy)]y_1 y_2]xa]$
By (11.1)
$= [\lambda x\, [\lambda y_1 y_2\, \neg[\lambda xy\; [\lambda x\, \neg\Box\neg E!x]x \;\&\; [\lambda x\, \neg\Box\neg E!x]y \;\&\; \Box\forall F(Fx \equiv Fy)]y_1 y_2]xa]$
By (7.4.e)
$= [\lambda x\, [\lambda y_1 y_2\, \neg[\lambda xy\; [\lambda x\, \neg\Box\neg E!x]x \;\&\; [\lambda x\, \neg\Box\neg E!x]y \;\&\; \Box\forall F((Fx \rightarrow Fy) \;\&$
$(Fy \rightarrow Fx))]y_1 y_2]xa]$ By (7.4.c)

Fortunately, the proof of those theorems below that involve the formula $x \neq_E y$ do not require that we dig quite as deep into its chain of definitions.

(**159**) **Theorem:** Equivalence of Non-identity$_E$ and Not Identical$_E$.

$$x \neq_E y \equiv \neg(x =_E y)$$

Note that this fact is not an immediate consequence of a single formula definition, but instead requires that we cite a number of term and formula definitions, as well as principles like β-Conversion.

(**160**) **Theorems:** Non-identity$_E$, Necessity, and Possibility. (.1) objects x and y are distinct$_E$ if and only if necessarily they are distinct$_E$, and (.2) objects x and y are possibly distinct$_E$ if and only if then they are distinct$_E$:

(.1) $x \neq_E y \equiv \Box x \neq_E y$

(.2) $\Diamond x \neq_E y \equiv x \neq_E y$

(**161**) **Theorems:** Identity$_E$, Identity, and Actuality.

(.1) $x =_E y \equiv \mathcal{A} x =_E y$

(.2) $x \neq_E y \equiv \mathcal{A} x \neq_E y$

(.3) $\alpha = \beta \equiv \mathcal{A}\alpha = \beta$, where α, β are variables of the same type

(.4) $\alpha \neq \beta \equiv \mathcal{A}\alpha \neq \beta$, where α, β are variables of the same type

It is important to observe here that these modally-strict theorems, provable *without* the necessitation-averse axiom (30)$\star$ of actuality.

(**162**) **Theorem:** A Distinguished Description Involving Identity. (161.3) plays a key role in the proof of the following theorem:

$$y = \imath x(x = y)$$

This asserts: y is identical to the individual that (in fact) is identical to y.

This is a modally-strict theorem and though it appears to be trivial, keep in mind that identity is a defined notion and that definite descriptions are rigid and axiomatized in terms of the actuality operator. Notice that Frege uses a somewhat more complicated version of this claim as an *axiom* governing his definite description operator. In Frege 1893 (§18), Frege asserts Basic Law VI, which we may write as: $y = \backslash \acute{\epsilon}(\epsilon = y)$. On Frege's understanding of the description operator, Law VI basically asserts that y is identical to the unique member of the extension of the concept *being identical to* y. Our theorem, by contrast asserts: y is identical to the unique object x that is identical to y. No reference to the extension of a concept is needed.

(**163**) **Definitions:** η-Variants. We now work our way towards theorems (164) and (165), which are further consequences of η-Conversion (36.3). Let ρ be a complex n-place relation term ($n \geq 0$). Then we say:

(.1) ρ is *elementary* if and only if ρ has the form $[\lambda \nu_1 \ldots \nu_n \, \Pi^n \nu_1 \ldots \nu_n]$, where Π^n is any n-place relation term and $\nu_1, \ldots, \nu_n$ are distinct individual variables none of which occur free in Π^n.

So when $n = 0$, $[\lambda \, \Pi^0]$ is elementary. Furthermore, we say:

(.2) ρ is an *η-expansion of* Π^n if and only if ρ is the elementary λ-expression $[\lambda \nu_1 \ldots \nu_n \, \Pi^n \nu_1 \ldots \nu_n]$

(.3) Π^n is the *η-contraction of* ρ if and only if ρ is the elementary λ-expression $[\lambda \nu_1 \ldots \nu_n \, \Pi^n \nu_1 \ldots \nu_n]$

Note that ρ may have many different η-expansions, depending on the choice of $\nu_1, \ldots, \nu_n$, but an elementary λ-expression ρ can have only one η-contraction.

Before we proceed, note that we again will be using the prime symbol $'$ to help us define η-variants. Thus, we are deploying this symbol in yet a third way. Previously, we've used the symbol to help us formulate the Substitution of Identicals (25) and to talk about alphabetic variants (35). But no confusion should arise, since the context shall make it clear how the prime symbol $'$ is being used.

Next, then, where ρ and ρ' are n-place relation terms ($n \geq 0$), we say:

(.4) ρ' is an *immediate* (i.e., one-step) *η-variant of* ρ just in case ρ' results from ρ either (a) by replacing one n-place relation term Π^n in ρ by an η-expansion $[\lambda \nu_1 \ldots \nu_n \, \Pi^n \nu_1 \ldots \nu_n]$ or (b) by replacing one elementary λ-expression $[\lambda \nu_1 \ldots \nu_n \, \Pi^n \nu_1 \ldots \nu_n]$ in ρ by its η-contraction Π^n.

Note that this definition even applies in the case where ρ and ρ' are the 0-place propositional formulas φ^* and $\varphi^{*\prime}$, since both are relation terms. Thus, $[\lambda \, \Pi^0]$ and Π^0 are immediate η-variants of each other. Clearly, in general, if ρ' is an immediate η-variant of ρ, then ρ is an immediate η-variant of ρ'.

Finally we say, for n-place relation terms ρ and ρ' ($n \geq 0$):

(.5) ρ' is an *η-variant of* ρ whenever there is a sequence of n-place relation terms $\rho_1, \ldots, \rho_m$ ($m \geq 1$) with $\rho = \rho_1$ and $\rho' = \rho_m$ such that every member of the sequence is an immediate η-variant of the preceding member of the sequence.

(.6) ρ is *η-irreducible* just in case ρ contains no λ-expressions within λ-expressions.

Thus, the metalinguistic relation ρ *is an* η*-variant of* ρ' is the transitive closure of the relation ρ *is an immediate* η*-variant of* ρ'.[116] We now illustrate these definitions with examples:

Examples of Elementary η-Variants (Expansion/Contraction Pairs):

- $[\lambda xyz\, F^3xyz]$ / F^3
- $[\lambda x\, [\lambda y\, \neg Fy]x]$ / $[\lambda y\, \neg Fy]$
- $[\lambda xy\, [\lambda uv\, \Box\forall F(Fu \equiv Fv)]xy]$ / $[\lambda uv\, \Box\forall F(Fu \equiv Fv)]$
- $[\lambda\, p]$ / p
- $[\lambda\, \neg Pa]$ / $\neg Pa$

Immediate η-Variants

- All of the above
- $[\lambda y\, [\lambda z\, Pz]y \rightarrow Say]$ / $[\lambda y\, Py \rightarrow Say]$
- $[\lambda y\, Py \rightarrow [\lambda uv\, Suv]ay]$ / $[\lambda y\, Py \rightarrow Say]$
- $[\lambda y\, [\lambda z\, Pz]y \rightarrow [\lambda uv\, Suv]ay]$ / $[\lambda y\, [\lambda z\, Pz]y \rightarrow Say]$
- $[\lambda y\, [\lambda z\, Pz]y \rightarrow [\lambda uv\, Suv]ay]$ / $[\lambda y\, Py \rightarrow [\lambda uv\, Suv]ay]$
- $[\lambda y\, [\lambda\, p]]$ / $[\lambda y\, p]$
- $[\lambda x_1 \ldots x_n\, [\lambda\, Pa]]$ / $[\lambda x_1 \ldots x_n\, Pa]$
- $[\lambda z\, Pz]y \rightarrow Say$ / $Py \rightarrow Say$
- $[\lambda\, [\lambda z\, Pz]y]$ / $[\lambda\, Py]$

η-Variant Pairs

- All of the above
- $[\lambda y\, [\lambda z\, Pz]y \rightarrow [\lambda uv\, Suv]ay]$ / $[\lambda y\, Py \rightarrow Say]$
- $[\lambda z\, Pz]y \rightarrow [\lambda uv\, Suv]ay$ / $Py \rightarrow Say$
- $[\lambda y\, [\lambda z\, Pz]y \rightarrow Say]$ / $[\lambda y\, Py \rightarrow [\lambda uv\, Suv]ay]$

With a thorough grip on the notion of η-variants, we may prove the following theorems.

(**164**) **Lemmas:** Useful Facts about η-Conversion.

other than an improper λ-expression; see p. 191

(.1) $[\lambda x_1 \ldots x_n\, \Pi^n x_1 \ldots x_n] = \Pi^n$, where Π^n is any n-place relation term $(n \geq 0)$ in which none of $x_1, \ldots, x_n$ occur free

[116] Intuitively, the transitive closure of R is that relation R' that relates any two elements in a chain of R-related elements. That is, for any elements x and y of a domain of R-related elements, $xR'y$ holds whenever there exist $z_0, z_1, \ldots, z_n$ such that (i) $z_0 = x$, (ii) $z_n = y$, and (iii) for all $0 \leq i < n$, $z_i R z_{i+1}$.

(.2) $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n] = \Pi^n$, where $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n]$ is any elementary λ-expression and $\nu_1, \dots, \nu_n$ are any distinct individual variables none of which occur free in Π^n.[117] *and Π is not an improper λ-expression; see p. 191*

(.3) $\rho = \rho'$, whenever ρ' is an *immediate* η-variant of ρ
ϱ and ϱ' don't contain improper λ-expressions; see p. 191

(**165**) **Theorems:** η-Conversion for arbitrary η-Variants. Recall that ρ is a metavariable ranging over n-place relation terms. Then we have:

$$\rho = \rho', \text{ where } \rho' \text{ is any } \eta\text{-variant of } \rho$$

ϱ and ϱ' don't contain improper λ-expressions; see p. 191

As an instance of this theorem schema, we have following identity claim, where the λ-expression on the left-hand side of the identity symbol contains two embedded λ-expressions:

$$[\lambda y\, [\lambda z\, Pz]y \to [\lambda uv\, Suv]ay] = [\lambda y\, Py \to Say]$$

Here is a proof of this claim:

1.	$[\lambda z\, Pz] = P$	ηC
2.	$[\lambda uv\, Suv] = S$	ηC
3.	$[\lambda y\, [\lambda z\, Pz]y \to [\lambda uv\, Suv]ay] = [\lambda y\, [\lambda z\, Pz]y \to [\lambda uv\, Suv]ay]$	ReflId
4.	$[\lambda y\, [\lambda z\, Pz]y \to [\lambda uv\, Suv]ay] = [\lambda y\, Py \to [\lambda uv\, Suv]ay]$	SubId, 1, 3
5.	$[\lambda y\, [\lambda z\, Pz]y \to [\lambda uv\, Suv]ay] = [\lambda y\, Py \to Say]$	SubId, 2, 4

(**166**) **Theorem:** Propositional Equations.

$$[\lambda\, p] = [\lambda\, q] \equiv p = q$$

9.10 The Theory of Objects

(**167**) **Theorems:** The Domain of Objects is Partitioned. To show the domain of objects is partitioned, we prove two theorems. First, every object is either ordinary or abstract:

(.1) $\forall x(O!x \vee A!x)$

Second, no object is both ordinary and abstract:

(.2) $\neg\exists x(O!x \,\&\, A!x)$

These are modally-strict theorems and, hence, necessary truths.

(**168**) **Theorems:** Identity$_E$ is an Equivalence Relation on Ordinary Objects:

[117] This last clause is, strictly speaking, unnecessary, since we've defined *elementary* λ-expressions above so that in the elementary expression $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n]$, none of the ν_i occur free in Π^n.

(.1) $O!x \rightarrow x =_E x$

(.2) $x =_E y \rightarrow y =_E x$

(.3) $(x =_E y \;\&\; y =_E z) \rightarrow x =_E z$

(**169**) **Theorems:** Ordinary Objects and Identity. It now follows that if either x or y is ordinary, the x is identical to y if and only if x is E-identical to y:

$$(O!x \vee O!y) \rightarrow (x = y \equiv x =_E y)$$

(**170**) **Theorem:** Ordinary Objects Obey Leibniz's Law. The identity$_E$ of indiscernible ordinary objects is a theorem:

$$(O!x \;\&\; O!y) \rightarrow (\forall F(Fx \equiv Fy) \rightarrow x =_E y)$$

This tells us that if we have two ordinary objects x, y, then we simply have to show that they exemplify the same properties to conclude they are identical$_E$. We don't have to show that they necessarily exemplify the same properties to establish their identity$_E$. By contraposition, if we know two ordinary objects x, y are distinct$_E$, then we know that there exists a property that distinguishes them.

(**171**) **Theorem:** Distinct Ordinary Objects Have Distinct Haecceities. Recall that $[\lambda y\; y =_E x]$ is the *haecceity of* x. Then it follows that ordinary objects x, z are distinct if and only if their haecceities $[\lambda y\, y =_E x]$ and $[\lambda y\, y =_E z]$ are distinct, i.e.,

$$(O!x \;\&\; O!z) \rightarrow (x \neq z \equiv [\lambda y\; y =_E x] \neq [\lambda y\; y =_E z])$$

(**172**) **Theorem:** Abstract Objects Obey a Variant of Leibniz's Law. A variant of Leibniz's Law is now derivable, namely, (.1) whenever abstract objects x, y encode the same properties, they are identical:

(.1) $(A!x \;\&\; A!y) \rightarrow (\forall F(xF \equiv yF) \rightarrow x = y)$

Thus to show abstract objects x, y are identical, it suffices to prove that they encode the same properties; we don't have to show that they necessarily encode the same properties. Similarly, it follows that (.2) whenever one abstract object encodes a property the other fails to encode, they are distinct:

(.2) $(A!x \;\&\; A!y) \rightarrow (\exists F(xF \;\&\; \neg yF) \rightarrow x \neq y)$

Thus, to show abstract objects x and y are distinct, it suffices to show that x encodes a property that y doesn't.

(**173**) **Theorem:** Ordinary Objects Necessarily Fail to Encode Properties.

$O!x \rightarrow \Box\neg\exists FxF$

(**174**) **Theorem:** Ordinary Objects Necessarily Exist. Our system implies that (.1) possibly, there exists an object that exemplifies *being concrete*; (.2) necessarily, there exists an object that exemplifies *being ordinary*; and (.3) necessarily, it is not the case that every object exemplifies *being abstract*:

(.1) $\Diamond\exists xE!x$

(.2) $\Box\exists xO!x$

(.3) $\Box\neg\forall xA!x$

It is important to note that none of (.1) – (.3) implies $\exists xE!x$, i.e., these theorems don't imply the existence of any concrete objects. The existence of concrete objects is an empirical matter, subject to *a posteriori* investigation and not subject to a derivation from axioms asserted *a priori*.

(**175**) **Theorem:** Abstract Objects Necessarily Exist. Our system implies that (.1) necessarily, there exists an object that exemplifies *being abstract*; (.2) necessarily, it is not the case that everything exemplifies *being ordinary*; (.3) necessarily, it is not the case that every object exemplifies *being concrete*:

(.1) $\Box\exists xA!x$

(.2) $\Box\neg\forall xO!x$

(.3) $\Box\neg\forall xE!x$

(**176**) **Theorem:** Objects that Encode Properties Are Abstract. It follows by contraposing (38) that if x encodes a property, then x is abstract:

$\exists FxF \rightarrow A!x$

The converse fails because of there exists an abstract *null* object, which encodes no properties. See theorem (192.1) below.

(**177**) **Theorems:** Strengthened Comprehension for Abstract Objects. The Comprehension Principle for Abstract Objects (39) and the definition of identity (15) jointly imply that there is a *unique* abstract object that encodes just the properties such that φ:

$\exists!x(A!x \,\&\, \forall F(xF \equiv \varphi))$, provided x doesn't occur free in φ

The proof is simplified by appealing to (172).

(**178**) **Theorems:** Abstract Objects via Strengthened Comprehension. Strengthened Comprehension principle (177) asserts the unique existence of a number of interesting abstract objects. There exists a unique abstract object that encodes all and only the properties F such that: (.1) y exemplifies F; (.2) y and z exemplify F; (.3) y or z exemplify F; (.4) y necessarily exemplifies F; (.5) F is identical to property G; and (.6) F is necessarily implied by G:

(.1) $\exists!x(A!x \,\&\, \forall F(xF \equiv Fy))$

(.2) $\exists!x(A!x \,\&\, \forall F(xF \equiv Fy \,\&\, Fz))$

(.3) $\exists!x(A!x \,\&\, \forall F(xF \equiv Fy \vee Fz))$

(.4) $\exists!x(A!x \,\&\, \forall F(xF \equiv \Box Fy))$

(.5) $\exists!x(A!x \,\&\, \forall F(xF \equiv F = G))$

(.6) $\exists!x(A!x \,\&\, \forall F(xF \equiv \Box\forall y(Gy \rightarrow Fy)))$

Many of the above objects (and others) will figure prominently in the theorems which follow.

(**179**) **Lemmas:** Actuality and Unique Existence. The actuality operator commutes with the unique existence quantifier:

(.1) $\mathcal{A}\exists!\alpha\varphi \equiv \exists!\alpha\mathcal{A}\varphi$

Furthermore, given (.1), it follows from (107.1) that:

(.2) $\exists y(y = \imath x\varphi) \equiv \mathcal{A}\exists!x\varphi$, provided y doesn't occur free in φ

(.2) is especially important. If we can establish a claim of the form $\exists!x\varphi$ by way of a modally strict proof, then the Rule of Actualization (RA) yields a modally-strict proof of $\mathcal{A}\exists!x\varphi$. Then, by (.2), we can derive $\exists y(y = \imath x\varphi)$ as a modally-strict theorem.

(**180**) **Theorems:** Descriptions Guaranteed to be Logically Proper. It now follows, by a modally-strict proof, that for any condition φ in which x doesn't occur free, there exists something which is identical to the individual that is both abstract and encodes just the properties such that φ:

$$\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))), \text{ provided } x, y \text{ don't occur free in } \varphi$$

Although the above theorem is an immediate consequence of (177) and (100)★, the resulting proof wouldn't be modally-strict. There is, however, a modally-strict proof that appeals to (179.2).

Since the above schema has a modally-strict proof, its necessitation follows by RN. If we think semantically for a moment, and treat possible worlds as semantically primitive entities, it becomes clear that the necessitation of our schema does *not* say that at every world $\boldsymbol{w}$, there exists something which is identical to the x such that, at $\boldsymbol{w}$, x both exemplifies being abstract and encodes all and only the properties satisfying φ at $\boldsymbol{w}$. Rather, the necessitation says that at every possible world, there exists something identical to the x such that, at the distinguished actual world $\boldsymbol{w}_0$, x both exemplifies being abstract and encodes exactly the properties satisfying φ at $\boldsymbol{w}_0$.

(**181**) **Metadefinitions:** Canonical Descriptions, Matrices, and Canonical Individuals. The previous theorem guarantees that descriptions of the form $\imath x(A!x \& \forall F(xF \equiv \varphi))$ are logically proper (when x isn't free in φ). We henceforth say:

> A definite description is *canonical* iff it has the form $\imath \nu(A!\nu \& \forall F(\nu F \equiv \varphi))$, for some formula φ in which the individual variable ν doesn't occur free.

We call the matrix of a canonical description a *canonical matrix*. By a slight abuse of language, we call the individuals denoted by such descriptions *canonical* individuals.

(**182**) ⋆**Theorems:** Canonical Individuals Encode Their Defining Properties. In general, we cannot give a modally strict proof of the claim: anything identical to a canonical individual both exemplifies *being abstract* and encodes all and only the individual's defining properties. But there is, at least, a non-modally strict proof:

> $y = \imath x(A!x \& \forall F(xF \equiv \varphi)) \rightarrow (A!y \& \forall F(yF \equiv \varphi))$,
> provided x doesn't occur free in φ and y is substitutable for x in φ

This is just an instance of (101.2)⋆.

(**183**) **Theorems:** Canonical Individuals are Abstract. It is an easy, but non-modally strict consequence of the previous theorem that anything identical to a canonical individual exemplifies *being abstract*. However, there is a modally strict proof of this claim:

> $y = \imath x(A!x \& \forall F(xF \equiv \varphi)) \rightarrow A!y$,
> provided x doesn't occur free in φ and y is substitutable for x in φ

(**184**) ⋆**Theorems:** The Abstraction Principle. Before we state the Abstraction Principle, note that when φ is a formula in which x doesn't occur free and in which G is substitutable for F, the formula φ^G_F (i.e., the result of substituting G for every free occurrence of F in φ) asserts that G is such that φ. Here, we may think of φ^G_F as a syntactic representation of the semantic claim that G *satisfies* φ, where this particular notion of satisfaction was defined in Section 5.5.1, footnote 45. It is now provable that: the abstract object encoding exactly the properties such that φ encodes a property G if and only if G is such that φ:

$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi^G_F$,
provided x doesn't occur free in φ and G is substitutable for F in φ

To see an example, let b be an arbitrary object. Then we have the following instance of the above theorem: $\imath x(A!x \,\&\, \forall F(xF \equiv Fb))G \equiv Gb$. This asserts: the abstract object that encodes exactly the properties that b exemplifies encodes G iff b exemplifies G. The reason we call this 'abstraction' should be clear: in the right-to-left direction, we've abstracted out, from the simple predication Gb, an encoding claim about a particular abstract object.

Intuitively, we may describe this example as follows. When φ is formula Gb, then φ describes a unique logical pattern of predications about b, namely, the logical pattern consisting of those properties G that are such that Gb. The Strengthened Comprehension Principle (177) *objectifies* this pattern and asserts its unique existence. Then (180) guarantees that the canonical description of the objectified pattern is logically proper. Finally, the Abstraction Principle then yields an important truth about this objectified pattern, namely, that it encodes G if and only G matches the pattern. This way of looking at the Abstraction Principle applies to arbitrarily complex formulas φ in which x doesn't occur free, since these express a condition on properties and thereby define a logical pattern of properties.

(**185**) **Remark:** The Abstraction Principle and Necessitation. Inspection shows that the proof of the Abstraction Principle depends upon Hintikka's schema (98)$\star$, which in turn depends on the fundamental theorem (97)$\star$ governing descriptions. So the Abstraction Principle is not a modally strict theorem and is not subject to the Rule of Necessitation. It would serve well to get a broad perspective on these important facts. In the discussion that follows, we provide such a perspective in the material mode and leave the explanation in the formal mode to a footnote.

To understand more fully why RN can't be applied to instances of Abstraction, suppose we've extended our theory in a natural way with the following facts: b exemplifies being a philosopher but b might not have been a philosopher. That is, where P is the property *being a philosopher*, suppose we've extended our theory with:

$\star$**Fact**: Pb

Modal Fact: $\Diamond\neg Pb$

Thus, the claim Pb is contingently true, by definition (148.1). Moreover, on pain of contradiction, it is necessitation-averse and can't be subject to the Rule of Necessitation. That is why we've labeled Pb as a $\star$Fact. It seems reasonable to suggest that the above two claims are part of the data, i.e., part of the body of truths we're trying to systematize with our theory. So let's suppose that we have asserted them as axioms. Then any reasoning that depends on $\star$Fact fails to be modally strict.

Now consider the following consequence of the Abstraction Principle (184)$\star$:

(a) $\imath x(A!x \,\&\, \forall F(xF \equiv Fb))P \equiv Pb$

It then follows from (a) and our $\star$Fact that:

(b) $\imath x(A!x \,\&\, \forall F(xF \equiv Fb))P$

Since the description in (b) is canonical, it is logically proper (180), and so may be instantiated, along with P, in our axiom for the rigidity of encoding (37), which asserts $xF \to \Box xF$. By doing so, we may infer:

(c) $\Box\imath x(A!x \,\&\, \forall F(xF \equiv Fb))P$

Of course, the proof of (c) fails to be modally strict, since it was derived from *two* $\star$-claims: the above contingent $\star$Fact (Pb) and (a), which is derived from (184)$\star$.

Nevertheless, (c) is a theorem in the extended theory we're considering. Now if we could apply RN to (a), we would obtain:

(d) $\Box(\imath x(A!x \,\&\, \forall F(xF \equiv Fb))P \equiv Pb)$

Then from (d), (c), and the relevant instance of (111.6), which asserts that $\Box(\varphi \equiv \psi) \to (\Box\varphi \equiv \Box\psi)$, it would follow by MP that:

(e) $\Box Pb$

But this would contradict our Modal Fact $\Diamond\neg Pb$, which is equivalent to $\neg\Box Pb$. This demonstrates why RN isn't applicable to (consequences of) the Abstraction Principle — our system would become inconsistent if it were extended with contingent truths.[118]

[118]In the formal mode, the reason why we can't generally apply RN to instances of Abstraction is that there are interpretations in which the necessitation of the Abstraction Principle fails to be valid. To see this, let us again help ourselves for the moment to the semantically primitive notion of a possible world. The semantic counterpart of the above $\star$Fact and Modal Fact is any interpretation of our language in which Pb is true at the actual world $\boldsymbol{w}_0$ but false at some other possible world, say, $\boldsymbol{w}_1$. In such an interpretation, our consequence (a) of the Abstraction Principle fails to be necessarily true. (a) fails to be necessarily true in the left-to-right direction by the following argument. Since Pb is true at $\boldsymbol{w}_0$, $\imath x(A!x \,\&\, \forall F(xF \equiv Fb))$ encodes P at $\boldsymbol{w}_0$. Since the properties an object encodes are necessarily encoded (126.2), $\imath x(A!x \,\&\, \forall F(xF \equiv Fb))$ encodes P at $\boldsymbol{w}_1$. But, by hypothesis, Pb is false at $\boldsymbol{w}_1$. Hence, $\boldsymbol{w}_1$ is a world that is the witness to the truth of the following possibility claim:

$$\Diamond(\imath x(A!x \,\&\, \forall F(xF \equiv Fb))P \,\&\, \neg Pb)$$

Since the left-to-right condition of (a) isn't necessary in this interpretation, the necessitation of this condition fails to be valid.

Similarly, if we consider the negation of P, namely $\overline{P}$, with respect to the interpretation described

It is important to note that in the foregoing discussion, we have a described a scenario of the kind mentioned in (52), where we outlined conditions under which the converse of RN fails to hold. The converse of RN asserts that if $\vdash \Box\varphi$, then $\vdash_{\Box} \varphi$. But if we let φ be sentence (b) in the above scenario, then we indeed have $\vdash \Box\varphi$ and not $\vdash_{\Box} \varphi$. For we saw that since (b) is an encoding formula and a theorem, its necessitation (c) is also a theorem. But there isn't a modally strict proof of (b), since any proof would depend on our ⋆Fact Pb as well as some non-modally strict fact about descriptions, such as the right-to-left direction of the Abstraction Principle (184)⋆.[119]

It is worth digressing a moment to note both that the converse of RN fails and that we can show this even *without* extending our system with contingent facts. To see that there is a φ such that both $\vdash \Box\varphi$ but not $\vdash_{\Box} \varphi$, recall that in (150.3) we proved that there are contingent truths, i.e., that there is a p such that both p and $\Diamond\neg p$. Suppose p_1 is such a contingent truth, so that we know p_1 and $\Diamond\neg p_1$. Note that p_1 is necessitation-averse; if we could apply RN to p_1 to conclude $\Box p_1$, we would contradict $\Diamond\neg p_1$, since this is equivalent to $\neg\Box p_1$; see the Exercise at the end of Remark (151). Now consider the abstract object:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \exists q(q \,\&\, F = [\lambda y\, q])))$$

This is the abstract object that encodes all and only properties F of the form $[\lambda y\, q]$ where q is a true proposition. By the Abstraction Principle (184)⋆, we know:

$$(\vartheta)\quad \imath x(A!x \,\&\, \forall F(xF \equiv \exists q(q \,\&\, F = [\lambda y\, q])))[\lambda y\, p_1] \equiv \exists q(q \,\&\, [\lambda y\, p_1] = [\lambda y\, q])$$

above, then the following consequence of the Abstraction Principle fails to be necessarily true in the right-to-left direction:

$$\text{(f)}\quad \imath x(A!x \,\&\, \forall F(xF \equiv Fb))\overline{P} \equiv \overline{P}b$$

To see that it is possible for the right condition of (f) to be true while the left condition false, we may reason as follows. Since Pb is true at $\boldsymbol{w}_0$ in the interpretation we've described, $\overline{P}b$ is false at $\boldsymbol{w}_0$ and so $\imath x(A!x \& \forall F(xF \equiv Fb))$ fails to encode $\overline{P}$ at $\boldsymbol{w}_0$. Since the properties an abstract object fails to encode are properties it necessarily fails to encode (126.7), it follows that $\imath x(A!x \,\&\, \forall F(xF \equiv Fb))$ fails to exemplify $\overline{P}$ at $\boldsymbol{w}_1$. But, by hypothesis, Pb is false at $\boldsymbol{w}_1$ and so $\overline{P}b$ is true at $\boldsymbol{w}_1$. Hence, $\boldsymbol{w}_1$ is a witness to the truth of the following possibility claim:

$$\Diamond(\overline{P}b \,\&\, \neg\imath x(A!x \,\&\, \forall F(xF \equiv Fb))\overline{P})$$

Since this shows that the right-to-left direction of (f) isn't necessarily true in this interpretation, the necessitation of this direction fails to be valid.

[119] Since (b) is not an axiom, any proof of (b) must infer it by Modus Ponens from two previous lines in the proof. One of those lines has to be our ⋆Fact Pb and the other has to be the right-to-left direction of the Abstraction Principle (184)⋆. In other words, any proof of (b) must either include the lines:

$$Pb \qquad \star\text{Fact}$$

$$Pb \to \imath x(A!x \,\&\, \forall F(xF \equiv Fb))P \qquad (184)\star\ \text{(right-to-left)}$$

or include lines that imply these lines. But all such lines are necessitation-averse. See the second part of footnote 118 for the reasoning that shows why the right-to-left direction of the Abstraction Principle is necessitation-averse.

But the right side of this biconditional is derivable as follows: we already know p_1 and so we may conjoin it with $[\lambda y\, p_1] = [\lambda y\, p_1]$, which is an instance of theorem (71.1). From this conjunction it follows that $\exists q(q \,\&\, [\lambda y\, p_1] = [\lambda y\, q])$. Hence it follows from (ϑ) that:

(ξ) $\imath x(A!x \,\&\, \forall F(xF \equiv \exists q(q \,\&\, F = [\lambda y\, q])))[\lambda y\, p_1]$

By (37), it then follows that:

(ζ) $\Box\imath x(A!x \,\&\, \forall F(xF \equiv \exists q(q \,\&\, F = [\lambda y\, q])))[\lambda y\, p_1]$

So if we let φ be the encoding formula labeled (ξ), then (ζ) shows that $\vdash \Box\varphi$. But there is no modally strict proof of φ since any proof of φ would have to appeal to the necessitation-averse fact that p_1, as well as to the right-to-left direction of (ϑ). Hence, the converse of RN fails.

(**186**) **Theorems:** Actualized Abstraction. If we strategically place an actuality operator in the Abstraction Principle, we may formulate a version that has a modally strict proof, namely, the abstract object encoding just the properties such that φ encodes a property G if and only if it is actually the case that G is such that φ:

> $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \mathcal{A}\varphi^G_F$, provided x doesn't occur free in φ and G is substitutable for F in φ

The reader is encouraged to show, without the benefit of the proof in the Appendix, that this theorem can be proved without appealing to (30)$\star$ or any other $\star$-theorem.

(**187**) **Theorems:** Properties That Are Necessarily Such That φ. There are modally strict proofs of the claims: (.1) if G is necessarily such that φ, then $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ encodes G, and (.2) if G is necessarily such that φ, then necessarily, $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ encodes G if and only if G is such that φ:

(.1) $\Box\varphi^G_F \to \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$, provided x doesn't occur free in φ

(.2) $\Box\varphi^G_F \to \Box(\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi^G_F)$

The proof of (.1) in the Appendix uses Actualized Abstraction (186), and the proof of (.2) utilizes (.1). Note that (.2) describes sufficient conditions under which instances of the Abstraction Principle, as formulated in (184)$\star$, become necessary truths, namely, when G is a property that is necessarily such that φ.

(**188**) **Metadefinitions:** Rigid Conditions and Strictly Canonicity. We now introduce terms into our metalanguage for the special conditions under which canonical descriptions can be considered *strictly* canonical. Let φ be any formula in which x doesn't occur free but in which F may occur free. Then we say:

(.1) φ is a *rigid condition* on properties if and only if $\vdash \Box\forall F(\varphi \to \Box\varphi)$.

(.2) The canonical description $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ is *strictly canonical* just in case φ is a rigid condition on properties.

Henceforth, we sometimes abuse language and speak of the strictly canonical individuals denoted by strictly canonical descriptions.

(**189**) **Theorems:** Facts about Strict Canonicity. It is an important, modally strict fact that (.1) every abstract individual that encodes exactly the properties such that φ is necessarily an abstract individual that encodes exactly the properties such that φ, provided φ is a rigid condition on properties:

(.1) $(A!x \& \forall F(xF \equiv \varphi)) \to \Box(A!x \& \forall F(xF \equiv \varphi))$, provided φ is a rigid condition on properties in which x doesn't occur free.

Moreover, when φ is a rigid condition on properties, it is a modally strict fact that (.2) anything identical to a strictly canonical individual both exemplifies *being abstract* and encodes all and only the properties such that φ:

(.2) $y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)) \to (A!y \,\&\, \forall F(yF \equiv \varphi))$, provided φ is a rigid condition on properties in which x doesn't occur free.

By comparing (.2) with (182)★, which asserts $z = \imath x\psi \to \psi^z_x$, we see that (.2) is a special case in which (182)★ can be proved by modally strict means, namely, when the description $\imath x\psi$ is strictly canonical.

Finally, it is worth noting that the Abstraction Principle formulated in (184)★ becomes a modally strict theorem when restricted to strictly canonical objects:

(.3) $\imath x(A!x \& \forall F(xF \equiv \varphi))G \equiv \varphi^G_F$, provided φ is a rigid condition on properties in which both x doesn't occur free and G is substitutable for F.

As it turns out, however, (.2) will be more useful than (.3): since many new terms will be introduced by way of strictly canonical descripitions, (.2) becomes immediately applicable to the identity statements to which these definitions give rise.

(**190**) **Remark:** Some Examples of Strictly Canonical Individuals. In item (178), we presented a series of instances of the Strengthened Comprehension Principle for Abstract Objects. The last three are examples of strictly canonical individuals. Consider the following canonical descriptions based on those last three examples:

(a) $\imath x(A!x \,\&\, \forall F(xF \equiv \Box Fy))$

(b) $\imath x(A!x \,\&\, \forall F(xF \equiv F = G))$

(c) $\imath x(A!x \,\&\, \forall F(xF \equiv \Box\forall y(Gy \to Fy)))$

These are all instances of $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ and, hence, logically proper, by (180). Moreover, in each case, the formula φ in question is a rigid condition on properties, i.e., $\vdash \Box\forall F(\varphi \to \Box\varphi)$. We can see this as follows:

- In example (a), φ is $\Box Fy$. But as an instance of the 4 schema (119.6), we know $\Box Fy \to \Box\Box Fy$, i.e., $\varphi \to \Box\varphi$. Hence, by applying GEN, we have $\forall F(\varphi \to \Box\varphi)$, and by RN, $\Box\forall F(\varphi \to \Box\varphi)$.

- In example (b), φ is $F = G$, and so by the left-to-right direction of (124), we know that $\varphi \to \Box\varphi$. So, again, by GEN and RN, we have $\Box\forall F(\varphi \to \Box\varphi)$.

- In example (c), φ is $\Box\forall y(Gy \to Fy)$. Hence we can reason as we did in (a), by way of the 4 schema, to establish $\Box\forall F(\varphi \to \Box\varphi)$.

Thus, (a) – (c) are examples of strictly canonical individuals.

(**191**) **Definitions:** Null and Universal Objects. We say: (.1) x is a *null* object just in case x is an abstract object that encodes no properties; and (.2) x is a *universal* object just in case x is an abstract object that encodes every property:

(.1) $Null(x) \;=_{df}\; A!x \,\&\, \neg\exists FxF$

(.2) $Universal(x) \;=_{df}\; A!x \,\&\, \forall FxF$

(**192**) **Theorems:** Existence and Uniqueness of Null and Universal Objects. It is now easily established that (.1) that there is a unique null object and (.2) that there is a unique universal object:

(.1) $\exists!xNull(x)$

(.2) $\exists!xUniversal(x)$

Consequently, it follows, by a modally strict proof (.3) that the null object exists, and (.4) that the universal object exists:

(.3) $\exists y(y = \imath xNull(x))$

(.4) $\exists y(y = \imath xUniversal(x))$

(**193**) **Definitions:** Notation for the Null and Universal Objects. We may therefore introduce the following new terms to designate the null object and the universal object:

(.1) $\boldsymbol{a}_{\varnothing} \;=_{df}\; \imath xNull(x)$

(.2) $\boldsymbol{a}_{\boldsymbol{V}} \;=_{df}\; \imath xUniversal(x)$

(**194**) **Theorems:** Facts about the Null and Universal Object. The following are facts about the conditions $Null(x)$ and $Universal(x)$: (.1) if x is a null object, then necessarily x is a null object, and (.2) if x is a universal object, then necessarily x is a universal object:

(.1) $Null(x) \to \Box Null(x)$

(.2) $Universal(x) \to \Box Universal(x)$

These facts allow us to show (.1) the null object is a null object, and (.2) the universal object is a universal object:

(.3) $Null(\boldsymbol{a}_{\varnothing})$

(.4) $Universal(\boldsymbol{a}_{\boldsymbol{V}})$

Though these theorems sound trivial, their proof by modally strict means is not.

(**195**) **Remark:** A Rejected Alternative. Now that we've seen how to obtain a modally strict proof of $Null(\boldsymbol{a}_{\varnothing})$, one might wonder: why didn't we define $\boldsymbol{a}_{\varnothing}$ directly as:

(ϑ) $\boldsymbol{a}_{\varnothing} =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv F \neq F))$

After all, a modally strict proof of $Null(\boldsymbol{a}_{\varnothing})$ could still be given.[120] A similar observation could be made about $Universal(x)$ and $\boldsymbol{a}_{\boldsymbol{V}}$, but we henceforth consider only $\boldsymbol{a}_{\varnothing}$, since the considerations are analogous.

The alternative definition (ϑ) is certainly legitimate, but we have a good reason for not deploying it. Since (ϑ) defines $\boldsymbol{a}_{\varnothing}$ as the abstract object that encodes all and only non-self-identical properties, $\boldsymbol{a}_{\varnothing}$ is *not* explicitly introduced as the null object *per se*. Of course, one can then show that given (ϑ), $\boldsymbol{a}_{\varnothing}$ is a *unique* null object (i.e., a unique abstract object that encodes no properties) and start referencing it as 'the null object'. But this fails to take advantage of the fact that our system allows us to properly define the condition $Null(x)$ as $A!x \,\&\, \neg\exists F xF$, prove that there is a unique such object, show that the description $\imath x Null(x)$ is logically proper, and then use the well-defined description $\imath x Null(x)$ to introduce the notation $\boldsymbol{a}_{\varnothing}$. If we are going to use the expression $\boldsymbol{a}_{\varnothing}$ to name the null object, the correct way to do so is to define $\boldsymbol{a}_{\varnothing}$ as $\imath x Null(x)$

[120]Let φ be the formula $F \neq F$ and let χ be the formula $A!x \,\&\, \forall F(xF \equiv F \neq F)$. Now by applying GEN and then RN to an appropriate instance of (124.2), we can establish $\Box\forall F(\varphi \to \Box\varphi)$. Since φ is therefore a rigid condition on properties (188.1), it follows by (189.2) that $y = \imath x\chi \to \chi^{y}_{x}$. By the alternative definition (ϑ) that we're considering in the text, we know both that $\boldsymbol{a}_{\varnothing}$ is logically proper and that $\boldsymbol{a}_{\varnothing} = \imath x\chi$. So it follows that $\chi^{\boldsymbol{a}_{\varnothing}}_{x}$, i.e., $A!\boldsymbol{a}_{\varnothing} \,\&\, \forall F(\boldsymbol{a}_{\varnothing}F \equiv F \neq F)$. But from this we can derive $Null(\boldsymbol{a}_{\varnothing})$ since all that remains to be shown is that the second conjunct implies $\neg\exists F \boldsymbol{a}_{\varnothing}F$ (exercise).

after having shown the latter is logically proper by a modally strict proof. But once we do this, it takes a little work to show that $\boldsymbol{a}_\varnothing$ can be instantiated into its own defining matrix by a modally strict proof.

(**196**) **Theorems:** Facts about the Granularity of Relations. The following facts govern the granularity of relational properties having abstract constituents: (.1) For any relation R, there are distinct abstract objects x and y for which *bearing R to x* is identical to *bearing R to y*, and (.2) For any relation R, there are distinct abstract objects x and y for which *being a z such that x bears R to z* is identical to *being a z such that y bears R to z*:

(.1) $\forall R \exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda z\, Rzx] = [\lambda z\, Rzy])$

(.2) $\forall R \exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda z\, Rxz] = [\lambda z\, Ryz])$

These theorems are to be expected if we think semantically for the moment and reconsider the Aczel models discussed in Chapter 3. Recall that for the purpose of modeling the theory, abstract objects were represented as sets of properties (though for the reasons pointed out in Section 3.6, we shouldn't confuse abstract objects with the sets that represent them in Aczel models). Now consider any relation R. Cantor's Theorem now tells us there can't be a distinct property *bearing R to s* for each distinct set s of properties, for then we would have a one-to-one mapping from the power set of the set of properties into a subset of the set of properties. This model-theoretic fact is captured by the above theorem: there can't be a distinct property *bearing R to x* for each distinct abstract object x. Thus, Cantor's Theorem isn't violated: there are so many abstract objects that for some distinct abstract objects x and y, the property *bearing R to x* collapses to the property *bearing R to y*. This is just what one expects given that two powerful principles, comprehension for abstract objects (39) and comprehension for properties (129.2), are true simultaneously.

It also follows that, for any property F, there are distinct abstract objects x, y such that *that-Fx* is identical to *that-Fy*:

(.3) $\forall F \exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda\, Fx] = [\lambda\, Fy])$

This is as expected, given (.1) and (.2).

(**197**) **Theorem:** Some Abstract Objects Not Strictly Leibnizian. The previous theorem, (196.1), has a rather interesting consequence, namely, that there exist distinct abstract objects that exemplify exactly the same properties:[121]

$$\exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, \forall F(Fx \equiv Fy))$$

[121] I am indebted to Peter Aczel for pointing out that this theorem results once we allow impredicatively-defined relations into our system.

In other words, there are distinct abstract objects that are indiscernible with respect to the properties they exemplify.[122] To prove this claim, let R in theorem (196.1) be the impredicatively-defined relation $[\lambda xy\, \forall F(Fx \equiv Fy)]$.

Thus classical Leibnizian indiscernibility doesn't imply the identity of abstract objects. Nevertheless, by (172), a somewhat different form of the Leibnizian principle of the identity of indiscernibles applies to abstract objects: such objects are identical whenever indiscernible with respect to properties they encode.

9.11 Propositional Properties

(**198**) **Definition:** Propositional Properties. Let us call a property F *propositional* iff for some proposition p, F is *being such that* p:

$$Propositional(F) \ =_{df} \ \exists p(F = [\lambda y\, p])$$

Warning: It is very important not to confuse the object-theoretic definition of a *propositional property* with the metatheoretic definition of a *propositional formula*. These are completely different notions. A propositional property is one constructed out of a proposition and has the form $[\lambda y\, p]$. A propositional formula is a formula with no encoding subformulas.

(**199**) **Theorems:** Existence of Propositional Properties. For any proposition (or state of affairs) p, the propositional property $[\lambda y\, p]$ exists.

(.1) $\forall p \exists F(F = [\lambda y\, p])$

In general, if F is *being such that* p, then necessarily, an object x exemplifies F iff p is true:

(.2) $F = [\lambda y\, p] \rightarrow \Box \forall x(Fx \equiv p)$

Finally, it is a theorem that if F is propositional, then necessarily F is propositional:

(.3) $Propositional(F) \rightarrow \Box Propositional(F)$

Propositional properties play an extremely important role in some of the applications of object theory in later chapters.

[122] If one recalls the structure of the Aczel models described in the Introduction, this result is to be expected. When abstract objects are modeled as sets of properties, where properties are modeled as sets of urelements, then the fact that an abstract object x exemplifies a property F is modeled by ensuring that the special urelement that serves as the proxy of x is an element of F. Consequently, since distinct abstract objects must sometimes be represented by the same proxy, some distinct abstract objects exemplify the same properties.

(**200**) **Definition:** Indiscriminate Properties. Let us say that a property F is *indiscriminate* if and only if necessarily, if something exemplifies F then everything exemplifies F:

$$Indiscriminate(F) =_{df} \Box(\exists xFx \rightarrow \forall xFx)$$

Exercise. Prove that if F is indiscriminate, then there aren't two objects such that one exemplifies F and the other doesn't, i.e., show $Indiscriminate(F) \rightarrow \neg\exists x\exists y(Fx \,\&\, \neg Fy)$.

(**201**) **Theorem:** Propositional Properties are Indiscriminate. This follows from our two previous definitions:

$$Propositional(F) \rightarrow Indiscriminate(F)$$

(**202**) **Theorem:** Other Facts about Indiscriminate Properties. Some of the following facts will prove useful in later chapters: (.1) necessary properties are indiscriminate; (.2) impossible properties are indiscriminate; (.3) $E!$, $\overline{E!}$, $O!$, and $A!$ are not indiscriminate; and (.4) $E!$, $\overline{E!}$, $O!$, and $A!$ are not propositional properties.

(.1) $Necessary(F) \rightarrow Indiscriminate(F)$

(.2) $Impossible(F) \rightarrow Indiscriminate(F)$

(.3) (.a) $\neg Indiscriminate(E!)$

(.b) $\neg Indiscriminate(\overline{E!})$

(.c) $\neg Indiscriminate(O!)$

(.d) $\neg Indiscriminate(A!)$

(.4) (.a) $\neg Propositional(E!)$

(.b) $\neg Propositional(\overline{E!})$

(.c) $\neg Propositional(O!)$

(.d) $\neg Propositional(A!)$

(**203**) **Theorems:** Propositional Properties, Necessity, and Possibility. The following claims about propositional properties can be established: (.1) if F might be a propositional property, then it is one; (.2) if F isn't a propositional property, then necessarily it isn't; (.3) if F is a propositional property, then necessarily it is; and (.4) if F might not be a propositional property, then in fact it isn't one:

(.1) $\Diamond\exists p(F = [\lambda y\, p]) \rightarrow \exists p(F = [\lambda y\, p])$

(.2) $\forall p(F \neq [\lambda y\, p]) \rightarrow \Box\forall p(F \neq [\lambda y\, p])$

(.3) $\exists p(F = [\lambda y\, p]) \rightarrow \Box\exists p(F = [\lambda y\, p])$

(.4) $\Diamond\forall p(F \neq [\lambda y\, p]) \rightarrow \forall p(F \neq [\lambda y\, p])$

(**204**) **Theorems:** Encoded Propositional Properties Are Necessarily Encoded. It is provable that: (.1) if it is possible that every property that x encodes is propositional, then in fact every property x encodes is propositional, and (.2) if every property that x encodes is propositional, then necessarily every property x encodes is propositional:

(.1) $\Diamond\forall F(xF \rightarrow \exists p(F = [\lambda y\, p])) \rightarrow \forall F(xF \rightarrow \exists p(F = [\lambda y\, p]))$

(.2) $\forall F(xF \rightarrow \exists p(F = [\lambda y\, p])) \rightarrow \Box\forall F(xF \rightarrow \exists p(F = [\lambda y\, p]))$

9.12 The Metatheory of Definitions

Reasoning with definitions has played, and will continue to play, an important part in the proof of almost every theorem, since few theorems are expressed using only primitive notation or rest solely on axioms that are expressed using only primitive notation. Before we go any further, it would serve well to articulate the underlying theory and inferential role of definitions in our system, so as to indicate more precisely what we mean when we say "It follows by definition that ... " within the context of a derivation or proof. With Frege being one of the few exceptions, most discussions of the theory of definition in formal systems have been framed with respect to a very simple language, namely, the language of the first-order predicate calculus, possibly extended by function terms.[123] By contrast, our language uses second-order modal syntax with 0-place relation terms and includes not only formulas that are themselves terms but also terms that result from applying variable binding operators to formulas (e.g., descriptions and λ-expressions). The theory of definitions for this language involves some interesting subtleties. In particular, we have to discuss how definitions behave with respect to complex terms, rigidly-denoting terms, non-denoting terms (i.e., some descriptions), and encoding formulas. Consequently, those familiar with the classical theory of definitions may still find the following discussion intriguing if not useful. It may be skipped, however, by those more interested in seeing how the foregoing core theorems of

[123]In writing this section, I consulted Frege 1879, §24; Padoa 1900; Frege 1903, §§56–67, §§139–144, and §§146–147; Frege 1914, 224–225; Suppes 1957; Mates 1972; Dudman 1973; Belnap 1993; Hodges 2008; Urbaniak & Hämäri 2012; and Gupta 2014. Hodges 2008 and Urbaniak & Hämäri 2012 provide insightful discussions of the contributions by Kotarbiński, Łukasiewicz, Leśniewski, Ajdukiewicz and Tarski to the elementary theory of definitions.

object theory can be further applied. But those who plan to skip this section should be aware that when we say "it follows by definition" in a proof, we are implicitly relying on one of the rules in items (207.1) – (207.4) and (208.1) – (208.6), since they constitute the principles that govern the inferential role of definitions in our system.

For the reason mentioned in Remark (19), we have adopted the position that definitions introduce new expressions into our language and are not mere metalinguistic abbreviations of our object language. Furthermore, we have assumed that in a properly formed definition, the variables having free occurrences in both the definiens and definiendum should be the same.[124] Finally, recall the distinction between *term definitions* and *formula definitions* introduced in Remark (19). Note that these two kinds of definitions are not mutually exclusive — definitions in which the definiens and definiendum are propositional formulas are both term and formula definitions. Nevertheless, we discuss the theory and inferential roles of term and formula definitions separately in what follows, in (207) and (208), respectively. However, we preface the discussion of these two items with two important Remarks.

(**205**) **Remark:** Unique Existence Not Sufficient for a New (Individual) Term. In some systems, it is legitimate to introduce, by definition, a new individual term κ when it is established that there is a unique x such that φ. That is, in some systems, if $\exists!x\varphi$ is an axiom or theorem, one may simply introduce a new term κ to designate any entity that is uniquely such that φ. Thus, in the classical theory of definitions, when $\vdash \exists!x\varphi$, one may introduce κ by stipulating (Suppes 1957, 159–60; Gupta 2014, Section 2.4):

$$\kappa = x \ =_{df} \ \varphi$$

This definition of κ is legitimate because (i) any formula ψ^{κ}_{x} in which κ occurs can be expanded by definition into the claim that $\exists!x\varphi \,\&\, \exists x(\varphi \,\&\, \psi)$, and (ii) no new theorems can be proved in the language with κ that aren't already provable in the language without κ.

But such a procedure would be incorrect (indeed, disastrous) for the present system. To see why, suppose that for some formula φ we were to assert, prove, or assume:

(a) $\exists!x\varphi$

(b) $\Diamond\neg\exists x\varphi$

[124] Though Suppes nicely explains why a definition must never allow free variables to occur in the definiens without occurring free in the definiendum, he does allow free variables in the definiendum that don't occur free in the definiens. But he notes that, in the latter case, we can trivially get the variables to match by adding dummy clauses to the definiens. Thus, in his example (Suppes 1957, 157), the definition $Q(x,y) =_{df} x > 0$ can be turned into $Q(x,y) =_{df} x > 0 \,\&\, y = y$. In what follows, we eschew definitions in which the free variables don't match, without loss of generality.

and, then, on the basis of (a), stipulate:

(c) $\kappa = x =_{df} \varphi$

Now since (b) immediately yields:

(d) $\neg\Box\exists x\varphi$

by (117.2), a contradiction becomes derivable. Since (a) implies $\exists x\varphi$, assume b is such an entity, so that we know φ^b_x. Now note, independently, that since κ is logically proper, we can instantiate theorem (75) to obtain:

(e) $\kappa = b \to \Box\kappa = b$

Now by substituting b for x, we obtain the following instance of (c):

(f) $\kappa = b =_{df} \varphi^b_x$

Note, independently, that we can convert (f) into an equivalence: from the tautology $\varphi^b_x \equiv \varphi^b_x$, it follows by definition (f) that $\kappa = b \equiv \varphi^b_x$. Since this last result was established by a modally strict proof, we can use it together with the Rule of Substitution to obtain the following from (e):

(g) $\varphi^b_x \to \Box\varphi^b_x$

But since we've already established φ^b_x, it follows that $\Box\varphi^b_x$. Hence, by $\exists$I, $\exists x\Box\varphi$. So by the Buridan formula (123.1), $\Box\exists x\varphi$, which contradicts (d).

Thus, the classical theory of definitions fails for our system because our logic assumes that terms are rigid designators, no matter whether they are simple or complex, primitive or defined. One may not define κ as $\iota x\varphi$ when $\exists!x\varphi$ *happens to be* true and, indeed, not even when $\exists!x\varphi$ is necessarily true. Intuitively, in the former case, if $\exists!x\varphi$ is a necessitation-averse axiom, a non-modally strict theorem, or a contingent premise, there may be different individuals, or no individuals, that uniquely satisfy φ at different (semantically-primitive) worlds. Moreover, in the latter case, if $\exists!x\varphi$ is necessary, there may be different individuals that are uniquely such that φ at different worlds. Consequently, if we really wanted to adapt the classical theory of definitions to the present system, we should stipulate that a definition of the form $\kappa = x =_{df} \varphi$ is permissible only when both $\vdash \exists!x\Box\varphi$ and $\vdash \exists!x\varphi$.[125]

[125]Note that $\vdash \exists!x\Box\varphi$ alone is not sufficient. For $\vdash \exists!x\Box\varphi$ doesn't guarantee $\vdash \exists!x\varphi$ and the latter is needed for κ to be well-defined. Intuitively, from the fact that there is exactly one thing that is φ at every possible world, it doesn't follow that there is exactly one thing that is φ at the actual world. There may be two or more things that are φ at the actual world even though there is exactly one thing that is φ at every world. So we need both $\vdash \exists!x\Box\varphi$ and $\vdash \exists!x\varphi$.

However, $\vdash \exists!x\Box\varphi$ is sufficient to block the case described above because one can't simultaneously assert $\exists!x\Box\varphi$ and $\Diamond\neg\exists x\varphi$. A contradiction would ensue without the mediation of any definitions. Moreover, it also blocks the case where, at each world, a *different* witness uniquely satisfies φ, i.e., blocks the case where $\Box\exists!x\varphi$ is true but not $\exists!x\Box\varphi$.

But we shall not adapt the classical theory of definitions in this way. That's because we have definite descriptions in our system and the logically correct way to introduce new individual terms is by explicitly defining them using definite descriptions (we discuss the further requirement that such descriptions be logically proper in the next Remark). That is, we shall require that a new individual term, say κ, be introduced by way of a definite description, using a definition of the form:

$$\kappa \ =_{df} \ \imath x\varphi$$

This introduces κ by way of a rigidly designating term (and, as stipulated below, one that provably designates), rather than by way of a formula that contingently asserts the unique existence of something.

Moreover, we shall require something similar for relation terms. It does not suffice to stipulate, when $\vdash \exists!\Box F^n\varphi^*$ and $\vdash \exists!F^n\varphi^*$, that $\Pi^n = F^n \ =_{df} \ \varphi^*$, for some new relation term Π^n. Instead, we prefer that the new term Π^n, for $n \geq 1$, be introduced by a definition of the form:

$$\Pi^n \ =_{df} \ [\lambda x_1 \ldots x_n \, \varphi^*]$$

This introduces Π^n by way of a rigidly designating λ-expression. In the case where $n = 0$, we have the option of introducing a new term Π^0 using either of the following forms:

$$\Pi^0 \ =_{df} \ [\lambda \, \varphi^*]$$

$$\Pi^0 \ =_{df} \ \varphi^*$$

Both of these forms introduce the new term Π^0 by way of a rigidly designating expression. Note that from a semantic point of view, even though φ^* rigidly denotes a proposition, the truth value of the proposition it denotes may vary from (semantically-primitive) possible world to possible world!

Finally, note that the above constraint on the introduction of new terms is independent of our rules for reasoning with *arbitrary names*. When we reason from $\exists\alpha\varphi$ to some conclusion ψ by showing, for some new (i.e., arbitrary) constant τ, that φ^τ_α implies ψ (i.e., in accordance with the conditions laid down in (85) for Rule $\exists$E), we are not introducing any new terms into the language when we reason (we're temporarily grabbing a fresh constant already in the language). We may still reason from $\exists\alpha\varphi$ by saying "let τ be an arbitrary such entity, so that we know φ^τ_α". The arbitrary term τ is chosen from our stock of primitive constants of the appropriate type, and as long as no special assumptions about τ are used in the proof, the reasoning is valid. We may not, for example, reason as follows:

> Consider $\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$, which is an of Comprehension for Abstract Objects (39). Assume a is such an object, so that we know $A!a \,\&$

$\forall F(aF \equiv \varphi)$. Hence, by RN, $\Box(A!a \,\&\, \forall F(aF \equiv \varphi))$. So by $\exists$E, $\exists x\Box(A!x \,\&\, \forall F(xF \equiv \varphi))$.

Exercise 1. Say what is wrong with the above reasoning. **Exercise 2**. Show how the conclusion of the incorrect reasoning displayed above implies that truth and necessity are equivalent, i.e., show how $\exists x\Box(A!x \,\&\, \forall F(xF \equiv \varphi))$, for arbitrary φ, implies that $\varphi \equiv \Box\varphi$.[126]

(**206**) **Remark:** Constraint on Term Definitions. Recall Remark (28), in which we noted that claims having the form $\exists\beta(\beta = \tau)$ ("there exists something identical with τ") informally tell us that term τ is *logically proper*. (In the formal mode, using semantic notions, this means that term τ has a denotation.) We have been, and will continue, observing the following strict constraint on term definitions generally:

> **Constraint on Term Definitions**
> Only terms τ that are provably logically proper (i.e., only terms τ for which $\vdash \exists\beta(\beta = \tau)$) may serve as definientia in individual term definitions.

Note, however, that since our system guarantees that all terms other than descriptions are logically proper, the above constraint really amounts to the following: only definite descriptions that are provably logically proper may serve as definientia in individual term definitions.

At this point, it is important to introduce and discuss a group of edge cases, namely, definitions in which the definiens is a definite description that is provably logically proper but not by a modally strict proof. For example, we may sometimes want to extend our theory by adding a contingent truth of the form $\exists!x\varphi$ as an axiom or premise. For example, suppose that we want to add the claim that there exists a unique moon of the Earth as an axiom to our system.

[126] Solution to Exercise 2: Suppose $\exists x\Box(A!x \,\&\, \forall F(xF \equiv \varphi))$, for an *arbitrary* formula φ. Then let a be such an object, so that we know $\Box(A!a \,\&\, \forall F(aF \equiv \varphi))$. Since a necessary conjunction implies that the conjuncts are necessary, it follows that $\Box A!a \,\&\, \Box\forall F(aF \equiv \varphi))$. From the second conjunct of this last result, it follows both that:

(ϑ) $\forall F\Box(aF \equiv \varphi)$ — by the CBF schema (122.2)

(ξ) $\forall F(aF \equiv \varphi)$ — by the T schema (32.2)

If we apply $\forall$E to both, we obtain, respectively:

(ϑ') $\Box(aF \equiv \varphi)$

(ξ') $aF \equiv \varphi$

From (ϑ') it follows by (111.6) that:

(ζ) $\Box aF \equiv \Box\varphi$

Now to see how this implies that truth and necessity are equivalent, we need only show $\varphi \to \Box\varphi$, since $\Box\varphi \to \varphi$ is an instance of the T schema. So assume φ. From this, it follows from (ξ') that aF. From this it follows that $\Box aF$, by axiom (37). So by (ζ), it follows that $\Box\varphi$.

If we use 'e' as the name of the Earth and represent this axiom as $\exists!xMxe$, then we would designate the axiom as an additional *necessitation-averse* axiom and we would annotate its item number with a $\star$.[127] Hence, this axiom would become a $\star$-theorem. So by theorem (100)$\star$, it would follow that $\exists y(y = \imath xMxe)$ is a $\star$-theorem.[128] Hence, $\imath xMxe$ would be provably logically proper, though the proof would not be modally strict. Nevertheless, by the Constraint on Term Definitions, we would be entitled to introduce a name, say m, to designate this unique object, as follows:

$$m =_{df} \imath xMxe \qquad (n)\star$$

Although we shall not actually include any such definitions like the above in what follows, the reader should remember that if one were to extend our system with a definition such as the above, one should:

- annotate the definition as **Definition**$\star$ and subsequently mark the item number with a $\star$, to indicate that the definition rests on a contingency, and
- treat any derivation or proof, in which one of the definiendum or definiens in a $\star$-definition is substituted for the other, as non-modally strict (and so mark the theorem number with a $\star$).

If the reason for this isn't already clear, then it will become clear in the next item, where we discuss the inferential role of term definitions.

(**207**) **Remark:** The Inferential Role of Term Definitions. We have often introduced term definitions under constraint (206) to extend our language with new terms and subsequently cited 'by definition' to (i) assert identities on the basis of such term definitions, and (ii) draw inferences in which we have substituted, within any formula or term, the definiens for the definiendum (or vice versa). We now explain and justify this practice. In our discussion, we won't be concerned with the fact that term definitions are independent of the choice of any free and bound variables that they may contain. We discuss this aspect of definitions in the next item (209).

Our inferential practices are grounded in the following:

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[127] Recall that a *necessitation-averse* axiom is simply one for which we are not taking modal closures. But since the claim $\exists!xMxe$ isn't just necessitation-averse but rather contingent, we could also assert $\Diamond\neg\exists!xMxe$ as an axiom. But the full assertion of contingency is not needed for the present discussion; it suffices that the axiom is marked as necessitation-averse.

[128] Of course, the application of (100)$\star$ to $\exists!x\varphi$ to obtain $\exists y(y = \imath x\varphi)$ would turn the latter into a $\star$-theorem even if $\exists!x\varphi$ had been a necessary axiom or a modally strict theorem. But in this case, we could apply the Rule of Actualization to $\exists!x\varphi$ to obtain $\mathcal{A}\exists!x\varphi$, which by (179.2) yields a modally strict proof of $\exists y(y = \imath x\varphi)$. Thus, in cases like this, a definition in which $\imath x\varphi$ is used as definien is not subject to s any of the steps discussed immediately below. We will see such definitions on occasion in what follows.

(.1) **Convention for Term Definitions**
The introduction of definitions having τ as a definiendum and τ' as a definiens under Constraint (206) is a convention for the following reformulation of our system:

(.a) extend the language of our system with τ as a new primitive term,

(.b) if, for some variable β not free in τ', there is a modally strict proof of $\exists\beta(\beta = \tau')$, then add the closures of the schemata

- $\exists\beta(\beta = \tau)$
- $\tau = \tau'$

as new, necessary axioms to our system; if, on the other hand, there is a proof of $\exists\beta(\beta = \tau')$ but not a modally-strict proof, add the $\Box$-free closures of these same schemata as new, necessitation-averse axioms to our system, with item numbers that are appropriately marked with a $\star$ whenever cited.

The second part of clause (.b) covers the example described in the latter half of Remark (206). If we were to extend our theory with the contingent axiom that there exists a unique moon of the Earth ($\exists!xMxe$), then by theorem (100)$\star$, $\exists y(y = \imath xMxe)$ would become a necessitation-averse $\star$-theorem. So by the second part of clause (.b) of the above Convention for Term Definitions, the definition:

$$m =_{df} \imath xMxe$$

would be a proxy for both extending the language of our system with the individual constant m and adding the $\Box$-free closures of the following:

$$\exists y(y = m)$$

$$m = \imath xMxe$$

as *necessitation-averse* axioms. Any derivation that depended on these axioms would fail to be modally-strict. Note, though, that by an instance of the modally-strict lemma (125), namely, $\exists y(y = \imath xMxe) \to \Box\exists y(y = \imath xMxe)$, it would follow that there is a non-modally strict proof of $\Box\exists y(y = \imath xMxe)$. This is as it should be: *given* the contingent axiom that there is a unique moon of Earth, then (i) necessarily, the thing that is in fact a moon of the Earth exists, and (ii) the proof of this necessity claim rests upon a contingency, as would the proof of the claim that necessarily, there is something that is m.

By contrast, if the definiens τ' is such that there is a modally strict proof of $\exists\beta(\beta = \tau')$, then the first part of clause (.b) in the Convention for Term Definitions applies. We know that such proofs are available in the case where τ' is any relation term, or any individual constant or variable, since such $\exists\beta(\beta = \tau')$

is then axiomatic. It remains to consider the case where the definiendum τ is introduced with the description $\imath x\varphi$ as definiens. Note that if we can produce a modally strict proof of $\exists!x\varphi$, then we can use the Rule of Actualization (RA) to get a modally strict proof and $\mathcal{A}\exists!x\varphi$, and then apply theorem (179.2) to obtain a modally strict proof of $\exists y(y = \imath x\varphi)$. This case is then governed by the first part of clause (.b) in the Convention for Term Definitions, even when there are free variables in $\exists!x\varphi$. In all of these cases, the definition becomes a convention for extending the language with τ and adding *all* the closures of $\exists\beta(\beta = \tau)$ and $\tau = \tau'$ as new, necessary axioms to our system.

Consider what the above convention implies in the interesting case where the definition has a free individual variable. Let us start with the fact that the following is a theorem of object theory (indeed, an instance of Strengthened Comprehension for abstract objects (177)), in which the variable z occurs free:

$$\exists!x(A!x \,\&\, \forall F(xF \equiv Fz))$$

By (180), we then know that there is a modally strict proof of:

$$\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv Fz)))$$

By GEN, it follows that:

$$\forall z\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv Fz)))$$

Hence, we've met Constraint (206) and may introduce the following functional term $\boldsymbol{c}_z$, where the variable z occurs free in both definiendum and definiens:

$$(\vartheta)\quad \boldsymbol{c}_z =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv Fz))$$

By the Convention for Term Definitions (.1), the above line is a convention for adding $\boldsymbol{c}_z$ as a primitive functional term and asserting the closures of the (necessary) axiom:

$$\boldsymbol{c}_z = \imath x(A!x \,\&\, \forall F(xF \equiv Fz))$$

Hence, it is an axiom that:

$$\forall z(\boldsymbol{c}_z = \imath x(A!x \,\&\, \forall F(xF \equiv Fz)))$$

Note that we can't instantiate this universal claim with an improper description; if $\imath yPy$ doesn't denote anything, our axioms and rules for universal instantiation don't allow us to infer:

$$(\xi)\quad \boldsymbol{c}_{\imath yPy} = \imath x(A!x \,\&\, \forall F(xF \equiv F\imath yPy))$$

from the previous universal claim. When $\imath yPy$ denotes nothing, $\imath x(A!x \,\&\, \forall F(xF \equiv F\imath yPy))$ denotes the null object, since no properties F are such that $F\imath yPy$. So although $\boldsymbol{c}_{\imath yPy}$ is well-formed and has a denotation even when $\imath yPy$

doesn't denote, we can't derive (ξ). Indeed (ξ) and examples like it don't count as proper instances of the definition.[129]

Given the Convention for Term Definitions, it immediately follows by (46.1) and (46.3) that definitions become assertible as theorems having the form of (the closures of) identity statements and are thus derivable from any (possibly empty) set of premises:

(.2) **Rules of Identity by Definition**
Where one of τ, τ' is the definiendum and the other definiens in any proper instance of a term definition obeying Contraint (206):

(.a) $\Gamma \vdash_{\Box} \varphi$, where φ is any closure of the identity $\tau = \tau'$ and provided there is a *modally strict* proof of $\exists\beta(\beta = \tau')$.

(.b) $\Gamma \vdash \varphi$, where φ is any $\Box$-free closure of $\tau = \tau'$

Consider clause (.2.a). If there is a modally strict proof of $\exists\beta(\beta = \tau')$ and the definition $\tau =_{df} \tau'$ is just a convention that, among other things, extends our system with new, necessary axioms $\tau = \tau'$ and their closures, as described in (.1), then by (46.1), all such formulas become theorems and, by (46.3), derivable from any set of premises. So in any reasoning context, we may assert $\tau = \tau'$, for any proper instance of the definition that obeys Constraint (206). If there is no modally-strict proof of $\exists\beta(\beta = \tau')$, then clause (.2.b) tells us that the new axioms $\tau = \tau'$ and their $\Box$-free closures also become theorems. But, in this case, the Convention for Term Definitions (.1) tells us that the axioms have been marked as necessitation-averse, and so the theorems inherit the $\star$-annotation.

Thus, in either case, we are justified in substituting, within any formula or term, the definiens for the definiendum and vice versa. The Rules of Identity by Definition (.2) imply the following, immediate special cases of Rule SubId:

(.3) **Substitution of Defined Terms** (**Rule SubDefTerms**)
If one of τ, τ' is a definiendum and the other its definiens in any proper instance of a term definition, and φ' is the result of substituting τ' for zero or more occurrences of τ in φ, then

[129] It is worth observing here what happens in the case where a non-denoting descriptions such $\imath yPy$ appears in a defined relation term. The resulting term may not denote what you expect. For example, in (136.2) defines the 0-place relation term, $\overline{p}$, as $[\lambda\, \neg p]$. Then even though $\imath yPy$ fails to denote and the formula $Q\imath yPy$ is false, the term $\overline{Q\imath yPy}$ (i.e., $[\lambda\, \neg Q\imath yPy]$) has a denotation. The λ-expression $[\lambda\, \neg Q\imath yPy]$ has well-defined denotation conditions that are grounded in the well-defined truth conditions of $\neg Q\imath yPy$. Indeed, given that $\imath yPy$ fails to denote anything, $[\lambda\, \neg Q\imath yPy]$ denotes a proposition that is necessarily true, since it denotes a proposition that is true at a world w just in case $\neg Q\imath yPy$ is true at w. But $\neg Q\imath yPy$ is true at every world, since $Q\imath yPy$ is false at every world. Hence, $\overline{Q\imath yPy}$ denotes a necessary truth and is therefore logically proper. We'll see further examples of this phenomenon below.

(.a) if $\Gamma \vdash_{\Box} \varphi$, then $\Gamma \vdash_{\Box} \varphi'$, provided the proof of $\exists\beta(\beta = \tau')$ is modally strict. [Variant: $\varphi \vdash_{\Box} \varphi'$]

(.b) if $\Gamma \vdash \varphi$, then $\Gamma \vdash \varphi'$. [Variant: $\varphi \vdash \varphi'$]

The proof is straightforward.[130]

The Variant versions of Rule SubDefTerms tell us that whenever τ is a definiendum and τ' its definiens, or vice versa, we may infer φ' from any premise φ containing term τ. Thus, given definition (11.1):

$$O! =_{df} [\lambda x\, \Diamond E!x]$$

let:

$$\begin{array}{lcl} \tau & = & O! \\ \tau' & = & [\lambda x\, \Diamond E!x] \\ \varphi & = & \Box O!a \\ \varphi' & = & \Box[\lambda x\, \Diamond E!x]a \end{array}$$

Then it follows from Rule SubDefTerms (.3.b) that:

- $\Box O!a \vdash \Box[\lambda x\, \Diamond E!x]a$

And by the symmetry of identity,

- $\Box[\lambda x\, \Diamond E!x]a \vdash \Box O!a$

Clearly, in this example, since the proof of $\exists F(F = [\lambda x\, \Diamond E!x])$ is modally strict, it follows by (.3.a) that:

- $\Box O!a \vdash_{\Box} \Box[\lambda x\, \Diamond E!x]a$
- $\Box[\lambda x\, \Diamond E!x]a \vdash_{\Box} \Box O!a$

Now consider an example of Rule SubDefTerms where the definiens is a description $\iota x\varphi$ for which the proof of $\exists y(y = \iota x\varphi)$ is not modally strict. Even in this case, the description $\iota x\varphi$ rigidly denotes the same object no matter what the context, and so any term, e.g., κ, that we define using this description, becomes substitutable for the description in every context, as indicated in (.3.b). However, the foregoing discussion in this Remark and the previous one (206) should have now made it clear that such a definition is a proxy for

[130]To show (.3.a), suppose, without loss of generality, that (i) τ is the *definiendum* and τ' the *definiens* in some proper instance of a term definition, (ii) φ' is the result of substituting τ' for zero or more occurrences of τ in φ, and (iii) there is a modally strict proof of φ from Γ. Then from (i), we know by the Rule of Identity by Definition (.2.a) that there is a modally strict proof of $\tau = \tau'$ from Γ. But from this, (ii), and (iii), it follows by Rule SubId (74.2) that there is a modally strict proof of φ' from Γ.

The proof of (.3.b) is similar.

a necessitation-averse axiom so any substitution of κ for $\imath x\varphi$ in a derivation immediately turns the derivation into one that is not modally strict.

Clearly, then, given these consequences of our Convention for Term Definitions, any theorem involving a definiendum implies a corresponding theorem about its definiens and vice versa. Moreover, the following are straightforward consequences of Rule SubDefTerms:

(.4) **Rules of Equivalence by Term Definition**
If one of τ, τ' is the definiendum and the other definiens in any instance of a term definition, and φ' is the result of substituting τ' for zero or more occurrences of τ in φ, then:[131]

(.a) $\Gamma \vdash_{\Box} \varphi \equiv \varphi'$, provided the proof of $\exists\beta(\beta = \tau')$ is modally strict.

(.b) $\Gamma \vdash \varphi \equiv \varphi'$

Note how the above convention and rules address the case where τ and τ' are propositional formulas φ^* and ψ^*. Consider, as an example, that the following instance of the formula definition (7.4.b) becomes a proper instance of a term definition:

$$Px \,\&\, Qx =_{df} \neg(Px \rightarrow \neg Qx)$$

The Rule of Identity by Definition (.2.a) yields the identity:

$$\vdash_{\Box} Px \,\&\, Qx = \neg(Px \rightarrow \neg Qx)$$

Moreover, Rule SubDefTerms (.3.a) yields:

$$[\lambda x\, Px \,\&\, Qx]a \vdash_{\Box} [\lambda x\, \neg(Px \rightarrow \neg Qx)]a$$

and a Rule of Equivalence by Term Definition (.4.a) yields:

$$\vdash_{\Box} [\lambda x\, Px \,\&\, Qx]a \equiv [\lambda x\, \neg(Px \rightarrow \neg Qx)]a$$

So our conventions and rules for term definitions govern the inferential role of those instances of formula definitions in which the definiens and definiendum are terms.[132]

[131] To show (.4.a), note that the Variant of Rule SubDefTerms asserts $\varphi \vdash \varphi'$. So it follows by the Deduction Theorem that $\vdash \varphi \rightarrow \varphi'$. By symmetry, if $\varphi \vdash \varphi'$ is an instance of Rule SubDefTerms, then so is $\varphi' \vdash \varphi$, and so again by the Deduction Theorem, $\vdash \varphi' \rightarrow \varphi$. Hence $\vdash \varphi \equiv \varphi'$, and by (46.3), it follows that $\Gamma \vdash \varphi \equiv \varphi'$. For (.4.b), the reasoning is the same, but proceeds in light of the additional assumption that the proof of $\exists\beta(\beta = \tau')$ is modally strict.

[132] Indeed, as pointed out in footnote 129, even if $\neg\exists y(y = \imath wKw)$, the following is an proper instance of (7.4.b) and so a genuine term definition:

$$P\imath wKw \,\&\, Q\imath wKw =_{df} \neg(P\imath wKw \rightarrow \neg Q\imath wKw)$$

The reason is that the definiens $\neg(P\imath wKw \rightarrow \neg Q\imath wKw)$ is a 0-place relation term, and so has a denotation even when the description $\imath wKw$ fails to denote.

The above facts help us to establish that our conventions and rules preserve the traditional way of understanding term definitions, although we shall not do more to prove it here. Our convention and rules preserve both (a) the *eliminability* criterion, and (b) the *conservativeness* criterion. For a term definition to satisfy the eliminability criterion, every formula φ of our extended language (containing the new term) must be provably equivalent in the extended language to a formula of the original language. For a term definition to satisfy the conservativeness criterion, every formula of our original language that is provable in the extended language must be provable in the original language (or, re-expressing this in the contrapositive, a conservative definition must not allow us to prove some previously unprovable formula). So rather than reformulate our system with new terms and axioms, we adopt the above conventions for definitions.

(**208**) **Remark:** The Inferential Role of Formula Definitions. We have often (i) introduced formula definitions to extend our language with new syncategorematic expressions and formulas, and (ii) drawn inferences 'by definition' in which we've substituted the definiens for the definiendum (or vice versa) within in any formula or term. Formula definitions can be distinguished from term definitions by the fact that they don't require a constraint analogous to Constraint (206): given any instance of the definition, we may substitute the definiens for definiendum in any context, or vice versa, no matter what terms (logically proper or otherwise) might appear in the instance of the definition (provided those terms are substitutable for any free variables they might replace). Thus, the inferential role of a definiens φ and definiendum ψ is the same, no matter whether the instances of φ and ψ contain logically proper or logically improper terms. So it does *not* suffice to regard a definition $\varphi =_{df} \psi$ as a proxy for the closures of new necessary axioms of the form $\varphi \equiv \psi$, for then we couldn't derive substitution instances that involve non-denoting terms: the universal generalizations of the axioms $\varphi \equiv \psi$ would only yield those instantiations to terms that are logically proper. We expand in detail on these facts below.

Before we lay out the theory of formula definitions, two observations are in order. First, recall that in (19) we distinguished two types of formula definitions: those that use metavariables and those that use object language variables. Examples of the former are (7.4.a), (7.4.d), and (7.4.e), while examples of the latter are (15) and (16.1). So when we talk about *instances* of a formula definition, we intend either of the following:

- the result of uniformly substituting formulas for the metavariables in the definition, or

- the result of uniformly substituting terms for any free object language variables in the definition (provided the terms being substituted are substitutable for the free variables they replace).

For now, we put aside the fact that formula definitions are independent of the choice of any free and bound variables that they may contain. We discuss this aspect of definitions in the next item (209).

Second, we remind the reader of a point made in Remark (19). When the definientia of formula definitions are non-propositional formulas, i.e., formulas with encoding subformulas, the definienda also have to be regarded as non-propositional formulas. In what follows, we see exactly how formula definitions are conventions for extending the object language with new formulas. These new formulas are regarded as non-propositional when their definientia are non-propositional, and the non-propositionality is inherited by any formula in which the new formulas occur as subformula. Thus, not only are the new formulas $x = y$, $F = G$, etc., non-propositional, but so are $x = y \vee \neg x = y$, $F = G \rightarrow \forall(Fx \equiv Gx)$, etc. In what follows, we'll therefore assume that since formula definitions are conventions for extending the language with new formulas, the definition of *propositional* formula is to be adjusted accordingly, so as to exclude any formula or subformula that has been introduced by definition by way of a non-propositional formula.

With these observations in mind, we may now say that our inferential practices involving formula definitions are grounded by the following:

(.1) **Convention for Formula Definitions**
The introduction of a formula definition $\varphi =_{df} \psi$ is a convention for the following reformulation of our system:

(.a) extend the language with the new syncategorematic expression appearing in φ and with a formation rule that stipulates φ is a new formula,[133]

(.b) assert the closures of all of the biconditionals $\varphi \equiv \psi$ as new necessary axioms.

Thus, the following rule is an immediate consequence of Convention (.1), (46.1) ("necessary axioms are modally strict theorems"), and (46.3):

(.2) **Simple Rule of Equivalence by Definition**
If (a) one of φ, ψ is the definiendum and the other its definiens in some instance of a definition and (b) χ is any closure of $\varphi \equiv \psi$, then $\Gamma \vdash_{\Box} \chi$.

Now here are two notable features of convention (.1) and rule (.2), with examples to follow:

[133] If the definiens ψ contains any encoding subformulas, then φ also fails to be propositional, so that any formula in which φ subsequently appears as a subformula will fail to be propositional.

- Since formulas with non-denoting descriptions can be instances of definitions, clause (.1.b) tells us that the new axioms for which the definition goes proxy include formulas containing such descriptions.

- When φ and ψ are the propositional formulas φ^* and ψ^*, then *both* Convention (.1) and Convention (207.1) apply to the definition $\varphi =_{df} \psi$, so that its introduction goes proxy for asserting, as new necessary axioms, the closures of the instances of $\varphi^* \equiv \psi^*$ and $\varphi^* = \psi^*$. Such axioms obey Constraint (206) even if the definition instance contains a non-denoting description: the instance relates 0-place relation terms that are always logically proper and provably so by a modally strict proof.

Here is an example that exhibits both notable features, since the following is an *instance* of definition (7.4.a):

(A) $P\imath x\varphi \,\&\, Qy \;=_{df}\; \neg(P\imath x\varphi \rightarrow \neg Qy)$

Even if $\imath x\varphi$ fails to be logically proper, Convention (.1) applies and the Simple Rule of Equivalence (.2) guarantees:

$$\vdash_{\Box} (P\imath x\varphi \,\&\, Qy) \equiv \neg(P\imath x\varphi \rightarrow \neg Qy)$$

Furthermore, since the definiens of (A) is a propositional formula (even if φ in $\imath x\varphi$ contains encoding subformulas), the definiendum is as well. Now it is a modally strict axiom (29.2) that:

$$\exists p(p = \neg(P\imath x\varphi \rightarrow \neg Qy))$$

Since this is therefore a modally strict theorem, Constraint (206) is satisfied and so by the Rule of Identity by Definition (207.2.a), we also know:

$$\vdash_{\Box} (P\imath x\varphi \,\&\, Qy) = \neg(P\imath x\varphi \rightarrow \neg Qy)$$

So even if the description fails to denote, a modally strict identity holds between the propositions denoted by the terms flanking the identity sign (no matter what individual is assigned to y).

Now given the Rule of Substitution (113), the above Simple Rule of Equivalence by Definition yields the following:

(.3) **Substitution of Defined Subformulas** (**Rule SubDefSubForm**)
If (a) one of φ, ψ is the definiendum and the other its definiens in some instance of a definition and (b) χ' is the result of substituting ψ for zero or more occurrences of φ where the latter is a subformula of χ, then if $\Gamma \vdash \chi$, then $\Gamma \vdash \chi'$ [Variant: $\chi \vdash \chi'$]

(.4) **Rule of Equivalence by Defined Subformulas**
$\Gamma \vdash \chi \equiv \chi'$, where χ and χ' are as in (.3)

The justification of (.3) is immediate: clause (a) implies $\vdash_{\Box} \varphi \equiv \psi$ by the Simple Rule of Equivalence by Definition (.2), and so (b) implies, by the Rule of Substitution, that if $\Gamma \vdash \chi$, then $\Gamma \vdash \chi'$. (.4) is justified, without loss of generality, by applying the Deduction Theorem to the Variant version of (.3) to obtain $\vdash \chi \to \chi'$, and appealing to (46.1) to obtain $\Gamma \vdash \chi \to \chi'$.

Here is a pair of examples demonstrating the Variant version of Rule SubDefSubForm, derived from the instance $P = Q =_{df} \Box\forall x(xP \equiv xQ)$ of the definition of property identity (16.1):

- $\Box P = Q \vdash \Box\Box\forall x(xP \equiv xQ)$
- $\Box\Box\forall x(xP \equiv xQ) \vdash \Box P = Q$

In the first member of the pair, we have:

$$\begin{array}{lcl} \varphi & = & P = Q \\ \psi & = & \Box\forall x(xP \equiv xQ) \\ \chi & = & \Box P = Q \\ \chi' & = & \Box\Box\forall x(xP \equiv xQ) \end{array}$$

The second member of the pair can be annotated analogously, given the commutativity of the biconditional.

It is interesting to note that although clause (b) in the antecedent of Rule SubDefSubForm requires φ to be a subformula of χ, we added this condition primarily so that we would have an intermediate stage from which we can introduce the final two rules:

(.5) **Substitution of Defined Formulas** (**Rule SubDefForm**)
If (a) one of φ, ψ is the definiendum and the other its definiens in some instance of a definition and (b) χ' is the result of substituting ψ for zero or more occurrences of φ *anywhere* in χ, then if $\Gamma \vdash \chi$, then $\Gamma \vdash \chi'$
[Variant: $\chi \vdash \chi'$]

(.6) **Rule of Equivalence by Defined Formulas**
$\Gamma \vdash \chi \equiv \chi'$, where χ and χ' are as in (.5)

The difference between (.3) and (.5) is that the latter relaxes the requirement that φ be a subformula of χ. (.5) guarantees that a definiendum φ and a definiens ψ can be exchanged for one another no matter how or where either appears in a formula χ; they need not be subformulas of χ. Whereas the justification of (.3) was immediate and easy, the justification of (.5) is much more complex and requires a structural induction based on the BNF. Though the justification is given in the Appendix to this chapter as **Metatheorem** $\langle$**9.5**$\rangle$, it would serve well to work through a series of examples that demonstrate why the rule works and what the justification covers. In the following examples,

φ is always the definiendum, ψ the definiens, χ the formula in which we plan to exchange ψ for φ, and χ' always has at least one occurrence of φ replaced by ψ.

Since (.3) already guarantees that definiendum and definiens can be exchanged in χ whenever one or the other is a subformula of χ, we only have to consider those cases where neither is a subformula of χ. The key to seeing that definiendum and definiens can be exchanged even if neither is a subformula of χ is this:

- The only way a formula φ can occur in χ *not* as a subformula is if, at some place in χ, φ occurs within the matrix of an n-place λ-expression ($n \geq 1$) or within the matrix of a definite description.

This fact can be established by inspection of the BNF in (6) and the definition of subformula (8). The definition of subformula tells us that φ is a subformula of χ if χ is $\neg\varphi$, $\varphi \to \theta$, $\theta \to \varphi$, $\forall\alpha\varphi$, $\mathcal{A}\varphi$, $\Box\varphi$, $[\lambda\,\varphi]$, or φ itself. By inspecting the BNF, it becomes clear that:

- The BNF "bottoms out" in two base cases in which φ has a non-subformula occurrence in χ:

  (a) χ is an exemplification formula $\Pi^n\kappa_1\ldots\kappa_n$ ($n \geq 1$) and φ occurs somewhere in one of the terms $\Pi^n, \kappa_1, \ldots, \kappa_n$, or

  (b) χ is an encoding formula $\kappa_1\Pi^1$ and φ occurs somewhere in either κ_1 or Π^1.

- In these cases where φ has a non-subformula occurrence in one of the terms of an exemplification or encoding formula, φ occurs somewhere in these terms within the matrix of an n-place λ-expression $[\lambda\nu_1\ldots\nu_n\,\theta^*]$ ($n \geq 1$) or within the matrix of a description $\imath\nu\theta$.

- Though the BNF also bottoms out in another case, namely when χ is a formula of the form Π^0, we need not consider this case here. Note if χ is a formula of the form Π^0, the base cases are when Π^0 is Σ^0 (i.e., a propositional constant) or Ω^0 (i.e., a propositional variable) which add no potential occurrences of φ. The other forms of Π^0, namely $[\lambda\,\theta^*]$ and θ^* are not base cases because θ^* occurs as a direct subformula of Π^0. That is, in these latter two cases, the BNF has not bottomed out.

Though (a) and (b) in the first bullet point above are the base cases, the second bulleted point above suggests that even these cases can be complex: φ can have a nonsubformula appearance in χ by being the matrix of one of the terms of χ or by being the matrix of a term within one of those terms, and so on, down to an arbitrarily finite depth.

In the very simplest cases, φ just is the matrix of a 1-place λ-expression or the matrix of a definite description and χ is an exemplification or encoding formula in which those complex terms occur, as in the following examples:

(a) Π^1 is $[\lambda x\, \varphi^*]$, κ is $\imath x\varphi$, and χ is an exemplification formula such as $[\lambda x\, \varphi^*]a$, $P\imath x\varphi$, or $[\lambda x\, \varphi^*]\imath x\varphi^*$.[134]

(b) Π^1 is $[\lambda x\, \varphi^*]$, κ is $\imath x\varphi$, and χ is an encoding formula such as $a[\lambda x\, \varphi^*]$, $\imath x\varphi P$ or $\imath x\varphi^*[\lambda x\, \varphi^*]$.

In these simplest cases, it is straightforward to see that we can derive χ' from χ if the definition $\varphi =_{df} \psi$ converts to the modally strict theorem $\vdash_\Box \varphi \equiv \psi$. Here is why.

Consider first the (a) and (b) simplest cases where φ appears only in the relation term. There are two such cases, namely, where χ is $[\lambda x \varphi^*]a$ or $a[\lambda x \varphi^*]$. In these two cases, we know that φ^* and ψ^* must be propositional formulas (no matter how complex). Hence they are terms, and so the definition $\varphi^* =_{df} \psi^*$ converts to the identity $\varphi^* = \psi^*$. Thus, we can use Rule SubId to substitute ψ^* for φ^* and so derive χ' from χ.

Now consider the (a) and (b) simplest cases where φ appears only in the individual term. There are again two such cases, namely, where χ is $P\imath x\varphi$ or $\imath x\varphi P$. We need only look at one of them, since the reasoning is similar in both. So suppose χ is $\imath x\varphi P$. Then we know by the modally-strict Russell analysis (106) that no matter whether the description $\imath x\varphi$ is logically proper or not, χ is, by a modally strict proof, equivalent to the following formula, where z is chosen, without loss of generality, to be substitutable in both φ and ψ:

(ϑ) $\exists x(\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \to z = x) \,\&\, xP)$

This puts φ and φ^z_x into subformula positions. Now we know by the Simple Rule of Equivalence by Definition (.2) that our definition $\varphi =_{df} \psi$ implies both of the following:

(ξ_1) $\vdash_\Box \varphi \equiv \psi$

(ξ_2) $\vdash_\Box \forall x(\varphi \equiv \psi)$

Since z is substitutable in both φ and ψ, it is substitutable in $\varphi \equiv \psi$, and so it follows from (ξ_2) by $\forall$E that:

(ζ) $\vdash_\Box \varphi^z_x \equiv \psi^z_x$

So, we can use the Rule of Substitution with (ξ_1) to substitute ψ for φ in (ϑ) and use the Rule of Substitution with (ζ) to substitute ψ^z_x for φ^z_x in (ϑ). The result of performing the substitutions simultaneously is:

[134] Note that we've marked φ as φ^* only when it must be propositional.

$\exists x(\psi \,\&\, \forall z(\mathcal{A}\psi^z_x \rightarrow z = x) \,\&\, Rbx)$

Thus, by a right-to-left direction of the modally-strict Russell analysis (106), it follows that $\imath x\psi P$, i.e., χ'. And analogous reasoning applies to other case, namely, where χ is $P\imath x\varphi$.

The two final (a) and (b) simplest cases are where χ is either $[\lambda x\, \varphi^*]\imath x\varphi^*$ or $\imath x\varphi^*[\lambda x\, \varphi^*]$. Here, we know that φ^* has to be propositional since it appears inside the λ-expression. Since our definition $\varphi^* =_{df} \psi^*$ therefore converts to the identity $\varphi^* = \psi^*$, we can use Rule SubId to substitute ψ^* for φ^* in the λ-expression. It may come as a surprise that we can also use the identity and Rule SubId to substitute ψ^* for φ^* in the description. That's because in this case, the description can be seen as having the form $\imath x\, p$, with the term φ^* substituted for the propositional variable p. The description $\imath x\, p$ is perfectly well-formed![135] Hence, in the case where φ^* is a propositional matrix of the description, we can apply the substitution of identicals. Hence, substituting ψ^* for both occurrences of φ^* in $[\lambda x\, \varphi^*]\imath x\varphi^*$, we obtain $[\lambda x\, \psi^*]\imath x\psi^*$, i.e., χ'.

Let us now increase the complexity in only one dimension. That is, let's hold χ fixed as an exemplification or encoding formula but now consider cases where the nonsubformula occurrence of φ is doubly-nested within χ, i.e., φ occurs in a term within a term. Consider first those cases in which φ is the matrix of a description that appears in a relation term of χ, for example:

(a) Π^1 is $[\lambda y\, Ry\imath x\varphi]$ and χ is $[\lambda y\, Ry\imath x\varphi]a$

(b) Π^1 is $[\lambda y\, Ry\imath x\varphi]$ and χ is $a[\lambda y\, Ry\imath x\varphi]$

Here the strategy of deriving χ' from χ bifurcates in the (a) and (b) cases. In the (a) case, we use β-Conversion on χ to obtain $Ra\imath x\varphi$. Now by familiar reasoning used above, we can appeal to the modally-strict Russell analysis (106) to infer, by a modally strict proof, that $Ra\imath x\psi$. Hence, by β-Conversion, $[\lambda y\, Ry\imath x\psi]a$, i.e., χ'.

In the (b) case, however, we may *not* use β-Conversion on χ. Instead, we first apply the Rule of Necessitation to the modally strict theorem $\varphi \equiv \psi$, to obtain $\Box(\varphi \equiv \psi)$. Then it follows that $\mathcal{A}(\varphi \equiv \psi)$, by (89). So by the $\imath$-Conversion axiom governing λ-expressions (36.4.a), we may infer $[\lambda y\, Ry\imath x\varphi] = [\lambda y\, Ry\imath x\psi]$. Hence, by Rule SubId, we can derive $a[\lambda y\, Ry\imath x\psi]$ from $a[\lambda y\, Ry\imath x\varphi]$, i.e., derive χ' from χ.

Consider next those cases in which φ is doubly-nested within χ by appearing as the matrix of a λ-expression that occurs within a description in χ, for example:

[135] By inspection, the BNF stipulates that the propositional variables $p, q, \ldots$, are well-formed instances of 0-place relation variables Ω^0. Hence, they are well-formed instances of 0-place relation terms Π^0. Hence they are well-formed instances of propositional formulas φ^*. Hence they are well-formed instances of formulas φ. So since $\imath x\varphi$ is well-formed for any instance of φ, $\imath x\, p$ is a well-formed description, with free variable p.

(a) κ is $\iota y([\lambda x\, \varphi^*]y)$ and χ is $P\iota y([\lambda x\, \varphi^*]y)$, or
κ is $\iota y(y[\lambda x\, \varphi^*])$ and χ is $P\iota y(y[\lambda x\, \varphi^*])$

(b) κ is $\iota y([\lambda x\, \varphi^*]y)$ and χ is $\iota y([\lambda x\, \varphi^*]y)P$, or
κ is $\iota y(y[\lambda x\, \varphi^*])$ and χ is $\iota y(y[\lambda x\, \varphi^*])P$

In all of these cases, however, the fact that φ^* is propositional means we can use Rule SubId to substitute ψ^* for φ^* inside the λ-expressions. That's because the definition $\varphi^* =_{df} \psi^*$ converts to the identity $\varphi^* = \psi^*$. Hence, we can derive χ' from χ.

Of course, there are other cases in which φ is doubly-nested within an exemplification or encoding formula χ, such as when χ is any of the following:

$$[\lambda z\, \neg[\lambda y\, Ry\iota x\varphi]z]a$$
$$a[\lambda z\, \neg[\lambda y\, Ry\iota x\varphi]z]$$
$$P\iota y(Ry\iota x\varphi)$$
$$\iota y(Ry\iota x\varphi)P$$

And of course, there are base cases that include triply-nested occurrences of φ within an exemplification or encoding formula χ. And so on.

Although we can't generalize just on the few examples given above, they have been sufficiently complex to suggest that the proof strategy will generalize to any case where the definiendum or definiens occurs deeply embedded, but not as a subformula, within an exemplification or encoding formula χ. And if the strategies work for all these base cases of exemplification and encoding formulas without any special assumptions, then a properly-formed inductive hypothesis should allow us to establish the result for the inductive cases. In particular, if we have a way of inferring χ' from χ in all the base cases in which χ is an exemplification or encoding formula, then we have, by symmetry, established a modally strict proof that $\chi \equiv \chi'$ for those base cases. This serves to ground all the inductive cases in which χ is complex. For when χ is more complex, the non-subformula occurrences of φ still have to occur with the terms of an exemplification or encoding formula within χ. So we can use the provably-strict equivalence established for exemplification and encoding formulas to bootstrap the induction.[136]

[136] For example, if given $\varphi =_{df} \psi$, one can show:

$$\iota x\varphi P \to \iota x\varphi P \vdash \iota x\psi P \to \iota x\psi P$$

as follows:

Proof. From the definition $\varphi =_{df} \psi$, our conventions and rules yield $\vdash_{\Box} (\varphi \equiv \psi)$. Now we prove a Lemma: $\iota x\varphi P \vdash \iota x\psi P$.

Assume $\iota x\varphi P$. By the modally strict Russell analysis: $\exists x(\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \to z = x) \,\&\, xP)$. So by the Rule of Substitution $\exists x(\mathcal{A}\psi \,\&\, \forall z(\mathcal{A}\psi^z_x \to z = x) \,\&\, xP)$. So by the right-to-left direction of the modally strict Russell analysis: $\iota x\psi P$.

As mentioned previously, the full justification of the Rule SubDefForm (.5) goes by way of a structural induction based on the BNF, and is proved in the Appendix to this chapter as **Metatheorem** ⟨9.5⟩. Then the Rule of Equivalence by Defined Formulas (.6) is a straightforward consequence of Rule SubDefForm.

The above discussion establishes that our convention and rules preserve the traditional theory and inferential role of formula definitions, although we shall not do more to prove it here. Our convention satisfies both the eliminability criterion and the conservativeness criterion formulated at the end of Remark (207). By using definitions under the above convention and reasoning with them in accordance with the above rules, the philosophical power of the primitive expressions of our language truly emerges.

(**209**) **Remark**. The Choice of Variables in Definitions. Recall that we used metavariables to state certain formula definitions (e.g., those in (7.4)) and used object language variables to state both term and formula definitions (e.g., the term and formula definitions in (11) – (16)). A few observations about the choice of object language variables in these latter definitions is in order. We divide the discussion into two parts, one concerning bound variables and the other free variables.

Here are three definitions that appear in Part II; we've seen the first and third already, but the middle one appears later, in item (219.1):

Since the proof of this Lemma is modally strict, it follows that $\vdash_\Box \imath x\varphi P \to \imath x\psi P$. Now by the analogous reasoning, we can also establish, $\vdash_\Box \imath x\psi P \to \imath x\varphi P$. Hence we know $\vdash_\Box \imath x\varphi P \equiv \imath x\psi P$. Thus, by the Rule of Substitution, we can substitute $\imath x\psi P$ for both occurrences of the subformula $\imath x\varphi P$ in the formula $\imath x\varphi P \to \imath x\varphi P$, to obtain $\imath x\psi P \to \imath x\psi P$.

Now, to increase complexity, note that we can also show:

$$\imath y(Ry\imath x\varphi)P \to \imath y(Ry\imath x\varphi)P \vdash \imath y(Ry\imath x\psi)P \to \imath y(Ry\imath x\psi)P$$

as follows:

Proof. First we prove the Lemma: $Ry\imath x\varphi \vdash Ry\imath x\psi$:

By strict Russell, $Ry\imath x\varphi$ implies $\exists x(\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \to z = x) \,\&\, xP)$. So by Rule of Substitution $\exists x(\mathcal{A}\psi \,\&\, \forall z(\mathcal{A}\psi^z_x \to z = x) \,\&\, xP)$. So by strict Russell, $Ry\imath x\psi$.

Since the proof of this Lemma is modally strict, we've established: $\vdash_\Box (Ry\imath x\varphi \to Ry\imath x\psi)$. By analogous reasoning, we obtain $\vdash_\Box (Ry\imath x\psi \to Ry\imath x\varphi)$. Hence, we've established:

(ω) $\vdash_\Box (Ry\imath x\varphi \equiv Ry\imath x\psi)$

(ω') $\vdash_\Box (Rz\imath x\varphi \equiv Rz\imath x\psi)$

Now assume $\imath y(Ry\imath x\varphi)P$. Then by the strict Russell analysis: $\exists y(\mathcal{A}Ry\imath x\varphi \,\&\, \forall z(\mathcal{A}Rz\imath x\varphi \to z = y) \,\&\, yP)$. So by the Rule of Substitution, (ω), and (ω'), it follows that $\exists y(\mathcal{A}Ry\imath x\psi \,\&\, \forall z(\mathcal{A}Rz\imath x\psi \to z = y) \,\&\, yP)$. So by the right-to-left direction of the Russell analysis: $\imath y(Ry\imath x\psi)P$.

Thus, our strategy seems to generalize to the inductive cases where χ is more complex than an exemplification or encoding formula. But **Metatheorem** ⟨9.5⟩ provides the guarantee that it does.

$$O! =_{df} [\lambda x \, \Diamond E!x] \tag{11.1}$$

$$\top =_{df} \imath x(A!x \,\&\, \forall F(xF \equiv \exists p(p \,\&\, F = [\lambda y \, p]))) \tag{219.1}$$

$$F = G =_{df} \Box \forall x(xF \equiv xG) \tag{16.1}$$

Compare the above with the following alternative definitions, which use different bound variables in the definiens:

$$O! =_{df} [\lambda y \, \Diamond E!y]$$

$$\top =_{df} \imath z(A!z \,\&\, \forall G(zG \equiv \exists q(q \,\&\, G = [\lambda y \, q])))$$

$$F = G =_{df} \Box \forall z(zF \equiv zG)$$

The choice of bound variables in these definitions is irrelevant, for the same theorems would have been derivable no matter which of the alternatives we had used. What is of interest, however, is that different reasoning is required to establish this fact in each of the above cases!

In the first example, (11.1), we have a term definition of $O!$. Here we need only cite the derived rule (68), which legitimizes the substitution of alphabetically-variant relation terms in any context, to conclude that the alternative definition of $O!$ as $[\lambda y \, \Diamond E!y]$ would have been an equipotent substitute for the stated definition. Rule (68) ensures generally that any theorem we can prove about a complex relation term τ corresponds to a theorem about the alphabetic variants of τ. So it doesn't matter which alphabetic variant of the definiens we use in a relation term definition.

In the second example, (219.1), we have a term definition of $\top$ (read: *The True*) with a canonical description as definiens. The individual term $\top$ is introduced in Chapter 10 as a name of the abstract object that encodes exactly the properties F constructed out of true propositions. (We prove that this object is one of two truth values.) The official definition of $\top$ uses a definiens with bound variables x, F, p, y while the alternative definiens has the corresponding bound variables z, G, q, y. Note that we do not have an axiom like α-Conversion equating *arbitrary* alphabetically-variant descriptions, for as we've seen, it is not generally valid that $\imath\nu\varphi = (\imath\nu\varphi)'$ for arbitrary alphabetically-variant descriptions $\imath\nu\varphi$ and $(\imath\nu\varphi)'$, since in interpretations where $\imath\nu\varphi$ fails to have a denotation, such identities aren't true (in such cases, the exemplification formulas that appear when the defined identity symbol is expanded into primitive notation fail to be true, making the whole formula false).

However, such identities hold whenever $\imath\nu\varphi$ is a *logically proper* description. This is a theorem, and was established as item (116.2), which asserts:

$$\exists y(y = \imath\nu\varphi) \rightarrow \imath\nu\varphi = (\imath\nu\varphi)'$$

for alphabetically-variant descriptions $\imath\nu\varphi$ and $(\imath\nu\varphi)'$. This perfectly suits our purposes, since we have adopted the constraint in Remark (206) that individual term definitions are allowed only when the definiens is logically proper. Since we can derive the identity of alphabetically-variant logically proper descriptions, it follows, by Rule SubId (74.2), that for any theorem we can prove about the logically proper description $\imath\nu\varphi$, there is a corresponding theorem about $(\imath\nu\varphi)'$, and vice versa. So it doesn't matter which one serves as the definiens when introducing a new individual term as definiendum. Thus, with theorem (116.2) in place, we are able to ignore the choice of bound variables in descriptions used as definientia.

In the final example, (16), the official definition of $F = G$ uses $\Box\forall x(xF \equiv xG)$ as definiens, whereas the alternative definition has an alphabetic variant of the definiens, namely, $\Box\forall z(zF \equiv zG)$. Since alphabetic variants are provably equivalent (116.1), we know:

$$\vdash \Box\forall x(xF \equiv xG) \equiv \Box\forall z(zF \equiv zG),$$

Since the proof of (116.1) is modally-strict, it follows by the Rule of Substitution (113) that for any theorem we can prove containing $\Box\forall x(xF \equiv xG)$ as a subformula, there is a corresponding theorem containing $\Box\forall z(zF \equiv zG)$ instead, and vice versa. So it doesn't matter which alphabetically-variant formula we use as the definiens in the formula definition (16.1).

Our reasoning here applies generally to the alphabetic variants of any definiens in a formula definition. Inferentially speaking, the choice of bound variable is arbitrary (as long as we pick a variable that is safe, i.e., won't get captured by some other variable-binding operator present in the formula). We've now seen that in the three examples of definitions that differ only by the bound variables in the definiens, different reasoning is required to justify that the choice of bound variables is inconsequential.

We discuss, finally, the choice of free variables in a definiendum and definiens. It is interesting that although the choice of free variables in a definition is arbitrary, the reason why depends on whether we're dealing with a formula definition or a term definition. Consider again the above example of a formula definition:

- $F = G \; =_{df} \; \Box\forall x(xF \equiv xG)$ (16.1)

Now consider this alternative:

- $G = H \; =_{df} \; \Box\forall x(xG \equiv xH)$

The reason there is no practical difference between (16.1) and its alternative, for inferential purposes, is rooted in our Convention (208.1), which tells us that definition (16.1) is shorthand for adding $F = G$ as a new formula to the language along with (the closures of) the axiom: $F = G \equiv \Box\forall x(xF \equiv xG)$. So the following would be an axiom:

φ: $\forall F\forall G(F = G \equiv \Box\forall x(xF \equiv xG))$

But we've already established that alphabetically-variant formulas φ and φ' are interderivable (115.1) and provably equivalent (116.1). Hence, where φ' is:

φ': $\forall G\forall H(G = H \equiv \Box\forall x(xG \equiv xH))$

it follows that $\vdash_\Box \varphi \equiv \varphi'$. Thus, the stated definition (16.1) and its alternative are proxies for provably equivalent axioms φ, φ', respectively. Any formula ψ we can derive from φ can be derived from φ' by first deriving φ from φ', and vice versa; similarly, any formula ψ from which we can derive φ is a formula from which we can derive φ', and vice versa. Thus, φ and φ' imply, and are implied by, the same formulas.[137] Given that our use of definitions is a convention we've adopted in lieu of adding new formulas and axioms, we can conclude that it makes no difference whether we deploy definition (16.1) or its alternative: both have the same inferential role.

Consider, finally, why the choice of free variables in term definitions is arbitrary. In item (321), we define $\boldsymbol{a}_G$ (the thin Form of G) as follows:

- $\boldsymbol{a}_G =_{df} \imath x ThinFormOf(x, G)$ (321)
  where $ThinFormOf(x, G) =_{df} A!x \,\&\, \forall F(xF \equiv F = G)$ (318)

Now consider an alternative that uses a different free variable:

- $\boldsymbol{a}_H =_{df} \imath x ThinFormOf(x, H)$
  where $ThinFormOf(x, H) =_{df} A!x \,\&\, \forall F(xF \equiv F = H)$

By having the free variable G, the definiens of (321), $\imath x ThinFormOf(x, G)$ is an open term: for each property that can serve as value for the variable G, the description takes on a denotation relative to that value. (Note that we are still under the constraint in Remark (206); if there is a value of G for which the description $\imath x ThinFormOf(x, G)$ fails to denote, we may not use this description as definiens. But, in the present case, we can and do establish that $\forall G\exists y(y =$

[137] Strictly speaking, the claims are: (a) $\Gamma, \varphi \vdash \psi$ if and only if $\Gamma, \varphi' \vdash \psi$ and (b) $\Gamma, \psi \vdash \varphi$ if and only if $\Gamma, \psi \vdash \varphi'$. To show the left-to-right direction of (a), assume $\Gamma, \varphi \vdash \psi$. By the Deduction Theorem (54), it follows that $\Gamma \vdash \varphi \rightarrow \psi$. Independently, from $\vdash_\Box \varphi' \rightarrow \varphi$ (i.e., from the right-to-left direction of (116.1)), we know $\vdash \varphi' \rightarrow \varphi$, by (45.1). So it follows by (46.3) that $\Gamma \vdash \varphi' \rightarrow \varphi$. By a corollary (55.1) to the Deduction Theorem, it follows that $\Gamma \vdash \varphi' \rightarrow \psi$ and, by (46.10), that $\Gamma, \varphi' \vdash \psi$. The right-to-left direction of (a) is established analogously.

The argument for (b) is also analogous. To show the left-to-right direction of (b), assume $\Gamma, \psi \vdash \varphi$. By the Deduction Theorem (54), it follows that $\Gamma \vdash \psi \rightarrow \varphi$. And from $\vdash_\Box \varphi \rightarrow \varphi'$ (i.e., the left-to-right direction of (116.1), we have both $\vdash \varphi \rightarrow \varphi'$, by (45.1), and $\Gamma \vdash \varphi \rightarrow \varphi'$, by (46.3). So by a corollary (55.1) to the Deduction Theorem, it follows that $\Gamma \vdash \psi \rightarrow \varphi'$ and, by (46.10), that $\Gamma, \psi \vdash \varphi'$. The right-to-left direction of (b) is established analogously.

$\imath x ThinFormOf(x,G)$).) Hence, the definiendum $\boldsymbol{a}_G$ is a functional term and behaves a lot like a classical function term; it is well-defined for every value of G.

Now recall our general discussion in Remark (207) of the inferential role of a term definition. If given that $\exists y(y = \imath x\varphi)$ is a modally-strict theorem, the inferential role of an individual term definition of the form:

$$\kappa =_{df} \imath x\varphi$$

is equivalent to adding κ as a new term along with the closures of the necessary axioms:

$$\exists\beta(\beta = \kappa)$$

$$\kappa = \imath x\varphi$$

In the case we're discussing, where $\kappa = \boldsymbol{a}_G$ and $\imath x\varphi = \imath x ThinFormOf(x,G)$, we have to consider the additional fact that κ and $\imath x\varphi$ have the free variable G. In particular, definition (321), given the modally-strict theorem $\forall G\exists y(y = \imath x ThinFormOf(x,G))$, is a proxy for adding $\boldsymbol{a}_G$ as a primitive term to our language along with the axioms:

$$\forall G\exists y(y = \boldsymbol{a}_G)$$

$$\forall G(\boldsymbol{a}_G = \imath x ThinFormOf(x,G))$$

Similarly, the *alternative* to definition (16.1), given the modally-strict theorem $\forall H\exists y(y = \imath x ThinFormOf(x,H))$, is a proxy for adding $\boldsymbol{a}_H$ as a primitive term to our language along with the axioms:

$$\forall H\exists y(y = \boldsymbol{a}_H)$$

$$\forall H(\boldsymbol{a}_H = \imath x ThinFormOf(x,H))$$

Since the stated term definition and its alternative are proxies for (pairwise) alphabetically-variant axioms, they have the same inferential role. So the choice of free variables in such term definitions is inconsequential.

With the above remarks, we've forestalled the concerns that Gödel (1944, 120) raised about Whitehead and Russell's *Principia Mathematica* in connection with the inferential role of incomplete symbols and defined symbols and expressions. We quoted their concerns at the end of Chapter 7 and noted there that they were to be addressed, in part, by the foregoing discussion of the inferential role of definitions.

Chapter 10

Basic Logical Objects

In this chapter, we prove the existence of some basic logical objects and some fundamental theorems about them. Such objects include the truth-value of proposition p (‘p°’), the class of Fs (‘$\{y|Fy\}$’), the direction of line a (‘$\vec{a}$’), the shape of figure c (‘$\tilde{c}$’), etc. We also generalize these applications to develop theorems governing any logical object abstracted from an equivalence condition or equivalence relation.[138]

10.1 Truth-Values

(**210**) **Remark:** On Truth-Values. Frege postulated truth-values in his lecture of 1891 (13), and they are the very first logical objects that he officially introduces in his *Grundgesetze der Arithmetik*; they appear in Volume I, §2, just after the section on functions (see Frege 1893/1903, I, p. 7). In what follows, we identify truth-values as abstract, logical objects, prove they exist, and further prove, among other things: (a) that necessarily there are exactly two truth-values, and (b) that the truth-value of p is identical to the truth-value of q if and only if p is equivalent to q. The theorems about truth-values proved below are often principles that Frege implicitly assumed. The key idea underlying truth-values of propositions is that, given a proposition p, the condition, q *is*

[138]Some of the definitions and theorems below are revised and enhanced versions of those developed in Anderson & Zalta 2004. The main difference is that, in the present work, we develop definitions of such notions as *truth-value*, *class*, etc., so as to make it easier to prove modally strict theorems about these notions. By contrast, in Anderson & Zalta 2004, the definitions often used rigid definite descriptions, thereby making it difficult to establish modally strict theorems. Compare, for example, the definition of x is a *truth-value* in Anderson & Zalta (2004, 14) with the one developed below in (221). There are modally strict facts about truth-values proved below for which the version in Anderson & Zalta 2004 not modally strict. For example, theorem (225), that there are exactly two truth-values, is modally strict in the present work, but the corresponding version in Anderson & Zalta (2004, 14) is not.

materially equivalent to p, intuitively defines a logical pattern relative to p. This pattern can be objectified into an abstract, logical object that encodes just those properties F having the form $[\lambda y\, q]$ for propositions q materially equivalent to p. Such a logical object is identified as the truth-value of p.

(**211**) **Definitions:** Truth-Value of a Proposition. Recall that in (198) we defined a propositional property to be any property F such that $\exists q(F = [\lambda y\, q])$. We may use this notion to define: *x is a truth-value of p* just in case x is an abstract object that encodes all and only those properties F such that for some proposition q materially equivalent to p, F is the propositional property $[\lambda y\, q]$:

$$\mathit{TruthValueOf}(x,p) \ =_{df}\ A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$$

If we say *F is constructed from q* whenever $F = [\lambda y\, q]$, then we may read the above definition as follows: x is a truth-value of p whenever x is an abstract object that encodes all and only propositional properties constructed from some proposition q materially equivalent to p.

(**212**) **Theorem:** There Exists a (Unique) Truth-Value of p.

(.1) $\exists x \mathit{TruthValueOf}(x,p)$

(.2) $\exists! x \mathit{TruthValueOf}(x,p)$

So, by GEN, for every proposition p, there is a (unique) truth-value of p.

(**213**) **Theorem:** The Truth-Value of p Exists. Given the theorem (212.2), it follows that there exists something which is *the truth-value of p*:

$$\exists y(y = \imath x \mathit{TruthValueOf}(x,p))$$

So, by GEN, for every proposition p, the truth-value of p exists.

(**214**) **Term Definition:** Notation for the Truth-Value of p. Since the previous theorem guarantees that the description $\imath x \mathit{TruthValueOf}(x,p)$ is logically proper, for every proposition p, we may introduce the notation p° to refer to p's truth-value:

$$p^\circ \ =_{df}\ \imath x \mathit{TruthValueOf}(x,p)$$

We may think of the expression p° either as a new functional term (the logical object it denotes is a function of the proposition p) or as a complex variable, since it denotes an abstract object relative to each value of the variable p. Such expressions were discussed at the end of Remark (10). Note that the definition of p° meets the constraint laid down in Remark (206).

(**215**) **Remark:** p° is Not Strictly Canonical. Clearly, by definitions (214) and (211):

$$p^\circ = \imath x(A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q])))$$

So by (181), p° is (identical to) a canonical individual. However, in (188.2), we stipulated that a canonical description $\imath x(A!x \& \forall F(xF \equiv \varphi))$ is *strictly* canonical just in case φ is a rigid condition on properties, i.e., by (188.1), just in case $\vdash \Box\forall F(\varphi \to \Box\varphi)$. If we let φ be $\exists q((q \equiv p) \& F = [\lambda y\, q])$, then it takes a little work to see that φ fails to be a rigid condition on properties. The argument is an indirect one: we show $\vdash \neg\Box\forall F(\varphi \to \Box\varphi)$, for some p, and conclude that φ fails to be rigid, on pain of inconsistency.

To show $\vdash \neg\Box\forall F(\varphi \to \Box\varphi)$ for some p, it suffices to show $\vdash \Diamond\exists F(\varphi \,\&\, \neg\Box\varphi)$ for some p, for by classical modal reasoning, $\neg\Box\forall F(\varphi \to \Box\varphi)$ and $\Diamond\exists F(\varphi \,\&\, \neg\Box\varphi)$ are equivalent. To show there is a proof of the latter, we have prove the following holds for some p:

$$\Diamond\exists F(\exists q((q \equiv p) \,\&\, F = [\lambda y\, q]) \,\&\, \neg\Box\exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$$

By the T$\Diamond$ schema (118), it suffices to prove that the following holds for some p:

(ζ) $\exists F(\exists q((q \equiv p) \,\&\, F = [\lambda y\, q]) \,\&\, \neg\Box\exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$

The proof of (ζ) is by cases, where the four mutually exclusive and jointly exhaustive cases are: (a) p is some contingent truth, (b) p is some contingent falsehood, (c) p is some necessary truth, and (d) p is some necessary falsehood. In each case, we have to prove there is a witness to the existential quantifier $\exists F$ in (ζ). Though we leave the full proof of all four cases an exercise, consider case (a), in which p is some contingent truth. Then, by definition (148.1), $p \,\&\, \Diamond\neg p$. So we choose our witness to be $[\lambda y\, r]$, where r is a necessarily true proposition; we know such a proposition exists by (145.1). Then we show both:

(ϑ) $\exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q]))$

(ξ) $\neg\Box\exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q]))$

Clearly, (ϑ) is easy to establish, since r is a witness: r is materially equivalent to p (given that r and p are both true) and $[\lambda y\, r] = [\lambda y\, r]$ is known by the reflexivity of identity (71.1). To establish (ξ), suppose, for reductio, that necessarily, some proposition materially equivalent to p, say s, is such that $[\lambda y\, r] = [\lambda y\, s]$. By definition of proposition identity, this is just to suppose that necessarily, s is both materially equivalent to p and identical to r. But now we have a contradiction: if necessarily, s is both materially equivalent to p and identical to a necessary truth, then p is a necessary truth, contradicting the hypothesis that

p is contingently true.[139] Thus, by establishing (ϑ) and (ξ), we may conjoin them, apply $\exists$I, and use T$\Diamond$ to conclude $\Diamond\exists F(\varphi \,\&\, \neg\Box\varphi)$, completing the proof of case (a). We leave the other cases (b) – (d) as exercises.[140]

Thus, on pain of inconsistency, the formula φ we're considering isn't a rigid condition on properties but rather a *modally fragile* one. In this case, $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ fails to be *strictly* canonical and so p° fails to be strictly canonical, since it is identical to a canonical individual that fails to be strictly canonical. This is instructive because it tells us that p° inherits the modal fragility of the formula φ — any general conclusions we draw about the properties p° encodes will appeal to contingent facts, and so such conclusions will fail to be modally strict. The theorems of (217)★ below constitute good examples. If the point comes as a surprise, a closer inspection of the proofs of the theorems that follow should make it clearer. It may also be worth noting that when we define *possible worlds* in Chapter 12, we shall be in a position to define, for each world w, the truth-value of p *with respect to* w. For more on this world-relativized notion of the truth-value of p, see (444) – (451).

(**216**) **Definition:** How Objects Encode Propositions. We now extend the notion of encoding. We say that object x *encodes* proposition p just in case x is an abstract object that encodes the property *being-such-that-p*, i.e., just in case x is an abstract object that encodes $[\lambda y\, p]$. If we use the notation to $x\Sigma p$ to assert that x encodes p, then we have the following definition:

$$x\Sigma p \;=_{df}\; A!x \,\&\, x[\lambda y\, p]$$

We henceforth adopt the convention that '$x\Sigma\ldots$' is to be interpreted with the smallest scope possible. For example, $x\Sigma p \to p$ is to be parsed as $(x\Sigma p) \to p$ rather than as $x\Sigma(p \to p)$.

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[139] If this argument for (ξ) isn't transparent, here is a formal proof. Before reasoning by reductio, we first establish the following theorem:

(A) $\Box[\exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q]) \to (p \equiv r)]$

Then from (A) and the obvious reductio hypothesis, it follows by the K axiom (32.1) that $\Box(p \equiv r)$. But by (111.6), this result and our assumption $\Box r$ imply $\Box p$, which contradicts the assumption that $\Diamond\neg p$. So it remains to show (A). By RN, it suffices to show:

$$\exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q]) \to (p \equiv r)$$

So assume $\exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q])$. Suppose s is such a proposition, so that we know both $s \equiv p$ and $[\lambda y\, r] = [\lambda y\, s]$. The latter implies $r = s$. Hence, $r \equiv p$, i.e., $p \equiv r$.

[140] As a hint, here is a choice of witness for each case. In case (b), where p is contingently false, then $\neg p \,\&\, \Diamond p$; so choose the witness to be $[\lambda y\, r]$, where r is some necessarily false proposition, which we know exists by (145.2). In case (c), where p is necessarily true, choose the witness to be $[\lambda y\, r]$, where r is some contingently true proposition, which we know exists by (150.3). In case (d), where p is necessarily false, choose the witness to be $[\lambda y\, r]$, where r is some contingently false proposition, which we know exists by (150.4). In each case, show that $\varphi \,\&\, \neg\Box\varphi$ when $[\lambda y\, r]$ is substituted for F.

(**217**) ⋆**Lemmas:** Lemmas Concerning Truth-Values of Propositions. The following lemmas are simple consequences of our definitions: (.1) x is a truth-value of p if and only if x is the truth-value of p; (.2) the truth-value of p is a truth-value of p; (.3) the truth-value of p encodes a property F just in case F is identical to *being such that* q, for some proposition q materially equivalent to p; (.4) the truth-value of p encodes proposition r iff r is materially equivalent to p; (.5) the truth-value of p encodes p. More formally:

(.1) $\mathit{TruthValueOf}(x,p) \equiv x = p^{\circ}$

(.2) $\mathit{TruthValueOf}(p^{\circ},p)$

(.3) $\forall F(p^{\circ}F \equiv \exists q((q \equiv p)\ \&\ F = [\lambda y\, q]))$

(.4) $p^{\circ}\Sigma r \equiv (r \equiv p)$

(.5) $p^{\circ}\Sigma p$

These consequences are not modally-strict.

(**218**) ⋆**Theorem:** The Fregean Biconditional Principle for Truth-Values of Propositions.

$$p^{\circ} = q^{\circ} \equiv p \equiv q$$

That is, the truth-value of p is identical to the truth-value of q if and only if p is materially equivalent to q.

(**219**) **Term Definitions:** The True and The False. We define *The True* (‘$\top$’) as the logical object that encodes all and only true propositions, and *The False* (‘$\bot$’) as the logical object that encodes all and only false propositions:

(.1) $\top =_{df} \iota x(A!x\ \&\ \forall F(xF \equiv \exists p(p\ \&\ F = [\lambda y\, p])))$

(.2) $\bot =_{df} \iota x(A!x\ \&\ \forall F(xF \equiv \exists p(\neg p\ \&\ F = [\lambda y\, p])))$

Clearly, both $\top$ and $\bot$ are canonical objects (181), but we leave it as an exercise to show that they are not strictly canonical (188.2).[141] The conditions under which $\top$ and $\bot$ encode properties are modally fragile and any general conclusions we draw about these two objects in particular will fail to be modally strict.

(**220**) **Lemmas:** Facts about Propositional Properties and Truths (or Falsehoods). Some basic facts about propositional properties are: (.1) p is true if and only if the propositional properties constructed from true propositions are precisely

[141] In a later chapter, when we introduce possible worlds and use w as a restricted variable to range over possible worlds, we will define The-True-at-w ($\top_w$) and The-False-at-w ($\bot_w$). These distinguished world-relativized truth-values are strictly canonical. See items (447.1) and (447.2).

the propositional properties constructed from propositions materially equivalent to p; and (.2) it is not the case that p if and only if the propositional properties constructed from false propositions are precisely the propositional properties constructed from propositions materially equivalent to p):

(.1) $p \equiv \forall F[\exists q(q \,\&\, F = [\lambda y\, q]) \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q])]$

(.2) $\neg p \equiv \forall F[\exists q(\neg q \,\&\, F = [\lambda y\, q]) \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q])]$

Note that these theorems are modally-strict.

(221) **Definition:** Truth-Values. When Frege introduced truth-values in 1891, he not only stipulated that they exist but also asserted that there are exactly two of them, naming them The True and The False. By contrast, now that we have *defined* the The True and The False as logical objects, we next define the notion of *truth-value*, prove that The True and The False are truth-values, and prove that there are *exactly* two truth-values. Accordingly, we begin by defining: x is a *truth-value* iff x is a truth-value of some proposition:

$$TruthValue(x) \;=_{df}\; \exists p(TruthValueOf(x, p))$$

(222) ⋆**Theorem:** The Truth-Value of q is a Truth-Value.

$$TruthValue(q^{\circ})$$

We next work our way to the modally-strict claim that there are exactly two truth-values.

(223) **Lemmas:** Abstract Objects That Encode Just The Truths (or Just The Falsehoods) Are Truth-Values.

(.1) $(A!x \,\&\, \forall F(xF \equiv \exists q(q \,\&\, F = [\lambda y\, q]))) \rightarrow TruthValue(x)$

(.2) $(A!x \,\&\, \forall F(xF \equiv \exists q(\neg q \,\&\, F = [\lambda y\, q]))) \rightarrow TruthValue(x)$

These facts are modally strict.

(224) ⋆**Theorems:** The True and The False Are Distinct Truth-Values:

(.1) $TruthValue(\top)$

(.2) $TruthValue(\bot)$

(.3) $\top \neq \bot$

Note that (.1) and (.2) are *not* trivialities. We haven't stipulated that The True and The False are truth-values.

(225) **Theorem:** There are Exactly Two Truth-Values. We could invoke theorems (224.1)⋆ – (224.3)⋆ to prove that there are exactly two truth-values, by

showing that anything that is a truth-value is identical to $\top$ or $\bot$. But the resulting proof is not be modally strict. Though there would be nothing wrong with such a proof, there is a modally strict proof of our claim:

$$\exists x \exists y[\mathit{TruthValue}(x) \,\&\, \mathit{TruthValue}(y) \,\&\, x \neq y \,\&\, \forall z(\mathit{TruthValue}(z) \rightarrow z = x \vee z = y)]$$

By RN, this is a necessary truth.

(226) ⋆**Lemmas:** TruthValueOf, The True, and The False.

(.1) $\mathit{TruthValueOf}(x,p) \rightarrow (p \equiv x = \top)$

(.2) $\mathit{TruthValueOf}(x,p) \rightarrow (\neg p \equiv x = \bot)$

(227) ⋆**Theorems:** Facts about p°, $\top$, and $\bot$. The following two principles governing truth-values are provable: (.1) a proposition is true iff its truth-value is The True; (.2) a proposition is false iff its truth-value is The False:

(.1) $p \equiv (p^\circ = \top)$

(.2) $\neg p \equiv (p^\circ = \bot)$

These two principles may have been thought so obvious that it is not worth asking whether one can prove them. But that would have been a mistake.

It is also straightforward to show that: (.3) p is true iff The True encodes p; (.4) p is true iff The False fails to encode p; (.5) $\neg p$ is true iff The False encodes p, and (.6) $\neg p$ is true iff The True fails to encode p, i.e.,

(.3) $p \equiv \top \Sigma p$

(.4) $p \equiv \neg \bot \Sigma p$

(.5) $\neg p \equiv \bot \Sigma p$

(.6) $\neg p \equiv \neg \top \Sigma p$

10.2 Extensions of Propositions

In this section, we investigate the extension of a proposition, and in the next, the extension of a property.

(228) **Definitions:** Extension of a Proposition. We define *x is an extension of* proposition p just in case x (a) is an abstract object, (b) encodes only propositional properties, and (c) encodes all and only propositions materially equivalent to p:

$$\mathit{ExtensionOf}(x,p) =_{df} A!x \,\&\, \forall F(xF \rightarrow \exists q(F = [\lambda y\, q])) \,\&\, \forall q((x\Sigma q) \equiv (q \equiv p))$$

(**229**) **Theorems:** An Equivalence. It now follows that $\mathit{ExtensionOf}(x,p)$ is equivalent to $\mathit{TruthValueOf}(x,p)$:

$$\mathit{ExtensionOf}(x,p) \equiv \mathit{TruthValueOf}(x,p)$$

Since this equivalence is established by a modally strict proof, the Rule of Substitution allows us to substitute either one for the other wherever one occurs as a subformula. Note also that if we apply the Rule of Actualization to this theorem, then it follows by (105) that something is identical to $\imath x\mathit{ExtensionOf}(x,p)$ iff it is identical to $\imath x\mathit{TruthValueOf}(x,p)$.

(**230**) **Theorems:** Fundamental Theorems of Extensions of Propositions. It is now provable that: (.1) there is a unique extension of p; (.2) the extension of p exists; (.3) the extension of proposition p is the truth-value of p, and (.4) the extension of proposition p is a truth-value:

(.1) $\exists! x\mathit{ExtensionOf}(x,p)$

(.2) $\exists y(y = \imath x\mathit{ExtensionOf}(x,p))$

(.3) $\imath x\mathit{ExtensionOf}(x,p) = p^{\circ}$

(.4) $\mathit{TruthValue}(\imath x\mathit{ExtensionOf}(x,p))$

With (.4), we've derived a version of Carnap's assertion (1947, 26) that the extension of a sentence is a truth-value. But on our reconstruction, it is the extension of a proposition, in the first instance, that is a truth-value.

10.3 Extensions of Properties: Classes

(**231**) **Remark:** In this section, we define the notion of a class in terms of the notion of an extension of a property. In the previous sections, we introduced the notion of an extension of a proposition after introducing the notion of a truth-value because Frege introduced the notion of a truth-value before Carnap asserted that the extension of a sentence is a truth-value. But in the case of extensions of properties and classes, the former is the prior notion; the notion of an extension of a general term goes back to medieval logic and is well-entrenched in the Port Royal Logic (1662). The modern notion of a class is then defined below in terms of the older philosophical notion of an extension of a property.[142] Before we develop our analysis, we begin with an important distinction.

[142] See Buroker 2014 (Section 3), for a discussion of the notion of extension of general term in the Port Royal Logic.

(232) **Remark:** Natural vs. Theoretical Mathematics. In what follows it is important to distinguish between *natural* mathematics and *theoretical* mathematics. Natural mathematics consists of ordinary, pretheoretic claims we make about mathematical objects, such as the following:

- The *number* of planets is eight.
- There are more individuals in the *class* of insects than in the *class* of humans.
- The lines on the pavement have the same *direction*.
- The figures drawn on the board have the same *shape*.

By contrast, the claims of theoretical mathematics are the axioms, theorems, hypotheses, conjectures, etc., asserted in the context of some explicit mathematical theory or in the context of some implicit or informal, but still theoretical, mathematical assumptions. Example of such claims are:

- (In Zermelo-Fraenkel set theory) the null set is an element of the unit set of the null set.
- (In Real Number Theory) 2 is less than or equal to π.

One distinguishing feature of pure theoretical mathematics is that the fundamental axioms and assumptions of those theories govern special, abstract mathematical relations and operations (e.g., membership, predecessor, less than, addition, etc.) and don't involve ordinary relations or individuals.[143]

We will develop an analysis of theoretical mathematics in Chapter 15, where mathematical objects and mathematical relations will be identified as abstract objects and abstract relations, respectively. Prior to that chapter, however, we shall have occasion to analyze various *natural* mathematical objects, by identifying them as abstractions from the body of ordinary (exemplification) predications that are independent of any mathematical theory. The first group of natural mathematical objects we examine are the natural classes. A natural class is not an abstraction based on the axioms of some mathematical theory of sets, but is rather an abstraction from the facts that a given property may be exemplified by different individuals and that different properties may be exemplified by the very same individuals. Intuitively, a natural class is the (exemplification) *extension* of a property.

[143]In impure or applied mathematics, there are often some non-mathematical relations and assumptions; these won't be subject to our analysis, but are subject to the investigations of natural science.

(**233**) **Remark:** The Naive/Logical Conception of Set. The conception of natural classes just described is very closely related to the *naive conception of set*. Boolos introduces the naive conception of set as follows (1971, 216):[144]

> Here is an idea about sets that might occur to us quite naturally, and is perhaps suggested by Cantor's definition of a set as a totality of definite elements that can be combined into a whole *by a law*.
>
> By the law of excluded middle, any (one-place) predicate in any language either applies to a given object or does not. So, it would seem, to any predicate there correspond two sorts of thing: the sort of thing to which the predicate applies (of which it is true) and the sort of thing to which it does not apply. So, it would seem, for any predicate there is a set of all and only those things to which it applies (as well as a set of just those things to which it does not apply). Any set whose members are exactly the things to which the predicate applies—by the axiom of extensionality, there cannot be two such sets—is called the *extension* of the predicate. Our thought might therefore be put: "Any predicate has an extension." We shall call this proposition, together with the argument for it, the *naive conception of set*.

Boolos goes on to contrast this conception with the *iterative* conception, where sets are structured in such a way that they fall into an ordered series of stages.[145] But what is important for our purposes here is that Boolos' description of the naive conception of set more fully articulates the conception of natural classes given above.

Now, given the background system of object theory, the most straightforward way of formally representing the claim that Boolos says captures naive

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[144]In the first paragraph of the following quotation, Boolos is referring to the opening lines of Cantor 1895, 481 (1915, 85; 1932, 282), where Cantor says "Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen." Boolos paraphrases this in English as "A set is any collection ... into a whole of definite, well-distinguished objects ... of our intuition or thought" (1971, 215).

[145]Boolos writes (1971, 221):

> At stage zero, there is a set for each possible collection of individuals (and if there are no individuals, there is only one set, namely, the null set). At stage one, there exists a set for each possible collection consisting of individuals and sets formed at stage zero. And so on, until one reaches stage omega, at which there exists a set for each possible collection consisting of individuals and sets formed at stages one, two, three, ... Of course, this is only the beginning.

We shall put aside further discussion of the iterative conception of set, since that conception informs our understanding of the theoretical mathematics of sets and, in particular, Zermelo-Fraenkel set theory (ZF). The philosophical analysis of the language and theorems of ZF will be discussed in Chapter 15, where we analyze theoretical mathematics generally.

set theory ("any predicate has an extension") is as follows, where S is the property *being a set*:

> **Fundamental Principle of Naive Set Theory**
> $\forall F \exists y(Sy \,\&\, \forall x(x \in y \equiv Fx))$

Thus, Boolos' talk of predicates and predicate application becomes represented by talk of properties and property exemplification, and so Boolos's formulation "any predicate has an extension" becomes represented as the principle: for every property F, there is a set whose members are precisely the individuals exemplifying F.

But this is not how Boolos goes on to represent the Fundamental Principle of Naive Set Theory. Instead, perhaps because of doubts about second-order logic, Boolos invokes open formulas φ in which y doesn't occur free (but which typically would have a free occurrence of x). Using $\mathcal{K}$ to denote a standard first-order language having (a) variables that range over both sets and individuals and (b) distinguished predicates S for *being a set* and $\in$ for *membership*, Boolos writes (1971, 217):

> If the naive conception of set is correct, there should (at least) be a set of just those things to which φ applies, if φ is a formula of $\mathcal{K}$. So (the universal closure of) $\ulcorner(\exists y)(Sy \,\&\, (x)(x \in y \equiv \varphi))\urcorner$ should express a truth about sets (if no occurrence of 'y' in φ is free).

Of course, Boolos takes it that he has properly represented the central claim ("any predicate has an extension") of the naive conception because he is assuming that every open formula φ with free variable x may be used (in a sufficiently strong logic) to introduce or define a predicate or property, either by way of the instance $\exists F \forall x(Fx \equiv \varphi)$ of property comprehension (and introducing an arbitrary name for a witness to this instance) or by way of the λ-expression $[\lambda x\, \varphi]$.

But this is an assumption that object theory *rejects*. In object theory, not every open formula with a free occurrence of an individual variable can be used to define a predicate or property; only *propositional* formulas with a free individual variable may be so used.[146] By solving the paradoxes of naive object theory (i.e., by placing the restriction that φ not have any encoding subformulas

[146]Recall that in his letter to Frege of 1902, Russell first formulates a paradox in terms of the predicate: to be a predicate that cannot be predicated of itself. But this involves a construction in which a predicate appears in an argument position, something which is not allowed in our typed second-order language: a predicate Π may not appear as one of the individual terms κ_i in the formulas $\Pi^n \kappa_1 \ldots \kappa_n$ or $\kappa_1 \Pi^n$. If we then focus on Russell's paradox as formulated with respect to the naive conception of sets, one might reasonably offer the following diagnosis. The mistake that Frege (and others holding the naive conception of set) may have made was to hang on to the intuition that "every predicate has an extension" after extending a safe second-order logic with the notions and principles needed to assert the existence of extensions. For though the 1-place

on property comprehension), we've forestalled paradoxes that were thought to infect naive set theory. The way is still open to preserving the central claim of the naive conception of set ('Every predicate has an extension'), namely, by deriving, as a theorem, that for every property F, there is a set whose members are precisely the individuals exemplifying F. And if this is a theorem, it becomes a metatheoretical fact that for every 1-place predicate, i.e., every expression that can be instantiated for a property variable, there is a corresponding set of individuals to which that predicate applies.

If this analysis is correct, then not only do the natural conception of a class and the naive conception of set nicely dovetail, but the central ideas of both do not, in and of themselves, lead to paradox, at least not without bringing a few more assumptions to their formalization. One of our goals in what follows, therefore, is to show that this analysis is indeed correct. To do this, we: (a) precisely define what it is for an abstract logical object to be an extension of a property, (b) define classes to be extensions of properties, and (c) prove that *every* property has an extension, (d) prove that every class is the extension of some property, (e) define membership in a class, and (f) prove that for every property F, there is a class whose members are precisely the individuals exemplifying F. Since these and other principles formulated below make the naive conception of set formally precise, we shall henceforth designate the 'naive' conception with the following, less rhetorical label: the *logical conception* of set. Thus, in what follows, natural classes may be identified with sets logically conceived.

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comprehension principle of second-order logic:

$\exists F \forall x(Fx \equiv \varphi)$, provided φ has no free occurrences of F

is consistent, new properties can be generated out of the new open formulas that must be present in the language of second-order logic to assert the Fundamental Principle of Naive Set Theory, i.e.,

$\forall F \exists y(Sy \,\&\, \forall x(x \in y \equiv Fx))$

For if we assume that the open formula $z \notin z$ can be used to define the property $[\lambda z\, z \notin z]$, we can instantiate the latter into the above claim to infer:

$\exists y(Sy \,\&\, \forall x(x \in y \equiv [\lambda z\, z \notin z]x))$

By β-Conversion and the Rule of Substitution, this is equivalent to:

$\exists y(Sy \,\&\, \forall x(x \in y \equiv x \notin x))$

Let a be such an object, so that we know:

$Sa \,\&\, \forall x(x \in a \equiv x \notin x))$

By instantiating a into the right conjunct, we get the contradiction:

$a \in a \equiv a \notin a$

Thus, if one holds onto the assumption that every open formula defines a property, when extending second-order logic with comprehension to assert the existence of sets, one sets up a 'feedback existence loop' that lead to contradiction. The safe domain of properties in standard second-order logic with comprehension becomes incremented with the property $[\lambda z\, z \notin z]$, which then feeds back into the Fundamental Principle of Naive Set Theory to produce a paradoxical set.

(234) **Definition:** An Extension of a Property. We say: x is an *extension of* property G iff x is an abstract object that encodes just the properties materially equivalent to G:

$$ExtensionOf(x,G) =_{df} A!x \,\&\, \forall F(xF \equiv \forall z(Fz \equiv Gz))$$

Before studying extensions of properties further, it would serve well to remark briefly upon the above definition.

(235) **Remark:** Intuitions about What Extensions Are. The previous definition might have come as a surprise one might have thought that an 'extension of G' should be an abstraction somehow defined in terms of its members, i.e., the things exemplifying G. But we have defined an extension as an abstraction over the properties F that are materially equivalent to G. As we shall soon see, our definition immediately yields identity conditions for extensions that don't require the notion of membership (which has neither been taken as primitive nor yet defined). Moreover, our definition will easily allow us to define *the* extension of G (ϵG) and derive a consistent version of Frege's Basic Law V ($\epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx)$). Finally, once we define classes as extensions and define membership ($y \in x$), we can then derive the principle of extensionality for classes, i.e., that classes (i.e., extensions) are identical whenever they have the same members.

Thus, the definition in (234) more closely follows Frege's conception of an extension, since it is defined not as an entity that gathers together all and only the things that fall under G, but rather by abstracting over all and only the properties F exemplified by all and only the objects that exemplify G. Intuitively, if we informally allow ourselves some notions from set theory for the moment, then we might say that the condition $\forall z(Fz \equiv Gz)$ partitions the domain of properties into equivalences of properties that are materially equivalent and that $ExtensionOf(x,G)$ is an abstraction over all and only the properties that are in the cell of the partition that contains G. Since the body of theorems below establish that our definition undergirds and preserves the naive and logical conception of a class without taking any notions from set theory as primitive and without asserting any axioms of set theory, we might claim that our definition provides the metaphysical underpinning for these fundamental notions of set theory.

(236) **Theorem:** Pre-Basic Law V. The following is a modally strict truth underlying Frege's Basic Law V:

$$(ExtensionOf(x,G) \,\&\, ExtensionOf(y,H)) \rightarrow (x = y \equiv \forall z(Gz \equiv Hz))$$

(237) **Theorems:** There is a (Unique) Extension of a Property.

(.1) $\exists x ExtensionOf(x,G)$

(.2) $\exists! x ExtensionOf(x,G)$

So, by GEN, for every property G, there is a (unique) extension of G. These theorems don't quite yet capture the intuition that Boolos (1971, 216) takes to be definitive of the naive conception of set ("Every property has an extension"), since we haven't yet defined membership or shown that all and only the objects x exemplifying G are members of the extension of G.

(**238**) **Theorem:** The Extension of a Property Exists. It now follows that there exists an individual that is the extension of property G:

$$\exists y(y = \imath x ExtensionOf(x,G))$$

(**239**) **Term Definition:** The Extension of a Property. Since we've now established that the description $\imath x ExtensionOf(x,G)$ is logically proper (and by a modally strict proof), we may use it to introduce the notation ϵG to rigidly refer to *the extension of G*:

$$\epsilon G \quad =_{df} \quad \imath x ExtensionOf(x,G)$$

Thus, the expression ϵG is a functional term or complex variable that denotes an abstract object for each property G taken as argument.

(**240**) **Theorem:** Material Equivalence and Contingency. Consider the impossible property $\overline{L}$, i.e., the negation of L, where L was defined in (140) as $[\lambda x\, E!x \rightarrow E!x]$. $\overline{L}$ is impossible in the sense of (138.2) — necessarily, nothing exemplifies it. Now consider the property $[\lambda x\, E!x \,\&\, \Diamond\neg E!x]$. This is the property *being contingently concrete*. Then it is a modally strict theorem that it is possible that both $\overline{L}$ is materially equivalent to $[\lambda x\, E!x \,\&\, \Diamond\neg E!x]$ and possibly it isn't:

$$\Diamond(\forall z(\overline{L}z \equiv [\lambda x\, E!x \,\&\, \Diamond\neg E!x]z) \,\&\, \Diamond\neg\forall z(\overline{L}z \equiv [\lambda x\, E!x \,\&\, \Diamond\neg E!x]z))$$

This theorem has the form $\Diamond(\varphi \,\&\, \Diamond\neg\varphi)$. It helps us to show that ϵG is not strictly canonical.[147]

(**241**) **Remark:** ϵG is Not Strictly Canonical. Clearly, by definitions (239) and (234):

[147] One can give an intuitive argument for this theorem if we think semantically for the moment and take the notion of a possible world as semantically primitive. Now axiom (32.4) implies that there is a possible world, say $\boldsymbol{w}_1$, at which there are contingently concrete objects and a possible world, say $\boldsymbol{w}_2$, at which there aren't. Now consider the property $\overline{L}$ and let P be the property of *being contingently concrete*. Since necessarily, nothing exemplifies $\overline{L}$, it is not materially equivalent to P at w_1 (since something is P there but nothing is $\overline{L}$ there). But at $\boldsymbol{w}_2$, $\overline{L}$ is materially equivalent to G (since nothing is either P or $\overline{L}$ there). Reversing these conclusions, we've established therefore that (a) possibly, $\overline{L}$ and P are materially equivalent, and (b) possibly, they aren't. But by (119.12), this implies possibly, both (a) they are materially equivalent and (b) possibly they are not.

$$\epsilon G = \imath x(A!x \,\&\, \forall F(xF \equiv \forall z(Fz \equiv Gz)))$$

So by (181), ϵG is (identical to) a canonical individual. However, it is instructive to see why ϵG is not strictly canonical, i.e., by (188.2), why the formula $\forall z(Fz \equiv Gz)$ is not a rigid condition on properties. Let φ be $\forall z(Fz \equiv Gz)$. Then we can use (240) to show that φ fails to be a rigid condition on properties, i.e., by (188.1), that there isn't a proof of $\Box\forall F(\varphi \to \Box\varphi)$. The argument for this is an indirect one: we find a property G for which $\vdash \neg\Box\forall F(\varphi \to \Box\varphi)$ and conclude that φ fails to be rigid, on pain of inconsistency.

Of course, it is tempting to simply note that once we *apply* the theory, it becomes clear that φ fails to be rigid condition on properties. Let G be the property *being a creature with a kidney*. Then we can show $\Diamond\exists F(\varphi \,\&\, \neg\Box\varphi)$. Choose our witness to be the property *being a creature with a heart* (H). It is intuitively true both that (a) all and only creatures with hearts are creatures with kidneys, and (b) it is possible, that not all and only creatures with a heart are creatures with kidneys (or vice versa), i.e.,

$$\forall z(Hz \equiv Gz) \,\&\, \Diamond\neg\forall z(Hz \equiv Gz)$$

If we extend our theory with the above axiom relating these two particular properties, then by $\exists$I and the T$\Diamond$ schema, we have a proof $\Diamond\exists F(\varphi \,\&\, \Diamond\neg\varphi)$, and so, by standard (modal) reasoning, a proof of $\neg\Box\forall F(\varphi \to \Box\varphi)$.

But we shouldn't rest with an intuitive example when there is a proof of $\neg\Box\forall F(\varphi \to \Box\varphi)$ without extending our theory with new axioms. Given the equivalence of $\neg\Box\forall F(\varphi \to \Box\varphi)$ and $\Diamond\exists F(\varphi \,\&\, \Diamond\neg\varphi)$, it suffices to show that there is a proof of the latter, for some property G. When we choose G to be $[\lambda x\, E!x \,\&\, \Diamond\neg E!x]$, then by (240), we know:

$$\Diamond(\forall z(\overline{L}z \equiv Gz) \,\&\, \Diamond\neg\forall z(\overline{L}z \equiv Gz))$$

So by $\exists$I:

$$\exists F\Diamond(\forall z(Fz \equiv Gz) \,\&\, \Diamond\neg\forall z(Fz \equiv Gz))$$

By CBF$\Diamond$ (122.4):

$$\Diamond\exists F(\forall z(Fz \equiv Gz) \,\&\, \Diamond\neg\forall z(Fz \equiv Gz))$$

i.e.,

$$\Diamond\exists F(\varphi \,\&\, \Diamond\neg\varphi)$$

So φ fails to be a rigid condition on properties, on pain of inconsistency. Thus, the description $\imath x(A!x \,\&\, \forall z(Fz \equiv Gz))$ fails to be strictly canonical, and so ϵG fails to be (identical to) a strictly canonical individual.

(242) ⋆**Lemmas:** Facts about the Extension of a Property. The previous definition, together with earlier theorems, straightforwardly yield, as non-modally strict theorems, that: (.1) x is an extension of G if and only if x is the extension of G; (.2) the extension of G is an extension of G; (.3) the extension of G encodes a property F iff F is materially equivalent to G; and (.4) the extension of G encodes G::

(.1) $\mathit{ExtensionOf}(x,G) \equiv x = \epsilon G$

(.2) $\mathit{ExtensionOf}(\epsilon G, G)$

(.3) $\forall F(\epsilon GF \equiv \forall z(Fz \equiv Gz))$

(.4) ϵGG

We've adopted here a convention of putting the symbol 'G', in formulas of the form ϵGF, in a slightly smaller font to make it easier to parse such formulas as encoding formulas of the form xF with the complex individual term ϵG substituted for x.

(243) ⋆**Theorem:** Frege's Basic Law V. A consistent version of Frege's Basic Law V is now derivable, namely, the extension of F is identical with the extension of G iff F and G are materially equivalent:[148]

$$\epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx)$$

After receiving the letter from Russell informing him of Russell's paradox, Frege added an Appendix to Frege 1903 and reconstructed the paradox within his formal system by deriving a contradiction from Basic Law V. This derivation, however, cannot be reproduced in our system. To see why, note that in Frege's system, the open formula "x is an extension of a concept under which x doesn't fall", i.e., $\exists F(x = \epsilon F \,\&\, \neg Fx)$, defines a concept. (Here we're using 'concept' in the way Frege conceived of them, namely, as properties.) Let K be this concept. Since Frege's system guarantees that every concept has an extension, he proceeds to show that the extension of K falls under K if and only if it does not. However, in our system, the open formula $\exists F(x = \epsilon F \,\&\, \neg Fx)$ doesn't define a property; it can't be used in the Comprehension Principle for Properties and $[\lambda x\, \exists F(x = \epsilon F \,\&\, \neg Fx)]$ is not well-formed; formulas of the form $x = y$ may not appear as subformulas of λ-expressions since they are defined in terms of encoding subformulas.[149] So even though every property has an extension, we

[148] cf. Frege, *Grundgesetze I*, §20. Note that our theorem is not modally strict. In a later chapter, we'll develop a modally strict proof of: the extension of F at w is identical to the extension of G at w if and only if F and G are materially equivalent at w. See (456).

[149] Note that a λ-expression of the form $[\lambda x\, \varphi]$ may legitimately contain the term ϵF, despite the fact that ϵF is defined in terms of a description, $\imath x \mathit{ExtensionOf}(x,F)$, whose matrix $\mathit{ExtensionOf}(x,F)$

can't assert the existence of the paradoxical property Frege used to derive the contradiction in his system.

Exercise. Develop a modally strict proof of $\epsilon F = \epsilon G \equiv \mathcal{A}\forall x(Fx \equiv Gx)$.

(244) **Definition:** Membership. We now define y is a *member of* x, or y is an *element of* x, if and only if x is an extension of some property that y exemplifies:

$$y \in x \ =_{df} \ \exists G(\mathit{ExtensionOf}(x,G) \,\&\, Gy)$$

Cf. Frege 1893, §34. In what follows, the reader should carefully distinguish the membership symbol $\in$ just introduced in defined formulas $x \in y$ from the functional symbol ϵ in the complex term ϵG.

(245) **Theorem:** Membership and Exemplification. A precise correlation between membership and exemplification now obtains for extensions: if x is an extension of H, then y is an element of x iff y exemplifies H (cf. Frege 1893, §55, Theorem 1):

$$\mathit{ExtensionOf}(x,H) \rightarrow \forall y(y \in x \equiv Hy)$$

(246) **Definition:** A Class Corresponding to a Property. We say: x is a *class* corresponding to property G (or x is a *class of* Gs) just in case x is an extension of G:

$$\mathit{ClassOf}(x,G) \ =_{df} \ \mathit{ExtensionOf}(x,G)$$

Later, we shall introduce restricted variables to range over classes.

(247) **Theorems:** Membership in a Class and Exemplification. It now follows easily that (.1) y is a member of x iff x is a class of some property that y exemplifies, and (.2) if x is a class of Hs, then y is an member of x iff y exemplifies H:

(.1) $y \in x \equiv \exists G(\mathit{ClassOf}(x,G) \,\&\, Gy)$

(.2) $\mathit{ClassOf}(x,H) \rightarrow \forall y(y \in x \equiv Hy)$

(248) **Theorems:** There is a (Unique) Class of Gs. It is also easy, in light of (246), to infer that there is a (unique) class of Gs from (237.1) and (237.2):

(.1) $\exists x \mathit{ClassOf}(x,G)$

(.2) $\exists! x \mathit{ClassOf}(x,G)$

is not propositional. The matrix $\mathit{ExtensionOf}(x,F)$ is defined as $A!x \,\&\, \forall G(xG \equiv \forall z(Gz \equiv Fz))$, and so fails to be propositional, since its definiens has an encoding subformula. But the non-propositional formula $\mathit{ExtensionOf}(x,F)$ is not a subformula of $[\lambda x\, \varphi]$ when ϵF, i.e., $\imath x \mathit{ExtensionOf}(x,F)$, is a term in φ. Hence, it really is only the presence of the identity claim $x = \epsilon F$ that prevents $[\lambda x \exists F(x = \epsilon F \,\&\, \neg Fx)]$ from being well-formed.

So, by GEN, for every property G, there is a unique class of Gs.

(**249**) **Definition:** Natural Classes. We say that x is a (natural) *class* if and only if x is a class of some property G:

$$Class(x) \ =_{df} \ \exists G(ClassOf(x,G))$$

For simplicity, we're using the defined notation $Class(x)$ instead of $NaturalClass(x)$. In light of our discussion in Remarks (232) and (233), the above definiens for $Class(x)$ could also be used to define $LogicalSet(x)$.

(**250**) **Theorem:** Classes and Extensions of Properties. Thus, the following philosophically significant claim becomes an easy theorem, namely, x is a class if and only if x is an extension of some property:

$$Class(x) \equiv \exists G(ExtensionOf(x,G))$$

By GEN, an important component of the logical conception of set is derived—all and only classes are extensions of a property.

(**251**) ⋆**Theorem:** The Extension of G is a Class. Similarly, the extension of a property is a class:

$$Class(\epsilon G)$$

Cf. Carnap 1947, 19 (item 4-14).

(**252**) **Theorem:** The Fundamental Theorem for Natural Classes and Logically-Conceived Sets. We saw in Remark (233) that the fundamental principle govening the conception of natural classes and logical sets is: for every property F, there is a natural class (logical set) whose members are precisely the individuals exemplifying F. We now have a proof of this claim:

$$\forall F \exists x(Class(x) \,\&\, \forall y(y \in x \equiv Fy))$$

The proof in the Appendix shows how this follows from previous theorems. Our next goal is to derive the Principle of Extensionality. But we before we do, we digress to discuss restricted variables.

10.4 Interlude: Restricted Variables

(**253**) **Remark:** Issues About Restricted Variables. In what follows, we shall start to make use of *restricted variables*, i.e., variables introduced to range only over individuals or relations that meeting a certain well-defined condition. Up to this point in our text, we have been using the letter c as a substitute for the individual constant a_3. But given the condition $Class(x)$ is well-defined, let us use the letters $c, c', c'', \ldots$ as restricted individual variables that range over individuals x such that $Class(x)$.

The use of restricted variables in the present system requires some care, as the following extended discussion will show. Let's first consider restricted variables bound by universal and existential quantifiers. If $c, c', c'', \ldots$ are to be restricted individual variables ranging classes, then the usual practice is to express a universal claim of the form $\forall x(Class(x) \rightarrow \ldots x \ldots)$ by using a claim of the form $\forall c(\ldots c \ldots)$. Thus, we may represent the (true) claim that not every class has members, i.e., $\neg\forall x(Class(x) \rightarrow \exists y(y \in x))$, as $\neg\forall c\exists y(y \in c)$, and represent the (false) claim that every class fails to have members, i.e.,

(A) $\forall x(Class(x) \rightarrow \neg\exists y(y \in x))$

as:

(B) $\forall c\neg\exists y(y \in c)$

Moreover, the usual practice is to express an existential claim of the form $\exists x(Class(x) \;\&\; \ldots x \ldots)$ by using a claim of the form $\exists c(\ldots c \ldots)$. So, the (true) claim that some class fails to have members, i.e.,

(C) $\exists x(Class(x) \;\&\; \neg\exists y(y \in x))$

could be more simply represented with restricted variables as:

(D) $\exists c\neg\exists y(y \in c)$

So to a first approximation, when a restricted variable is bound by a quantifier, it should be clear how the restricted variable is to be eliminated.

But what are we to make of a formula containing a *free* occurrence of the restricted variable c, for example:

(E) $\neg\exists y(y \in c)$

How is (E) to be interpreted? Note that (E) is a subformula of both (B) and (D). However, since (B) is short for (A), (E) appears to be going proxy for the formula:

(F) $Class(x) \rightarrow \neg\exists y(y \in x)$

but since (D) is short for (C), (E) appears to be going proxy for the formula:

(G) $Class(x) \;\&\; \neg\exists y(y \in x)$

So are we to regard (E) as shorthand for (F) or for (G)? Should the fact that we face such a question lead us to conclude that we should use restricted variables only when they are bound?

Let us say that (F) is the 'conditional' interpretation of (E) and say that (G) is the 'conjunctive' interpretation of (E). Our examples above suggest that (E) should be given the conditional interpretation in some contexts, while given the conjunctive interpretation in others. Thus, we shall suppose that the interpretation of formulas with free restricted variables depends on the context. In what follows, we examine other examples of this phenomenon before we lay out the conventions we'll deploy for using restricted variables.

First, let's examine how we reason from premises containing free restricted variables by considering the Principle of Extensionality, which is formulated with standard variables as the claim that two classes are identical whenever they have the same members, i.e.,

(H) $(Class(x) \,\&\, Class(y)) \rightarrow (x = y \equiv \forall z(z \in x \equiv z \in y))$

Though we shall derive this principle in item (257) in the next section, focus on the fact that with restricted variables, we can formulate the principle more efficiently as:

(I) $c = c' \equiv \forall z(z \in c \equiv z \in c')$

Now if we were to attempt to we prove the (more interesting) right-to-left direction of (I), we would begin by assuming the antecedent $\forall z(z \in c \equiv z \in c')$ with the goal of showing $c = c'$. But what exactly have we assumed when we assume $\forall z(z \in c \equiv z \in c')$? If we express the answer without using restricted variables, then the answer is clear; we've assumed the conjunction:

(J) $Class(x) \,\&\, Class(y) \,\&\, \forall z(z \in x \equiv z \in y)$

with the goal of showing $x = y$. After all, in a standard conditional proof of (H), you first assume $Class(x) \,\&\, Class(y)$ and, then, for the right-to-left direction of the consequent, assume $\forall z(z \in x \equiv z \in y)$ and attempt to prove $x = y$. So in the context of reasoning from premises with free restricted variables such as $\forall z(z \in c \equiv z \in c')$, we use the conjunctive interpretation of those variables.

Next, consider the practice of using free restricted variables in definitions. For example, in advanced set theory, mathematicians introduce $\alpha, \beta, \gamma, \delta$ as restricted variables ranging over *ordinals*. They then define properties or functions of ordinals by using these restricted variables.[150] Consider how we might

[150] In standard texts on set theory, one typically finds an *ordinal* to be defined as any set strictly well-ordered with respect to $\in$ and such that every element is also a subset. Then, using restricted variables, one might find the following definition of the function term *the successor of* α and the notion α *is a limit ordinal*:

$$Suc(\alpha) =_{df} \alpha \cup \{\alpha\}$$

$$LimitOrdinal(\alpha) =_{df} \alpha \neq 0 \,\&\, \neg\exists\beta(\alpha = Suc(\beta))$$

In Drake 1974 (41), for example, we find ordinal addition defined basically as follows, where λ

do something similar in the present theory. Using c as a free restricted variable, we might say that a class c is *empty* just in case c has no members:

(L) $Empty(c) =_{df} \neg\exists y(y \in c)$

How are we to understand this definition, given the Conventions laid down in (208); i.e., how are we to formulate (L) without the restricted variables?

The correct way to eliminate the restricted variables in (L) is as follows:

(M) $Empty(x) =_{df} Class(x) \,\&\, \neg\exists y(y \in x)$

Once we do this, Convention (208.1) applies, and so our definition becomes shorthand for adding $Empty(x)$ as a new formula and taking the closures of the axioms:

(N) $Empty(x) \equiv (Class(x) \,\&\, \neg\exists y(y \in x))$

By contrast, it would be a mistake to interpret (L) as a kind of conditional definition, i.e., as shorthand for adding $Empty(x)$ to the language and taking the closures of the axioms:

(O) $Class(x) \rightarrow (Empty(x) \equiv \neg\exists y(y \in x))$

(N) and (O) aren't equivalent. (N) implies (O) but not vice versa, since $\varphi \equiv (\psi \,\&\, \chi)$ implies $\psi \rightarrow (\varphi \equiv \chi)$, but the converse doesn't hold.[151] If (O) were the correct way to eliminate the restricted variables in (L), then $Empty(x)$ would not be generally defined but rather defined only when x is a class. Of course, this might lead one to suggest that we simply consider $Empty(x)$ to be defined only when $Class(x)$ is known to be true either by hypothesis or by proof, for if ψ is known to be true, then $\varphi \equiv (\psi \,\&\, \chi)$ and $\psi \rightarrow (\varphi \equiv \chi)$ become equivalent.[152] Indeed, if ψ is known to be true, then $\varphi \equiv (\psi \,\&\, \chi)$ becomes equivalent

ranges over limit ordinals and 0 is defined earlier (25) as $\varnothing$:

$$\alpha + 0 = \alpha$$

$$\alpha + Suc(\beta) = Suc(\alpha + \beta)$$

$$\alpha + \lambda = \bigcup_{\delta<\lambda}(\alpha + \delta)$$

Though Drake says how we should interpret restricted variables when they appear as bound variables (1974, 22, 26), he never explains what is meant when they occur free in definitions.

[151] To see that $\varphi \equiv (\psi \,\&\, \chi)$ implies $\psi \rightarrow (\varphi \equiv \chi)$, assume (a) $\varphi \equiv (\psi \,\&\, \chi)$ and (b) ψ. To show $\varphi \equiv \chi$, we show both directions. ($\rightarrow$) Assume φ. Then by (a), $\psi \,\&\, \chi$, and hence χ. ($\leftarrow$) Assume χ. Then by (b) we have $\psi \,\&\, \chi$. Hence, φ, by (a).

To see the converse doesn't hold, consider any scenario in which ψ is false but either (i) φ and χ are both true or (ii) φ is true and χ is false. Then $\psi \rightarrow (\varphi \equiv \chi)$ is true, but $\varphi \equiv (\psi \,\&\, \chi)$ is false.

[152] Suppose ψ is known to be true, either by assumption or by proof. Then we show the equivalence by showing both directions. ($\rightarrow$) This direction was already established, by the argument in the previous footnote. ($\leftarrow$) Assume $\psi \rightarrow (\varphi \equiv \chi)$. Then since ψ is known, it follows that (a) $\varphi \equiv \chi$. Now we have to show $\varphi \equiv \psi \,\&\, \chi$. ($\rightarrow$) Assume φ. Then by (a), χ. Hence $\psi \,\&\, \chi$. ($\leftarrow$) Assume $\psi \,\&\, \chi$. Then from χ and (a), it follows that φ.

to $\varphi \equiv \chi$.[153]

In the present work, however, we prefer to avoid introducing formula definitions that are not defined in all contexts. Whereas we can tolerate the presence of terms defined only in certain contexts (see (266) below), we consider it preferable to eliminate the restricted variables in (L) as we've done in (M), so that $Empty(x)$ is generally defined. Then once we introduce the restricted variable c to range over classes, we have the following desirable consequences:

(a) $Class(c)$ becomes axiomatic and hence known to be true by a proof;

(b) $Empty(c) \equiv Class(c) \,\&\, \neg\exists y(y \in c)$ becomes an instance of the biconditional resulting from definition (M), and

(c) the equivalence resulting from definition (L), i.e., $Empty(c) \equiv \neg\exists y(y \in c)$, becomes equivalent to (b), by the reasoning in footnote 153.

Consequently, when we use free restricted variables in a formula definition in what follows, we give them the conjunctive interpretation. Formula definitions, therefore, constitute another context where we can't take the conditional interpretation of a formula containing a free occurrence of a restricted variable.

The reader might also wonder here about the use of restricted variables in descriptions and λ-expressions? For example, how would we understand the bound restricted variable in $\imath c\varphi$ and $[\lambda c\, \varphi]$. And what if φ were to contain free restricted variables? How would we interpret term definitions in which the definiens has a bound restricted variable, or in which the definiendum and definiens both have free restricted variables? We shall try to answer such questions over the course of the next few items. We begin by considering how to eliminate bound restricted variables and then consider how to eliminate free restricted variables.

(**254**) **Metadefinitions:** Bound Occurrences of Restricted Variables. Clearly, in light of these issues, restricted variables in formulas and terms have to be introduced carefully and under tightly-circumscribed conditions. We discuss bound occurrences of restricted variables here and discuss free occurrences in the next item. In our discussion of these conditions, we suppose:

- α is an unrestricted individual variable or an n-place relation variable (for some n),
- ψ is some formula in which α has a free occurrence (ψ may of course be defined notation, in which case our conventions for formula definitions are in effect),

[153] Suppose ψ is known to be true, either by assumption or by proof. Then we show the equivalence by showing both directions. ($\rightarrow$) Assume $\varphi \equiv (\psi \,\&\, \chi)$. So if φ, then $\psi \,\&\, \chi$ and hence χ. And if χ, then since we now know $\psi \,\&\, \chi$, it follows that φ. ($\leftarrow$) Assume $\varphi \equiv \chi$. Now to establish $\varphi \equiv \psi \,\&\, \chi$, follow the proof from (a) in the previous footnote.

- γ is a variable of the same type as α but restricted to range over those entities α such that ψ, and
- φ is any formula in which γ is substitutable for α.

First, then, we may introduce, by definition, formulas in which restricted variables are bound by quantifiers:

(.1) $\forall\gamma\varphi^{\gamma}_{\alpha} =_{df} \forall\alpha(\psi \to \varphi)$

(.2) $\exists\gamma\varphi^{\gamma}_{\alpha} =_{df} \exists\alpha(\psi \,\&\, \varphi)$

Notice an immediate difference between the inferential role of these two definitions. From any theorem of the form $\forall\alpha\chi$, it follows that $\forall\gamma\chi^{\gamma}_{\alpha}$.[154] For example, if every individual is ordinary or abstract ($\forall x(O!x \vee A!x)$), then every set is ordinary or abstract ($\forall c(O!c \vee A!c)$). However, from $\exists\alpha\chi$, it does not follow that $\exists\gamma\chi^{\gamma}_{\alpha}$. From the fact that something is F ($\exists xFx$), it doesn't follow that some set is F ($\exists cFc$).

Second, we may introduce, by definition, definite descriptions in which restricted variables are bound by the description operator. Where γ and α are individual variables, we also define:

(.3) $\imath\gamma\varphi^{\gamma}_{\alpha} =_{df} \imath\alpha(\psi \,\&\, \varphi)$

Notice that if we know that $\imath\alpha(\psi \,\&\, \varphi)$ is logically proper and we introduce:

$$\tau =_{df} \imath\alpha(\psi \,\&\, \varphi)$$

then we may infer by (101.2)★ that $\psi^{\tau}_{\alpha} \,\&\, \varphi^{\tau}_{\alpha}$. Consequently, if we instead introduce:

$$\tau =_{df} \imath\gamma\varphi^{\gamma}_{\alpha}$$

then we already know ψ^{τ}_{α} when we use (101.2)★ to infer φ^{τ}_{α}.

Finally, we may introduce, by definition, λ-expressions in which restricted variables are bound by the λ. Of course, the matrices of λ-expressions must be propositional formulas and so for any propositional formulas φ^* and ψ^*, we define:

(.4) $[\lambda\gamma\, {\varphi^*}^{\gamma}_{\alpha}] =_{df} [\lambda\alpha\, \psi^* \,\&\, \varphi^*]$

We leave the generalization to n-place λ-expressions for a footnote.[155]

Conventions (.1) – (.4) are all used in the following examples. If ψ is $Class(x)$ and c is a restricted individual variable ranging over individuals x such that $Class(x)$, then given the above definitions:

[154] Assume $\forall\alpha\chi$. Then, χ, by $\forall$E. Hence, $\psi \to \chi$, by axiom (21.1). So by GEN, $\forall\alpha(\psi \to \chi)$. Hence, by our definition, it follows that $\forall\gamma\chi^{\gamma}_{\alpha}$.

[155] If $\gamma_1,\ldots,\gamma_n$ are all restricted variables ranging over the entities α such that ψ^* and $\alpha_1,\ldots,\alpha_n$ have a free occurrence in φ^*, then we may define:

$$[\lambda\gamma_1\ldots\gamma_n\, {\varphi^*}^{\gamma_1,\ldots,\gamma_n}_{\alpha_1,\ldots,\alpha_n}]$$

- $\forall c \forall c'(\forall z(z \in c \equiv z \in c') \to c = c')$ is defined as:
  $\forall x \forall y(Class(x) \to (Class(y) \to (\forall z(z \in x \equiv z \in y) \to x = y)))$
- $\exists c \neg \exists y(y \in c)$ is defined as $\exists x[Class(x) \,\&\, \neg \exists y(y \in x)]$
- $\imath c(\neg \exists y(y \in c))$ is defined as $\imath x(Class(x) \,\&\, \neg \exists y(y \in x))$
- $[\lambda c \neg \exists y(y \in c)]$ does not abbreviation a λ-expression, since the expression $[\lambda x Class(x) \& \neg \exists y(y \in x)]$ is not a legitimate term of the language and can't serve as definiens — the matrix $Class(x) \& \neg \exists y(y \in x)$ is not a propositional formula.

We turn next to the conditions under which we may deploy free occurrences of restricted variables in formulas and terms.

(255) **Conventions:** Free Occurrences of Restricted Variables. Again, let α, ψ, γ, and φ be as in (254). Then we may adopt the following conventions about the use of formulas and terms containing free occurrences of γ in the certain contexts:

(.1) *In axioms and theorems*. Instead of asserting or proving the conditional formula $\psi \to \varphi$, we may instead assert or prove the simpler formula $\varphi^{\gamma}_{\alpha}$, provided γ is not already free in φ.

To give an example (albeit one that requires two applications of this convention), consider again the Principle of Extensionality (257). If formulated with unrestricted free variables, we would assert :

(PE) $Class(x) \to (Class(y) \to (\forall z(z \in x \equiv z \in y) \to x = y))$

However, we can simplify (PE) in two steps. First, we may reformulate (PE) as:

(ϑ) $Class(y) \to (\forall z(z \in c \equiv z \in y) \to c = y))$

In this first step of the simplification process, we've applied our convention by taking ψ to be the antecedent $Class(x)$ of (PE), φ to be the consequent of (PE), α be x and γ be c. Thus, (ϑ) is $\varphi^{\gamma}_{\alpha}$. But we may simplify (PE) further by reformulating (ϑ) as:

(ζ) $\forall z(z \in c \equiv z \in c') \to c = c')$

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as the expression:

$$[\lambda \alpha_1 \ldots \alpha_n \, \psi^{*}{}^{\alpha_1}_{\alpha} \,\&\, \ldots \,\&\, \psi^{*}{}^{\alpha_n}_{\alpha} \,\&\, \varphi^{*}]$$

So, for example, if ψ is $A!x$, $z_1, \ldots, z_n$ are taken to be a restricted variables ranging over individuals x such that $A!x$, and φ^* is $\neg R^n x_1 \ldots x_n$, then $[\lambda z_1 \ldots z_n \neg R^n z_1 \ldots z_n]$ is defined to be the expression: $[\lambda x_1 \ldots x_n \, A!x_1 \,\&\, \ldots \,\&\, A!x_n \,\&\, \neg R x_1 \ldots x_n]$.

In this second step of the simplification process, we've applied our convention by taking ψ be the antecedent $Class(y)$ of (ϑ), φ be the consequent of (ϑ), α be y and γ be c' (we have to choose a new restricted variable because c already occurs free in φ). Thus (ζ) is $\varphi^{\gamma}_{\alpha}$.

(.2) *In premises (i.e., assumptions) in proofs*. We may take $\varphi^{\gamma}_{\alpha}$ as a premise in a proof with the understanding that ψ^{γ}_{α} is already known. Again, we've already seen an example of this with *two* variables in (253), when we were discussing how to prove the right-to-left direction of the Principle of Extensionalty formulated with restricted variables: we derive $c = c'$ from the assumption $\forall z(z \in c \equiv z \in c')$, but the assumption already presupposes $Class(c)$ and $Class(c')$.

(.3) *In formula definitions*. A definition of the following form (in which the definiendum χ^{γ}_{α} has γ substituted for α):

$$\chi^{\gamma}_{\alpha} =_{df} \varphi^{\gamma}_{\alpha}$$

is to be regarded as the definition:

$$\chi =_{df} \psi \,\&\, \varphi$$

Again, we've seen example of this in (253), where definition (L) was interpreted as definition (M). In that case, α is the variable x, ψ is the formula $Class(x)$, χ is the expression $Empty(x)$, and φ is the formula $\neg\exists y(y \in x)$, so that (L) is obtained by setting γ to the variable c (the definiendum χ^{γ}_{α} then becomes $Empty(c)$ and the definiens $\varphi^{\gamma}_{\alpha}$ becomes $\neg\exists y(y \in c)$).

It should be noted that we can generalize this to the case where the new expression being defined has multiple argument places filled by restricted variables. So where $\gamma_1, \ldots, \gamma_n$ are variables of the same type as $\alpha_1, \ldots, \alpha_n$ and all of the γ_i range over those α such that ψ, then

$$\chi^{\gamma_1,\ldots,\gamma_n}_{\alpha_1,\ldots,\alpha_n} =_{df} \varphi^{\gamma_1,\ldots,\gamma_n}_{\alpha_1,\ldots,\alpha_n}$$

is to be regarded as the definition:

$$\chi =_{df} \psi^{\alpha_1}_{\alpha} \,\&\, \ldots \,\&\, \psi^{\alpha_n}_{\alpha} \,\&\, \varphi$$

We can see how this definition works in an example we've already considered (though which is officially presented as item (262) the next section). If α is x, ψ is $Class(x)$, χ is $UnionOf(x,z,w)$, φ is $\forall y(y \in x \equiv y \in z \vee y \in w)$, $\alpha_1, \alpha_2, \alpha_3$ are x, z, w, and $\gamma_1, \gamma_2, \gamma_3$ are c, c', c'', respectively, then the definition:

$$UnionOf(c,c',c'') =_{df} \forall y(y \in c \equiv y \in c' \vee y \in c'')$$

has the form:

$$\chi^{\gamma_1,\gamma_2,\gamma_3}_{\alpha_1,\alpha_2,\alpha_3} =_{df} \varphi^{\gamma_1,\gamma_2,\gamma_3}_{\alpha_1,\alpha_2,\alpha_3}$$

Thus, our definition is to be regarded as shorthand for:

$$UnionOf(x,z,w) =_{df} Class(x) \,\&\, Class(z) \,\&\, Class(w) \,\&\, \forall y(y \in x \equiv y \in z \vee y \in w)$$

which has the form:

$$\chi =_{df} \psi^{\alpha_1}_{\alpha} \,\&\, \psi^{\alpha_2}_{\alpha} \,\&\, \psi^{\alpha_3}_{\alpha} \,\&\, \varphi$$

(.4) *In term definitions*. We postpone discussion of this until Remark (266), in which we discuss the use of free restricted variables in *individual* term definition (265), which is where such variables are officially used for the first time. We shall not need to use free restricted variables in relation term definitions and so we omit this case. Note that, even if it becomes convenient to introduce such variables into relation term definitions, the problem they pose in individual term definitions does not arise, since relation terms are always logically proper. See the discussion in (266).

Finally, note that we shall sometimes introduce *restricted constants* and use them for reasoning via Rule ∃E. Suppose the constant τ has been designated as a restricted constant for entities satisfying the condition ψ and suppose we know $\exists\gamma\varphi^{\gamma}_{\alpha}$ (which contains γ as a restricted variable ranging over entities α such that ψ). Then when we assume, as a premise, that τ is an arbitrarily chosen such γ, so that we know φ^{τ}_{α}, we also know ψ^{τ}_{α}, since τ is a restricted constant.

(**256**) **Remark:** Restricted Variables and Empty Conditions. One more cautionary note about the use of restricted variables is called for. If a restricted variable γ is introduced to range over entities that meet a well-defined condition ψ (with free variable α), and it isn't established whether or not $\exists\alpha\psi$, then the quantifiers $\forall\gamma$ and $\exists\gamma$ aren't guaranteed to exhibit the classical behavior expressed by the theorem $\forall\alpha\varphi \to \exists\alpha\varphi$ (86.1). Indeed, this behavior fails when ψ is a well-defined but *empty* condition in which the variable α is free, i.e., the claim:

(ϑ) $\forall\gamma\varphi^{\gamma}_{\alpha} \to \exists\gamma\varphi^{\gamma}_{\alpha}$

fails to hold in such a scenario. To see why, note that by the conventions for bound restricted variables officially stated in (254), (ϑ) is shorthand for:

(ξ) $\forall\alpha(\psi \to \varphi) \to \exists\alpha(\psi \,\&\, \varphi)$

Now if ψ is empty, i.e., $\neg\exists\alpha\psi$, then we can derive the antecedent of (ξ): from $\neg\exists\alpha\psi$, it follows that $\forall\alpha\neg\psi$, which by ∀E implies $\neg\psi$; hence $\psi \to \varphi$, and so

$\forall\alpha(\psi \to \varphi)$, by GEN. But the consequent of (ξ) provably fails to hold: if $\neg\exists\alpha\psi$, then $\neg\exists\alpha(\psi \,\&\, \varphi)$.

So if we introduce restricted variables γ to range over ψ without having a proof that ψ is non-empty, we can't assume $\forall\gamma\varphi^{\gamma}_{\alpha} \to \exists\gamma\varphi^{\gamma}_{\alpha}$. We are, of course, safe in the case of *Class*(x). Since we know that properties exist, it follows immediately from (251)★ that $\exists x Class(x)$. So we can prove $\forall c\varphi^{c}_{x} \to \exists c\varphi^{c}_{x}$:

> Since we know $\exists x Class(x)$, suppose a is an arbitrary such individual, so that we know *Class*(a). Now assume $\forall c\varphi^{c}_{x}$. Then by our convention, $\forall x(Class(x) \to \varphi)$. In particular, $Class(a) \to \varphi^{a}_{x}$. Hence, φ^{a}_{x}. Conjoining what we know, we may infer $Class(a) \,\&\, \varphi^{a}_{x}$. So by $\exists$I, $\exists x(Class(x) \,\&\, \varphi)$, which conclusion remains once we discharge our assumption that *Class*(a) by $\exists$E. So by our convention, $\exists c\varphi^{c}_{x}$.

We may generalize on this example to conclude that quantifiers binding restricted variables introduced for non-empty conditions behave, in this regard, like quantifiers binding unrestricted variables.

10.5 The Laws of Natural Classes and Logical Sets

We now make use of restricted variables $c, c', c'', \ldots$ ranging over (natural) classes.

(257) **Theorem:** The Principle of Extensionality. The principle of extensionality asserts that whenever classes have the same members, they are identical:

$$c = c' \equiv \forall z(z \in c \equiv z \in c')$$

We've seen, in the discussion of convention (255.1), that this is shorthand for the claim:

$$(Class(x) \,\&\, Class(y)) \to (x = y \equiv \forall z(z \in x \equiv z \in y))$$

This theorem a key component of the theory of natural classes (logical sets).

(258) **Definition:** Empty Classes. We say that a class c is *empty* just in case it has no members:

$$Empty(c) \;=_{df}\; \neg\exists y(y \in c)$$

Clearly, when we apply the convention (255.3) for restricted variables, this definition is to be understood as the following:

$$Empty(x) \;=_{df}\; Class(x) \,\&\, \neg\exists y(y \in x)$$

So by the discussion in (253), where we justify our interpretation of free restricted variables in definitions, we know that in the presence of the fact that

$Class(c)$, we can derive the equivalence $Empty(c) \equiv Class(c) \mathbin{\&} \neg\exists y(y \in c)$ and $Empty(c) \equiv \neg\exists y(y \in c)$.

(**259**) **Theorems:** There Exists a (Unique) Empty Class.

(.1) $\exists c Empty(c)$

(.2) $\exists! c Empty(c)$

The restricted variables are to be understood according to our metadefinitions in (254.2).[156]

Since (.2) is established by a modally strict proof, we may apply the Rule of Actualization to it and use (179.2) to yield the modally strict conclusion:[157]

(.3) $\exists y(y = \imath c Empty(c))$

Hence, should the need arise, we can introduce the null class symbol as follows:

- $\varnothing =_{df} \imath c Empty(c)$

The bound occurrence of the restricted variable c in the definiens of such a definition would be understood according to our metadefinition in (254.3).

(**260**) **Definition:** Universal* Classes. We've already defined the notion of a universal object; in (191.2) we defined: $Universal(x)$ iff x is an abstract object that encodes every property. To avoid clash of notation, we introduce a different expression, to define a universal class. We say that a class c is *universal** just in case every individual is a member of c:

$$Universal^*(c) =_{df} \forall y(y \in c)$$

(**261**) **Theorems:** There Exists a (Unique) Universal* Class.

(.1) $\exists c Universal^*(c)$

(.2) $\exists! c Universal^*(c)$

[156] Since (.2) uses both the defined unique-existence quantifier and restricted variables, it might serve well to say explicitly that this theorem asserts:

(.2) $\exists x[Class(x) \mathbin{\&} \neg\exists y(y \in x) \mathbin{\&} \forall z((Class(z) \mathbin{\&} \neg\exists y(y \in z)) \to z = x)]$

I.e., there is a class x with no members such that every class z with no members is identical to x.

[157] We have assumed that the following is an instance of (179.2):

$$\exists y(y = \imath c Empty(c)) \equiv \mathcal{A}\exists! c Empty(c)$$

To see that it is, note that the above is valid shorthand for:

$$\exists y(y = \imath x(Class(x) \mathbin{\&} Empty(x))) \equiv \mathcal{A}\exists! x(Class(x) \mathbin{\&} Empty(x))$$

This latter is a genuine instance of (179.2) if we set φ to $Class(x) \mathbin{\&} Empty(x)$.

(**262**) **Definition:** Unions. We say that a class c is a *union of* c' and c'' just in case the elements of c are precisely the elements of c' supplemented by the elements of c'':

$$UnionOf(c,c',c'') =_{df} \forall y(y \in c \equiv (y \in c' \vee y \in c'')$$

We saw how to eliminate the free restricted variables in this particular definition in (255.3).

(**263**) **Theorems:** Existence of (Unique) Unions. It now follows that (.1) there is a class c that is a union of classes c' and c''; and (.2) there is a unique class c that is a union of classes c' and c'':

(.1) $\exists c UnionOf(c,c',c'')$

(.2) $\exists! c UnionOf(c,c',c'')$

If we eliminate the unique-existence quantifier and all the restricted variables, then (.2) is short for:

$$Class(z) \,\&\, Class(w) \rightarrow \exists x[Class(x) \,\&\, UnionOf(x,z,w) \,\&$$
$$\forall u(Class(u) \,\&\, UnionOf(u,z,w) \rightarrow u = x)]$$

I.e., for any two classes z and w, there is a unique class containing the members of z as well as the members of w.

(**264**) **Theorem:** Existence of The Union of Two Classes. From the previous theorem, it follows that the class that is a union of c' and c'' exists:

$$\exists y(y = \imath c UnionOf(c,c',c''))$$

Notice that this is a modally strict theorem. We do *not* infer the above from (263.2) by using (100)$\star$. Note also that since c' and c'' are free restricted variables, the above is shorthand for:

$$(Class(z) \,\&\, Class(w)) \rightarrow \exists y(y = \imath c UnionOf(c,z,w))$$

Thus, the existence of a class that is a union of z and w is guaranteed only when z and w are classes.

(**265**) **Restricted Term Definition:** Notation for the Union of Two Classes. Given the previous theorem, we may use the standard notation $c' \cup c''$ to denote the union of classes c' and c'':

$$c' \cup c'' =_{df} \imath c UnionOf(c,c',c'')$$

Since we proved that $\imath c UnionOf(c,c',c'')$ exists (264) by a modally strict proof, we don't have to annotate this definition with a $\star$. Note that we've labeled the above definition a *restricted term definition*. A few observations about this are in order.

(**266**) **Remark:** On Restricted Individual Term Definitions. The use of free restricted variables in individual term definitions requires that we place certain constraints on such definitions to avoid violation of Constraint (206). The latter Constraint requires that only logically proper terms may serve as definientia in term definitions. The Constrant is relevant because in (264) we noted that:

$$\exists y(y = \imath c UnionOf(c, c', c''))$$

is really the conditional theorem:

$$(Class(z) \,\&\, Class(w)) \rightarrow \exists y(y = \imath c UnionOf(c, z, w))$$

Thus, we haven't generally established:

$$\exists y(y = \imath c UnionOf(c, z, w))$$

as a theorem, and so we cannot infer that $\imath c UnionOf(c, z, w)$ is logically proper for every z and w. So, by the Constraint, we may not use this description to define $z \cup w$ generally. The logical propriety of $\imath c UnionOf(c, z, w)$ is only *conditional* on the proof or hypothesis that z and w are classes.

Since the introduction of c and c' as restricted variables allows us to assume they are classes, $\imath c UnionOf(c, c', c'')$ is always logically proper and so $c' \cup c''$ is always both well-formed and logically proper as well. Indeed, as long as we know, either by hypothesis or by proof, that κ_1 and κ_2 are individual terms such that $Class(\kappa_1)$ and $Class(\kappa_2)$, then $\kappa_1 \cup \kappa_2$ is both well-formed and logically proper. We may legitimately use $\kappa_1 \cup \kappa_2$ under such conditions because they comply with Constraint (206). We'll therefore assume that relevantly similar conditions hold when we introduce and deploy other restricted term definitions in what follows.

As an aside, note that in the case of $c' \cup c''$, it just so happens that the restricted variables c' and c'' range over entities of the same kind as that the value of $c' \cup c''$, namely, classes. But it should be apparent that we may define new restricted individual terms even when the free restricted variables used in the definition range over entities different in kind from the entity denoted by the term. For example, in item (285) below, we use u as a restricted variable ranging over ordinary objects and introduce $\{u\}$ to denote the singleton class (i.e., unit class) of u. Here, the restricted variable u ranges over entities different in kind from the entity denote by the term. If we were to introduce the constant s to denote Socrates, under the assumption or axiom that Socrates is ordinary, then $\{s\}$ is not only well-formed, but denotes the class whose sole member is

Socrates. For a nice example of a restricted *relation* variable that occurs free in an individual term definition, see definition (315).

(**267**) **Definition** Class Complements. We may define c' is a *class complement* of c just in case c' contains all and only those individuals that fail to be in c:

$$ClassComplementOf(c',c) =_{df} \forall y(y \in c' \equiv y \notin c)$$

(**268**) **Theorems:** Existence of (Unique) Complements. It now follows that (.1) there is a class c' that is a complement of class c, and (.2) there is a unique class c' that is a complement of class c:

(.1) $\exists c'\, ClassComplementOf(c',c)$

(.2) $\exists! c'\, ClassComplementOf(c',c)$

Since (.2) is modally strict, we may, in the usual way, infer the modally strict conclusion:

(.3) $\exists y(y = \imath c' ClassComplement(c',c))$

Hence, should the need arise, we can introduce 'overbar' notation for the class complements by formulating the following restricted term definition:

- $\bar{c} =_{df} \imath c' ClassComplement(c',c)$

Our discussion in (266) makes it clear that overbar notation is defined only for terms that are known to be classes, either by hypothesis or by proof.

(**269**) **Definition:** Intersections. We say that c is an *intersection* of c' and c'' just in case c has as members all and only the individuals that c' and c'' have in common:

$$IntersectionOf(c,c',c'') =_{df} \forall y(y \in c \equiv y \in c' \,\&\, y \in c'')$$

(**270**) **Theorem:** Existence of (Unique) Intersections. It follows that: (.1) there is a class c that is an intersection of c' and c''; and (.2) there is a unique class c that is an intersection of c' and c'':

(.1) $\exists c IntersectionOf(c,c',c'')$

(.2) $\exists! c IntersectionOf(c,c',c'')$

(**271**) **Theorem:** Existence of The Intersection of Two Classes. By now familiar reasoning, it follows that the class that is an intersection of c' and c'' exists:

$$\exists z(z = \imath c IntersectionOf(c,c',c''))$$

This is a modally strict theorem.

(272) **Restricted Term Definition:** Notation for the Intersection of Two Classes. Thus, for now familiar reasons, we are entitled to introduce the standard notation $c' \cap c''$ to denote the intersection of c' and c'':

$$c' \cap c'' =_{df} \iota c IntersectionOf(c, c', c'')$$

In the usual way, we may regard $\cap$ as a binary functional symbol that is well-formed and logically proper only when both of its arguments are known to be classes, either by hypothesis or by proof.

(273) ⋆**Theorem:** Intersection Membership Principle. It is now straightforward to show that an individual z is a member of $c' \cap c''$ if and only if z is a member of both c' and c'':

$$z \in c' \cap c'' \equiv (z \in c' \,\&\, z \in c'')$$

(274) **Theorems:** Class Comprehension. When logicians formulate naive set theory so as to avoid quantification over properties, they assert the following schema as an axiom: for every formula φ in which x doesn't occur free, there is a set x that has as members all and only the objects such that φ (see, e.g., Boolos 1971, 217). Of course, this classical 'naive' comprehension principle uses a primitive notion of membership and is subject to Russell's paradox.

By contrast, if we limit the comprehension schema to propositional formulas φ^* (i.e., formulas with no encoding subformulas), we may derive the following class comprehension principles as theorem schemata that are immune to Russell's paradox by the defined notion of membership: (.1) there is a class whose members consist of all and only the individuals such that φ^*, and (.2) there is a unique class whose members consist of all and only the individuals such that φ^*:

(.1) $\exists c \forall y (y \in c \equiv \varphi^*)$, provided c doesn't occur free in φ^*

(.2) $\exists! c \forall y (y \in c \equiv \varphi^*)$, provided c doesn't occur free in φ^*

Observe that the following particular formula:

$$\exists c \forall y (y \in c \equiv y \notin c),$$

is not an instance of Class Comprehension, since the $y \notin c$ violates both restrictions: (a) c occurs free in φ, and (b) $y \notin c$ fails to be a propositional formula. We leave it as an exercise to show that $y \notin c$ fails to be propositional. By the same token, when we eliminate the restricted variables, the formula:

$$\exists x (Class(x) \,\&\, \forall y (y \in x \equiv y \notin x)),$$

is not an instance of Class Comprehension, for the similar reasons.

Note that the following formulas all fail to be instances of Class Comprehension:

$$\exists c \forall y(y \in c \equiv Class(y)), \text{ i.e.,}$$
$$\exists x(Class(x) \,\&\, \forall y(y \in x \equiv Class(y))$$

$$\exists c \forall y(y \in c \equiv Class(y) \,\&\, y \notin y), \text{ i.e.,}$$
$$\exists x(Class(x) \,\&\, \forall y(y \in x \equiv Class(y) \,\&\, y \notin y))$$

In all these cases, the formula to the right of the biconditional sign fails to be propositional. Hence, there is no back door route here to the Russell paradox.

(**275**) **Theorem:** The Class of Individuals-Such-That-φ^* Exists. In virtue of (274.2), the Rule of Actualization, and theorem (179.2), we may derive, as a modally-strict theorem, that the class of individuals-such-that-φ^* exists:

$$\exists z(z = \imath c \forall y(y \in c \equiv \varphi^*)), \text{ provided } c \text{ doesn't occur free in } \varphi^*$$

(**276**) **Term Definition:** Class Abstracts. (275) entitles us to introduce and use the standard, and simpler, 'class abstract' notation, $\{y \,|\, \varphi^*\}$, to denote the class of all and only objects y such that φ^*:

$$\{y \,|\, \varphi^*\} \;=_{df}\; \imath c \forall y(y \in c \equiv \varphi^*), \text{ provided } c \text{ doesn't occur free in } \varphi^*$$

Thus, the expression $\{y \,|\, \ldots\}$ is a term-forming operator that operates on a propositional formula φ^*. Moreover, it is also a variable-binding operator that binds y with scope $\{y \,|\, \ldots\}$.

In light of the observations at the end of (274), the following expressions are not well-formed, since the matrix of the class abstract is not a propositional formula:

$$\{y \,|\, y \notin y\}$$
$$\{y \,|\, Class(y)\}$$
$$\{y \,|\, Class(y) \,\&\, y \notin y\}$$

So one may not formulate the class abstract for the Russell class in our system.

(**277**) ⋆**Theorems:** Some Facts about Class Abstracts, Including the Class Abstraction Principle. It now follows that (.1) $\{y \,|\, \varphi^*\}$ is a class whose members are precisely the individuals such that φ^*. From this, the Class Abstraction Principle immediately follows: (.2) an individual z is an element of the class of individuals such that φ^* just in case z is such that φ^*. From this, it follows that (.3) the class of individuals such that φ^* is identical to the extension of the property $[\lambda y\, \varphi^*]$. And, finally, from (.3) it follows that (.4) the class of Gs is the extension of the property G. That is, where φ^* is any appropriate formula (i.e., the variable bound by the $\imath$ in the definiens of $\{y \,|\, \varphi^*\}$ does not appear free in φ^*), we have:

(.1) $y \in \{y \mid \varphi^*\} \equiv \varphi^*$

(.2) $z \in \{y \mid \varphi^*\} \equiv \varphi^{*z}_{y}$

(.3) $\{y \mid \varphi^*\} = \epsilon[\lambda y\, \varphi^*]$

(.4) $\{y \mid Gy\} = \epsilon G$

With (.2), we have a proof of a claim taken to be a definition in *Principia Mathematica* (see the discussion surrounding ∗20·01 and ∗20·02). Note that we can't infer its necessitation; intuitively, the objects that are φ^* at other worlds may not be members of the class of individuals that are in fact such that φ^*, and vice versa. Note that although the proofs of (.3) and (.4) given in the Appendix aren't modally strict, we may nevertheless infer their necessitations by the necessity of identity principle, i.e., the left-to-right direction of (75). Thus, we have additional cases of necessary truths derived by means of non-modally strict proofs.

(**278**) **Theorems:** The Separation Schema. Let φ^* be any propositional formula in which c doesn't occur free. Then (.1) there is a class c whose elements are precisely the members of some given class c' and such that φ^*; and (.2) there is a unique class c whose elements are precisely the members of some given class c' and such that φ^*:

(.1) $\exists c \forall y(y \in c \equiv y \in c' \,\&\, \varphi^*)$, where c doesn't occur free in φ^*

(.2) $\exists! c \forall y(y \in c \equiv y \in c' \,\&\, \varphi^*)$, where c doesn't occur free in φ^*

(**279**) **Theorem:** Existence of the Class of Individuals in c' such that φ^*. By now familiar reasoning, it follows that there exists something that is the class of all individuals that are both in c' and such that φ^*:

$$\exists z(z = \imath c \forall y(y \in c \equiv y \in c' \,\&\, \varphi^*)),\ \text{provided } c \text{ doesn't occur free in } \varphi^*$$

In the usual manner, the presence of the free restricted variable c' means that this theorem is really a conditional.

(**280**) **Restricted Term Definition:** Class Separation Abstracts. In light of (279), we may expand the class abstract notation introduced previously, to allow class abstracts of the form $\{y \mid y \in c' \,\&\, \varphi^*\}$. This denotes the unique class of objects in a given class c' that are such that φ^*:

$$\{y \mid y \in c' \,\&\, \varphi^*\} =_{df} \imath c \forall y(y \in c \equiv y \in c' \,\&\, \varphi^*)$$

This is another restricted term definition because the notation is defined using the free restricted variable c'.

(**281**) ⋆**Theorem:** Separation Abstraction Principle.

$$z \in \{y \mid y \in c' \,\&\, \varphi^*\} \equiv (z \in c' \,\&\, \varphi^{*z}_{\ y})$$

(**282**) ⋆**Theorem:** Consequence of Class Separation and Intersection. It now follows that the unique class of objects in a given class c' which are such that φ^* is identical to the intersection of c' and the class of all objects such that φ^* :

$$\{y \mid y \in c' \,\&\, \varphi^*\} = c' \cap \{y \mid \varphi^*\}$$

(**283**) **Theorem:** Quasi-Replacement. Where R is any 2-place relation and c' is any class, it follows that there is a class c whose members are precisely those objects to which the members of c' bear R, i.e.,

$$\exists c \forall y(y \in c \equiv \exists z(z \in c' \,\&\, Rzy))$$

This is a form of replacement since, *a fortiori*, the theorem holds for those relations R that are functions, i.e., such that $\forall x \forall y \forall z(Rxy \,\&\, Rxz \to y = z)$.

(**284**) **Theorems:** Singletons of Ordinary Objects. The theory of natural classes does not guarantee that there are singletons of abstract objects, since identity is defined in terms of encoding subformulas and may not be used in comprehension. However, it does guarantee that (.1) for any ordinary object z, there is a class whose members are the objects identical$_E$ to z, and (.2) for any ordinary object z, there is a unique class whose members are the objects identical$_E$ to z:

(.1) $O!z \to \exists c \forall y(y \in c \equiv y =_E z)$

(.2) $O!z \to \exists! c \forall y(y \in c \equiv y =_E z)$

No similar principles are derivable when $=$ is substituted for $=_E$. See the discussion below in Remark (285).

In virtue of (.2), we know that if z is an ordinary object, then the class of objects identical$_E$ to z exists:

(.3) $O!z \to \exists x(x = \iota c \forall y(y \in c \equiv y =_E z))$

Note that if we let u be a restricted variable ranging over ordinary objects, then we may reformulate (.3) as follows, in which u occurs free:

(.3) $\exists x(x = \iota c \forall y(y \in c \equiv y =_E u))$

In the next remark, we discuss why we don't take the trouble to formulate and prove stronger versions of (.1) – (.3), which assert the existence of such singletons for every individual whatsoever. These stronger versions are indeed provable, but there are reasons why they aren't very useful.

(**285**) **Remark:** Observations About Singletons. If we continue to use u as a restricted variable ranging over ordinary individuals, then on the basis of the variant version of (284.3) that uses u, we can introduce a *restricted* individual term, $\{u\}$, for the *singleton* (or *unit class*) of u:

$$\{u\} =_{df} \{y \mid y =_E u\}$$

Given our discussion in (266), terms of the form $\{\kappa\}$ are well-formed *only when* κ is known to be an ordinary individual, either by hypothesis or by proof.

Observe also that the definiens in the above definition, $\{y \mid y =_E u\}$, is itself defined in (276) as $\imath c\forall y(y \in c \equiv y =_E u)$. Clearly, we have legitimately defined $\{u\}$; since $y =_E u$ is a propositional formula, $\{y \mid y =_E u\}$ is a legitimate term of the form $\{y \mid \varphi^*\}$, by definition (276).

We noted briefly in (284) that our theory provides no guarantee that there are singletons definable using identity instead of identity$_E$. We cannot establish, for any individual z, that $\exists c\forall y(y \in c \equiv y = z)$. The formula $y = z$ is defined in terms of encoding subformulas and thereby fails to be propositional. So we cannot use $y = z$ in an instance of Class Comprehension (274). Nor is there a property $[\lambda y\, y = z]$ that we could substitute for F to produce an instance of the Fundamental Theorem for Natural Classes (252). Indeed, none of the other existence principles we've derived for natural classes allow us to assert the existence of a class such as $\{y \mid y = z\}$.[158]

Finally, it is important to observe that (284.1) – (284.3) can be weakened to the following, in which the antecedent $O!z$ is eliminated:

$$\exists c\forall y(y \in c \equiv y =_E z)$$

$$\exists! c\forall y(y \in c \equiv y =_E z)$$

$$\exists x(x = \imath c\forall y(y \in c \equiv y =_E z))$$

These are theorems because the property $[\lambda y\ y =_E z]$ exists even if z is an abstract object! However, when z is abstract, then the class c such that $\forall y(y \in c \equiv y =_E z)$ has no members. Given definitions (12) and (13), $y =_E z$ holds just in case both y and z are both ordinary objects and necessarily exemplify the same properties. Hence, no y is such that $y =_E z$ when z is abstract. So even though $\{z\}$, if defined as $\{y \mid y =_E z\}$, would be well-formed and logically proper when z is abstract, it would misleadingly refer to the empty class. Indeed, the singleton principle, $x \neq y \to \{x\} \neq \{y\}$, would fail, for when x and y are distinct abstract individuals, both $\{x\}$ and $\{y\}$ would both be identical to the empty class and so identical. But for ordinary objects u, $\{u\}$ provably isn't empty and contains only u as an element, and the principle $u \neq v \to \{u\} \neq \{v\}$, for ordinary u, v, is derivable (exercise).

[158] Questions about the existence of singletons have played an important role in metaphysics (e.g., Lewis 1991). If we treat the axioms of set theory that asser or imply the existence of singletons as part of theoretical mathematics, then the analysis of those axioms will be addressed in Chapter 15. However, any axioms that assert or imply the existence of singletons outside the context of set theory should be evaluated in the larger context of theories that assert the existence of abstract individuals generally. Our theory finds no guarantee that every abstract individual has a singleton, but only a guarantee that every ordinary object has a singleton.

(**286**) **Theorems:** Conditional Existence of Pair Classes. For any *ordinary* objects x and z, there is a (unique) class whose members are precisely x and y:

(.1) $(O!x \,\&\, O!z) \to \exists c \forall y(y \in c \equiv y =_E x \vee y =_E z)$

(.2) $(O!x \,\&\, O!z) \to \exists! c \forall y(y \in c \equiv y =_E x \vee y =_E z)$

Since the property $[\lambda y\, y =_E x \vee y =_E z]$ is critical to the proof, it becomes clear why we need to appeal to $=_E$ rather than $=$ to obtain this theorem, namely, the expression $[\lambda y\, y = x \vee y = z]$ isn't well-formed.

(**287**) **Theorems:** Conditional Class Adjunction. We now have (.1) if x is an ordinary object, there is a class c whose elements are the members of a given class c' along with x; (.2) if x is an ordinary object, there is a unique class c whose elements are the members of a given class c' along with x:

(.1) $O!x \to \exists c \forall y(y \in c \equiv y \in c' \vee y =_E x)$

(.2) $O!x \to \exists! c \forall y(y \in c \equiv y \in c' \vee y =_E x)$

(**288**) **Exercises:** Anti-Extensions. Let us say that x is an *anti-extension of* property G if and only if x is an abstract object that encodes exactly the properties F that are exemplified by all and only those objects that fail to exemplify G. Formulate and prove some interesting theorems about anti-extensions.

10.6 Abstraction via Equivalence Conditions

(**289**) **Remark:** Logical Objects Abstracted from Equivalence Conditions. We now observe a general pattern that has emerged in the previous two sections. Let α, β be distinct n-place relation variables, for some n, and consider any formula φ in which there are free occurrences of α and free occurrences of β. Let us write $\varphi(\gamma, \delta)$ for the result of simultaneously substituting γ for the free occurrences of α and δ for the free occurrences of β, where γ and δ are *any* two n-place relation variables, so that $\varphi(\alpha, \beta)$ just is φ. Then we say that such a formula φ is an *equivalence condition* on n-place relations whenever the following are all provable:

$$\varphi(\alpha, \alpha) \qquad \text{(Reflexivity)}$$

$$\varphi(\alpha, \beta) \to \varphi(\beta, \alpha) \qquad \text{(Symmetry)}$$

$$\varphi(\alpha, \beta) \to (\varphi(\beta, \gamma) \to \varphi(\alpha, \gamma)) \qquad \text{(Transitivity)}$$

For example, it is easy to show that the formulas $q \equiv p$ is an equivalence condition on propositions, and that $\forall z(Fz \equiv Gz)$ is an equivalence condition on properties, i.e., to prove:

- $q \equiv q$
  $(q \equiv p) \rightarrow (p \equiv q)$
  $((q \equiv p) \,\&\, (p \equiv r)) \rightarrow q \equiv r$

- $\forall z(Fz \equiv Fz)$
  $\forall z(Fz \equiv Gz) \rightarrow \forall z(Gz \equiv Fz)$
  $(\forall z(Fz \equiv Gz) \,\&\, \forall z(Gz \equiv Hz)) \rightarrow \forall z(Fz \equiv Hz)$

Generally speaking, an equivalence condition with free n-place relation variables α, β partitions the domain of n-place relations over which α, β range in much the same way that an equivalence relation on individuals partitions the domain of individuals. The formula $q \equiv p$ partitions the domain of propositions into mutually exclusive and jointly exhaustive cells of materially equivalent propositions. The formula $\forall z(Fz \equiv Gz)$ partitions the domain of properties into mutually exclusive and jointly exhaustive cells of materially equivalent properties.

We can now observe that the definitions of truth-values and extensions of properties (i.e., classes) are both based on instances of comprehension for abstract objects involving equivalence conditions:

- We used the equivalence condition $q \equiv p$ on propositions in (211) to define $\mathit{TruthValueOf}(x, p)$.

- We used the equivalence condition $\forall z(Fz \equiv Gz)$ on properties in (234) to define $\mathit{ExtensionOf}(x, G)$.

These definitions were in turn used to define the canonical objects p° and ϵG, respectively. This process can be generalized, as the next series of definitions and theorems show.

(**290**) **Definitions:** Abstractions from Equivalence Conditions. For the next sequence of definitions and theorems we adopt the following conventions:

- Let $\varphi(q, p)$ be any given equivalence condition on propositions.

- Let $\psi(F, G)$ be any given equivalence condition on properties.

Then we can define abstractions of p and G with respect to these equivalence conditions:

(.1) $\varphi\text{-}\mathit{AbstractionOf}(x, p) \ =_{df} \ A!x \,\&\, \forall F(xF \equiv \exists q(\varphi(q, p) \,\&\, F = [\lambda y\, q]))$

(.2) $\psi\text{-}\mathit{AbstractionOf}(x, G) \ =_{df} \ A!x \,\&\, \forall F(xF \equiv \psi(F, G))$

(**291**) **Theorems:** Existence of Unique Abstractions. For each of the preceding definitions, Strengthened Comprehension (177) yields the existence of a unique abstraction:

(.1) $\exists!x(\varphi\text{-}AbstractionOf(x,p))$

(.2) $\exists!x(\psi\text{-}AbstractionOf(x,G))$

So by (180), we can prove the logical propriety of corresponding canonical descriptions:

(.3) $\exists y(y = \imath x(\varphi\text{-}AbstractionOf(x,p)))$

(.4) $\exists y(y = \imath x(\psi\text{-}AbstractionOf(x,G)))$

(**292**) **Definitions:** Notation for Abstractions of Equivalence Conditions. Hence, in each case, we may legitimately introduce notation for canonical objects:

(.1) $\widehat{p}_{\varphi} =_{df} \imath x(\varphi\text{-}AbstractionOf(x,p))$

(.2) $\widehat{G}_{\psi} =_{df} \imath x(\psi\text{-}AbstractionOf(x,G))$

Thus, if given any equivalence condition on propositions or properties, our theory distinguishes a canonical logical object for each cell of the partition. For any proposition p, $\widehat{p}_{\varphi}$ encodes all and only the propositions q such that $\varphi(q,p)$ and, for any property G, $\widehat{G}_{\psi}$ encodes all and only the properties F such that $\psi(F,G)$.

(**293**) ⋆**Theorems:** Frege's Principle for Abstractions. Our results now justify one classical form of *definition by abstraction*, since the above definitions and theorems yield the following non-modally strict theorems:

(.1) $\widehat{p}_{\varphi} = \widehat{q}_{\varphi} \equiv \varphi(p,q)$

(.2) $\widehat{F}_{\psi} = \widehat{G}_{\psi} \equiv \psi(F,G)$

The other classical form of definition by abstraction is justified in the next section.[159]

Finally, note that the preceding theorems are derivable without giving rise to the Julius Caesar problem (see Chapter 18, Section 18.1). In the next section, we similarly justify the other classical form of definition by abstraction, involving equivalence *relations* among individuals.

10.7 Abstraction via Equivalence Relations

We now turn from abstractions over equivalence conditions on relations to abstractions over equivalence relations on individuals.

[159]See Mancosu ms., for an excellent discussion of the history of definition by abstraction in mathematics.

10.7.1 Directions and Shapes

In this subsection, we shall be analyzing directions and shapes as natural mathematical objects rather than as objects systematized by some mathematical theory. We shall be assuming only an ordinary, pretheoretical understanding of the geometrical properties *being a line* and *being a figure*, as well as an ordinary, pretheoretical understanding of the geometrical relations *being parallel to* and *being similar to*. Consequently, we're not assuming any theoretical axioms of geometry in what follows.[160]

(**294**) **Remark:** Pretheoretic Conception of Lines. Consider the pretheoretical property *being a line* familiar to us from ordinary language. In what follows, we use L to denote this property. We don't actually make use of any assumptions governing this property other than the assumptions governing the relation of *being parallel to* described in this Remark.[161]

Now the ordinary 2-place relation *being parallel to* ($\|$), is an equivalence relation restricted to individuals that exemplify the property *being a line*. That is, we assume the following principles governing *being a line* and *being parallel to*:

$$Lx \rightarrow x\|x$$

[160] We are assuming one can distinguish the pretheoretic geometrical properties of *being a line*, *being a figure*, etc., and the pretheoretic geometrical relations of *being parallel to*, *being similar to*, etc., from their theoretical counterparts. The theoretical counterparts are governed by the axioms of some well-defined mathematical theory, but the pretheoretical properties and relations are not. For now, we shall discuss only the pretheoretical properties and relations, as understood in ordinary, everyday thoughts.

By contrast, the theoretical counterparts will be subject to our analysis of the relations of *theoretical* mathematics in Chapter 15. In that later chapter, we assert the existence of abstract relations (including abstract properties) in addition to abstract individuals. Abstract relations will be distinguished from ordinary relations; the former, but not the latter, may encode properties of relations. Similarly, abstract properties will be distinguished from ordinary properties; the former may encode properties of properties. In Chapter 15, theoretical mathematical properties axiomatized in some mathematical theory, such as *being a number*, *being a set*, and *being a line*, will be identified as particular abstract properties (relative to the theory in question).

[161] I take it, however, that the pretheoretical geometrical property *being a line* is governed by the principles that (ordinary) lines are concrete objects (i.e., $\forall x(Lx \rightarrow E!x)$) and that it is possible that lines exist (i.e., $\Diamond\exists xLx$). From the latter it follows by BF$\Diamond$ (122.3) that there exist objects x that are possibly lines (e.g., from the fact that it is possible there exists a vertical line running along the right edge of the text on this page, it follows that there is an x such that possibly x is a vertical line running along the right edge of the text on this page). I also suppose that the pretheoretical property *being a line* allows for lines that consist of discrete parts (e.g., a line of people) and for lines that are continuous at a certain level of granularity (e.g., a line of ink on paper). Finally, I take it that, pretheoretically, lines have some (approximate) physical length and (average) thickness. Of course, if the reader finds any of this controversial, she is entitled to ignore it. Strictly speaking, none of these pretheoretic claims are needed for understanding the analysis that follows. But I take it that Frege's example of a particular line, *the Earth's axis*, is an abstraction and would be analyzed as a theoretical entity of science, and hence as an abstract object, not an ordinary one.

$$Lx \,\&\, Ly \to (x \| y \to y \| x)$$

$$Lx \,\&\, Ly \,\&\, Lz \to (x \| y \,\&\, y \| z \to x \| z)$$

Over the course of the next few items, let us use the variables u, v, w as restricted variables ranging over lines (we continue to use the variables x, y, z as variables for any kind of object). Hence, the principles displayed immediately above may be written as follows:

$$u \| u$$

$$u \| v \to v \| u$$

$$u \| v \,\&\, v \| w \to u \| w$$

(**295**) **Lemma:** Fact About *Being Parallel To*. From the assumption that *being parallel to* is an equivalence relation on lines, we can establish that the properties *being a line parallel to u* and *being a line parallel to v* are materially equivalent if and only if u is parallel to v:

$$\forall z([\lambda w\, w \| u]z \equiv [\lambda w\, w \| v]z) \equiv u \| v$$

This fact plays a key role in what follows.

(**296**) **Definition:** Directions of Lines. Using a familiar method of definition by abstraction described by Frege in 1884, we may define: x is a *direction* of y just in case x is an extension of the property *being a line and parallel to y*:

$$\mathit{DirectionOf}(x, y) =_{df} Ly \,\&\, \mathit{ExtensionOf}(x, [\lambda z\, Lz \,\&\, z \| y])$$

Using restricted variables, we can rewrite this definition as: x is a *direction* of line u just in case x is an extension of the property *being a line parallel to u*:

$$\mathit{DirectionOf}(x, u) =_{df} \mathit{ExtensionOf}(x, [\lambda w\, w \| u])$$

(**297**) **Theorems:** The Existence of Directions. It now follows that (.1) there is a direction of line u; (.2) there is a unique direction of line u; and (.3) the direction of line u exists:

(.1) $\exists x \mathit{DirectionOf}(x, u)$

(.2) $\exists! x \mathit{DirectionOf}(x, u)$

(.3) $\exists y(y = \imath x \mathit{DirectionOf}(x, u))$

Note that (.1) – (.3) are really conditionals, since the free variable u is restricted. Consequently, these theorems don't presuppose the existence of ordinary lines.

(**298**) **Restricted Term Definition:** The Direction of Line u. We are therefore justified in introducing the following notation for the direction of line u:

$\vec{u} =_{df} \imath x DirectionOf(x,u)$

This is a restricted term definition because u is a free restricted variable. By (266), we may regard the expression $\vec{\kappa}$ as well-formed and logically proper only when κ is known to be a line, either by hypothesis or by proof.

(**299**) ⋆**Theorem:** Fregean Biconditional for Directions. It now follows that the direction of line u is identical to the direction of line v iff u and v are parallel:

$\vec{u} = \vec{v} \equiv u \| v$

(Cf. Frege 1884, §65.) We have therefore established the Fregean biconditional principle for directions.

(**300**) **Definition:** Directions. A *direction* is any object that is a direction of some line:

$Direction(x) =_{df} \exists u DirectionOf(x,u)$

cf. Frege 1884, §66.

(**301**) **Theorem:** Conditional Existence of Directions. It now follows that if lines exist, then directions exist:

$\exists y Ly \rightarrow \exists x Direction(x)$

(**302**) **Remark:** Pretheoretic Conception of Shapes. Consider the pretheoretical property *being a figure*, and note that *being similar to* is an equivalence relation among figures. We may then apply our theory by abstracting over similar figures to define shapes. Let the variables u, v, w now range over figures, and let $u \sim v$ assert that figure u is similar to figure v. So, we may assume:

$u \sim u$

$u \sim v \rightarrow v \sim u$

$u \sim v \;\&\; v \sim w \rightarrow u \sim w$

(**303**) **Lemma:** Fact About Similarity. By the same reasoning we used in (295), we know that the properties of *being a shape similar to u* and *being a shape similar to v* are materially equivalent if and only if u is similar to v:

$\forall z([\lambda w\, w \sim u]z \equiv [\lambda w\, w \sim v]z) \equiv u \sim v$

We now derive a Frege-style analysis of shapes.

(**304**) **Definition:** Shapes of Figures. Given the principles in the preceding remark, we may define: x is a *shape* of figure u just in case x is an extension of the property *being a figure similar to u*:

$$ShapeOf(x,u) =_{df} ExtensionOf(x,[\lambda v\ v\sim u])$$

(**305**) **Theorems:** The Existence of Shapes. It now follows that (.1) there is a shape of figure u; (.2) there is a unique shape of figure u; and (.3) the shape of figure u exists:

(.1) $\exists x ShapeOf(x,u)$

(.2) $\exists! x ShapeOf(x,u)$

(.3) $\exists y(y = \imath x ShapeOf(x,u))$

Again, these are all implicitly conditionals, given the presence of the free restricted variable u.

(**306**) **Restricted Term Definition:** The Shape of a Figure. We are therefore justified in introducing the following notation for the shape of figure u:

$$\tilde{u} =_{df} \imath x ShapeOf(x,u)$$

The free restricted variable u makes this a restricted term definition.

(**307**) ⋆**Theorem:** Fregean Biconditional for Shapes. It now follows that the shape of figure u is identical to the shape of figure v iff u and v are similar:

$$\tilde{u}=\tilde{v} \equiv u\sim v$$

(**308**) **Definition:** Shapes. A *shape* is any object that is a shape of some figure:

$$Shape(x) =_{df} \exists u ShapeOf(x,u)$$

(**309**) **Theorem:** Conditional Existence of Shapes. It now follows that if figures exist, then shapes exist. Where P denotes the property *being a figure*, we have:

$$\exists y Py \rightarrow \exists x Shape(x)$$

This theorem parallels that of theorem (301).

10.7.2 General Abstraction via Equivalence Relations

In this subsection, we revert to using w as an unrestricted individual variable. Hence x,y,z,w are all unrestricted individual variables.

(**310**) **Definition:** Equivalence Relations. When R is a 2-place relation that is reflexive, symmetric, and transitive, we say that R is an *equivalence* relation on individuals:

$$Equivalence(R) =_{df} \forall x Rxx \,\&\, \forall x,y(Rxy \rightarrow Ryx) \,\&\, \forall x,y,z(Rxy \,\&\, Ryz \rightarrow Rxz)$$

Since *Equivalence*(R) is a well-defined condition on relations, we henceforth use $\widetilde{R}$ as a restricted variable ranging over equivalence relations.

(**311**) **Theorem:** Example of an Equivalence Relation. It is straightforward to establish that the relation of *exemplifying the same properties* is an equivalence relation:

$$Equivalence([\lambda xy\, \forall F(Fx \equiv Fy)])$$

Since above theorem implies us that the condition *Equivalence*(R) is non-empty, we know that the quantifiers $\forall\widetilde{R}$ and $\exists\widetilde{R}$ behave classically; i.e., that we can prove $\forall\widetilde{R}\varphi \to \exists\widetilde{R}\varphi$. Cf. the discussion in (256).

(**312**) **Lemmas:** Fact About Equivalence Relations. For any equivalence relation $\widetilde{R}$, we can establish that (.1) the properties *bearing* $\widetilde{R}$ *to* x and *bearing* $\widetilde{R}$ *to* y are materially equivalent if and only if x bears $\widetilde{R}$ to y:

(.1) $\forall w([\lambda z\, \widetilde{R}zx]w \equiv [\lambda z\, \widetilde{R}zy]w) \equiv \widetilde{R}xy$

Note also that for any equivalence relation $\widetilde{R}$, (.2) the individuals that bear $\widetilde{R}$ to x are precisely the individuals to which x bears $\widetilde{R}$, i.e.,

(.2) $\forall y([\lambda z\, \widetilde{R}zx]y \equiv [\lambda z\, \widetilde{R}xz]y)$

As a fact about properties, (.2) says that $[\lambda z\, \widetilde{R}zx]$ is materially equivalent to $[\lambda z\, \widetilde{R}xz]$. Hence, (.3) the properties materially equivalent to $[\lambda z\, \widetilde{R}zx]$ are precisely the properties materially equivalent to $[\lambda z\, \widetilde{R}xz]$, i.e.,

(.3) $\forall F(\forall y(Fy \equiv [\lambda z\, \widetilde{R}zx]y) \equiv \forall y(Fy \equiv [\lambda z\, \widetilde{R}xz]y))$

(**313**) **Definition:** Abstractions from Equivalence Relations. Where $\widetilde{R}$ is any equivalence relation, we define: w is an $\widetilde{R}$*-abstraction* of x just in case w is an extension of the property $[\lambda z\, \widetilde{R}zx]$:

$$\widetilde{R}\text{-}AbstractionOf(w,x) =_{df} ExtensionOf(w,[\lambda z\, \widetilde{R}zx])$$

(**314**) **Theorems:** The Existence of $\widetilde{R}$-Abstractions. It now follows that (.1) there is an $\widetilde{R}$-abstraction of individual x; (.2) there is a unique $\widetilde{R}$-abstraction of individual x; and (.3) the $\widetilde{R}$-abstraction of individual x exists:

(.1) $\exists w \widetilde{R}\text{-}AbstractionOf(w,x)$

(.2) $\exists! w \widetilde{R}\text{-}AbstractionOf(w,x)$

(.3) $\exists y(y = \imath w \widetilde{R}\text{-}AbstractionOf(w,x))$

These are all really conditionals, given that $\widetilde{R}$ is a free restricted variable ranging over equivalence relations.

(**315**) **Restricted Term Definition:** Notation for the $\widetilde{R}$-Abstraction of x. Whenever $\widetilde{R}$ is an equivalence relation, we notate the $\widetilde{R}$-abstraction of x as follows:

$$\widehat{x_{\widetilde{R}}} =_{df} \imath w \widetilde{R}\text{-}AbstractionOf(w, x)$$

(Cf. Frege 1884, §68.) This is a restricted term definition because $\widetilde{R}$ is a restricted variable that occurs free.

(**316**) ★**Theorem:** The Fregean Biconditional for Definition by Abstraction. To establish that the $\widetilde{R}$-abstraction of x is indeed defined by classical abstraction over an equivalence relation, we prove that it obeys the classical principle, namely, that the $\widetilde{R}$-abstraction of x is identical to the $\widetilde{R}$-abstraction of y if and only if x bears $\widetilde{R}$ to y:

$$\widehat{x_{\widetilde{R}}} = \widehat{y_{\widetilde{R}}} \equiv \widetilde{R}xy$$

Cf. Frege 1884, §66.

Chapter 11

Platonic Forms

(**317**) **Remark:** Platonic Forms. Plato is well-known for having postulated *Forms* to explain why it is that, despite an ever-changing reality, we can truly say that *different* objects have something in common, such as when we say that distinct objects x and y are both red spheres, or beautiful paintings, virtuous persons, etc. Plato thought that since concrete objects are always undergoing change and have many of their characteristics only temporarily, there must be something that is universal and unchanging if we can truly say that different objects are both F. Plato called the aspects of reality that are universal and unchanging the *Forms* and supposed that objects acquire their characteristics by participating in, or partaking of, these Forms.

Plato's fundamental principle about the Forms is the One Over Many Principle. This principle is most famously stated in *Parmenides* 132a, and though we shall discuss it in more detail in what follows, a study of the seminal papers in Plato scholarship suggests that the following statement of the principle is accurate:[162]

(OM) **One Over Many Principle**
If x and y are both F, then there exists something that is the Form of F, or F-ness, in which they both participate.

In this principle, we are to substitute predicate nouns or adjectives for the symbol 'F', so that we have instances like the following:

> If x and y are both human, then there exists something that is the Form of humanity, or humanness, in which they both participate.
>
> If x and y are both red, then there exists something that is the Form of redness in which they both participate.

[162]See, e.g., Vlastos 1954, principles (A1) and (B1); Vlastos 1969, principle (1); and Strang 1963, principle (OM).

Though 'The Form of F' and 'F-ness' are two traditional ways of referring to the same thing, we must remember that in the expression 'F-ness', the letter 'F' is representing an arbitrary predicate noun or adjective and must be replaced by such an expression to produce a term denoting a Form. By contrast, in the theoretical/technical expression 'The Form of F', the symbol 'F' is a variable ranging over properties and so requires that 'F' be replaced by a gerund or abstract noun to yield a term denoting a Form.[163]

Now questions about (OM) immediately arise:

- What is *The Form of F*?
- What is participation?
- Is (OM) to be regarded as an axiom or can it be derived from more general principles?

To answer these questions, some philosophers have been tempted to identify *The Form of F* with the property F and to analyze: an object x *participates in*, or *partakes of*, The Form of F just in case x exemplifies F. We can formulate these analyses as follows:

(A) $\mathit{The\ Form\ of\ F} =_{df} F$

(B) $\mathit{ParticipatesIn}(x, F) =_{df} Fx$

Given (A) and (B), there is a natural formal representation of (OM):

(C) $(Fx \,\&\, Fy \,\&\, x \neq y) \rightarrow \exists G(G = F \,\&\, Gx \,\&\, Gy)$

This formula is a simple theorem of the second-order predicate calculus with identity.[164] So, (A), (B) and (C) provide answers, respectively, to the bulleted questions above.

Of course, some philosophers (e.g., the followers of Quine, nominalists, logical positivists, etc.) would object that the above analysis assumes second-order logic and its ontology of properties. But object theory has a theory of properties as rigorous as any mathematical theory and so provides a precise framework in which the above analysis can be put forward.

[163]For example, we may substitute 'human' for 'F' to produce the term 'humanness' (i.e., 'humanity') and substitute 'red' for 'F' to produce the term 'redness'. But we must substitute 'being human' or 'humanity' for 'F' to produce the term 'The Form of Being Human' or 'The Form of Humanity', and must substitute 'being red' or 'redness' to produce the term 'The Form of Being Red' or 'The Form of Redness'. A good rule of thumb is that if the symbol 'F' is being used in a natural language context, such as when we say 'x is F', then the symbol 'F' may be replaced by a predicate noun or adjective, but when the symbol 'F' is being used in a theoretical/technical or formal context, then the symbol 'F' is being used as a variable ranging over properties.

[164]Suppose $Fx \,\&\, Fy \,\&\, x \neq y$. Then by &E we have $Fx \,\&\, Fy$. Moreover, the laws of identity yield $F = F$. Hence, by &I, we obtain $F = F \,\&\, Fx \,\&\, Fy$. Hence, by $\exists$I, it follows that $\exists G(G = F \,\&\, Gx \,\&\, Gy)$.

Despite its precision, however, analysis (A) – (C) is problematic as a theory of Platonic Forms. We can begin to see why by considering a passage in Vlastos (1954), where he reformulates Plato's One Over Many Principle, as it occurs in *Parmenides* 132a1–b2 (1954, 320):

> This is the first step of the Argument, and may be generalized as follows:
>
> (A1) If a number of things, a, b, c are all F, there must be a single Form, F-ness, in virtue of which we apprehend a, b, c, as all F.
>
> Here 'F' stands for any discernible character or property. The use of the same symbol, 'F,' in 'F-ness,' the symbolic representation of the "single Form,"[5] records the identity of the character discerned in the particular ("large") and conceived in the Form ("Largeness") through which we see that this, or any other, particular has this character.

The footnote numbered 5 in this passage is rather interesting, for it includes the claim "That F and F-ness are logically and ontologically distinct is crucial to the argument" (1954, 320). So Vlastos is distinguishing the Form of F from the property F, though in other passages, he calls the property F "the predicative function of the same Form" (1954, note 39).

Geach (1956) also balks at the suggestion that, for Plato, the Form of F is the attribute or property F, at least in connection with Forms corresponding to 'kind terms' such as 'man' and 'bed'. He notes (1956, 74):

> Surely his [Plato's] chosen way of speaking of these Forms suggests that for him a Form was nothing like what people have since called an "attribute" or a "characteristic." The bed in my bedroom is to the Bed, not as a thing to an attribute or characteristic, but rather as a pound weight or yard measure in a shop to the standard pound or yard.

Geach thus takes the Form of F to be a "paradigm" individual that exemplifies F (1956, 76).[165] He uses *individual* variables $x, y, \ldots$ to range over the Forms. He uses a different style of variable, namely $F, G, \ldots$, for properties (or attributes). Similarly, Strang (1963) formulates (OM) by using the notation 'A' to refer to the property A and '$F(A)$' to refer to a Form of A.

[165] In 1956, 76, Geach says that a *paradigm F* is something that *is F*; it is a *standard F* insofar as it is an exemplar of *F par excellence*. But it is important to note that a paradigm exemplar of F, say b, has all manner of properties that are completely incidental to its being F and that undermine the suggestion that b is a paradigm. A paradigm sphere will have a radius of some particular length l, be constructed of some particular material m, reflect light of some particular color c, etc. Intuitively, however, the paradigm sphere, in so far as it is to serve as the Form of Sphericity, should be something abstracted from all these particular, incidental properties, though it surely exemplifies all the properties implied by *being a sphere*. This understanding of a *paradigm*, as something that 'has', in the sense of *encodes*, all and only the properties implied by *being a sphere*, will be captured by the 'thick' conception of the Forms discussed in the Section 11.2. Consequently, we postpone further discussion of this issue until then, and specifically until Remark (347).

Allen (1960) offers a second reason why analysis (A) – (C) should not be adopted, namely, it doesn't make sense of a principle to which Plato often appeals:

(SP) **Self-Predication Principle**
The Form of F is F.

Allen explicitly notes (1960, 148):[166]

> Plato obviously accepts the following thesis: some (perhaps all) entities which may be designated by a phrase of the form "the F Itself," or any synonyms thereof, may be called F. So the Beautiful Itself will be beautiful, the Just Itself just, Equality equal.[3]

Allen assumed these facts in the following lines from the opening passage of his article (1960, 147):

> The significance—or lack of significance—of Plato's self-predicative statements has recently become a crux of scholarship. Briefly, the problem is this: the dialogues often use language which suggests that the Form is a universal which has itself as an attribute and is thus a member of its own class, and, by implication, that it is the one perfect member of that class. The language suggests that the Form *has* what it *is*: it is self-referential, self-predicable.
>
> Now such a view is, to say the least, peculiar. Proper universals are not instantiations of themselves, perfect or otherwise. Oddness is not odd; Justice is not just; Equality is equal to nothing at all. No one can curl up for a nap in the Divine Bedsteadity; not even God can scratch Doghood behind the Ears.
>
> The view is more than peculiar; it is absurd. ...

With this, we are in a position to appreciate a later passage about (SP) in this same paper. Allen writes (1960, 148–9):

> But this thesis [SP] does not, by itself, imply self-predication; for that, an auxiliary premise is required. This premise is that a function of the type "... is F" may be applied univocally to F particulars and to the F Itself, so that when (for example) we say that a given act is just, and that Justice is just, we are asserting that both have identically the same character. But this premise would be false if the function were systematically equivocal, according as the subject of the sentence was a Form or a particular. In that case, to say that Justice is just and that any given act is just would be to

[166]The footnote numbered 3 in the following quotation provides his documentation. Allen cites the following passages where Plato offers a version of the claim in question: *Protagoras*, 330c, 331b; *Phaedo*, 74b, d, 100c; *Hippias Major*, 289c, 291e, 292e, 294a-b; *Lysis*, 217a; *Symposium*, 210e-211d.

> say two quite different (though perhaps related) things, and the difficulties inherent in self-predication could not possibly arise. ...I propose to show that functions involving the names of Forms exhibit just this kind of ambiguity.

Though I do not endorse many of the subsequent conclusions Allen draws in his paper, the conclusion in the above passage strikes me as insightful. The idea that there is ambiguity in predication and, indeed, that Plato saw the ambiguity, has been picked up by other Plato scholars, notably Frede (1967) and Meinwald (1992).[167]

Thus, the problem with the analysis (A) – (C) above is that it doesn't really make sense of Plato's text: it doesn't distinguish the property F from The Form of F and it can't make sense of the Self-Predication Principle. Vlastos' suggestion that the property F and the Form of F are ontologically as well as logically distinct, and Allen's suggestion that the language in the Self-Predication Principle involves a systematic ambiguity, are central to the analyses developed in this chapter.

Object theory distinguishes the property F from the abstract, logical individuals encoding F that might serve as *The Form of F*. Two such individuals immediately come to mind: (1) the abstract object that encodes just F and no other properties, and (2) the abstract object that encodes all and only the properties necessarily implied by F. The former offers a 'thin' conception on which The Form of F is the 'pure', objectified form of the property F ('thin' in the sense that it encodes a single property and 'pure' in the sense that it encodes no other property). The latter offers a 'thick' conception on which The Form of F encodes exactly what all F-exemplifiers necessarily exemplify in virtue of exemplifying F, i.e., those properties G such that $\Box\forall x(Fx \to Gx)$. The thin conception of Forms was developed in Zalta 1983 (Chapter II, Section 1), whereas the thick conception was developed in Pelletier and Zalta 2000. These conceptions are discussed below in some detail, in Sections 11.1 and 11.2, respectively. Though the thick conception may be a more considered and scholarly approach to Plato, the thin conception is not without interest, for it already yields theorems that provide plausible readings of (OM) and (SP).

11.1 The Thin Conception of Forms

(**318**) **Definition:** A Thin Form of G. Let us say that x is a *thin Form of G* iff x is an abstract object that encodes just the property G:

$$\mathit{ThinFormOf}(x,G) =_{df} A!x \,\&\, \forall F(xF \equiv F = G)$$

[167] For a more complete history of Plato scholarship in which it is proposed that there is an ambiguity in predication, see the Appendix to Pelletier & Zalta 2000.

(**319**) **Theorems:** There Exists a (Unique) Thin Form of G.

(.1) $\exists x(\mathit{ThinFormOf}(x,G))$

(.2) $\exists!x(\mathit{ThinFormOf}(x,G))$

(**320**) **Theorem:** The Thin Form of G Exists. Our definitions now imply, by a modally strict proof, that the thin Form of G exists:

$$\exists y(y = \imath x(\mathit{ThinFormOf}(x,G)))$$

(**321**) **Term Definition:** Notation for the Thin Form of G. We may therefore introduce notation, $\boldsymbol{a}_G$, to designate the thin Form of G:

$$\boldsymbol{a}_G =_{df} \imath x \mathit{ThinFormOf}(x,G)$$

From this and (318), we know $\boldsymbol{a}_G = \imath x(A!x \,\&\, \forall F(xF \equiv F = G))$. So $\boldsymbol{a}_G$ is (identical to) a canonical individual.

(**322**) **Theorem:** $\boldsymbol{a}_G$ is Strictly Canonical. The condition $F = G$ is a rigid condition on properties, as this was defined in (188.1):

$$\Box\forall F(F = G \rightarrow \Box F = G)$$

Since $\boldsymbol{a}_G = \imath x(A!x \,\&\, \forall F(xF \equiv F = G))$, it follows that $\boldsymbol{a}_G$ is (identical to) a strictly canonical individual, as this was defined in (188.2). So the theorems in (189) apply to $\boldsymbol{a}_G$.

(**323**) **Theorems:** Facts about the Thin Form of G. It now follows, by modally strict proofs, that: (.1) the thin Form of G is abstract and for every F, encodes F if and only if F just is G; and (.2) The thin Form of G is a thin Form of G:

(.1) $A!\boldsymbol{a}_G \,\&\, \forall F(\boldsymbol{a}_G F \equiv F = G)$

(.2) $\mathit{ThinFormOf}(\boldsymbol{a}_G, G)$

The interesting fact about (.2) is that it is derived without appeal to (101.2)$\star$.

(**324**) **Theorem:** A Thin Form of G Encodes G. It is a simple consequence of our definition that if x is a thin Form of G, then x encodes G:

$$\mathit{ThinFormOf}(x,G) \rightarrow xG$$

(**325**) **Definition:** Participation. We now say: *y participates in x* iff there is a property F such that x is a thin Form of F and y exemplifies F:

$$\mathit{ParticipatesIn}(y,x) =_{df} \exists F(\mathit{ThinFormOf}(x,F) \,\&\, Fy)$$

This definition will be refined in section (11.2), where we discuss the thick conception of Forms and distinguish two kinds of participation corresponding to the two kinds of predication. But the above definition serves well enough for our purposes in the present section.[168]

(**326**) **Lemma:** Thin Forms, Predication, and Participation. It is an immediate consequence of the previous definition that if x is a thin Form of G, then an individual y exemplifies G iff y participates in x:

$$\mathit{ThinFormOf}(x,G) \rightarrow \forall y(Gy \equiv \mathit{ParticipatesIn}(y,x))$$

(**327**) **Theorem:** The Equivalence of Exemplification and Participation. It is now a consequence that an object x exemplifies a property G iff x participates in the thin Form of G:

$$Gx \equiv \mathit{ParticipatesIn}(x, \boldsymbol{a}_G)$$

This theorem verifies that, under our analysis, Plato's notion of participation is equivalent to the modern notion of exemplification.

(**328**) **Theorem:** The One Over the Many Principle. As noted in Remark (317), Plato's most important principle governing the Forms is (OM): if there are two distinct individuals exemplifying G, then there exists something that is *the* Form of G in which they both participate. (OM) is validated by the following theorem:

$$Gx \,\&\, Gy \,\&\, x \neq y \rightarrow \exists z(z = \boldsymbol{a}_G \,\&\, \mathit{ParticipatesIn}(x,z) \,\&\, \mathit{ParticipatesIn}(y,z))$$

So we've preserved the main principle of Plato's theory without collapsing the distinction between the property G and the thin Form of G.

(**329**) **Theorems:** Facts about Thin Forms. Consider the property *being ordinary*, $O!$. It follows that (.1) the thin Form of G fails to exemplify $O!$; and hence (.2) the thin Form of $O!$ fails to exemplify $O!$:

(.1) $\neg O!\boldsymbol{a}_G$

(.2) $\neg O!\boldsymbol{a}_{O!}$

Since (.1) implies that $\neg O!\boldsymbol{a}_G$ for every G, (.2) is an immediate consequence. (.2) will play a role in the Remark that follows the next theorem.

(**330**) **Theorem:** The Thin Form of G Encodes G and Exactly One Property.

(.1) $\boldsymbol{a}_G G$

[168] This definition is a bit more specific than the definition formulated in Zalta 1983 (Ch. II, Section 1). In 1983, $\mathit{ParticipatesIn}(y,x)$ was defined as: $\exists F(xF \,\&\, Fy)$. By reformulating the definition as we've done in here, an object participates only in those abstract objects that are thin Forms.

This is an encoding formula in which the individual term, $\boldsymbol{a}_G$, is complex. It also follows that the Thin Form of G encodes exactly one property:

(.2) $\exists!H\boldsymbol{a}_G H$

(**331**) **Remark:** The 'Self-Predication' Principle. Recall that (SP) was formulated in Remark (317) as "The Form of G is G". As we saw in that remark, Allen (1960) suggests that in Plato's work, the context "… is G" does not apply univocally to both individuals that exemplify G and the Form of G. Furthermore, as noted in Section 1.4, Meinwald (1992, 378) argues that the second half of Plato's *Parmenides* leads us "to recognize a distinction between two kinds of predication, marked … by the phrases 'in relation to itself' (*pros heauto*) and 'in relation to the others' (*pros ta alla*)." (Intuitively, the Form of F *is* F in relation to itself, but ordinary things that exemplify F *are* F in relation to some other thing, namely, the Form of F.) Thus, Meinwald also takes (SP) to exhibit an ambiguity, and she suggests that it is true only if interpreted as a *pros heauto* predication.

Once we represent *pros heauto* predications as encoding predications and represent *pros ta alla* predications as exemplification predications, the ambiguity in (SP) can be resolved within the present system. The *pros heauto* reading is $\boldsymbol{a}_G G$, i.e., the thin Form of G encodes G. This is derivable as theorem (330.1) and hence true. The *pros ta alla* reading is $G\boldsymbol{a}_G$, i.e., the thin Form of G exemplifies G. We can prove in object theory that this is not generally true, since a counterexample is derivable. Indeed, we've already seen the counterexample, namely theorem (329), which asserts that $\neg O!\boldsymbol{a}_{O!}$. The thin Form of *being ordinary* fails to exemplify *being ordinary*. Hence $\neg\forall F(F\boldsymbol{a}_F)$.

Note also that $\neg G\boldsymbol{a}_G$ follows from the assumption that G is a concreteness-entailing property. A property F is *concreteness-entailing* just in case $\Box\forall x(Fx \rightarrow E!x)$. Intuitively, one might suppose that the following are such properties: *being extended in spacetime*, *being colored*, *having mass*, *being human*, etc. (Indeed, the property *being concrete* ($E!$) is provably concreteness-entailing.) So we can prove that if G is concreteness-entailing, then $\neg G\boldsymbol{a}_G$. To see this, assume that P is a concreteness-entailing property and, for reductio, that $P\boldsymbol{a}_P$. Now since P is concreteness-entailing, it follows from the definition by the T-schema that $\forall x(Px \rightarrow E!x)$. Hence $P\boldsymbol{a}_P \rightarrow E!\boldsymbol{a}_P$. Since $P\boldsymbol{a}_P$ is our reductio assumption, it follows that $E!\boldsymbol{a}_P$. But by by (323.1), we know $A!\boldsymbol{a}_P$, which by definition (11.2) is the claim $[\lambda x\, \neg\Diamond E!x]\boldsymbol{a}_P$. By β-Conversion, it follows that $\neg\Diamond E!\boldsymbol{a}_P$, i.e., $\Box\neg E!\boldsymbol{a}_P$. By the T schema, this implies $\neg E!\boldsymbol{a}_P$, at which point we've reached a contradiction.

Although we've now seen that the general form of (SP) is provably false when we represent the copula 'is' as *exemplification*, we shall see that there are special cases where, for some properties F, $\boldsymbol{a}_F$ does exemplify F.

(**332**) **Definition:** Thin Forms. We define: x is a *thin Form* if and only if x is a thin Form of F, for some F:

$$ThinForm(x) =_{df} \exists F(ThinFormOf(x,F))$$

(**333**) **Theorem:** A Fact about Thin Forms. Clearly, the thin Form of G is a thin Form:

$$ThinForm(\boldsymbol{a}_G)$$

By GEN, this holds for any property G.

(**334**) **Remark:** Thin Forms Aren't Paradoxical. Of course, one might wonder whether there is some Russell paradox lurking here: is there a property corresponding to the expression "being a thin Form that doesn't participate in itself" and if so, is there a thin Form of this property? The answer, on both counts, is no. First, the condition $ThinForm(x)$ is defined as $\exists G(ThinFormOf(x,G))$, and this in turn is defined as:

$$\exists G(A!x \,\&\, \forall F(xF \equiv F = G))$$

Given that the above formula has an encoding subformula, it fails to be propositional and so we cannot use it either to formulate an instance of the Comprehension Principle for Properties (129.2) or use it as the matrix of a λ-expression. That is, our system neither asserts $\exists F \forall x(Fx \equiv ThinForm(x))$ nor does it countenance the expression $[\lambda x\, ThinForm(x)]$.

Second, even if one could consistently assert $\exists F \forall x(Fx \equiv ThinForm(x))$, a Russell-style paradox requires that the system assert the following existence claim:

$$(\vartheta) \quad \exists F \forall x(Fx \equiv ThinForm(x) \,\&\, \neg ParticipatesIn(x,x))$$

For if there were such a property, say P, then $\boldsymbol{a}_P$ would exist and one could derive the Russell-style paradox by showing that $\boldsymbol{a}_P$ participates in itself if and only if it doesn't. But here again, (ϑ) is not an axiom even if $ThinForm(x)$ were to denote a property, since the definiens of $ParticipatesIn(x,y)$ is another non-propositional formula. That is, the following is not an instance of (129.1):

$$\exists R \forall x \forall y(Rxy \equiv ParticipatesIn(x,y))$$

nor is the λ-expression $[\lambda xy\, ParticipatesIn(x,y)]$ well-formed. Consequently, the formula $ThinFormx(x) \,\&\, ParticipatesIn(x,x)$ is not propositional and that is why (ϑ) is not an instance of Comprehension for Properties (129.2). Without (ϑ), one cannot assert the existence of a paradoxical thin Form of the kind needed for the Russell-style paradox to get off the ground.

(**335**) **Theorems:** Facts about Thin Forms and Platonic Being. Suppose that instead of reading the defined predicate $A!$ as *being abstract*, we temporarily read it as *Platonic Being*. Then we can prove: (.1) Thin Forms exemplify Platonic Being; (.2) thin Forms exemplify every property necessarily implied by Platonic Being; (.3) the thin Form of Platonic Being exemplifies Platonic Being; (.4) thin Forms participate in the thin Form of Platonic Being; and (.5) thin Forms participate in the thin Form of F, for every property F necessarily implied by Platonic Being:

(.1) $ThinForm(x) \rightarrow A!x$

(.2) $ThinForm(x) \rightarrow \forall F(\Box\forall y(A!y \rightarrow Fy) \rightarrow Fx)$

(.3) $A!\boldsymbol{a}_{A!}$

(.4) $ThinForm(x) \rightarrow ParticipatesIn(x, \boldsymbol{a}_{A!})$

(.5) $ThinForm(x) \rightarrow (\forall F(\Box\forall y(A!y \rightarrow Fy) \rightarrow ParticipatesIn(x, \boldsymbol{a}_F))$

(**336**) **Theorems:** Thin Forms and Participation. (.1) There exists a thin Form that participates in itself; and (.2) there exists a thin Form that doesn't participate in itself:

(.1) $\exists x(ThinForm(x) \,\&\, ParticipatesIn(x, x))$

(.2) $\exists x(ThinForm(x) \,\&\, \neg ParticipatesIn(x, x))$

Note that (.1) answers the question of whether there are any Forms that can participate in themselves in the positive. Also, we can't abstract out a Russell-style paradoxical property from (.2), for the reasons noted in (334).

(**337**) **Remark:** The Third Man Argument. Plato puts forward an argument in *Parmenides* (132a) that has come to be known as the Third Man Argument (TMA). This argument raises a concern as to whether the theory of Forms involves an infinite regress. As Vlastos (1954, 321) develops the argument, Plato appears to draw an inference from (A1) to (A2), both of which are supposed to be instances of (OM):

(A1) If a number of things, a, b, c are all F, there must be a single Form, F-ness, in virtue of which we apprehend a, b, c, as all F.

(A2) If a, b, c, and F-ness are all F, there must be another Form, F_1-ness, in virtue of which we apprehend a, b, c, and F-ness as all F.

If this inference is valid and F_1-ness is distinct from F-ness, then it raises the concern that the inference is only the first step of a regress that commits Plato to an infinite number of Forms corresponding to the single property F. But

Vlastos then notes that for the inference from (A1) to (A2) to be valid, there seems to be two implicit assumptions, the Self-Predication Principle (SP) discussed earlier and the following:

(NI) **Non-Identity Principle**
If x is F, x is not identical with the Form of F.

Vlastos then notes (1954, 326) that these two tacit assumptions are jointly inconsistent. This was a trenchant observation, for if the two claims are formally represented as:

(SP)′ $F\boldsymbol{a}_F$

(NI)′ $Fx \to x \neq \boldsymbol{a}_F$

then the inconsistency becomes manifest; we can instantiate $\boldsymbol{a}_F$ into (NI)′ to obtain $F\boldsymbol{a}_F \to \boldsymbol{a}_F \neq \boldsymbol{a}_F$, and this, together with (SP)′, yields the odious conclusion that $\boldsymbol{a}_F \neq \boldsymbol{a}_F$. Of course, the inconsistency of (SP)′ and (NI)′ involves a system in which there are: (a) principles that assert the existence and uniqueness of $\boldsymbol{a}_F$ and (b) principles governing definite descriptions in general and $\boldsymbol{a}_F$ in particular.

The literature that developed in response to Vlastos 1954 focused on (a) how to reformulate TMA so as to avoid the above inconsistency, and (b) whether there is textual support for the revised TMA. Papers were contributed by Sellars 1955, Geach 1956, with rejoinders in Vlastos 1955 and 1956. Vlastos 1969 (footnote 2) includes a list of papers that were subsequently published on the TMA, though see Cohen 1971, Meinwald 1992, and Pelletier & Zalta 2000 for further discussion and additional bibliography.

We conclude our discussion of TMA, as well as this section on the thin conception of the Forms, by noting that both of the tacit assumptions that Vlastos formulated for the TMA, as he understood them, are provably false on the present theory. We've already seen that the above reading (SP)′ of (SP) has a counterexample. Object theory also implies that the reading (NI)′ of (NI) has a counterexample. By (335.3), we know $A!\boldsymbol{a}_{A!}$, and by the logical propriety of $\boldsymbol{a}_{A!}$ and the reflexivity of identity (70.1), we know $\boldsymbol{a}_{A!} = \boldsymbol{a}_{A!}$. But the conjunction of these two conclusions, $A!\boldsymbol{a}_{A!}$ & $\boldsymbol{a}_{A!} = \boldsymbol{a}_{A!}$ is a counterexample to (NI)′, as formally represented above. Hence, we shouldn't accept either of the two tacit assumptions that Vlastos thought were needed for the TMA, at least not if the predication involved in (SP) and in the antecedent of (NI) are understood as exemplification predication.

11.2 The Thick Conception of Forms

(**338**) **Remark:** The Need for a Thick Conception of the Forms. We've seen how our work thus far validates (a) the view that truly predicating F of a Form of F is different from truly predicating F of ordinary things that exemplify F, and (b) Meinwald's (1992) idea that in the *Parmenides*, Plato is attempting to get his audience (i) to appreciate a distinction in predication, namely, between saying that x is G *pros heauto* (i.e., in relation to itself) and saying that x is G *pros ta alla* (i.e., in relation to the others),[169] and (ii) to recognize that the Self-Predication principle (SP) is generally true only if read as a *pros heauto* predication. We've also seen that, in object theory, where encoding predication formally represents *pros heauto* predication and the Form of G is conceived as the 'thin' abstract object that encodes just the single property G, (SP) is preserved as theorem (330.1), which asserts that the Form of G encodes G.

In what follows, however, we consider some new data that our analysis can't yet accommodate. According to Meinwald, there are also true *pros heauto* predications such as "the Form of G is F", where F is a property distinct from G. Meinwald lists the following such claims as ones Plato would endorse (1992, 379):[170]

> The Just is virtuous (*pros heauto*).
> Triangularity is three-sided (*pros heauto*).
> Dancing moves (*pros heauto*).

She explains these as follows (1992, 379–80):[171]

> The Just is Virtuous
>
> holds because of the relationship between the natures associated with its subject and predicate terms: Being virtuous is part of what it is to be just. Or we can describe predication as holding because Justice is a kind of Virtue. If we assume that to be a triangle is to be a three-sided plane

[169] We remind the reader again that Meinwald (1992, 381) relates the two kinds of predication in the *Parmenides* to the *kath' hauto* and *pros allo* uses of 'is' in the *Sophist*, as these are distinguished in Frede 1967, 1992.

[170] Meinwald writes:

> It is clear that such sentences come out true in Plato's work, as well as fitting our characterizations of predication of a subject in relation to itself.

Unfortunately, though, Meinwald doesn't identify any passages where Plato discusses examples like the ones to follow.

[171] In the following quotation, Meinwald refers to a *tree predication*. The *tree* in question is a genus-species structure, in which the characteristics associated with a genus are inherited by, and thus predicable of, the members of the species covered by the genus. Thus, a true tree predication is one in which a member of a species is F in virtue of the fact that F is one of the characteristics associated with the covering genus. In Meinwald's example, since dancing is a species of motion, 'Dancing moves' is a true tree predication.

figure (i.e., that Triangle is the species of the genus Plane Figure that has the differentia Three-Sided), then

Triangularity is three-sided.

holds too. We can also see that

Dancing moves.

is a true tree predication, since Motion figures in the account of what Dancing is.

Clearly, if the Form of G is identified as it was in the preceding section, i.e., as the abstract object that encodes G and no other property, then we cannot as yet interpret the above *pros heauto* predications as encoding predications about the Forms in question. For on such a 'thin' conception, the Form of Justice, $\boldsymbol{a}_J$, encodes only the property *being just* (J) and so does not encode the distinct property *being virtuous* (V). Similarly, the Form of Triangularity ($\boldsymbol{a}_T$) encodes only the property *being triangular* (T) and not the distinct property *being three-sided* ($3S$), and so isn't three-sided *pros heauto*. And similarly for the Form of Dancing.

However, on the thick conception, the Form of G is the abstract object that encodes all the properties necessarily implied by G, i.e., encodes all and only the properties F such that, necessarily, anything that exemplifies G exemplifies F. In what follows shall introduce the functional term Φ_G to designate this thick Form of G. So if the Form of Justice (Φ_J) encodes all of the properties necessarily implied by *being just*, then from the premise that *being virtuous* (V) is necessarily implied by *being just*, it follows that Φ_J encodes V. Thus, the encoding claim $\Phi_J V$ ("the Form of Justice encodes being virtuous") provides a reading of the *pros heauto* predication "The Just is virtuous".

Similarly, from the facts that:

Being triangular necessarily implies *being three-sided*.

Dancing necessarily implies *being in motion*.

it follows, on the thick conception of the Forms of Triangularity (Φ_T) and Dancing (Φ_D), that:

The Form of Triangularity encodes *being three-sided*.
$\Phi_T 3S$

The Form of Dancing encodes *being in motion*.
$\Phi_D M$

So the thick conception of the Forms is distinguished from the thin conception by the fact that it can provide an analysis of *pros heauto* predications of the form "The Form of G is F" when supplemented by facts about property implication and property distinctness. Under the thin conception, we cannot derive $\boldsymbol{a}_G F$ given the premises that F is distinct from G and necessarily implied by G. Under the thick conception, we can derive $\Phi_G F$ from G necessarily implies F; this is a consequence of theorem (346.2) established below. *A fortiori*, we can derive $\Phi_G F$ from the conjunction of $F \neq G$ and G necessarily implies F. These derivations, we suggest, are the best way to understand and represent the conditions under which "The Form of G is F *pros heauto*" is true.

(**339**) **Definition:** Property Implication and Equivalence. To define the thick conception of Plato's Forms, we need some preliminary definitions. We first define G *necessarily implies* F, written '$G \Rightarrow F$', as: necessarily, everything that exemplifies G exemplifies F:

(.1) $G \Rightarrow F \ =_{df} \ \Box\forall x(Gx \rightarrow Fx)$

For example, one might reasonably take the following as premises: the ordinary property *being triangular* necessarily implies the properties *being three-sided*, *having a shape*, *being concrete*, etc.

We may also say that properties G and F are *necessarily equivalent* just in case they necessarily imply each other:

(.2) $G \Leftrightarrow F \ =_{df} \ G \Rightarrow F \ \& \ F \Rightarrow G$

(**340**) **Theorems:** Facts about Necessary Equivalence. As simple consequences of the preceding definitions, we have (.1) G and F are necessarily equivalent just in case necessarily, all and only the individuals exemplifying G exemplify F; (.2) necessarily equivalence is reflexive, symmetric, and transitive; and (.3) properties are necessarily equivalent if and only they necessarily imply the same properties:

(.1) $G \Leftrightarrow F \ \equiv \ \Box\forall x(Gx \equiv Fx)$

(.2) $\Leftrightarrow$ is an equivalence condition, i.e.,
$G \Leftrightarrow G$ ($\Leftrightarrow$ is reflexive)
$G \Leftrightarrow F \rightarrow F \Leftrightarrow G$ ($\Leftrightarrow$ is symmetric)
$(G \Leftrightarrow F \ \& \ F \Leftrightarrow H) \rightarrow G \Leftrightarrow H$ ($\Leftrightarrow$ is transitive)

(.3) $G \Leftrightarrow F \equiv \forall H(G \Rightarrow H \ \equiv \ F \Rightarrow H)$

Moreover, recall that by definition (138.1), a property F is necessary just in case $\Box\forall x Fx$, and by definition (138.2), a property F is impossible just in case $\Box\forall x \neg Fx$. Then we have: (.4) if G and F are necessary properties, then G and F are necessarily equivalent; and (.5) if G and F are impossible properties, then G and F are necessarily equivalent:

(.4) $(Necessary(G)\,\&\,Necessary(F)) \rightarrow G \Leftrightarrow F$

(.5) $(Impossible(G)\,\&\,Impossible(F)) \rightarrow G \Leftrightarrow F$

(**341**) **Definition:** A (Thick) Form of G. We define: x is a *Form of* G iff x is an abstract object that encodes all and only the properties necessarily implied by G:

$$FormOf(x,G) \ =_{df} \ A!x\,\&\,\forall F(xF \ \equiv \ G \Rightarrow F)$$

(**342**) **Theorems:** There Exists a (Unique) Form of G.

(.1) $\exists x FormOf(x,G)$

(.2) $\exists ! x FormOf(x,G)$

(**343**) **Theorem:** The Form of G Exists. From (342), it follows that the Form of G exists:

$$\exists y(y = \imath x FormOf(x,G))$$

(**344**) **Term Definition:** Notation for the Form of G. Given the previous theorem, we introduce the notation Φ_G to designate The Form of G:

$$\Phi_G \ =_{df} \ \imath x FormOf(x,G)$$

From this and definition (341), we know $\Phi_G = \imath x FormOf(x,G)$. So Φ_G is (identical to) a canonical individual.

(**345**) **Theorem:** Φ_G is Strictly Canonical. The condition $G \Rightarrow F$ is a rigid condition on properties, as this was defined in (188.1):

$$\Box\forall F(G \Rightarrow F \ \rightarrow \ \Box G \Rightarrow F)$$

Since $\Phi_G = \imath x(A!x\&\forall F(xF \equiv G \Rightarrow F))$, it follows that Φ_G is (identical to) a strictly canonical individual, as this was defined in (188.2). So the theorems in (189) apply to Φ_G.

(**346**) **Theorems:** Facts about Φ_G. It now follows, by modally strict proofs, that: (.1) The Form of G is abstract and encodes exactly the properties F necessarily implied G; and (.2) The Form of G is a Form of G:

(.1) $A!\Phi_G\,\&\,\forall F(\Phi_G F \equiv G \Rightarrow F)$

(.2) $FormOf(\Phi_G,G)$

It follows *a fortiori* from the second conjunct of (.1) that $(G \Rightarrow F) \rightarrow \Phi_G F$. Since we have established that $\vdash (G \Rightarrow F) \rightarrow \Phi_G F$, it follows by (46.10) that $G \Rightarrow F \vdash \Phi_G F$. And by (46.7), $F \neq G, G \Rightarrow F \vdash \Phi_G F$. Thus, our theorems validate the distinguishing feature of the thick conception of the Forms discussed at the end of Remark (338).

(**347**) **Remark:** Thick Forms as Paradigms. It is worth pausing to note the sense in which the thick conception of Forms validates Geach's suggestion that Forms are *paradigms*. Let S denote the ordinary property *being spherical* (as this is ordinarily applied to concrete objects), and consider the Form of Being Spherical, Φ_S. On the thick conception, Φ_S encodes all and only those properties necessarily implied by *being spherical*. Thus, Φ_S encodes *having a radius of some particular length*, since that is implied by *being spherical*, although there isn't a particular length l such that Φ_S encodes *being of length* l. Similarly, Φ_S encodes *being constructed of some particular material*, since that is implied by *being spherical*, but there won't be a particular material m such that Φ_S encodes *being constructed of material* m. And so on. Thus, Forms (thickly conceived) don't encode any of the incidental properties that prevent them from being true paradigms. The above conception avoids the problem, discussed in footnote 165, that afflicts Geach's suggestion that the Form of F is a paradigm *exemplifier* of F.

(**348**) **Definitions:** Two Kinds of Participation. We may introduce two kinds of participation to correspond to the two kinds of predication. We say: (.1) an object y participates *pros ta alla* in x (written $ParticipatesIn_{\text{PTA}}(y,x)$) iff there is a property F such that x is a Form of F and y exemplifies F, and (.2) an object y participates *pros heauto* in x (written $ParticipatesIn_{\text{PH}}(y,x)$) iff there is a property F such that x is a Form of F and y encodes F:

(.1) $ParticipatesIn_{\text{PTA}}(y,x) =_{df} \exists F(FormOf(x,F) \,\&\, Fy)$

(.2) $ParticipatesIn_{\text{PH}}(y,x) =_{df} \exists F(FormOf(x,F) \,\&\, yF)$

(**349**) **Lemmas:** Forms, Predication, and Participation. It is an immediate consequence of the previous definitions that (.1) if x is a Form of G, then every individual y exemplifies G iff y participates$_{\text{PTA}}$ in x; (.2) if x is a Form of G, then every individual y that encodes G participates$_{\text{PH}}$ in x:

(.1) $FormOf(x,G) \rightarrow \forall y(Gy \equiv ParticipatesIn_{\text{PTA}}(y,x))$

(.2) $FormOf(x,G) \rightarrow \forall y(yG \rightarrow ParticipatesIn_{\text{PH}}(y,x))$

(**350**) **Remark:** Why the Consequent of (349.2) is not a Quantified Biconditional. One might wonder why the consequent of (349.2) is just a quantified conditional and not a quantified biconditional. Specifically, when $FormOf(x,G)$, why doesn't $ParticipatesIn_{\text{PH}}(y,x)$ imply yG? The key fact behind an answer to this question is that the Forms of distinct, but necessarily equivalent, properties are identical to one another, and this creates the conditions for a counterexample to the right-to-left direction of the consequent of (349.2). An extended discussion is required to fully document such a counterexample.

Consider the following two properties:

$$P = [\lambda x\, Qx \,\&\, \neg Qx] \qquad (Q \text{ any property})$$

$$T = [\lambda x\, Bx \,\&\, \forall y(Sxy \equiv \neg Syy)] \qquad (B \text{ any property, } S \text{ any relation})$$

If, say, Q is *being round*, B is *being a barber*, and S is the relation *x shaves y*, then P is the property *being round and not round* while T is the property *being a barber who shaves all and only those individuals that don't shave themselves*. It is reasonable to assert that these are distinct properties, i.e., that $P \neq T$.[172] But note that P and T are provably both *impossible* properties, for it is straightforward to show:

$$\Box\forall x \neg Px$$

$$\Box\forall x \neg Tx$$

Hence by (340.5), it follows that P and T are necessarily equivalent, and so by (340.3), that P and T necessarily imply the same properties, i.e.,

(ϑ) $\forall F(P \Rightarrow F \;\equiv\; T \Rightarrow F)$

Now consider the following two instances of theorem (342.1):

$$\exists x FormOf(x,P)$$

$$\exists x FormOf(x,T)$$

Let b and c be such objects, respectively, so that we know $FormOf(b,P)$ and $FormOf(c,T)$. By definition (341), we know all of the following:

(A) $A!b$

(B) $\forall F(bF \;\equiv\; P \Rightarrow F)$

(C) $A!c$

(D) $\forall F(cF \;\equiv\; T \Rightarrow F)$

172 There are various *pretheoretic* reasons one might give for this: (a) the informal argument that shows an object couldn't possibly exemplify P must appeal to the property Q rather than to the property B and the relation S, whereas the informal argument that shows an object couldn't possibly exemplify T must appeal to B and S rather than to Q; and (b) one can tell a story about an object that is T without thereby telling a story about an object that is P.

Despite the fact that $P \neq T$, it can be shown that b and c are identical, and this is crucial to the construction of our counterexample. To show that b and c are identical, it suffices, by (A), (C), and (172), to show $\forall F(bF \equiv cF)$. But this last claim follows straightforwardly from (ϑ), (B) and (D). Hence by definition of abstract object identity, we've established:

(G) $b = c$

Now to see how this leads to a counterexample to (ζ), consider the fact that by the Comprehension Principle for Abstract Objects (39), there is an abstract object that encodes just T and no other properties:

$$\exists x(A!x \,\&\, \forall F(xF \equiv F = T))$$

Let d be such an object, so that we know:

(H) $A!d \,\&\, \forall F(dF \equiv F = T)$

Now we can show that the following elements of the counterexample to (ζ) are all true:

(i) $\mathit{FormOf}(b, P)$

(ii) $\mathit{ParticipatesIn}_{\text{PH}}(d, b)$

(iii) $\neg dP$

(i) is already known. To show (ii), i.e., $\mathit{ParticipatesIn}_{\text{PH}}(d, b)$, we have to show:

$$\exists F(\mathit{FormOf}(b, F) \,\&\, dF)$$

By $\exists$I, it suffices to show $\mathit{FormOf}(b, T) \& dT$. But $\mathit{FormOf}(c, T)$ is already known, and from (G) we know $b = c$. Hence $\mathit{FormOf}(b, T)$, by Rule SubId. Moreover, dT follows immediately from (H), by instantiating the second conjunct of (H) to T and applying the reflexivity of identity. This establishes (ii). It remains to show (iii), i.e., $\neg dP$. Note that by the right conjunct of (H), $dP \equiv P = T$. But by hypothesis, $P \neq T$. Hence, $\neg dP$.

Thus, we've established the elements of the counterexample to (ζ). Consequently, under the reasonable hypothesis that there are distinct, impossible properties (alternatively, in any model in which there are distinct, impossible properties), $\mathit{ParticipatesIn}_{\text{PH}}(y, x)$ doesn't imply yG when $\mathit{FormOf}(x, G)$.[173]

> **Exercise:** Show that $\mathit{ThinFormOf}(x, G) \rightarrow \forall y(yG \equiv \mathit{ParticipatesIn}(y, x))$ is a theorem when *ThinFormOf* is defined as in (318) and *ParticipatesIn* is defined as in (325).

[173] This counterexample described in this Remark was discovered computationally, using PROVER9, and reported in Fitelson and Zalta 2007, thereby correcting an error in Pelletier and Zalta 2000.

(**351**) **Theorems:** Exemplification, Participation$_{\text{PTA}}$, Encoding, and Participation$_{\text{PH}}$. It is now a consequence that (.1) x exemplifies G iff x participates$_{\text{PTA}}$ in The Form of G; and (.2) if x encodes G, then x participates$_{\text{PH}}$ in the Form of G:

(.1) $Gx \equiv \mathit{ParticipatesIn}_{\text{PTA}}(x, \Phi_G)$

(.2) $xG \rightarrow \mathit{ParticipatesIn}_{\text{PH}}(x, \Phi_G)$

The discussion in Remark (350) explains why (.2) is a conditional and not a biconditional. Note also that these theorems are modally strict, despite the fact that they are conditionals governing a term defined by a rigid definite description.

(**352**) **Theorem:** *ParticipatesIn*$_{\text{PTA}}$ Fact. It is a consequence of the foregoing definitions that if y participates$_{\text{PTA}}$in x, then y exemplifies every property x encodes:

$$\mathit{ParticipatesIn}_{\text{PTA}}(y, x) \rightarrow \forall F(xF \rightarrow Fy)$$

(**353**) **Theorems:** Two Versions of the One Over Many Principle. (OM) may now be derived in two forms: (.1) if there are two distinct individuals exemplifying G, then there exists something that is *the* Form of G in which they both participate$_{\text{PTA}}$; (.2) if there are two distinct individuals encoding G, then there exists something that is *the* Form of G in which they both participate$_{\text{PH}}$:

(.1) $Gx \,\&\, Gy \,\&\, x \neq y \rightarrow \exists z(z = \Phi_G \,\&\, \mathit{ParticipatesIn}_{\text{PTA}}(x, z) \,\&\, \mathit{ParticipatesIn}_{\text{PTA}}(y, z))$

(.2) $xG \,\&\, yG \,\&\, x \neq y \rightarrow \exists z(z = \Phi_G \,\&\, \mathit{ParticipatesIn}_{\text{PH}}(x, z) \,\&\, \mathit{ParticipatesIn}_{\text{PH}}(y, z))$

So we've preserved the main principle of Plato's theory not only in a version that governs exemplification and participation$_{\text{PTA}}$, but also in a version that governs encoding and participation$_{\text{PH}}$.

(**354**) **Theorems:** Facts about (the Form of) Being Ordinary. Consider the property *being ordinary*, $O!$. It follows that (.1) the Form of G fails to exemplify $O!$; (.2) the Form of $O!$ fails to exemplify $O!$; and (.3) for some property G, the Form of G fails to exemplify G:

(.1) $\neg O!\Phi_G$

(.2) $\neg O!\Phi_{O!}$

(.3) $\exists G \neg G \Phi_G$

The first two are the counterparts of theorems (329.1) and (329.2) governing the thin conception of Forms. (.3) can be stated in an equivalent form: it isn't universally the case that Φ_G is G *pros ta alla*. Thus, an observation similar to the one discussed in Remark (331) holds for the thick conception of Forms: (354.3) establishes that the exemplification reading of (SP) fails to be universally true.

(**355**) **Theorem:** The Form of G is *G Pros Heauto*.

$$\Phi_G G$$

Thus we can derive the reading of the Self-Predication Principle (SP) discussed in (330.1).

(**356**) **Definition:** Forms. Forms, under the thick conception, are defined in the same general way as under the thin conception: x is a *Form* if and only if x is a Form of G, for some G:

$$Form(x) =_{df} \exists G(FormOf(x, G))$$

(**357**) **Theorems:** Some Facts about Forms. (.1) The Form of G is a Form; (.2) there exists a Form that doesn't participate$_{\text{PTA}}$ in itself; (.3) every Form participates$_{\text{PH}}$ in itself.

(.1) $Form(\Phi_G)$

(.2) $\exists x(Form(x) \,\&\, \neg ParticipatesIn_{\text{PTA}}(x, x))$

(.3) $\forall x(Form(x) \rightarrow ParticipatesIn_{\text{PH}}(x, x))$

As an exercise, explain why the existence of Forms that don't participate$_{\text{PTA}}$ in themselves doesn't give rise to paradox.

(**358**) **Theorems:** Facts about 'Self-Predication' *Pros Ta Alla* and Self-Participation. Again, if we suppose that the property *being abstract* is the property *Platonic Being*, then we have the following theorems: (.1) The Form of *Platonic Being* exemplifies *Platonic Being*; (.2) The Form of *Platonic Being* participates$_{\text{PTA}}$ in itself; (.3) there exists a Form that participates$_{\text{PTA}}$ in itself; (.4) if *Platonic Being* necessarily implies H, then for every property G, The Form of Φ_G exemplifies H; hence, (.5) if *Platonic Being* necessarily implies H, then The Form of Φ_H exemplifies H. Moreover, (.6) if *Platonic Being* necessarily implies the negation of H, then for every property G, $\neg H\Phi_G$, and hence (.7) if *Platonic Being* necessarily implies the negation of H, then $\neg H\Phi_H$:

(.1) $A!\Phi_{A!}$

(.2) $ParticipatesIn_{\text{PTA}}(\Phi_{A!}, \Phi_{A!})$

(.3) $\exists x(Form(x) \,\&\, ParticipatesIn_{\text{PTA}}(x, x))$

(.4) $(A! \Rightarrow H) \rightarrow \forall G(H\Phi_G)$

(.5) $(A! \Rightarrow H) \rightarrow H\Phi_H$

(.6) $(A! \Rightarrow \overline{H}) \to \forall G(\neg H\Phi_G)$

(.7) $(A! \Rightarrow \overline{H}) \to \neg H\Phi_H$

Theorem (.1) is a special case where a Form, namely The Form of Platonic Being, unconditionally exemplifies its defining property *pros ta alla*. This immediately implies (.2), that this Form participates$_{\text{PTA}}$ in itself. (.3) is then an immediate consequence of (.2). As an example of (.4), we now know that if *Platonic Being* implies *being eternal*, then for every property G, Φ_G exemplifies being eternal. In particular, (.5) if *Platonic Being* implies *being eternal* (E), then the Form of Eternality, Φ_E, exemplifies being eternal. This constitutes a conditional 'self-predication' *pros ta alla*. Note also that theorem (.7) is a simple consequence of (.6); as an example of (.7), if *Platonic Being* necessarily implies *not being in motion* ($\overline{M}$), then The Form of *being in motion*, Φ_M, fails to exemplify *being in motion*.

(**359**) **Theorems:** Necessary Implication and Participation.

(.1) $(A! \Rightarrow H) \to \forall G(\mathit{ParticipatesIn}_{\text{PTA}}(\Phi_G, \Phi_H))$

(.2) $(G \Rightarrow H) \to \mathit{ParticipatesIn}_{\text{PH}}(\Phi_G, \Phi_H)$

As an example of (.1), if *being abstract* necessarily implies *being at rest* (R), then for any G, The Form of G participates$_{\text{PTA}}$ in The Form of Rest Φ_R. As an example of (.2), if *being just* (J) necessarily implies *being virtuous* (V), then The Form of Justice, Φ_J, participates$_{\text{PH}}$ in The Form of Virtue, Φ_V; and if *being a triangle* (T) necessarily implies *being three-sided* ($3S$), then Φ_T participates$_{\text{PH}}$ in Φ_{3S}.

(**360**) **Remark:** Platonic Derivation of a Syllogism. Our definitions of participation and our definition of The Form of G offer us a Platonic analysis of a classic form of syllogism. Consider:

Humans are mortal.
Socrates is a human.

Socrates is mortal.

On a Platonic analysis of this argument, the conclusion is validly derivable from the premises. Both the minor premise ('Socrates is a human') and the conclusion can be analyzed as asserting that Socrates participates$_{\text{PTA}}$ in a certain Form. The major premise ('Humans are mortal') can be analyzed as asserting that The Form of Humanity is mortal *pros heauto*. Formally, where 's' denotes Socrates, 'H' denotes *being human*, and 'M' denotes *being mortal*, these analyses can be captured as follows:

$\Phi_H M$
$ParticipatesIn_{\text{PTA}}(s, \Phi_H)$

$ParticipatesIn_{\text{PTA}}(s, \Phi_M)$

It is straightforward to show that the conclusion follows from the premises. By the second conjunct of theorem (346.1), the first premise implies that $H \Rightarrow M$, i.e., $\Box\forall x(Hx \rightarrow Mx)$. By the T schema, we may infer $\forall x(Hx \rightarrow Mx)$. By theorem (351.1), the second premise of the argument implies Hs. Hence, it follows that Ms. But, again by theorem (351.1), it now follows that $ParticipatesIn_{\text{PTA}}(s, \Phi_M)$.[174]

(**361**) **Remark:** The Non-Identity Principle and the Third Man Argument. Recall that in the section on the thin conception of the Forms, in Remark (337), we discussed the fact that in formulating the Third Man Argument, Vlastos noted that the following principle was implicitly required:

(NI) **Non-Identity Principle**
If x is F, x is not identical with the Form of F.

However, as part of our investigation of the thick conception of the Forms the question arises, what happens if we replace the antecedent of (NI) with a claim about participation? The answer is that we obtain two interesting versions of (NI):

(NIa) If x participates$_{\text{PTA}}$ in The Form of F, x is not identical with that Form.
$ParticipatesIn_{\text{PTA}}(x, \Phi_F) \rightarrow x \neq \Phi_F$

(NIb) If x participates$_{\text{PH}}$ in The Form of F, x is not identical with that Form.
$ParticipatesIn_{\text{PH}}(x, \Phi_F) \rightarrow x \neq \Phi_F$

These are interesting because each leads to a Third Man Argument.

Recall the version of the Self-Predication Principle that provably fails to be generally true:

(SPa) $F\Phi_F$

Put aside for the moment the fact that (SPa) is not generally true, and also put aside the fact that (NIa) and (SPa) can't both be true given (351.1) (and the theorems upon which (351.1) rests) and the reflexivity of identiy.[175] Then the

[174] This discussion corrects another error in Pelletier and Zalta 2000. In that paper, we misanalyzed the major premise as $ParticipatesIn_{\text{PH}}(\Phi_H, \Phi_M)$, mistakenly believing that this implied that $\Phi_H M$, i.e., that this implied that The Form of Humanity is mortal *pros heauto*. But the discussion in (350) explains why this is an error. The error is corrected in the analysis above: "Humans are mortal" is analyzed as the claim that The Form of Humanity is mortal *pros heauto*. *Given* this as a premise, we may infer via the second conjunct of theorem (346.1) that $H \Rightarrow M$.

[175] From (SPa) it follow by (351.1) that $ParticipatesIn_{\text{PTA}}(\Phi_F, \Phi_F)$, and from this it follows by (NIa) that $\Phi_F \neq \Phi_F$.

first Third Man Argument begins with the premise that there are two distinct F-things *pros ta alla* and leads to a contradiction as follows:

> Suppose $Fx \& Fy \& x \neq y$. Then by (353.1), it follows that $\exists z(z = \Phi_F \& \mathit{ParticipatesIn}_{\text{PTA}}(x,z) \& \mathit{ParticipatesIn}_{\text{PTA}}(y,z))$. Assume that a is such an object, so that we know $a = \Phi_F \& \mathit{ParticipatesIn}_{\text{PTA}}(x,a) \& \mathit{ParticipatesIn}_{\text{PTA}}(y,a)$. The first and second conjunct of this result imply $\mathit{ParticipatesIn}_{\text{PTA}}(x,\Phi_F)$, and so by (NIa) above, it follows that $x \neq \Phi_F$. By (SPa) above, we know $F\Phi_F$. Since we now know that Fx, $F\Phi_F$, and $x \neq \Phi_F$, it follows by (353.1) that $\exists z(z = \Phi_F \& \mathit{ParticipatesIn}_{\text{PTA}}(x,z) \& \mathit{ParticipatesIn}_{\text{PTA}}(\Phi_F,z))$. Assume that b is such an object, so that we know $b = \Phi_F \& \mathit{ParticipatesIn}_{\text{PTA}}(x,b) \& \mathit{ParticipatesIn}_{\text{PTA}}(\Phi_F,b)$. The first and third conjunct of this last result imply $\mathit{ParticipatesIn}_{\text{PTA}}(\Phi_F,\Phi_F)$, and so by (NIa) above, it follows that $\Phi_F \neq \Phi_F$. But this contradicts $\Phi_F = \Phi_F$, which we know by Rule ReflId (74.1).

Given that theorem (354.3) establishes that the universal generalization of (SPa) is false, we already know that the above argument isn't sound. But the argument also fails to be sound because (NIa) is false; a counterexample is immediately forthcoming. By conjoining (358.2), i.e., $\mathit{ParticipatesIn}_{\text{PTA}}(\Phi_{A!},\Phi_{A!})$, with the instance $\Phi_{A!} = \Phi_{A!}$ of Rule ReflId, we have a counterexample to (NIa). Indeed, though (SPa) is not generally true, every Form that yields a true substitution instance of (SPa) will provide a counterexample to (NIa).

Turning next to (NIb), the second Third Man Argument begins with the premise that there are two distinct F-things *pros heauto* and leads to a contradiction as follows:[176]

> Suppose xF, yF, and $x \neq y$. Then it follows by (353.2) that $\exists z(z = \Phi_F \& \mathit{ParticipatesIn}_{\text{PH}}(x,z) \& \mathit{ParticipatesIn}_{\text{PH}}(y,z))$. Assume c is such an object, so that we know:
>
> $$c = \Phi_F \;\&\; \mathit{ParticipatesIn}_{\text{PH}}(x,c) \;\&\; \mathit{ParticipatesIn}_{\text{PH}}(y,c)$$
>
> The first two conjuncts of this result imply $\mathit{ParticipatesIn}_{\text{PH}}(x,\Phi_F)$, which by (NIb), implies $x \neq \Phi_F$. Moreover, by (355), we know $\Phi_F F$. Hence, we know xF, $\Phi_F F$, and $x \neq \Phi_F$. So by (353.2), it follows that $\exists z(z = \Phi_F \& \mathit{ParticipatesIn}_{\text{PH}}(x,z) \& \mathit{ParticipatesIn}_{\text{PH}}(\Phi_F,z))$. Assume d is such an object, so that we know:
>
> $$d = \Phi_F \;\&\; \mathit{ParticipatesIn}_{\text{PH}}(x,d) \;\&\; \mathit{ParticipatesIn}_{\text{PH}}(\Phi_F,d)$$
>
> The first and third conjunct of this result imply $\mathit{ParticipatesIn}_{\text{PH}}(\Phi_F,\Phi_F)$, which by (NIb) implies $\Phi_F \neq \Phi_F$. But this contradicts $\Phi_F = \Phi_F$, which we know by Rule ReflId.

[176] See Pelletier and Zalta 2000, 174, and Frances 1996, 59.

In this case, the version of the Self-Predication Principle used in the argument is theorem (355), and so the way to undermine the soundness of the argument is to challenge (NIb). But it is easy to show that (NIb) and $\Phi_F F$ (355) can't both be true together given (351.2), the theorems upon which (351.2) rests, and the reflexivity of identity. For from $\Phi_F F$, it follow by (351.2) that $ParticipatesIn_{\text{PH}}(\Phi_F, \Phi_F)$, and from this it follows by (NIb) that $\Phi_F \neq \Phi_F$. We've established by reductio, therefore, that (NIb) has to be rejected; everything else used to derive the contradiction is a theorem.

Since the object-theoretic analysis of Plato's Forms entails the rejection of (SPa), (NIa), and (NIb), the Third Man Argument is gone for good.

Chapter 12

Situations, Worlds, and Times

In this chapter we develop the theory of situations, worlds (both possible and impossible), moments of time, and world-states. Along the way we introduce world-relativized extensions of two kinds: world-relativized truth-values and world-relativized classes. Throughout this chapter, we do not distinguish propositions and states of affairs. For our purposes in what follows, we shall often refer to 0-place relations as *states of affairs*, since that more closely follows the traditional language of situation theory.

12.1 Situations

(**362**) **Remark:** On the Nature of Situations. In a series of papers (1980, 1981a, 1981b) that culminated in their book of 1983, Barwise and Perry argued against views that were widely held in the field of natural language semantics, such as: (a) that possible worlds, taken as primitive, constitute a fundamental semantic domain, (b) that properties (and relations) are analyzable as functions from possible worlds to sets of (sequences of) individuals, (c) that the denotation of a sentence is a truth-value, and (d) that the denotation of a sentence shifts when the sentence appears in indirect, intensional contexts. Barwise and Perry suggested that a better semantic theory of language could be developed if possible worlds were replaced with *situations*, i.e., parts of the world in which one or more states of affairs hold, where states of affairs consist of objects standing in relations.

Barwise and Perry (1984, 23) subsequently realized that their book of 1983 offered a *model* of situations rather than a *theory* of them and, consequently, changed the direction of their research. In their early attempts to develop a theory of situations, they brought to bear certain intuitions they had previously had about the nature of situations. The fundamental intuition never wavered

(Barwise 1985, 185):

> By a situation, then, we mean a part of reality that can be comprehended as a whole in its own right—one that interacts with other things. By interacting with other things, we mean that they have properties or relate to other things. They can be causes and effects, for example, as when we see them or bring them about. Events are situations, but so are more static situations, even eternal situations involving mathematical objects. We use $s, s', s'', \ldots$ to range over real situations. There is a binary relation $s \models \sigma$, read "σ holds in s", that holds between various situations s and states of affairs σ; that is, situations and states of affairs are the appropriate arguments for this relation of holding in.

One of the guiding intuitions was the distinction between the internal and external properties of situations. In Barwise and Perry 1981b (388), we find:

> Situations have properties of two sorts, internal and external. The cat's walking on the piano distressed Henry. Its doing so is what we call an external property of the event. The event consists of a certain cat performing a certain activity on a certain piano; these are its internal properties.

This distinction between the internal and external properties appears throughout the course of publications on situation theory. For example, Barwise writes (1985, 185):

> If $s \models \sigma$, then the fact σ is called a fact of s, or more explicitly, a fact about the internal structure of s. There are also other kinds of facts about s, facts external to s, so the difference between being a fact that holds in s and a fact about s more generally must be borne in mind.

And in Barwise 1989a (263–4), we find:

> The facts determined by a particular situation are, at least intuitively, intrinsic to that situation. By contrast, the information a situation carries depends not just on the facts determined by that situation but is relative to constraints linking those facts to other facts, facts that obtain in virtue of other situations. Thus, information carried is not usually (if ever) intrinsic to the situation.
>
> The objects which actual situations make factual thus play a key role in the theory. They serve to characterize the intrinsic nature of a situation.

Interestingly, the intuitive distinction between the intrinsic, internal properties of a situation and its extrinsic, external properties, never made it to the level of theory; situation theorists never formally regimented the distinction. However, we take it to be the key to the analysis of situations in what follows. If we identify a situation s to be an abstract object that encodes only properties of the form $[\lambda y\, p]$ (where p is some state of affairs or proposition), so that we can say p is *encoded in s* whenever s encodes $[\lambda y p]$, then the intrinsic, internal properties of s are its encoded properties, while the extrinsic, external properties of s will be any exemplification facts about s or any logical or natural-law-based exemplification generalizations either about s or connecting the relations in the states of affairs encoded in s.

This analysis of situations was worked out more fully in Zalta 1993. In what follows, we reprise the most important definitions and theorems from that paper, as well as many new ones. Readers familiar with situation theory should note that the *infons* of later situation theory:

$$\langle\langle R^n, a_1, \ldots, a_n; 1\rangle\rangle$$

$$\langle\langle R^n, a_1, \ldots, a_n; 0\rangle\rangle$$

represent states of affairs in which objects $a_1, \ldots, a_n$ do or do not stand in relation R^n, depending on whether the polarity is 1 or 0. We have no need of infon notation, since we not only have the standard notation $R^n a_1 \ldots a_n$ and $\neg R^n a_1 \ldots a_n$ but also the λ-notation $[\lambda\, R^n a_1 \ldots a_n]$ and $[\lambda\, \neg R^n a_1 \ldots a_n]$, both of which denote states of affairs. Moreover, we shall represent the following situation-theoretic claims asserting that an infon holds in situation s:

$$s \models \langle\langle R^n, a_1, \ldots, a_n; 1\rangle\rangle$$

$$s \models \langle\langle R^n, a_1, \ldots, a_n; 0\rangle\rangle$$

more simply as follows:

$$s \models R^n a_1 \ldots a_n$$

$$s \models \neg R^n a_1 \ldots a_n$$

These claims will be defined, respectively, as:

$$s[\lambda y\, R^n a_1 \ldots a_n]$$

$$s[\lambda y\, \neg R^n a_1 \ldots a_n]$$

We shall, on occasion, point out (a) where axioms central to situation theory are derived as theorems, and (b) where unsettled questions of situation theory, as collated in Barwise 1989a, are settled by a theorem of object theory. Our work thus far has already resolved Choices 14–17 in Barwise 1989a (270–1): the comprehension principle for propositions (i.e., states of affairs), theorem (133), guarantees that we can freely form states of affairs out of any objects and relations (Choice 14); that not every state of affairs is basic (Choice 15, Alternative 15.2); that there is a rich algebraic structure on the space of states of affairs (Choice 16); and that every state of affairs has a dual (Choice 17).

Moreover, theorem (196.3) resolves Choice 13 in favor of Alternative 13.2: for each property F, there are *distinct* abstract objects a, b such that $[\lambda\, Pa] = [\lambda\, Pb]$. In such cases, the states of affairs $[\lambda\, Pa]$ and $[\lambda\, Pb]$ are identical, but it is not the case that $a = b$. In Barwise 1989a (270), Alternative 13.2 is in effect when $\langle\langle R, a; i\rangle\rangle = \langle\langle S, b; j\rangle\rangle$ fails to imply $R = S$, $a = b$, and $i = j$.

(**363**) **Definition:** Situations. Using the notion of propositional property defined in (198), we may say that x *is a situation* just in case x is an abstract object that encodes only propositional properties:

$$Situation(x) \ =_{df} \ A!x \,\&\, \forall F(xF \rightarrow Propositional(F))$$

By the definition (198) of propositional properties, it follows that situation is an abstract object x such that every property F that x encodes is a property of the form *being-a-y-such-that-p*, for some proposition p. Note that the definition of situation decides Choice 9 in Barwise 1989 (267) since it is easy to show that there objects (e.g., ordinary objects and abstract objects that encode non-propositional properties) that are not situations.

Since $Situation(x)$ is a well-defined condition, we may henceforth use the symbols $s, s', s'', \ldots$ as restricted variables ranging over situations and $s_1, s_2, s_3, \ldots$ as constants denoting situations.

As a simple example of a situation, let R be any relation, and a and b be any objects, and consider the following instance of comprehension (39):

Example:
$\exists x(A!x \,\&\, \forall F(xF \equiv F = [\lambda y\, Rab] \vee F = [\lambda y\, \neg Rba]))$

Let c be such an object. Clearly, it is provable that c encodes just the properties $[\lambda y\, Rab]$ and $[\lambda y\, \neg Rba]$. So every property c encodes is a propositional property. Hence, c is a situation; intuitively, c is the smallest situation that encodes $[\lambda y\, Rab]$ and $[\lambda y\, \neg Rba]$.

(**364**) **Theorem:** Some Known Situations. Given our definitions, it follows that truth-values are situations:

$$TruthValue(x) \rightarrow Situation(x)$$

From this and the theorems that $TruthValue(p^\circ)$ (222)★, $TruthValue(\top)$ (224.1)★, and $TruthValue(\bot)$ (224.2)★, we can infer, as non-modally strict theorems, that p°, $\top$, and $\bot$ are all situations! Furthermore, since this establishes that there are situations, we also know that the quantifiers $\forall s$ and $\exists s$ behave classically in the sense that $\forall s\varphi \rightarrow \exists s\varphi$ is a theorem; cf. Remark (256).

(**365**) **Definition:** Truth In a Situation. Recall that in (216) we defined: x encodes p, written $x\Sigma p$, just in case x is abstract and x encodes $[\lambda y\, p]$. We now say that proposition p is *true in* x, written $x \models p$, just in case x is a situation that encodes p:

$$x \models p \ =_{df}\ Situation(x) \,\&\, x\Sigma p$$

In what follows, we always read '$\models$' with smallest possible scope. So, for example, $x \models p \rightarrow p$ is to be parsed as $(x \models p) \rightarrow p$ rather than $x \models (p \rightarrow p)$. Moreover, given our conventions in (255.3), we may use restricted variables to recast the above definition as:

$$s \models p \ =_{df}\ s\Sigma p$$

Note that we may alternatively read '$s \models p$' in situation-theoretic terms as follows, in which the variable p is interpreted as a variable ranging over states of affairs (understood as 0-place relations):

State of affairs p *holds in* situation s

State of affairs p *is a fact in* situation s

Situation s *makes* p *true*

(**366**) **Lemmas:** Rigidity of *Situation*(x). The following lemmas prove useful. (.1) x is a situation if and only if necessarily x is a situation; (.2) possibly x is a situation if and only if x is a situation; (.3) possibly x is a situation if and only if necessarily x is a situation; and (.4) actually x is a situation if and only if x is a situation:

(.1) $Situation(x) \equiv \Box Situation(x)$

(.2) $\Diamond Situation(x) \equiv Situation(x)$

(.3) $\Diamond Situation(x) \equiv \Box Situation(x)$

(.4) $\mathcal{A}Situation(x) \equiv Situation(x)$

(**367**) **Lemmas:** Rigidity of Truth In a Situation. Our definitions and theorems also guarantee that: (.1) p is true in s if and only if it is necessary that p is true in s; (.2) it is possible that p is true in s if and only if p is true in s; (.3) it is possible that p is true in s if and only if it is necessary that p is true in s; and actually p is true in s if and only if p is true in s:

(.1) $s \models p \equiv \Box s \models p$

(.2) $\Diamond s \models p \equiv s \models p$

(.3) $\Diamond s \models p \equiv \Box s \models p$

(.4) $\mathcal{A}s \models p \equiv s \models p$

(.5) $\neg s \models p \equiv \Box\neg s \models p$

Notice here that the proofs of the above don't depend on (366.1). To better see why, consider (.1). If we eliminate the restricted variable, (.1) becomes: $Situation(x) \rightarrow (x \models p \equiv \Box x \models p)$. To prove this, we assume x is a situation and to prove the more interesting left-to-right direction of the consequent, we further assume $x \models p$ (the right-to-left direction follows immediately from the T schema). Our two assumptions imply, by definition (365), that $x\Sigma p$. But situations are abstract, by definition (363). Hence from $A!x$ and $x\Sigma p$, it follows that $x[\lambda y\, p]$, by definition (126.2). Since this is an encoding formula, axiom (37) applies, and yields $\Box x[\lambda y\, p]$. So by reversing the definitions, we obtain $\Box x \models p$. None of this reasoning requires that we know that x is necessarily a situation whenever x is a situation, i.e., (366.1) is not required.

(**368**) **Theorem:** Situation Identity. A fundamental fact concerning situation identity is now derivable, namely, that situations s and s' are identical if and only if they make the same propositions true:

$$s = s' \equiv \forall p(s \models p \equiv s' \models p)$$

This decides Choice 5 in Barwise 1989 (264) in favor of Alternative 5.1.

(**369**) **Definition:** Parts of Situations. By applying our conventions (255.3) for introducing formula definitions with restricted variables, we may say that situation *s is a part of* situation s', written $s \trianglelefteq s'$, just in case every proposition true in s is true in s':

$$s \trianglelefteq s' =_{df} \forall p(s \models p \rightarrow s' \models p)$$

This definition determines Choice 2 in Barwise 1989 (261), since it requires that every part of a situation be a situation. By our conventions for free variables in definitions, the above definition is short for:

$$x \trianglelefteq y =_{df} Situation(x) \,\&\, Situation(y) \,\&\, \forall p(x \models p \rightarrow y \models p)$$

(**370**) **Theorems:** *Part of* is a Partial Order on Situations. *Part of* is reflexive, anti-symmetric, and transitive on the situations:

(.1) $s \trianglelefteq s$

(.2) $s \trianglelefteq s' \,\&\, s \neq s' \rightarrow \neg(s' \trianglelefteq s)$

(.3) $s \trianglelefteq s' \,\&\, s' \trianglelefteq s'' \rightarrow s \trianglelefteq s''$

Whereas a partial ordering of situations by $\trianglelefteq$ is assumed in situation theory (cf. Barwise 1989, 185, 259), such an ordering falls out as a theorem of object theory.

(**371**) **Theorems:** Parts and Identity. Two other constraints on the identity of situations are derivable, namely, (.1) that situations s and s' are identical if and only if each is part of the other, and (.2) that situations s and s' are identical if and only if they have the same parts:

(.1) $s = s' \equiv s \trianglelefteq s' \;\&\; s' \trianglelefteq s$

(.2) $s = s' \equiv \forall s''(s'' \trianglelefteq s \equiv s'' \trianglelefteq s')$

In the usual way, when $s \trianglelefteq s'$ and $s \neq s'$, it may be useful to say that s is a *proper* part of s'.

(**372**) **Definition:** Persistency. In situation theory, a state of affairs p is *persistent* if and only if whenever p holds in a situation s, p holds in every situation s' of which s is a part:

$$Persistent(p) =_{df} \forall s(s \models p \rightarrow \forall s'(s \trianglelefteq s' \rightarrow s' \models p))$$

Cf. Barwise 1989 (265).

(**373**) **Theorem:** Propositions are Persistent. The following is therefore an immediate consequence of the definitions of $\trianglelefteq$ and *Persistent*:

$$\forall p\, Persistent(p)$$

Thus, our theory implies Alternative 6.1 at Choice 6 in Barwise 1989 (265). The theory also resolves Choice 11 (Barwise 1989, 268) in favor of Alternative 11.1: no relations are perspectival; if a relation R appears in a state of affairs that is true in situation s, then the argument places of R remain the same in any situation s' of which s is a part.

(**374**) **Definitions:** Null and Trivial Situations. We define: (.1) x is a *null situation* if and only if x is a situation and no propositions are true in x, and (.2) x is a trivial situation iff x is a situation and every proposition is true in x:

(.1) $NullSituation(x) =_{df} Situation(x) \;\&\; \neg\exists p(x \models p)$

(.2) $TrivialSituation(x) =_{df} Situation(x) \;\&\; \forall p(x \models p)$

(**375**) **Theorems:** Existence and Uniqueness of Null and Trivial Situations. It is now easily established that (.1) that there is a unique null situation and (.2) that there is a unique trivial situation:

(.1) $\exists! x NullSituation(x)$

(.2) $\exists! x TrivialSituation(x)$

Consequently, it follows, by a modally strict proof (.3) that the null situation exists, and (.4) that the trivial situation exists:

(.3) $\exists y(y = \imath x NullSituation(x))$

(.4) $\exists y(y = \iota x TrivialSituation(x))$

(**376**) **Definitions:** Notation for The Null Situation and The Trivial Situation. We now introduce:

(.1) $\boldsymbol{s}_{\varnothing} =_{df} \iota x NullSituation(x)$

(.2) $\boldsymbol{s}_{\boldsymbol{V}} =_{df} \iota x TrivialSituation(x)$

In the notation $\boldsymbol{s}_{\varnothing}$ and $\boldsymbol{s}_{\boldsymbol{V}}$, we use a boldface, italic $\boldsymbol{s}$ as part of the name. This boldface, italic $\boldsymbol{s}$ is to be distinguished from the (nonbold, italic) restricted variable s. The reason for this should be clear: expressions such as $s_{\varnothing}$ and s_V (with nonbold italic s) should be used only when we are introducing functional terms having denotations the vary with the value of s. By contrast, we are here introducing new names for distinguished objects.

(**377**) **Theorems:** Facts about $\boldsymbol{s}_{\varnothing}$ and $\boldsymbol{s}_{\boldsymbol{V}}$. It is now to be established that (.1) if x is a null situation x is necessarily so, and (.2) if x is a trivial situation, x is necessarily so:

(.1) $NullSituation(x) \rightarrow \Box NullSituation(x)$

(.2) $TrivialSituation(x) \rightarrow \Box TrivialSituation(x)$

From these, we may produce modally strict proofs of the following:

(.3) $NullSituation(\boldsymbol{s}_{\varnothing})$

(.4) $TrivialSituation(\boldsymbol{s}_{\boldsymbol{V}})$

(**378**) **Theorems:** Further Facts about Null and Trivial Situations. Recall that we defined the null object $\boldsymbol{a}_{\varnothing}$ in (193.1) as $\iota x Null(x)$, where $Null(x)$ was defined in (191.1) as $A!x \,\&\, \neg\exists F xF$. Recall that we also defined the universal object $\boldsymbol{a}_{\boldsymbol{V}}$ in (193.2) as $\iota x Universal(x)$, where $Universal(x)$ was defined in (191.2) as $A!x \& \forall F xF$. It then follows that: (.1) $NullSituation(x)$ and $Null(x)$ are equivalent conditions; (.2) the null situation is identical to the null object; and (.3) the trivial situation is not identical to the universal object:

(.1) $NullSituation(x) \equiv Null(x)$

(.2) $\boldsymbol{s}_{\varnothing} = \boldsymbol{a}_{\varnothing}$

(.3) $\boldsymbol{s}_{\boldsymbol{V}} \neq \boldsymbol{a}_{\boldsymbol{V}}$

The proof of (.3) is especially interesting, since to show that the trivial situation and the universal object fail to be identical, we must find a property that one encodes that the other doesn't. Since the universal object encodes every property whatsoever and the trivial situation encodes all and only propositional properties, we know the former encodes every property the latter encodes. So one has to show there is a property that the universal object encodes that the trivial situation doesn't. It suffices to prove the existence of a property that is not a propositional property, the proof of (.3) is interesting because it does just that.

(**379**) **Metadefinition:** Conditions on Propositional Properties. In what follows, let φ be a formula in which x doesn't occur free. Then we shall say that φ is a *condition on propositional properties* if and only if there is a proof that necessarily, every property such that φ is a propositional property, i.e.,

φ is *condition on propositional properties* if and only if
$\vdash \Box\forall F(\varphi \to \mathit{Propositional}(F))$

By requiring a proof that *necessarily*, every property such that φ is propositional, we avoid spurious counterexamples.[177]

(**380**) **Theorems:** An Important Equivalence. Whenever φ is a condition on propositional properties, then x is a situation that encodes all and only the properties F such that φ if, and only if, x is an abstract object that encodes all and only the properties F such that φ:

$(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi)) \equiv (A!x \,\&\, \forall F(xF \equiv \varphi))$,
provided φ is a condition on propositional properties in which x doesn't occur free.

This theorem makes it easier to derive comprehension conditions for situations.

(**381**) **Theorems:** Comprehension Conditions for Situations. Where φ is a condition on propositional properties in which x doesn't occur free, there exists a (unique) situation x that encodes all and only those properties F such that φ:

(.1) $\exists x(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi))$, provided φ is a condition on propositional properties in which x doesn't occur free

(.2) $\exists! x(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi))$, provided φ is a condition on propositional properties in which x doesn't occur free

[177] For example, let ψ be the condition $F = F \,\&\, \neg Pa$, where Pa is some contingently true fact taken as an axiom. Then one can prove $\psi \to \mathit{Propositional}(F)$: in the case described ψ is false because its second conjunct is false, and so $\psi \to \mathit{Propositional}(F)$, by failure of the antecedent. So, by GEN, $\forall F(\psi \to \mathit{Propositional}(F))$. But ψ doesn't yet qualify as a condition on propositional properties: what we've established thus far appeals to the contingently true fact Pa and so fails to be modally strict. Hence, we can't apply RN to obtain a proof of $\Box\forall F(\psi \to \mathit{Propositional}(F))$. This squares with our intuition that ψ shouldn't count as a condition on propositional properties: at worlds where Pa is indeed false, every ordinary, non-propositional property is such that ψ.

Thus, whereas the classic works in situation theory (e.g., Barwise and Perry 1983, 7–8; Barwise 1989a, 261) assume the existence of situations, object theory yields, as theorems, existence principles that comprehend the domain of situations.

Notice also that with these theorems, we have developed a precise *theory* of situations, since (.1) and (368), respectively, constitute fully general situation comprehension and identity principles.

(**382**) **Theorems:** Canonical Situation Descriptions. It now follows that when φ is a condition on propositional properties, there exists something which is *the* situation that encodes exactly the properties F such that φ:

(.1) $\exists y(y = \imath x(Situation(x) \& \forall F(xF \equiv \varphi)))$, provided φ is a condition on propositional properties in which x doesn't occur free

Moreover, it is straightforward to show:

(.2) $\imath x(Situation(x) \& \forall F(xF \equiv \varphi)) = \imath x(A!x \& \forall F(xF \equiv \varphi))$, provided φ is a condition on propositional properties in which x doesn't occur free

Hence, when φ is a condition on propositional properties, we may say that descriptions of the form $\imath x(Situation(x) \& \forall F(xF \equiv \varphi))$ and $\imath s \forall F(sF \equiv \varphi)$ are *canonical situation descriptions*. By abuse of language, the individuals these canonical descriptions denote are *canonical situations* (cf. (181)).

(**383**) **Theorems:** Facts About Canonical Situations. Since we've defined what it is for φ to be a condition on propositional properties (379) and what it is for φ to be a rigid condition on properties (188.1), we may combine the two to talk about formulas φ that are rigid conditions on propositional properties. Consequently, whenever φ is a rigid condition on propositional properties, then it is a modally strict fact that if something is identical to a canonical situation, then it encodes exactly the properties such that φ:

$y = \imath s \forall F(sF \equiv \varphi) \rightarrow \forall F(yF \equiv \varphi)$, provided φ is a rigid condition on propositional properties in which x isn't free.

(**384**) **Metadefinitions:** Strict Canonicity and Situations. Given the preceding results, we say that $\imath s \forall F(sF \equiv \varphi)$ is a *strictly canonical situation description* whenever φ is a rigid condition on propositional properties. In the usual way, we sometimes abuse language to say that $\imath s \forall F(sF \equiv \varphi)$ is a *strictly canonical situation*.

Exercises:

1. Show that p° (214), $\top$ (219.1), and $\bot$ (219.2) are (identical to) canonical situations.

2. Consider the following situation description:

$$\imath s \forall F(sF \equiv F = [\lambda y\, Rab] \vee F = [\lambda y\, \neg Rba])$$

(a) Show that this is a canonical situation description; i.e., show that the condition $F = [\lambda y\, Rab] \vee F = [\lambda y\, \neg Rba]$ is a condition on propositional properties. (b) Show that this is a strictly canonical situation description.

3. Show that p°, $\top$, and $\bot$ are not (identical to) strictly canonical situations.

(**385**) **Exercise:** A Bounded Lattice of Situations. Let us define binary *join* ($\vee$) and *meet* ($\wedge$) operations on situations as follows: $s' \vee s''$ is the situation s that encodes a property F just in case F is encoded by s' or encoded by s'', while $s' \wedge s''$ is the situation that encodes a property F just in case F is encoded by s' and encoded by s''. Note that in the following formal representations, the symbol $\vee$, which has heretofore been used as a binary, formula-forming operation symbol for disjunction, is being *reused* for the *join* operation on situations; the context should always make it clear which way the symbol is intended, since when used as a join operation, $\vee$ connects two *individual* terms as opposed to two formulas:

$$s' \vee s'' =_{df} \imath s \forall F(sF \equiv s'F \vee s''F))$$

$$s' \wedge s'' =_{df} \imath s \forall F(sF \equiv s'F \,\&\, s''F))$$

These new terms have the form discussed in (382), and so to confirm that $s' \vee s''$ and $s' \wedge s''$ are (identical to) canonical situations, one has to show:

- if $\varphi = s'F \vee s''F$, then $\vdash \Box\forall F(\varphi \to \mathit{Propositional}(F))$
- if $\varphi = s'F \,\&\, s''F$, then $\vdash \Box\forall F(\varphi \to \mathit{Propositional}(F))$

We leave this as an exercise. Furthermore, to confirm that $s' \vee s''$ and $s' \wedge s''$ are (identical to) strictly canonical situations, one has show, respectively, that:[178]

- if $\varphi = sF \vee s'F$, then $\vdash \Box\forall F(\varphi \to \Box\varphi)$
- if $\varphi = sF \,\&\, s'F$, then $\vdash \Box\forall F(\varphi \to \Box\varphi)$

We shall leave it as an exercise for the reader to prove that the following laws for join and meet, which characterize *lattices* algebraically, are derivable as modally strict theorems governing situations:

[178] In the following cases, it suffices to show $\varphi \to \Box\varphi$, by GEN and RN. In the first case, assume φ, i.e., $s'F \vee s''F$. Reasoning by cases: (a) If $s'F$, then $\Box s'F$, by (37). So by $\vee$I, $\Box s'F \vee \Box s''F$. By (117.7), $\Box(s'F \vee s''F)$. Hence $\Box\varphi$. (b) If $s''F$, then by analogous reasoning, $\Box\varphi$.

In the second case, assume φ, i.e., $s'F \,\&\, s''F$. Then by &E, both $s'F$ and $s''F$. Thus, both $\Box s'F$ and $\Box s''F$, given (37). Hence, by &I and (111.3), $\Box(s'F \,\&\, s''F)$, i.e., $\Box\varphi$.

$$s \vee s = s$$
$$s \wedge s = s \qquad \text{(Laws of Idempotence)}$$

$$s \vee s' = s' \vee s$$
$$s \wedge s' = s' \wedge s \qquad \text{(Laws of Commutativity)}$$

$$s \vee (s' \vee s'') = (s \vee s') \vee s''$$
$$s \wedge (s' \wedge s'') = (s \wedge s') \wedge s'' \qquad \text{(Laws of Associativity)}$$

$$s \vee (s \wedge s') = s$$
$$s \wedge (s \vee s') = s \qquad \text{(Laws of Absorption)}$$

We leave it as a further exercise to show that the situations form a *bounded* lattice, i.e., that $\boldsymbol{s}_{\varnothing}$ and $\boldsymbol{s}_{\boldsymbol{V}}$ serve as the identity elements for $\vee$ and $\wedge$, respectively, i.e., that:

$$s \vee \boldsymbol{s}_{\varnothing} = s$$

$$s \wedge \boldsymbol{s}_{\boldsymbol{V}} = s$$

These, too, are derivable as modally strict theorems.

We leave the proofs of the above as exercises because later we will encounter analogous but even more general results. The study of (Leibnizian) concepts in Chapter 13 will establish that there are operations $\oplus$ and $\otimes$ that structure the (Leibnizian) concepts (indeed, the domain of abstract objects generally) as a bounded lattice and, indeed, a Boolean algebra.

(**386**) **Definition:** Actual Situations. We say that a situation *s is actual* iff every proposition true in s is true:

$$Actual(s) \ =_{df} \ \forall p(s \models p \rightarrow p)$$

It is important to remember that this particular condition on situations is *not* defined in terms of the actuality operator $\mathcal{A}$.

(**387**) **Theorem:** Some Actual Situations Are Possibly Not Actual. It is a consequence of our definition of *Actual* that some actual situations might fail to be actual:

$$\exists s(Actual(s) \,\&\, \Diamond \neg Actual(s))$$

Of course, the proof of this theorem makes use of a witness situation in which a contingent truth is true. But if s is a situation that is actual because every proposition true in s is a necessary truth, then s couldn't fail to be an actual sitation.

(**388**) **Remark:** Actual vs. Actual* Situations. We just defined an actual situation s to be one such that every proposition true in s is true. By contrast, let us say that a situation s is *actual** just in case every proposition true in s is actually true:

$$Actual^*(s) =_{df} \forall p(s \models p \rightarrow \mathcal{A}p)$$

Whereas we've just seen that it is provable that there are actual situations that might not be actual (387), it is now provable that every *actual** situation is necessarily *actual**:

$$Actual^*(s) \rightarrow \Box Actual^*(s)$$

The proof is left to a footnote.[179] When we deploy the notion of an actual situation in what follows, it is important to keep in mind how it differs from the notion of an actual* situation.

(**389**) **Theorems:** Actual and Nonactual Situations. It is an immediate consequence that there are both actual and nonactual situations:

(.1) $\exists s Actual(s)$

(.2) $\exists s \neg Actual(s)$

The theory therefore decides Choice 4 in Barwise 1989 (262) in favor of Alternative 4.2. Note, moreover, that some propositions are not true in any actual situation:

(.3) $\exists p \forall s(Actual(s) \rightarrow \neg s \models p)$

(**390**) **Lemma:** Embedding Situations in Another Situation. Where s' and s'' are any situations, there exists a situation of which both are a part:

$$\exists s(s' \trianglelefteq s \ \&\ s'' \trianglelefteq s)$$

It is tempting to call any such s an *upper bound* on s' and s'', but as noted earlier, we are going to postpone further discussion of algebraic notions until Chapter 13, when we discuss (Leibnizian) concepts.

(**391**) **Theorems:** Facts about Actual Situations. (.1) If p is true in an actual situation s, then s exemplifies being such that p, and (.2) for any two actual situations, there exists an actual situation of which both are a part:

(.1) $Actual(s) \rightarrow (s \models p \rightarrow [\lambda y\ p]s)$

[179] Assume $Actual^*(s)$, i.e.,

(ϑ) $\forall p(s \models p \rightarrow \mathcal{A}p)$

We have to show $\Box \forall p(s \models p \rightarrow \mathcal{A}p)$. By BF (122.1), it suffices to show $\forall p \Box(s \models p \rightarrow \mathcal{A}p)$. By GEN, it suffices to show $\Box(s \models p \rightarrow \mathcal{A}p)$, i.e., $\Box(\neg s \models p \vee \mathcal{A}p)$. By (117.7), it suffices to show:

(ξ) $\Box \neg s \models p \vee \Box \mathcal{A}p$

But by (ϑ), we know $s \models p \rightarrow \mathcal{A}p$, i.e., $\neg s \models p \vee \mathcal{A}p$. So we can show ($\xi$) by disjunctive syllogism from this last fact. If $\neg s \models p$, then it follows by (367.5), $\Box \neg s \models p$. Hence (ξ), by $\vee$I. If $\mathcal{A}p$, then $\Box \mathcal{A}p$, by (33.1). Hence (ξ), by $\vee$I.

(.2) $(Actual(s') \,\&\, Actual(s'')) \rightarrow \exists s(Actual(s) \,\&\, s' \trianglelefteq s \,\&\, s'' \trianglelefteq s)$

(.2) establishes what Barwise (1989, 235) calls the Compatibility Principle.

(**392**) **Definition:** Consistency. A situation s is *consistent* iff there is no proposition p such that both p is true in s and $\neg p$ is true in s:

$$Consistent(s) \;=_{df}\; \neg\exists p(s \models p \,\&\, s \models \neg p)$$

(**393**) **Remark:** Types of Consistency. We take the preceding definition of consistency to be a standard one.[180] However, some readers might wonder how the above definition differs from the proposal that a situation s is consistent just in case there is no proposition p such that the contradiction $p \,\&\, \neg p$ is true in s:

$$Consistent^{*}(s) \;=_{df}\; \neg\exists p(s \models (p \,\&\, \neg p))$$

It is important to recognize that *Consistent* and *Consistent** are independent conditions; there are situations s such that $Consistent(s)$ and $\neg Consistent^{*}(s)$, and there are situations s' such that $Consistent^{*}(s')$ and $\neg Consistent(s')$.

To see the former, let q_1 be any proposition and let s_1 be the following situation:

$$\iota s\forall F(sF \equiv F = [\lambda y\; q_1 \,\&\, \neg q_1])$$

Clearly, s_1 is (identical to) a canonical situation; it encodes just one property, namely $[\lambda y\; q_1 \,\&\, \neg q_1]$, and so every property it encodes is a propositional property. Moreover, it is (identical to) a strictly canonical situation, as this was defined in (384); if we let φ be the formula $F = [\lambda y\; q_1 \,\&\, \neg q_1]$, then there is a proof that $\Box\forall F(\varphi \rightarrow \Box\varphi)$. So by theorem (383), it follows that s_1 encodes a single property, namely, $[\lambda y\; q_1 \,\&\, \neg q_1]$. Hence, a single proposition, namely $q_1 \,\&\, \neg q_1$, is true in s_1. Now since every proposition is provably distinct from its negation, it follows that $Consistent(s_1)$, i.e., there is no proposition p such that $s_1 \models p$ and $s_1 \models \neg p$. However, clearly, $\neg Consistent^{*}(s_1)$, since there is a proposition p, namely q_1, such that $s_1 \models (p \,\&\, \neg p)$.

Now to see that there are situations s' such that $Consistent^{*}(s')$ and $\neg Consistent(s')$, let q_0 be $\exists x(E!x \,\&\, \Diamond\neg E!x)$ and let s_2 be the following situation:

$$\iota s\forall F(sF \equiv F = [\lambda y\; q_0] \vee F = [\lambda y\; \neg q_0])$$

Again, s_2 is (identical to) a strictly canonical situation (exercise). So, by (383), s_2 encodes just two properties: $[\lambda y q_0]$ and $[\lambda y \neg q_0]$. This, only propositions q_0 and $\neg q_0$ are true in s_2. Furthermore, given axiom (32.4), which asserts $\Diamond q_0 \,\&\, \Diamond\neg q_0$,

180 Our definition is clearly an object-theoretic counterpart of the classical syntactic definition of a consistent set of sentences Γ. For example, in Enderton 1972 (128) (= 2001, 134), we find that a set of sentences Γ is consistent iff there is no formula φ such that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

it is easy to show that there is no proposition p such that $q_0 = (p \& \neg p)$, and that there is no proposition p such that $\neg q_0 = (p \& \neg p)$.[181] Hence, it follows that $Consistent^*(s_2)$, since there is no proposition p such that $s_2 \models (p \& \neg p)$. But $\neg Consistent(s_2)$, since there is a proposition p, namely, q_0, such that both $s_2 \models p$ and $s_2 \models \neg p$.

Thus, *Consistent* and *Consistent** are independent notions. When we come across situations s such that both $Consistent(s)$ and $Consistent^*(s)$, we shall sometimes just note this fact in passing or leave the proof of $Consistent^*(s)$ as an exercise.

(**394**) **Theorem:** Actual Situations are Consistent. If a situation s is actual, then it is consistent:

$$Actual(s) \rightarrow Consistent(s)$$

Thus, we've derived what Barwise (1989, 235) calls the Coherency Principle for actual situations. (We'll introduce a different notion of coherency below.) Clearly, actual situations are also *Consistent**, as defined in Remark (393).

(**395**) **Definition:** Possibility of Situations. We say that a situation s *is possible* iff it is possible that s is actual:

$$Possible(s) =_{df} \Diamond Actual(s)$$

(**396**) **Theorem:** Facts about Possible Situations. Clearly, (.1) if a situation s is actual, then it is possible; and (.2) if p is true in s and it is not possible that p, then s is not possible:

(.1) $Actual(s) \rightarrow Possible(s)$

(.2) $((s \models p) \& \neg\Diamond p) \rightarrow \neg Possible(s)$

(**397**) **Lemmas:** Facts about Consistency. Note also that: (.1) if a situation s fails to be consistent, then necessarily s fails to be consistent; and (.2) if a situation s is possibly consistent, then s is consistent:

(.1) $\neg Consistent(s) \equiv \Box\neg Consistent(s)$

(.2) $\Diamond Consistent(s) \equiv Consistent(s)$

(**398**) **Theorems:** The Consistency and Possibility of Situations. (.1) Possible situations are consistent; and (.2) there are situations that are consistent but not possible:

[181] There is no proposition p, say p_1, such $q_0 = (p_1 \& \neg p_1)$ because $\Diamond q_0$ and $\neg\Diamond(p_1 \& \neg p_1)$. Nor is there a proposition p, say p_2, such that $\neg q_0 = (p_2 \& \neg p_2)$ because $\Diamond\neg q_0$ and $\neg\Diamond(p_2 \& \neg p_2)$.

(.1) $Possible(s) \rightarrow Consistent(s)$

(.2) $\exists s(Consistent(s) \,\&\, \neg Possible(s))$

One can see why (.2) is a theorem by considering situation s_1 discussed in Remark (393), in which a single proposition, namely $q_1 \,\&\, \neg q_1$, is true. s_1 is consistent, for inconsistency requires that at least two propositions be true in s_1, namely, some proposition and its (distinct) negation. But s_1 is not possible; it couldn't be the case that every proposition true in s_1 is true. A fuller proof is in the Appendix.

12.2 Possible Worlds

(**399**) **Remark:** On the Nature of Possible Worlds. It is only relatively recently that philosophers and logicians have started to think seriously and systematically about the nature of possible worlds. Of course, Leibniz mentioned them in both his *Theodicy* (T 128 = G.vi 107) and in the *Monadology* §53 (PW 187 = G.vi 615–616). But one might take a skeptical attitude towards certain aspects of Leibniz's conception of possible worlds. Stalnaker nicely formulates the skeptic's attitude in the following passage (1976, 65):[182]

> According to Leibniz, the universe—the actual world—is one of an infinite number of possible worlds existing in the mind of God. God created the universe by actualizing one of these possible worlds—the best one. It is a striking image, this picture of an infinite swarm of total universes, each by its natural inclination for existence striving for a position that can be occupied by only one, with God, in his infinite wisdom and benevolence, settling the competition by selecting the most worthy candidate. But in these enlightened times, we find it difficult to take this metaphysical myth any more seriously than the other less abstract creation stories told by our primitive ancestors. Even the more recent expurgated versions of the story, leaving out God and the notoriously chauvinistic thesis that our world is better than all the rest, are generally regarded, at best, as fanciful metaphors for a more sober reality.

After Leibniz, the notion of *possible world* languished until Wittgenstein famously characterized the actual world, in the opening lines of his *Tractatus* (1921), with the claims "The world is all that is the case" and "The world is the totality of facts, not of things" (1921, 7, Propositions 1 and 1.1). It is unclear whether Wittgenstein had a view about nonactual possible worlds; subsequent propositions in the *Tractatus* mention *possible states of affairs* but the

[182]One shouldn't conclude on the basis of this passage that Stalnaker is a possible worlds skeptic. He goes on to employ possible worlds in his work on the analysis of counterfactual conditionals.

text doesn't explicitly say whether any are so total and complete as to constitute a possible world (see, e.g., Propositions 2.012, 2.0124, and 2.013 in the *Tractatus*). Interestingly, Carnap (1947, 9) interpreted Wittgenstein's text as if some are.

In what follows, we shall be attempting to prove theorems about the *nature* of possible worlds; by contrast, we shall not be interested in *models* of them. We take it that possible worlds are *none of the following*: (a) *complete and consistent sets of sentences*, of the kind described in Lindenbaum's Lemma (Tarski 1930, 34, fn. ‡), (b) *state descriptions*, as defined by Carnap (1947, 9), (c) *model sets*, as defined by Hintikka (1955), (d) *variable assignments* agreeing on the values of the individual variables, as defined by Kripke (1959, 2–3), or (e) *Tarski models*, as put to use in Montague 1960. Similarly, I take the following works to contain mathematical models, not theories, of possible worlds: (f) Pollock 1967 (317), in which a possible world is identified as any *set* of states of affairs that is maximal and consistent, (g) Quine 1968 (14–16), in which possible worlds are identified with *sets* of quadruples of real numbers representing the coordinates of spacetime points occupied by matter, (h) Cresswell 1972a (6), in which possible worlds are identified as *sets* of basic particular situations, (i) Adams 1974 (225), in which talk of possible worlds is reduced to talk about maximal, consistent *sets* of propositions ('world-stories'), and (j) Menzel 1990 (371ff), in which appropriately structured Tarski models represent possible worlds in virtue of having the modal property *possibly being a representation of the way things are*. The proposals mentioned in (a) – (j) fail to be theories of possible worlds because possible worlds, whatever they are, are not mathematical objects.[183] They are not sets and so not sets of formal sentences, sets of propositions or states of affairs, formal models, model sets, or assignments to variables. Of course, for some purposes, these mathematical objects might serve to *represent* possible worlds, but our interest is in the worlds themselves.

Although the notion of *possible world* gained currency in the 1950s, the philosophers and logicians in that decade weren't especially interested their nature as entities in their own right. Copeland (2006, 381) describes a letter from Carew Meredith to Arthur Prior, dated 10 October 1956, in which Mered-

[183]The physicist Mark Tegmark (2008) asserts that the physical universe is a mathematical structure. Of course, this modern-day Pythagoreanism requires an analysis of what a mathematical structure is, something Tegmark doesn't provide. By contrast, we shall be analyzing mathematical structures as certain abstract objects. See Chapter 15. Hence, even if he is right about what the physical universe is, the present theory offers a further analysis of the mathematical structure in question.

For the present purposes, however, it suffices to note that Tegmark's conception of a physical universe as a mathematical structure doesn't lead to a theory of worlds as mathematical structures. Possible worlds are objects whose identity is, in part, tied to the nonmathematical propositions that are true or false at them; mathematical structures are just not the kind of thing at which nonmathematical propositions can be true or false.

ith uses the term 'possible world' when demonstrating how to falsify a certain formula. Copeland also describes a 1957 lecture handout from Timothy Smiley at the University of Cambridge, which indicates that necessary truths are true in all 'possible worlds', and a proposition is possible if it is true in *some* 'possible world' (2002, 121; 2006, 385; quotes in the original).[184] Moreover, Hintikka (1957, 61–62) suggested that models and model sets:

> ... correspond to the different situations we want to consider in modal logic, and they are interconnected, in the first place, by a rule saying (roughly) that whatever is necessarily true in the actual state of affairs must be (simply) true in all the alternative states of affairs.

The text doesn't make it clear whether "alternative states of affairs" can be partial (i.e., similar to situations) or are always total (and hence alternative possible worlds). Finally, though Kripke (1959) interprets modal logic in terms of a set of alternative assignments to variables, he then says (1959, 2):

> The basis of the informal analysis which motivated these definitions is that a proposition is necessary if and only if it is true in all "possible worlds". (It is not necessary for our present purposes to analyze the notion of a "possible world" any further.)

These invocations of possible worlds played an important role in the development of the semantics of modal logic in the early 1960s (Kripke 1963a, 1963b; Prior 1963). In all of these cases, however, it is fair to say that modal logicians weren't yet interested in the *nature* of the possible worlds, even though they thought that some such notion helps us to understand formal frameworks for interpreting modal language.

The study of the nature of possible worlds began in earnest when Lewis (1968) introduced *axioms* governing worlds. Though Lewis was, in the 1968 paper, primarily interested in formulating his 'counterpart theory' of possibilia, his axioms used variables ranging over worlds and implicitly defined properties he took worlds to have. The goal of precisely specifying the nature of possible worlds took root, however, and theories of them were subsequently developed in a variety of other works: Lewis 1973; Plantinga 1974, 1976; Fine 1977; Chisholm 1981; Zalta 1983; Pollock 1984; Lewis 1986; Armstrong 1989, 1997; and Stalnaker 2012.[185]

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[184] In Copeland 2002, footnote 25, and again in Copeland 2006, footnote 18, we find an acknowledgment to David Shoesmith for providing a copy of the lecture handout. I've not been able to acquire a copy.

[185] Interestingly, Stalnaker's position, until 2012, was that one can assume the existence of possible worlds in semantic analysis without endorsing any particular theory about their nature. In 1984 (Chapter 3), the final paragraph of his 1976 paper is revised and expanded, to include the following (57):

It is worthwhile recalling a famous passage in the work of David Lewis that contains a kind of *credo* and justification for the belief in possible worlds (Lewis 1973, 84):

> I believe that there are possible worlds other than the one we happen to inhabit. If an argument is wanted, it is this. It is uncontroversially true that things might be otherwise than they are. I believe, and so do you, that things could have been different in countless ways. But what does this mean? Ordinary language permits the paraphrase: there are many ways things could have been besides the way they actually are. On the face of it, the sentence is an existential quantification. It says that there exist many entities of a certain description, to whit 'ways things could have been'. I believe things could have been in countless ways; I believe permissible paraphrases of what I believe; taking the paraphrase at its face value, I therefore believe in the existence of entities that might be called 'ways things could have been'. I prefer to call them possible worlds.

Lewis, in a later work, offers a further justification for belief in possible worlds. He writes (1986, 3):

> Why believe in a plurality of worlds? – Because the hypothesis is serviceable, and that is a reason to think that it is true. The familiar analysis of necessity as truth at all possible worlds was only the beginning. In the last two decades, philosophers have offered a great many more analyses that make reference to possible worlds, or to possible individuals that inhabit possible worlds. I find that record most impressive. I think it is clear that talk of *possibilia* has clarified questions in many parts of the philosophy of logic, mind, of language, and of science – not to mention metaphysics

> . . . the moderate realism I want to defend need not take possible worlds to be among the ultimate furniture of the world. Possible worlds are primitive notions of the theory, not because of their ontological status, but because it is useful to theorize at a certain level of abstraction, a level that brings out what is common in a certain range of otherwise diverse activities. The concept of possible worlds that I am defending is not a metaphysical conception, although one application of the notion is to provide a framework for metaphysical theorizing. The concept is a formal or functional notion, like the notion of *individual* presupposed by the semantics for extensional quantification theory. . . .
>
> Similarly, a possible world is not a particular kind of thing or place. The theory leaves the *nature* of possible worlds as open as extensional semantics leaves the nature of individuals. A possible worlds is what truth is relative to, what people distinguish between in their rational activities. To believe in possible worlds is to believe only that those activities have a certain structure, the structure which possible worlds theory helps to bring out.

But in 2012 (8), Stalnaker puts forward the suggestion that possible worlds are properties, i.e., *ways a world might be*. So a possible world is "the kind of thing that is, or can be, instantiated or exemplified" (2012, 8).

> itself. Even those who officially scoff often cannot resist the temptation to help themselves abashedly to this useful way of speaking.

And on the page before, Lewis states what seems to be the most important principle governing worlds, namely, "*every* way that a world could possibly be is a way that some world *is*" (1986, 2).

It is not surprising that Lewis' suggestion, that possible worlds are physically-disconnected concrete entities inhabiting some logical space, was controversial. Van Inwagen 1986 (185–6) contrasts two of the leading conceptions of worlds that emerged:

> Lewis did not content himself with saying that there were entities properly called 'ways things could have been'; . . . He went on to say that what most of us would call 'the universe', the mereological sum of all the furniture of earth and the choir of heaven, is one among others of these 'possible worlds' or 'ways things could have been', and that the others differ from it "not in kind but only in what goes on in them" (Lewis 1973, 85).
>
> Whether or not the existence of a plurality of universes can be so easily established, the thesis that possible worlds are universes is one of the two 'concepts of possible worlds' that I mean to discuss. . . . The other concept I shall discuss is that employed by various philosophers who would probably regard themselves as constituting the Sensible Party: Saul Kripke, Robert Stalnaker, Robert Adams, R.M. Chisholm, John Pollock, and Alvin Plantinga.[3] These philosophers regard possible worlds as abstract objects of some sort: possible histories of *the* world, for example, or perhaps properties, propositions or states of affairs.

Van Inwagen goes on to call Lewis a 'concretist' about worlds while the members of the 'Sensible Party' are called 'abstractionists'. The main difference between these two conceptions is whether worlds are to be defined primarily in terms of a part-whole relation (concretists) or in terms of the propositions true at them (abstractionists).

Given this opposition, it seems that the abstractionist conception is a kind of generalization of Wittgenstein's view of the actual world in the *Tractatus*, mentioned above. But Menzel (2015) notes that there is a third important conception of worlds, namely, the 'combinatorialist' conception, on which possible worlds are taken to be "recombinations, or rearrangements, of certain metaphysical simples," where "both the nature of simples and the nature of recombination vary from theory to theory" (Section 2.3). This view is exhibited in the work of Quine, Cresswell, and Armstrong, *op. cit.*. On Menzel's analysis, Wittgenstein's view is more closely allied with the combinatorialist conception than the abstractionist one.[186]

[186] We also cited above Menzel's interesting paper of 1990, where he suggests a way that philoso-

With these introductory remarks, we may turn to the subtheory of possible worlds that is developed in object theory. I take the theory described below to be unique in that the principles governing worlds are precisely derived as theorems rather than stipulated. Some of the theorems below may already be familiar, having appeared in Zalta 1983 (78–84), 1993, Fitelson & Zalta 2007, and Menzel & Zalta 2014. Even though some of these have been reworked and enhanced in several ways, the basic idea has remained the same: possible worlds are defined to be situations of a certain kind and, hence, abstract individuals.[187] Thus, our conception of worlds can be traced back to Wittgenstein, but with the added insight about the distinction between the internal and external properties of situations discussed in Remark (362). The Tractarian conception "the world is all that is the case" will be validated by the theorem that p is true if and only if p is true at the actual world (426)★, since this theorem implies that the actual world *encodes* all that is the case. The Tractarian conception will be preserved in our definition of a world (400) as a situation s that might be such that all and only true propositions are true in s, since this definition implies that a possible world is a situation that might be such that it *encodes* everything that is the case.

(**400**) **Definition:** Possible Worlds. Recall (a) that an object x encodes a proposition p whenever x is an abstract object and x encodes $[\lambda y\, p]$, and (b) that situations are abstract objects. We may, in terms of these notions, now say that an object x is a *possible world* iff x is a situation which is possibly such that all and only the propositions x encodes are true:

$$PossibleWorld(x) =_{df} Situation(x) \,\&\, \Diamond\forall p(x\Sigma p \equiv p)$$

---

phers might do without possible worlds altogether. He provides a homophonic interpretation of modal language that may suffice for the purposes of analyzing our modal beliefs. Menzel suggests that we use certain Tarski models to represent the possible worlds of Kripke models and then he attributes modal properties to them. On Menzel's proposal, a Tarski model meeting certain conditions *could have represented* the world as it would have been had things been different. One could put the view more simply as follows: each such Tarski model might have been a model of the actual world. So not only does Menzel avoid claiming that Tarski models are possible worlds, strictly speaking he also avoids the claim that the former represent the latter, since on his view, we can simply rest with the primitive modal properties of Tarski models and thereby provide a precise, but homophonic, interpretation of modality.

Linsky and I (1994, 444) raised some concerns about this interesting suggestion. I only add here that if the purpose of doing without possible worlds is to avoid the committment to 'possible but non-actual objects', then the theory of possible worlds developed below satisfies this desideratum, given the Quinean interpretation of the quantifiers of our formalism. For on that interpretation, our possible worlds are existing, actual (abstract) objects and are 'possible' only in a defined sense that is consistent with actualism; see theorem (403) below and the definitions in terms of which it is cast.

[187] Our theory therefore reconciles situation theory and world theory. In the early 1980s, situation theory was thought to be incompatible with world theory. See the early publications of Barwise and Perry, *op. cit.*

Let us continue to use $s, s', s'', \ldots$ as restricted variables for situations. Now recall that we can say that p is *true in s*, written $s \models p$, whenever $s\Sigma p$. Then we may simplify the above definition to:

$$PossibleWorld(s) =_{df} \Diamond \forall p(s \models p \equiv p)$$

That is, a situation s is a possible world just in case it might be such that all and only the propositions true in s are true. Remember here that the definiens, $\Diamond \forall p(s \models p \equiv p)$, is to be parsed $\Diamond \forall p((s \models p) \equiv p)$. Note that our definition decides Choice 3 in Barwise 1989 (261) in favor of Alternative 3.1: worlds are situations.

The question may arise, why haven't we defined possible worlds as maximal and consistent situations? We take up this question in (409), after we formulate the definition of *maximality* and prove, in (408) and (404) below, that possible worlds are maximal and consistent.

(**401**) **Remark:** Restricted Variables for Possible Worlds. Let $w, w', w'', \ldots$ be restricted variables that range over the objects x such that $PossibleWorld(x)$. Since this is a well-defined condition, we may use these variables according to the conventions described in (254) – (255).

Note that our restricted w-variables present us with two interpretive options and we are free to exercise either option when it is convenient to do so. We may regard $w, w', \ldots$ either as singly-restricted variables ranging over objects x such that $PossibleWorld(x)$ or we may regard them as doubly restricted variables that range over the situations s such that $PossibleWorld(s)$. This is convenient for the following reasons:

- We can immediately regard notions defined on situations (using a free restricted variable s) as defined on possible worlds w. For example, since $Consistent(s)$ is defined on situations (392), the claim $Consistent(w)$ doesn't need to be defined.

- We now have *two* ways of eliminating the bound restricted variable w. For example, if we focus just on the variable-binding sentence-forming operators, we may expand the claims:

  (A) $\forall w Consistent(w)$

  (B) $\exists w Actual(w)$

  either in the usual way as:

  (C) $\forall x(PossibleWorld(x) \rightarrow Consistent(x))$

  (D) $\exists x(PossibleWorld(x) \,\&\, Actual(x))$

or as the claims:[188]

(E) $\forall s(PossibleWorld(s) \rightarrow Consistent(s))$

(F) $\exists s(PossibleWorld(s) \,\&\, Actual(s))$

And similarly for the variable-binding definite description operator. (The λ operator can't bind either of the restricted s or w variables since they range over objects satisfying conditions that contain encoding subformulas.)

- By treating (A) and (B) as shorthand, respectively, for (E) and (F) instead of as shorthand for (C) and (D), we can often simplify the reasoning process: the consequences of (E) and (F) obtained by Rules $\forall$E and $\exists$E will involve terms that are known or assumed to refer to situations.

Consequently, in what follows, we sometimes prefer to reason as if our w-variables are doubly restricted. Of course, in some cases, we continue to reason by eliminating the restricted w-variables in the usual way, as singly restricted. We do this on those occasions where we have to appeal to a previous principle (axiom, theorem, or definition) for which it may not be immediately evident whether it applies to restricted variables.

Moreover, in what follows, the reader may see a series of principles (i.e., definitions and theorems) in which the definition introduces some new notion defined on situations using the variable s while the subsequent theorem governs possible worlds (and may thus be stated using the variable w). The reason for this should now be clear: in general, we always choose the variable, restricted or otherwise, that is the most appropriate for the occasion, i.e., the variable that either (a) yields the simplest formulation of the definition and the simplest statement of the theorem (without compromising generality), or (b) minimizes the amount of reasoning that has to be done to prove the theorem (thereby minimizing the chances of introducing reasoning errors and maximizing the clarity of the justification).

(**402**) **Remark:** Truth At A Possible World. We noted above that w-variables may be interpreted as doubly restricted, i.e., as ranging over situations s such that *PossibleWorld*(s). We saw that, as a result, notions defined on situations s apply to possible worlds w without having to be redefined. An important

[188] Of course, (E) and (F) are themselves claims that utilize restricted variables. In turn, they respectively expand to:

$\forall x(Situation(x) \rightarrow (PossibleWorld(x) \rightarrow Consistent(x)))$

$\exists x(Situation(x) \,\&\, PossibleWorld(x) \,\&\, Actual(x))$

Note, however, that we shall not have occasion to regard (A) and (B), respectively, as short for the above, since the clause *Situation*(x) is redundant in both.

example is the notion of *truth in* a situation, i.e., $s \models p$. By interpreting w as a doubly restricted variable, one can see that $w \models p$ is defined. Note, however, that whereas we read $s \models p$ as 'p is true *in s*', we shall, for historical reasons, read $w \models p$ as 'p is true *at w*'. Thus, *truth at* a possible world is simply a special case of the notion *truth in* a situation. Moreover, since worlds are a special type of situation, it follows by definition (365) that $w \models p$ is equivalent to $w\Sigma p$, and since situations are abstract objects, it follows by (216) that both of these, in turn, are equivalent to $w[\lambda y\, p]$. Hence, when a proposition p is true at w, the property $[\lambda y\, p]$ *characterizes* w by way of an encoding predication.

(**403**) **Theorem:** Possible Worlds Are Possible Situations. It is now straightforward to show that possible worlds are possible situations:

$$Possible(w)$$

The reader is encouraged to think twice about this theorem. It is not a triviality; the claim to be established is not a tautology of the form $(\varphi \,\&\, \psi) \rightarrow \varphi$. We're not proving $Possible(s) \& World(s) \rightarrow Possible(s)$. A second look reveals that given the definition of *Possible*, the theorem captures a fundamental intuition about possible worlds, namely, that they are situations that *might be actual*. This claim is so fundamental to our conception of possible worlds that we tend to overlook the suggestion that it is capable of proof. But, indeed, it is provable, though the proof is relatively easy.

(**404**) **Theorem:** Possible Worlds Are Consistent and Non-Trivial. Clearly, (.1) possible worlds are consistent, and (.2) non-trivial situations:

(.1) $Consistent(w)$

(.2) $\neg TrivialSituation(w)$

Exercises: (a) Prove that possible worlds are consistent*, as the latter was defined in Remark (393). (b) Prove that possible worlds are not null situations, as the latter was defined in (374.1).

(**405**) **Theorem:** Rigidity of $PossibleWorld(x)$. Our definitions imply that (.1) x is a possible world if and only if necessarily x is a possible world; (.2) possibly x is a possible world if and only if x is a possible world; and (.3) possibly x is a possible world if and only if necessarily x is a possible world; and (.4) actually x is a possible world if and only if x is a possible world:

(.1) $PossibleWorld(x) \equiv \Box PossibleWorld(x)$

(.2) $\Diamond PossibleWorld(x) \equiv PossibleWorld(x)$

(.3) $\Diamond PossibleWorld(x) \equiv \Box PossibleWorld(x)$

(.4) $\mathcal{A}PossibleWorld(x) \equiv PossibleWorld(x)$

(**406**) **Lemmas:** Rigidity of Truth At a Possible World. It is straightforward to establish that (.1) p is true at w if and only if necessarily p is true at w; (.2) possibly p is true at w if and only if p is true at w; (.3) possibly p is true at w if and only if necessarily p is true at w; and actually p is true at w if and only if p is true at x:

(.1) $w \models p \equiv \Box w \models p$

(.2) $\Diamond w \models p \equiv w \models p$

(.3) $\Diamond w \models p \equiv \Box w \models p$

(.4) $\mathcal{A}w \models p \equiv w \models p$

An observation analogous to the one at the end of (367) is in order: (.1) – (.3) are derivable without the help of (405.1).

(**407**) **Definition:** Maximality. A situation s is *maximal* iff for every proposition p, either p is true in s or $\neg p$ is true in s:

$$Maximal(s) \;=_{df}\; \forall p(s \models p \vee s \models \neg p)$$

(**408**) **Theorem:** Possible Worlds Are Maximal.

$$Maximal(w)$$

We've therefore now established that possible worlds are maximal and consistent.

(**409**) **Remark:** Possible Worlds as MaxCon Situations. It would serve well to remark upon the difference between defining possible worlds as in (400) and defining them as *maxcon* situations, i.e., as situations that are maximal, consistent, and consistent*. The simple answer is that the definitions aren't equivalent; whereas situations satisfying (400) are maxcon, it doesn't follow that situations that are maxcon satisfy (400). For one can consistently assert the existence of a maxcon situation in which it is true that Socrates might have been a chunk of rock or might have had different parents. But if certain essentialist principles are true and it is a metaphysical fact that Socrates couldn't have been a chunk of rock and couldn't have had different parents, then a maxcon situation in which Socrates is a chunk of rock or had different parents is one that fails to be a possible world in the sense of (400). In other words, there may be modal facts that place constraints on what maxcon situations are possible worlds. (400) defines *possible world* in a way that respects those modal truths, should there be any.

Moreover, (400) defines *possible world* in a way that respects the observations of the philosophers discussed at the outset of this section—possible worlds are, in some sense, ways things might be, where the modality 'might' is metaphysical possibility. This is not captured by saying that that possible worlds are maximal and free of contradictions. Finally, the body of theorems derivable from (400) and its supporting definitions provide evidence of its power and correctness. This evidence will continue to accumulate as we go.

(**410**) **Theorem:** Maximality, Possible Worlds, and Possibility.

(.1) $Maximal(s) \rightarrow \Box Maximal(s)$

(.2) $PossibleWorld(s) \equiv (Maximal(s) \,\&\, Possible(s))$

(**411**) **Definition:** Coherence. We say that a situation s is *coherent* just in case for all propositions p, $\neg p$ is true in s if and only if p fails to be true in s:

$$Coherent(s) \ =_{df} \ \forall p(s \models \neg p \ \equiv \ \neg s \models p)$$

(**412**) **Theorem:** Fact About Coherence. A situation s is coherent if and only if, for all propositions p, p is true at s iff $\neg p$ fails to be true at s:

$$Coherent(s) \equiv \forall p(s \models p \equiv \neg s \models \neg p)$$

This theorem shows that there is an equivalent alternative definition of coherency.

Note that neither direction of the proof of this theorem proceeds by instantiating $\neg q$ for the quantified variable. The reason is that neither $s \models q$ nor $s \models \neg\neg q$ follows from the other. To see this, note that $s \models q$ is defined as $s[\lambda y\, q]$, and $s \models \neg\neg q$ is defined as $s[\lambda y\, \neg\neg q]$. But since our theory doesn't imply that $[\lambda y \neg\neg q] = [\lambda y\, q]$, we cannot infer $s[\lambda y\, q]$ from $s[\lambda y \neg\neg q]$, or vice versa. It is true that $[\lambda y\, q]$ and $[\lambda y\, \neg\neg q]$ are logically equivalent in the sense that $\Box\forall x([\lambda y\, q]x \equiv [\lambda y\, \neg\neg q]x)$. But situations aren't closed under necessary implication: from the facts that sF and $\Box\forall x(Fx \equiv Gx)$, it doesn't follow that sG.

However, as we shall see, this kind of reasoning does apply to possible worlds; $w \models q$ (i.e., $w[\lambda y\, q]$) does follow from $w \models \neg\neg q$ (i.e., $w[\lambda y\, \neg\neg q]$), and vice versa. The reason is that worlds are provably closed under necessary implication and since $[\lambda y\, q]$ and $[\lambda y\, \neg\neg q]$ necessarily imply one another, $w \models q$ and $w \models \neg\neg q$ are interderivable. See items (417) and (419) below.

(**413**) **Theorem:** The Equivalence of Coherence and Maximal Consistency. Indeed, there is another, equivalent way to define coherency given the next fact: a situation s is coherent if and only if it is both maximal and consistent:

$$Coherent(s) \equiv (Maximal(s) \,\&\, Consistent(s))$$

(**414**) **Theorem:** Possible Worlds Are Coherent.

$$Coherent(w)$$

(**415**) **Definition:** Necessary Implication and Necessary Equivalence. We say: p *necessarily implies* q, written $p \Rightarrow q$, just in case necessarily, if p then q:

(.1) $p \Rightarrow q \ =_{df} \ \Box(p \rightarrow q)$

We may also define: p is *necessarily equivalent* to q, written $p \Leftrightarrow q$, just in case necessarily, p if and only if q:

(.2) $p \Leftrightarrow q \ =_{df} \ \Box(p \equiv q)$

(**416**) *Theorem:* Necessary Equivalence and Necessary Implication. It is therefore a simple consequence of the previous definition that p is necessarily equivalent to q if and only if both p necessarily implies q and q necessarily implies p:

$$p \Leftrightarrow q \equiv ((p \Rightarrow q) \,\&\, (q \Rightarrow p))$$

(**417**) **Definition:** Modal Closure, i.e., Closure Under Necessary Implication. For the present definition and the theorems immediately following, let us use $p_1, \ldots, p_n$ (which we've heretofore taken to be constants) as *variables* ranging over propositions. We say that a situation s is *modally closed* or *closed under necessary implication* iff whenever a proposition q is necessarily implied by the conjunction $p_1 \,\&\, \ldots \,\&\, p_n$ each conjunct of which is true in s, then q is also true in s:

$$\begin{aligned} &ModallyClosed(s) \ =_{df} \\ &\quad [s \models p_1 \,\&\, \ldots \,\&\, s \models p_n \,\&\, ((p_1 \,\&\, \ldots \,\&\, p_n) \Rightarrow q)] \ \rightarrow \ s \models q \end{aligned}$$

Strictly speaking, it is the notion of *truth in a situation* that is modally closed when the definiens obtains, but it does no harm if we speak as if it is the situation that is closed in that case.

(**418**) **Lemmas:** Facts about Situations. It is helpful to establish two preliminary lemmas in preparation for introducing a notion of modal closure. It is a theorem that: (.1) if $p_1, \ldots, p_n$ are all true in situation s and the conjunction of $p_1, \ldots, p_n$ implies q, then if all and only true propositions are true in s, then q is true in s:

(.1) $[s \models p_1 \,\&\, \ldots \,\&\, s \models p_n \,\&\, ((p_1 \,\&\, \ldots \,\&\, p_n) \rightarrow q)] \rightarrow (\forall p(s \models p \equiv p) \rightarrow s \models q)$

Since this is a modally strict theorem, Rule RN applies. Hence the necessitation of (.1) is a theorem. So by the K axiom and other principles of modality, we can establish: (.2) if necessarily p_1 is true in situation s and … and necessarily p_n is true in s and $p_1 \,\&\, \ldots \,\&\, p_n$ necessarily implies q, then necessarily, q is true in s if all and only truths are true in s:

(.2) $[\Box s \models p_1 \mathbin{\&} \ldots \mathbin{\&} \Box s \models p_n \mathbin{\&} ((p_1 \mathbin{\&} \ldots \mathbin{\&} p_n) \Rightarrow q)] \rightarrow$
$\Box(\forall p(s \models p \equiv p) \rightarrow s \models q)$

(**419**) **Theorem:** Possible Worlds are Modally Closed. Given (418), it is easy to prove that possible worlds are modally closed:

$$ModallyClosed(w)$$

(**420**) **Theorem:** World-Relative Conjunctions and Conditionals Behave Classically. The preceding theorem helps us to more simply establish that (.1) $p \mathbin{\&} q$ is true at a possible world w if and only if both p is true at w and q is true at w; (.2) if p is true at a possible world, and the conditional $p \rightarrow q$ is true at that possible world, then q is true at that possible world; (.3) $p \equiv q$ is true at a possible world w if and only if: p is true at w iff q is true at w:

(.1) $w \models (p \mathbin{\&} q) \equiv (w \models p) \mathbin{\&} (w \models q)$

(.2) $w \models p \mathbin{\&} w \models (p \rightarrow q) \rightarrow w \models q$

(.3) $w \models (p \equiv q) \equiv ((w \models p) \equiv (w \models q))$

Thus, we may reason classically with respect to conjunctions and conditionals within a possible world. **Exercise:** Use (.1) to show that possible worlds are *consistent* iff and they are *consistent**.

(**421**) **Theorem:** There is a Unique Actual World.

$$\exists! w Actual(w)$$

This is a modally strict theorem. Note that since this theorem implies, *a fortiori*, that there are possible worlds, we know that the restricted quantifiers $\forall w$ and $\exists w$ behave classically in the sense that $\forall w \varphi \rightarrow \exists w \varphi$; cf. Remark (256).

After a short digression concerning the significance of the present theorem, we'll work our ways towards a definition of a new term that refers to the unique object that is an actual world.

(**422**) **Remark:** On the Uniqueness of the Actual World. On the present theory, the claim that there is a unique actual world is provable, necessary, and known *a priori*. By contrast, on the concretist conception of *possible world*, the claim that there is a unique actual world appears to be a contingent fact, known only *a posteriori* and hence, not susceptible to proof. It may be that nothing like the previous theorem is available to one adopting the concretist conception of possible world.

Moreover, as far as I can tell, philosophers adopting a conventional abstractionist theory of possible worlds can't offer a proof of the existence of a unique actual world without requiring that propositions or properties be identical when necessarily equivalent, i.e., without requiring that $\Box(p \equiv q) \rightarrow p = q$ or that $\Box\forall x(Fx \equiv Gx) \rightarrow F = G$. For if one admits, for any given proposition (or property), a distinct but necessarily equivalent proposition (or property), then when possible worlds are identified as maximal and possible propositions, states of affairs, or properties, one can prove the existence of multiple actual worlds, at least given the following standard definition: p is a possible world iff (i) $\forall q((p \Rightarrow q) \vee (p \Rightarrow \neg q))$ and (ii) $\Diamond p$.[189] Now suppose (a) p_1 is such a possible world, and (b) p_1 is actual, either in the sense that p_1 is true or in the sense that every proposition necessarily implied by p_1 is true. Then consider any proposition p_2 that is necessarily equivalent to p_1 (in the sense that $p_1 \Leftrightarrow p_2$) but *distinct* from p_1. Then it follows that p_1 and p_2 necessarily imply the same propositions.[190] So since p_1 satisfies clause (i), so does p_2 (exercise). Moreover, if p_1 is actual in the sense of being true, so is p_2, and if p_1 is actual in the sense that every proposition necessarily implied by p_1 is true, then so is p_2. So, there are two distinct actual worlds on this theory. The abstractionist can either derive the existence of a unique actual world by collapsing necessarily equivalent propositions or both allow for distinct but necessarily equivalent propositions and give up the claim that there is a unique actual world. Such a dilemma is not faced on the present theory. For a fuller discussion, see Zalta 1988 (73–4), 1993 (393–94), and McNamara 1993.

(**423**) **Theorem:** A Logically Proper Description. It follows immediately from (421) that there exists something which is the object that is an actual world:

$$\exists y(y = \imath w Actual(w))$$

The proof of this theorem is modally strict because it rests on (179.2), not on (100)$\star$.

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[189]Clause (i) is a maximality condition and clause (ii) guarantees metaphysical possibility. Of course, philosophers who would prefer to analyze possible worlds as properties of some sort would have to define things differently and would have to make that that theory precise (cf. Stalnaker 2012).

Moreover, since possible worlds aren't sets, I've put aside the suggestion that the existence of a unique actual world is provable on the grounds that one can prove the existence of a unique set of propositions that is both maximal and possible. This is not an alternative theory of possible worlds but rather a model of them, for reasons mentioned previously.

[190]The proof is a variant of left-to-right direction of (340.3). Assume $p_1 \Leftrightarrow p_2$. Then by (416), we know both:

(ϑ) $\Box(p_1 \rightarrow p_2)$

(ξ) $\Box(p_2 \rightarrow p_1)$

To show $\forall q(p_1 \Rightarrow q \equiv p_2 \Rightarrow q)$, it suffices by GEN and the definition of $\equiv$ to show $p_1 \Rightarrow q \equiv p_2 \Rightarrow q$. ($\rightarrow$) Suppose $p_1 \Rightarrow q$, i.e., $\Box(p_1 \rightarrow q)$. From (ξ) and this last result, it follows that $\Box(p_2 \rightarrow q)$, i.e., $p_2 \Rightarrow q$. ($\leftarrow$) Suppose $p_2 \Rightarrow q$, i.e., $\Box(p_2 \rightarrow q)$. then from (ϑ) and this last result, it follows that $\Box(p_1 \rightarrow q)$, i.e., $p_1 \Rightarrow q$.

(**424**) **Term Definition:** The Actual World. Since we know that $\imath wActual(w)$ is logically proper, we can use it to define the simple term $\boldsymbol{w}_{\alpha}$:

$$\boldsymbol{w}_{\alpha} =_{df} \imath wActual(w)$$

Note that the symbol $\boldsymbol{w}_{\alpha}$ is a boldface, italic $\boldsymbol{w}$ decorated by a Greek α to highlight the special character of the object denoted as the first among possible worlds. We do not use a restricted possible worlds variable w indexed by a Greek α, for that would be a complex variable whose value would depend on the value of w.

(**425**) ⋆**Theorems:** The True and The Actual World. Recall that we defined The True ($\top$) in (219.1) as the abstract object that encodes all and only those properties F of the form $[\lambda y\, p]$ for some true proposition p. It may come as a surprise that The True is identical with the actual world:

(.1) $\top = \boldsymbol{w}_{\alpha}$

We have therefore derived a fact about The True and the actual world suggested by Dummett in the following passage (1981, 180):

> If we take seriously Frege's manner of speaking in 'Uber Sinn und Bedeutung', the True must contain within itself the referents of the parts of all true sentences, and will admit a decomposition corresponding to each true sentence. It thus becomes, in effect, an immensely complex structure, as it were the single all-inclusive Fact, which is how Kluge conceives of it, making it virtually indistinguishable from the world.

Moreover, from (.1) and theorems (227.1)⋆ and (230.3) it follows that (.2) a proposition p is true if and only if the actual world $\boldsymbol{w}_{\alpha}$ is the extension of p:

(.2) $p \equiv \boldsymbol{w}_{\alpha} = \imath xExtensionOf(x,p)$

This is an important and unheralded truth that relates true propositions, the actual world, and extensions.

(**426**) ⋆**Theorem:** Truth and Truth At the Actual World. A proposition p is true iff p is true at the actual world:

$$p \equiv \boldsymbol{w}_{\alpha} \models p$$

The proof shows that an earlier fact about The True (227.3)⋆, namely $p \equiv \top\Sigma p$, can be transformed, to take on new significance.

(**427**) **Theorem:** The Actual World Is a Possible World and Maximal. While it is trivial to prove that $\boldsymbol{w}_{\alpha}$ is both a possible world and maximal from (101.2)⋆ and (408), the claims in fact have modally strict proofs:

(.1) $PossibleWorld(\boldsymbol{w}_\alpha)$

(.2) $Maximal(\boldsymbol{w}_\alpha)$

Note, that also it is easy to show $Actual(\boldsymbol{w}_\alpha)$ by *non*-modally strict means, this isn't a modally strict theorem. However, we'll soon see that there is a modally strict theorem that expresses a sense in which $\boldsymbol{w}_\alpha$ is actual.

(**428**) **Theorem:** Actually True and Truth at the Actual World. The previous theorem allows us to prove a modally strict counterpart of (426)★, namely, that it is actually the case that p if and only if p is true at the actual world:

$$\mathcal{A}p \equiv \boldsymbol{w}_\alpha \models p$$

Note that given the notion of actual* defined in Remark (388), the right-to-left direction of the the present theorem implies, by GEN, $Actual^*(\boldsymbol{w}_\alpha)$ as a modally strict theorem.

(**429**) ⋆**Theorem:** Possible Worlds Not $\boldsymbol{w}_\alpha$ Aren't Actual. It follows immediately that any possible world that fails to be $\boldsymbol{w}_\alpha$ fails to be an actual world:

$$w \neq \boldsymbol{w}_\alpha \rightarrow \neg Actual(w)$$

(**430**) ⋆**Theorem:** Parts of the Actual World. Actual situations are part of the actual world:

$$Actual(s) \equiv s \trianglelefteq \boldsymbol{w}_\alpha$$

Note that in Barwise 1989 (261), actual situations are *defined* to be the ones that are part of the actual world. For us, this is a theorem.

(**431**) ⋆**Theorems:** Facts about the Actual World. (.1) A proposition p is true at the actual world iff the actual world exemplifes being such that p. Moreover, (.2) a proposition p is true iff the proposition *that the actual world exemplifies being such that* p is true at the actual world:

(.1) $\boldsymbol{w}_\alpha \models p \equiv [\lambda y\, p]\boldsymbol{w}_\alpha$

(.2) $p \equiv \boldsymbol{w}_\alpha \models [\lambda y\, p]\boldsymbol{w}_\alpha$

(.2) is especially interesting because when p is true, we may derive from (.2) a statement of the form $\sigma \models \varphi^*(\sigma)$, where $\varphi^*(\sigma)$ is a propositional formula in which the situation term σ occurs. In situation theory, statements of this form indicate that situation σ is *nonwellfounded*, since σ *makes true* a fact about itself. Note that when p is true, then (.2) implies that $[\lambda y\, p]\boldsymbol{w}_\alpha$ is true at $\boldsymbol{w}_\alpha$. And when $\neg p$ is true, then (.2) implies that $[\lambda y\, \neg p]\boldsymbol{w}_\alpha$ is true at $\boldsymbol{w}_\alpha$. So, in either case, $\boldsymbol{w}_\alpha$ is nonwellfounded in the above sense: a fact of the form $\varphi^*(\boldsymbol{w}_\alpha)$ is true at $\boldsymbol{w}_\alpha$. Thus, (.1) and (.2) decide Choices 8 (266) and 10 (268) in Barwise

1989: situations can be constituents of facts and at least some situations are nonwellfounded.[191]

(**432**) **Lemmas:** Some Basic Facts about Modality, Situations, Possible Worlds, and Truth At. (.1) If it is possible that p, then there might be a possible world at which p is true; (.2) if there might be a possible world at which p is true, then there is a possible world at which p is true; (.3) if p is true, then for all situations s, if all and only true propositions are true in s then p is true in s; (.4) if p is necessarily true, then it is necessary that for all situations s, if all and only true propositions are true in s, p is true in s; (.5) if necessarily every situation is such that φ, then every situation is necessarily such that φ; (.6) if p is true at every possible world w, then this claim is necessary; and (.7) if the claim that p is true at every possible world is necessary, then p is necessary:

(.1) $\Diamond p \rightarrow \Diamond\exists w(w \models p)$

(.2) $\Diamond\exists w(w \models p) \rightarrow \exists w(w \models p)$

(.3) $p \rightarrow \forall s(\forall q(s \models q \equiv q) \rightarrow s \models p)$

(.4) $\Box p \rightarrow \Box\forall s(\forall q(s \models q \equiv q) \rightarrow s \models p)$

(.5) $\Box\forall s\varphi \rightarrow \forall s\Box\varphi$

(.6) $\forall w(w \models p) \rightarrow \Box\forall w(w \models p)$

(.7) $\Box\forall w(w \models p) \rightarrow \Box p$

These lemmas are used to simplify the proofs of the fundamental theorems of possible world theory, to which we now turn. Note that (.5) establishes that the Converse Barcan Formula (122.2) holds when restricted to situations.

(**433**) **Theorems:** Fundamental Theorems of Possible World Theory. The foremost principles of possible world theory are that (.1) it is possible that p iff there is a possible world at which p is true; (.2) it is necessary that p iff p is true at all possible worlds; (.3) it is not possible that p iff there is no possible world at which p is true; and (.4) it is not necessary that p iff there is a possible world at which p fails to be true:

(.1) $\Diamond p \equiv \exists w(w \models p)$

(.2) $\Box p \equiv \forall w(w \models p)$

(.3) $\neg\Diamond p \equiv \neg\exists w(w \models p)$

[191] Of course, $\boldsymbol{w}_\alpha$ doesn't make any encoding facts *true at* $\boldsymbol{w}_\alpha$, since there can't be encoding subformulas in the propositional formula φ^* used in constructions of the form $w \models \varphi^*$. So $\boldsymbol{w}_\alpha$ isn't nonwellfounded in *this* sense.

(.4) $\neg\Box p \equiv \exists w \neg(w \models p)$

In the Appendix, there is a proof of (.1) that makes use of (432.1) and (432.2). In addition, there are two proofs of (.2), one that makes use of (432.4) – (432.7) and a simpler one that makes use of (.1). The proofs of (.3) and (.4) are left as simple exercises.

Note that (.1) is a strengthened version of the claim that every way that a world could possibly be is a way that some world is (Lewis 1986, 2, 71, 86). When we analyze *ways that a world might be* as propositions that might be the case, then Lewis's principle is the left-to-right direction of (.1). For a fuller discussion, see Menzel & Zalta 2014.

(**434**) **Remark:** Reconciling Lewis Worlds with Possible Worlds. There may be a way to reconcile the above theory of possible worlds with Lewis's conception of worlds. Note that the present theory of possible worlds doesn't imply anything about which ordinary objects are concrete at which worlds. One might well be tempted to define:

> x is a *physical universe at* w if and only if (a) x is concrete at w, (b) every object y that is concrete at w is a part of x, and (c) x exemplifies exactly the propositional properties $[\lambda y\, p]$ such that p is true at w.

That is, given a part-whole relation $\sqsubseteq$ for concrete objects, one could define:[192]

$$\begin{aligned}&PhysicalUniverseAt(x,w) =_{df}\\ &\quad w \models E!x \;\&\; \forall y(w \models E!y \to y \sqsubseteq x) \;\&\; \forall p([\lambda y\, p]x \equiv w \models p)\end{aligned}$$

Given this definition and the assumption that $\sqsubseteq$ obeys a standard mereological principle, namely, that for any two objects x, y that are concrete at w there is an object z that is concrete at w of which x and y are a part, one might reasonably assert that for each possble world w at which there are concrete objects, there is a universe at w, i.e.,

$$\forall w(w \models \exists x E!x \to \exists y PhysicalUniverseAt(y,w))$$

Furthermore if $\sqsubseteq$ obeys the principle that for each w there is at most one concrete object at w for which every concrete object at w is a part, then it would

[192] It is important to understand why the third conjunct is included in the definition of *PhysicalUniverseAt*(x,w). Without this clause, a physical universe at w would be defined solely in terms of the objects that are concrete at w. Consider the case in which there are two possible worlds w_1 and w_2 such that (i) exactly the same objects are concrete at w_1 and w_2 and (ii) different propositions are true at w_1 and w_2. Thus, as we know, clause (ii) implies $w_1 \neq w_2$. But without the third clause in the definition, any physical universe at w_1 would be identified with any physical universe at w_2, since they have the same concrete parts. Of course, Lewis doesn't worry about such a result, since individuals are all world bound; his view doesn't allow for different possible worlds with exactly the same individuals. But it would be a problem for the above reconstruction of Lewis worlds without the third clause in the definition.

follow that for each possble world w at which there are concrete objects, there is a unique universe at w, i.e.,

$$\forall w(w \models \exists x E!x \rightarrow \exists! y PhysicalUniverseAt(y, w))$$

Indeed, from these claims, one should be able to prove that if there is a physical universe for w, there is a unique physical universe for w, i.e.,

$$\exists x PhysicalUniverseAt(x, w) \rightarrow \exists! x PhysicalUniverseAt(x, w)$$

Now if one understands Lewis's notion of possible worlds in terms of universes, then principles akin to those that govern his theory of possible worlds would govern universes. For it would follow that there is a universe for the actual world (if one assumes the existence of concrete objects), and that universes at other possible worlds are no different in kind from the universe of the actual world – all universes are ordinary objects. This captures Lewis's view that there are universes just like our own and that they are mereological sums of concrete objects. (Though in contrast to Lewis's view, the universes of nonactual possible worlds are, from our point of view, only possibly concrete, not concrete.) But the above may go a long way towards reconciling the Lewis conception of worlds with the abstractionist one.

(**435**) **Theorems:**. Existence of Non-Actual Possible Worlds. The Fundamental Theorems imply: (.1) if some proposition is contingently true, then there exists a nonactual possible world; (.2) if some proposition is contingently false, then there exists a nonactual possible world:

(.1) $\exists p(ContingentlyTrue(p)) \rightarrow \exists w(\neg Actual(w))$

(.2) $\exists p(ContingentlyFalse(p)) \rightarrow \exists w(\neg Actual(w))$

So the existence of nonactual possible worlds depends upon there being a contingent truth or a contingent falsehood.

Since we know by (150.1) that q_0, i.e., $\exists x(E!x \& \Diamond \neg E!x)$, is either a contingent truth or a contingent falsehood, one could reason by disjunctive syllogism to establish that (.3) there are nonactual possible worlds, and hence, that (.4) there are at least two possible worlds:

(.3) $\exists w \neg Actual(w)$

(.4) $\exists w \exists w'(w \neq w')$

But (.3) can also be established more directly, either from the pair of theorems (150.3) and (.1) above, or from the pair (150.4) and (.2) above. Both pairs immediately imply that there are non-actual possible worlds.

(**436**) **Remark:** Nonactual Possible Worlds. Our results show not only that our theory implies a claim stronger than Alternative 1.2 ("There may be more than one world") in Barwise 1989 (260), but also that those philosophers who have claimed that there are nonactual possible worlds are provably correct. Note, though, that even a Humean could accept the above theory of possible worlds as long as she rejects axiom (32.4). Without axiom (32.4), we can't reach the conclusion that there are nonactual possible worlds. Nevertheless, the Fundamental Theorems (433.1) – (433.4) remain derivable and true even if the actual world is the only possible world (Menzel & Zalta 2014). But given the value of (32.4), I would suggest that anyone who rejects it must already be in the grip of some other philosophical theory.[193] If so, theory comparison is in order.

(**437**) **Remark:** Iterated Modalities. It is worth spending some time showing that we can derive the correct possible worlds analysis of *iterated* modalities within the language of object theory itself. Let us represent the fact that *Obama might have had a son who might have become president* as:

$$(\varphi)\quad \Diamond\exists x(Sxo \,\&\, \Diamond Px)$$

By the $\Diamond$ form of the Barcan formula (122.3), this is equivalent to: there is an object x such that, possibly, both x is Obama's son and possibly x is president, i.e.,

$$\exists x\Diamond(Sxo \,\&\, \Diamond Px)$$

Suppose b is such an object, so that we know:

$$\Diamond(Sbo \,\&\, \Diamond Pb)$$

This is equivalent, by Fundamental Theorem (433.1), to:

$$\exists w(w \models (Sbo \,\&\, \Diamond Pb))$$

Suppose w_1 is such a world, so that we know:

$$w_1 \models (Sbo \,\&\, \Diamond Pb)$$

By (420.1), we know that this last fact implies and, indeed, is equivalent to:

$$(\vartheta)\quad (w_1 \models Sbo) \,\&\, (w_1 \models \Diamond Pb)$$

[193]Later, in the theory of natural numbers derived in Chapter 14, we assert an even stronger axiom, namely, that for any property G, if there is a natural number n such that n is the number of Gs, then there might have been a concrete object distinct from all the objects that are actually G. This material mode claim helps us to prove that every natural number has a successor, and it is justified by the fact that it represents a formal mode claim that philosophers and logicians already accept, namely, that "the domain (of individuals) might be of any size".

Now the question arises, how do we analyze the modal operator in the right conjunct of (ϑ) in terms of possible worlds? We may not immediately appeal to an instance of Fundamental Theorem (433.1) to substitute $\exists w(w \models Pb)$ for $\Diamond Pb$ in the right conjunct of (ϑ).[194] Instead, we next infer $\exists w(w \models \Diamond Pb)$ from the second conjunct of (ϑ). Then, by Fundamental Theorem (433.1), it follows that $\Diamond\Diamond Pb$, which by (119.9) is equivalent to $\Diamond Pb$. So, again, by Fundamental Theorem (433.1), we know that $\exists w'(w' \models Pb)$. So let us put together this last fact with the first conjunct of (ϑ), to obtain:

$$(w_1 \models Sbo) \,\&\, \exists w'(w' \models Pb)$$

From this it follows that:

$$\exists x((w_1 \models Sxo) \,\&\, \exists w'(w' \models Px))$$

And by a second application of $\exists$I (and the usual applications of $\exists$E to discharge our assumptions about w_1 and b), we obtain:

(ψ) $\exists w \exists x((w \models Sxo) \,\&\, \exists w'(w' \models Px))$

In other words, for some possible world w, there is an object x such that (a) the proposition Sxo is true at w and (b) the proposition that Px is true at some possible world w'. With our modally strict derivation of (ψ) from (φ), we have derived the possible-worlds truth conditions for the modal claim that $\Diamond\exists x(Sxo \,\&\, \Diamond Px)$.

Of course, before it is fair to call (ψ) *truth conditions* for (φ), we have to show that (ψ) is equivalent to (φ). It remains, therefore, to derive (φ) from (ψ). Here is a sketch of a modally strict proof:

(1)	$\exists w \exists x((w \models Sxo) \,\&\, \exists w'(w' \models Px))$	(ψ)
(2)	$(w_2 \models Sco) \,\&\, \exists w'(w' \models Pc)$	Premise for $\exists$E, w_2 and c arbitrary
(3)	$(w_2 \models Sco) \,\&\, \Diamond Pc$	From (2), by Fund. Thm. (433.1)
(4)	$\Box\Diamond Pc$	From 2nd conjunct of (3), by (32.3)
(5)	$\forall w(w \models \Diamond Pc)$	From (4), by (433.2)
(6)	$w_2 \models \Diamond Pc$	From (5), by $\forall$E
(7)	$w_2 \models (Sco \,\&\, \Diamond Pc)$	From 1st conjunct (2), (6), by (420.1)
(8)	$\exists w(w \models (Sco \,\&\, \Diamond Pc))$	From (7), by $\exists$I
(9)	$\Diamond(Sco \,\&\, \Diamond Pc)$	From (8), by (433.1)
(10)	$\exists x\Diamond(Sxo \,\&\, \Diamond Px)$	From (9), by $\exists$I
(11)	(φ)	From (10), by CBF$\Diamond$ (122.4)

[194]There are several reasons why we cannot, in this case, use the Rule of Substitution to substitute $\exists w(w \models Pb)$ for $\Diamond Pb$ on the basis of the fact that $\vdash_\Box \Diamond Pb \equiv \exists w(w \models Pb)$. First, $\Diamond Pb$ is not a subformula of $w_1 \models \Diamond Pb$; if we expand the latter by its definition to $w_1[\lambda y\, \Diamond Pb]$, then it becomes clear why $\Diamond Pb$ is not one of its subformulas. Second, if one were to make the proposed substitution, the resulting formula, $w_1 \models \exists w(w \models Pb)$, would not be well-formed. $w_1 \models \exists w(w \models Pb)$ expands to $w_1[\lambda y\, \exists w(w \models Pb)]$. This latter formula isn't well-formed because the relation term $[\lambda y\, \exists w(w \models Pb)]$ fails to be well-formed; its matrix $\exists w(w \models Pb)$ expands to $\exists w(w[\lambda y\, Pb])$, which has an encoding subformula.

This example therefore shows us how to use the Fundamental Theorems to derive the philosophically correct possible-world truth conditions of propositions with iterated modalities within the language of object theory.[195] It is not too far afield to suggest that in addition to the Fundamental Theorem, the keys to the equivalence of (φ) and (ψ) are, for the left-to-right direction, the Barcan Formula and $\Diamond\Diamond\varphi \to \Diamond\varphi$, and for the right-to-left direction, $\Diamond\varphi \to \Box\Diamond\varphi$ and the Converse Barcan Formula.

(**438**) **Theorem:** A Useful Equivalence Concerning Worlds and Objects. The following will prove to be a useful consequence of our theory of possible worlds when we investigate Leibniz's modal metaphysics: a proposition p is true at a world w iff the proposition *that x exemplifies being such that p* is true at w:

$$w \models p \equiv w \models [\lambda y\, p]x$$

This theorem plays a role in the development of the theory of Leibnizian concepts in Chapter 13.

(**439**) **Theorem:** Possible Worlds and *Ex Contradictione Quodlibet*. We now prove a few theorems about possible worlds that will serve as a contrast for our work in Section 12.4 on impossible worlds. Note that it follows immediately from (404) by GEN that every possible world is consistent. By applying definitions and quantification theory to this universal claim, it can be transformed into the equivalent claim that there is no possible world w and proposition p such that p and $\neg p$ are both true at w. It also follows that there is no possible world w and proposition p such that $p \,\&\, \neg p$ is true at w:

---

[195]The example discussed in this Remark was chosen because it has a form that is relevantly similar to an example that has figured prominently in the literature. In McMichael 1983 (54), we find:

> Consider the sentence:
>
> (5) It is possible that there be a person X who does not exist in the actual world, and who performs some action Y, but who might not have performed Y.
>
> This sentence is surely true. For example, John F. Kennedy could (logically) have had a second son who becomes a Senator, although he might have chosen to become an astronaut instead.

Clearly, we can simplify McMichael's example to:

> John F. Kennedy could have had a second son who becomes a Senator but might not have.

Then where Sxy represents x is a son of y, S_2x represents x is a senator, and k represents Kennedy, we could represent this example as:

$$\Diamond\exists x(Sxk \,\&\, S_2x \,\&\, \Diamond\neg S_2x)$$

The example discussed in our Remark further simplifies McMichael's example in two ways: (1) we have eliminated the conjunct S_2x, and (2) we've made the embedded modal claim into a positive statement instead of a negative one. Thus, it should be clear that our simplifications of the example have not changed its essential features. If one represents McMichael's original example without the simplifications, the proof that the modal claim and its possible-worlds truth conditions are equivalent goes basically the same way, though the details do become more complex.

(.1) $\neg\exists w \exists p(w \models (p \,\&\, \neg p))$

If we recall Remark (393), where we discussed the notion of *Consistency** and showed that it is independent of the notion of *Consistency*, then the above theorem can easily be transformed into the claim: every world is *consistent**.

This theorem has interesting consequences related to the traditional logical law *ex contradictione quodlibet*, which is almost always formulated in the formal mode as: any formula ψ is derivable from a contradiction of the form $\varphi \,\&\, \neg\varphi$, i.e., $\varphi \,\&\, \neg\varphi \vdash \psi$. This formal principle clearly governs our system.[196] However, when we formulate *ex contradictione quodlibet* in the material mode so that it applies to propositions, it becomes the easily-established theorem $(p\&\neg p) \Rightarrow q$. Hence by the fact that possible worlds are modally closed and the definition of modal closure, it follows that (.2) if $w \models (p \,\&\, \neg p)$, then $w \models q$:

(.2) $w \models (p \,\&\, \neg p) \to w \models q$

(.2) is easily proved by the failure of the antecedent, as established by (.1).

(**440**) **Theorem:** Truth at a Possible World and Disjunctions. (.1) A disjunction is true at a possible world if and only if at least one of the disjuncts is true at that possible world, and (.2) if $p \vee q$ is true at w and $\neg p$ is true at w, then q is true at w:

(.1) $w \models (p \vee q) \equiv (w \models p \vee w \models q)$

(.2) $(w \models (p \vee q) \,\&\, (w \models \neg p)) \to w \models q$

Thus, (.1) the laws of disjunction and (.2) disjunctive syllogism all hold with respect to truth at a possible world.

(**441**) **Remark:** Final Observations on the Theory of Possible Worlds. The foregoing theory of possible worlds requires none of Leibniz's theological doctrines, such as his claims about what goes on in God's mind or his theodicy to explain the existence of evil. Indeed, I think the opening paragraph of Stalnaker 1976 (65), which we quoted at the beginning of this section, should be re-evaluated in light of the foregoing analysis. However, Leibniz's structural vision about the space of worlds is no mere metaphor. It has been reconstructed as a scientifically and mathematically precise theory.

As we've seen, worlds are objects that can be abstracted from ordinary predication and possibility, and it doesn't matter whether we take the locus of predication and possibility to be the physical world, our minds, or language. One need only accept that there is in fact a corpus of ordinary predications and possibilities, that within this corpus there are special patterns of propositions, and that possible worlds are nothing more than those patterns.

[196]From the premise $\varphi \,\&\, \neg\varphi$, we may infer both φ and $\neg\varphi$, by &E. From φ, we may infer $\varphi \vee \psi$, for any ψ, by $\vee$I. But from $\varphi \vee \psi$ and $\neg\varphi$ it follows that ψ, by disjunctive syllogism (64.4.b).

As abstractions, possible worlds have an intrinsic nature, defined by their encoded properties. As noted previously, the propositions that are true at world w characterize w. That is, if $w \models p$, then w *is* such that p, in the sense that w encodes $[\lambda y\, p]$. Possible worlds, as we've defined them, don't model or represent anything, despite being abstract; they are not *ersatz* worlds (Lewis 1986, 136ff). Possible worlds just *are* abstract logical objects characterized by the propositions true at them.[197]

Finally, note that once we extend our system by adding particular contingent truths,[198] we do not require any special, further evidence for believing in the existence of each nonactual possible world implied by (433.1). Epistemologically, we don't have to justify our knowledge of each possible world or provide some information pathway from each world back to us that explains and justifies our belief in its existence. Instead, given contingently true propositions as data, we can cite (433.1) as the principle that guarantees the existence of the relevant nonactual worlds and thereby justify our belief in them. In turn, the justification of (433.1) rests on the fact that it is derivable from a very general theory with both inferential and explanatory power. Since the theorems help to demonstrate the inferential and explanatory power of the theory, the justification also goes in the other direction: the theory itself receives justification from the fact that the above principles governing possible worlds are derivable from it.

12.3 World-Relativized Truth-Values and Extensions

12.3.1 World-Relativized Truth-Values

(442) **Definition:** Truth-Value of p At w. We may now revise and enhance definition (211) by relativizing it to possible worlds. We define: x is a *truth-value of p at* possible world w if and only if x is an abstract object that encodes all

[197] Lewis (1986, Chapter 3) gives no credence to the notion of a possible world that Wittgenstein employs in such claims as "The world is all that is the case" and "The world is the totality of facts, not of things" (1921, 7, Propositions 1 and 1.1). Lewis doesn't count this as a genuine notion of a possible world. But as soon as one allows for possible worlds that are defined by the propositions true at them, then then it is hard to understand why they should be thought *representations* of something and thereby *ersatz*. The possible worlds defined in the present text are *not* ersatz, given the facts just noted in the text. To repeat, encoding is a mode of predication, and so when $w \models p$ and *being such that p* is thereby predicated of w, the property in question *characterizes* w. So w *is* such that p, and doesn't represent something that is such that p.

[198] I am referring here to such claims such as: Obama doesn't have a son but might have, Obama has two daughters but might not have, there aren't million carat diamonds but there might have been, etc.

and only those properties that are propositional properties constructed from propositions q that are materially equivalent to p at w:

$$TruthValueOfAt(x,p,w) =_{df} A!x \,\&\, \forall F(xF \equiv \exists q(w \models (q \equiv p) \,\&\, F = [\lambda y\, q]))$$

Although a truth-value of p at w is a situation (exercise), we use our standard unrestricted variables $x, y, \ldots$ instead of $s, s', \ldots$ so as to match the presentation of unrelativized truth-values of propositions in (211).

(**443**) **Theorems:** Unique Existence of Truth-Values of Propositions at Worlds. It now follows that: (.1) there is a unique truth-value of p at w, and (.2) the truth-value of p at w exists:

(.1) $\exists! x TruthValueOfAt(x,p,w)$

(.2) $\exists y(y = \imath x TruthValueOfAt(x,p,w))$

Recall that since these theorems involve the free restricted variable w, they are implicitly conditionals. Though since we've proved the existence of possible worlds, we can derive unconditional existence claims.

(**444**) **Restricted Term Definition:** Notation for The Truth-Value of p at w. Given (443.2) and our conventions for definitions, we may introduce notation for the truth-value of p at w as follows:

$$p^{\circ}_{w} =_{df} \imath x TruthValueOfAt(x,p,w)$$

This introduces p°_{w} as a binary functional term free variable p and free restricted variable w. Hence, an expression of the form p°_{κ} may be regarded as well-formed and logically proper only when κ is known to be a possible world, either by proof or by hypothesis.

(**445**) **Theorems:** Strict Canonicity of p°_{w}. Clearly, by definitions (444) and (442), we know that p°_{w} is (identical to) a canonical object, namely:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \exists q(w \models (q \equiv p) \,\&\, F = [\lambda y\, q])))$$

Now if we let φ be $\exists q(w \models (q \equiv p) \,\&\, F = [\lambda y\, q])$, then it follows that φ is a rigid condition on properties, as this was defined in (188.1):

(.1) $\Box \forall F(\varphi \to \Box \varphi)$

So we know that p°_{w} is strictly canonical, by (188.2), and is subject to theorem (189.2). So it is easy to establish, as a modally strict theorem, that (.2) p°_{w} is an abstract object that encodes exactly the properties F constructed out of propositions that are equivalent to p at w, and (.3) p°_{w} is a truth-value of p at w:

(.2) $A!p^{\circ}_{w} \,\&\, \forall F(p^{\circ}_{w}F \equiv \exists q(w \models (q \equiv p) \,\&\, F = [\lambda y\, q]))$

(.3) $\mathit{TruthValueOfAt}(p^{\circ}_{w}, p, w)$

Finally, it follows relatively quickly from the second conjunct of (.2) that (.4) the truth-value of p at w encodes p:

(.4) $p^{\circ}_{w}\Sigma p$

(**446**) **Remark:** Digression on Restricted Terms and (Strict) Canonicity. p°_{w} is the first strictly canonical term that we've officially introduced involving a restricted variable. We did introduce such terms once before, but only in the context of an Exercise — in Remark (385), we introduced $s' \vee s''$ and $s' \wedge s''$ as the join and meet operations on situations s' and s''. And we suggested, as exercises, that one should show that $s' \vee s''$ and $s' \wedge s''$ are both canonical situations and strictly canonical situations. To show $s' \vee s''$ is a canonical situation, for example, one has to show that $\Box\forall F((s'F \vee s''F) \to \mathit{Propositional}(F))$, i.e., that $s'F \vee s''F$ is a condition on propositional properties. And to show that $s' \vee s''$ is strictly canonical, one has to show $\Box\forall F((s'F \vee s''F) \to \Box(s'F \vee s''F))$, i.e., that $s'F \vee s''F$ is a rigid condition on properties.

The presence of such restricted variables in canonical and strictly canonical terms gives rise to some questions, namely, what affect, if any, do these variables have on the proofs that such terms are canonical and strictly canonical? Let's first focus on the example of p°_{w}, and then later consider analogous questions for the cases of $s' \vee s''$ and $s' \wedge s''$.

Consider, then, (445.1), which helps to establish that p°_{w} is strictly canonical. The restricted variable w appears in the formula φ in question, i.e., $\exists q(w \models (q \equiv p)\ \&\ F = [\lambda y\, q])$. One might wonder here: how does the presence of the variable w affect the reasoning that shows that φ is a rigid condition? Does the reasoning require theorem (405.1), i.e., $\mathit{PossibleWorld}(x) \to \Box\mathit{PossibleWorld}(x)$?

To answer these questions, one could simply inspect the proof. But the proof in the Appendix avails itself of the restricted variable and so takes advantage of the shortcuts afforded by its use. Instead, to the answer the questions, we should first remember that theorem (445.1) is really a conditional and to see this, we first have to expand φ and then eliminate the restricted variable. When we expand φ in (445.1), the theorem reads:

$$\Box\forall F(\exists q(w \models (q \equiv p)\ \&\ F = [\lambda y\, q]) \to \Box\exists q(w \models (q \equiv p)\ \&\ F = [\lambda y\, q]))$$

Now if we eliminate the restricted variable w, the theorem becomes:

$$\begin{aligned}&\mathit{PossibleWorld}(x) \to \\ &\quad \Box\forall F(\exists q(x \models (q \equiv p)\ \&\ F = [\lambda y\, q]) \to \Box\exists q(x \models (q \equiv p)\ \&\ F = [\lambda y\, q]))\end{aligned}$$

So the real work in the proof of (445.1) is to show that φ is a rigid condition relative to any given possible world x. The key to the proof is that both conjuncts of the claim $x \models (q \equiv p)$ & $F = [\lambda y\, q]$ are necessary truths if true. In particular, when x is a possible world, then:

$$x \models (q \equiv p) \rightarrow \Box x \models (q \equiv p),$$

by the rigidity of truth at a possible world (406.1). The fact that x is necessarily a possible world whenever it is a possible world (405.1) doesn't play a role. So the proof of (445.1) in the Appendix happily uses the restricted variable w without taking any special steps. The presence of the restricted variable neither affects the proof of $\Box\forall F(\varphi \rightarrow \Box\varphi)$ nor in any way undermines the resulting fact that φ is a rigid condition on properties. This example shows that one can introduce strictly canonical terms with restricted variables, such as p°_{w}, without any special precautions.

By analogous considerations, the proof that $s' \vee s''$ and $s' \wedge s''$ are canonical situations and strictly canonical, requires no appeal to (366.1), i.e., no appeal to the fact that $Situation(x) \rightarrow \Box Situation(x)$. As noted above, to show that $s' \vee s''$ is strictly canonical, one has to show that the formula $s'F \vee s''F$ is a rigid condition on properties, i.e., show $\Box\forall F((s'F \vee s''F) \rightarrow \Box(s'F \vee s''F))$. The reasoning used in footnote 178 doesn't appeal to (366.1). Even if we eliminate the restricted variable s' and s'', to show instead that:

$$Situation(x)\, \&\, Situation(y) \rightarrow \Box\forall F((xF \vee yF) \rightarrow \Box(xF \vee yF))$$

a revised proof based on the proof in footnote 178 wouldn't appeal to (366.1) either.

(**447**) **Restricted Term Definitions:** The-True-at-w and The-False-at-w. We define: (.1) The True-at-w ($\top_w$) to be the abstract object that encodes exactly the properties F of the form $[\lambda y\, q]$ constructed from some proposition q true at w; (.2) The False-at-w ($\bot_w$) to be the abstract object that encodes exactly the properties F of the form $[\lambda y\, q]$ constructed from some proposition q false at w:

(.1) $\top_w =_{df} \imath x(A!x\, \&\, \forall F(xF \equiv \exists q(w \models q\, \&\, F = [\lambda y\, q])))$

(.2) $\bot_w =_{df} \imath x(A!x\, \&\, \forall F(xF \equiv \exists q(w \models \neg q\, \&\, F = [\lambda y\, q])))$

Given the free restricted variable w in these definitions, we may regard $\top_\kappa$ and $\bot_\kappa$ as well-formed and logically proper only when κ is known to be a possible world, by hypothesis or proof.

(**448**) **Theorems:** Strict Canonicity of The-True-at-w and The-False-at-w. By inspection, $\top_w$ and $\bot_w$ are (identical to) canonical individuals. We now show that both of them are (identical to) strictly canonical individuals. When φ is $\exists q(w \models q\, \&\, F = [\lambda y\, q])$, then (.1) φ is a rigid condition on properties, as this was defined in (188.1):

(.1) $\Box\forall F(\varphi \rightarrow \Box\varphi)$

And when ψ is $\exists q(w \models \neg q \ \& \ F = [\lambda y\, q])$, it also follows that (.2) ψ is a rigid condition on properties:

(.2) $\Box\forall F(\psi \rightarrow \Box\psi)$

So by (188.2), $\top_w$ and $\bot_w$ are (identical to) strictly canonical individuals. Hence, they are subject to theorem (189.2) and so, by a modally strict proof, can be instantiated into their own defining descriptions:

(.3) $A!\top_w \,\&\, \forall F(\top_w F \equiv \exists q(w \models q \,\&\, F = [\lambda y\, q]))$

(.4) $A!\bot_w \,\&\, \forall F(\bot_w F \equiv \exists q(w \models \neg q \,\&\, F = [\lambda y\, q]))$

(**449**) **Theorems:** Facts about The-True-at-w and The-False-at-w. Recall definition (216), in which we stipulated that an object x encodes a proposition p ($x\Sigma p$) just in case x is an abstract object that encodes $[\lambda y\, p]$. Furthermore, recall definition (365), in which we stipulated that a proposition p is true in x ($x \models p$) just in case x is a situation and $x\Sigma p$. And we noted in (402) that truth at a world is a special case of truth in a situation. Consequently it follows from the definitions in (447) that (.1) The-True-at-w encodes a proposition p if and only if p is true at w; (.2) The-False-at-w encodes a proposition p if and only if $\neg p$ is true at w; and (.3) The-True-at-w just is w:

(.1) $\top_w \Sigma p \equiv w \models p$

(.2) $\bot_w \Sigma p \equiv w \models \neg p$

(.3) $\top_w = w$

Note that (.3) is not only more general than the fact that $\top = \boldsymbol{w}_\alpha$ (425.1)$\bigstar$ but is, by contrast, a modally strict theorem. We cannot use (.3), however, to produce a modally strict proof of $\top = \boldsymbol{w}_\alpha$. The following argument is not modally strict:

> By instantiating (.3) to $\boldsymbol{w}_\alpha$, we may conclude $\top_{\boldsymbol{w}_\alpha} = \boldsymbol{w}_\alpha$. But then, one can independently establish $\top_{\boldsymbol{w}_\alpha} = \top$ (exercise). Hence, by substitution of identicals, $\top = \boldsymbol{w}_\alpha$.

Any proof of the middle step (the one left as an exercise) would have to appeal to necessitation-averse principles and $\bigstar$-theorems to draw any conclusions about what properties $\top$ encodes. That's because $\top$ is not strictly canonical, and so only theorem (182)$\bigstar$, not (189.2), can be used to determine the properties it encodes. Nevertheless, the above argument is a perfectly-good non-modally strict derivation of (425.1)$\bigstar$.

(**450**) **Theorems:** Truth at w, The Truth-Value of p at w, and The True at w. (.1) p is true at w if and only if the truth-value of p at w is The-True-at-w; and (.2) p is false at w if and only if the truth-value of p at w is The-False-at-w.

(.1) $w \models p \equiv p^{\circ}_{w} = \top_{w}$

(.2) $w \models \neg p \equiv p^{\circ}_{w} = \bot_{w}$

(**451**) **Theorems:** Relativized Truth-Values and Modalities. It now follows that: (.1) p is necessary if and only if for every world w, the truth-value of p at w is (identical to) The-True-at-w; (.2) p is necessarily false if and only if for every world w, the truth-value of p at w is The-False-at-w; (.3) p is possible if and only if for some world w, the truth-value of p at w is The-True-at-w; and (.4) p is possibly false if and only if for some world w, the truth-value of p at w is The-False-at-w:

(.1) $\Box p \equiv \forall w(p^{\circ}_{w} = \top_{w})$

(.2) $\Box \neg p \equiv \forall w(p^{\circ}_{w} = \bot_{w})$

(.3) $\Diamond p \equiv \exists w(p^{\circ}_{w} = \top_{w})$

(.4) $\Diamond \neg p \equiv \exists w(p^{\circ}_{w} = \bot_{w})$

12.3.2 World-Relativized Extensions

(**452**) **Definitions:** An Extension of a Property at a World. We may now relativize definition (234) as follows: x is an extension of G at w if and only if x encodes just those properties F materially equivalent to G at w:

$$ExtensionOfAt(x, G, w) =_{df} A!x \,\&\, \forall F(xF \equiv w \models \forall y(Fy \equiv Gy))$$

(**453**) **Theorem:** The Extension of a Property at a World Exists. It now follows that (.1) there is a unique extension of G at w, and hence, that (.2) there is something which is *the* extension of G at w:

(.1) $\exists! x ExtensionOfAt(x, G, w)$

(.2) $\exists y(y = \imath x ExtensionOfAt(x, G, w))$

Since both G and w are free variables in these theorems, it follows By GEN, that they hold for all properties and worlds.

(**454**) **Restricted Term Definition:** Notation for The Extension of G at w. We may therefore introduce notation for the extension of G at w as follows:

$$\epsilon G_{w} =_{df} \imath x ExtensionOfAt(x, G, w)$$

Thus, we may regard expressions of the form ϵG_κ as binary functional terms that are well-formed and logically proper only when κ is known to be a possible world.

(**455**) **Theorems:** Strict Canonicity of ϵG_w. Clearly, by definitions (452) and (454), we know that ϵG_w is (identical to) a canonical object, namely:

$$\iota x(A!x \,\&\, \forall F(xF \equiv w \models \forall y(Fy \equiv Gy)))$$

If we now let φ be $w \models \forall y(Fy \equiv Gy)$, then it follows that φ is a rigid condition on properties, as this was defined in (188.1):

(.1) $\forall F(\varphi \rightarrow \Box\varphi)$

So we know that ϵG_w is strictly canonical, by (188.2). Hence, it is subject to theorem (189.2). It is therefore easy to establish, by modally strict means, that (.2) ϵG_w is an abstract object that encodes exactly those properties F that are materially equivalent to G at w, and (.3) ϵG_w is an extension of G at w:

(.2) $A!\epsilon G_w \,\&\, \forall F(\epsilon G_w F \equiv w \models \forall y(Fy \equiv Gy))$

(.3) $ExtensionOfAt(\epsilon G_w, G, w)$

Finally, it follows relatively quickly from the second conjunct of (.2) that (.4) the extension of G at w encodes G:

(.4) $\epsilon G_w G$

This is an encoding claim in which the individual term is the restricted functional term ϵG_w.

(**456**) **Theorem:** World-Relativized Pre-Law V and World-Relativized Law V. It now follows that (.1) if x is the extension of G at w and y is the extension of H at w, then $x = y$ if and only if it is true at w that G and H are materially equivalent; and (.2) the extension of F at w is identical to the extension of G at w if and only if it is true at w that F and G are materially equivalent:

(.1) $(ExtensionOfAt(x, G, w) \,\&\, ExtensionOfAt(y, H, w)) \rightarrow$
$(x = y \equiv w \models \forall z(Gz \equiv Hz))$

(.2) $\epsilon F_w = \epsilon G_w \equiv w \models \forall z(Fz \equiv Gz)$

So though Frege's Basic Law V (243)★ is not modally-strict, its world-relativized version is.

(**457**) **Remark:** Suggestions for Further Research. Define: x is a *class of* Gs *at* w if and only if x is an extension of G at w. Then define world-relative membership: y is an *element of* x *at* w iff x is a class of Gs at w and the proposition Gy is true at w:

$$y \in_w x =_{df} \exists G(ClassOfAt(x, G, w) \,\&\, w \models Gy)$$

Though we shall not develop these ideas further, the reader should be able to formulate and prove a host of interesting claims, as well as define further notions (such as *the* class of Gs at w) and prove facts about them.

12.4 Impossible Worlds

(**458**) **Remark:** On Impossible Worlds. From the 1960s through the 1990s, we start to find, in the literature, discussions of 'non-normal worlds' (Kripke 1965, Cresswell 1967, Rantala 1982, Priest 1992, and Priest & Sylvan 1992), 'non-classical worlds' (Cresswell 1972b), 'non-standard worlds' (Rescher & Brandom 1980, Paśniczek 1994) and 'impossible worlds' (Morgan 1973, Hintikka 1975, Routley 1980, Yagisawa 1988, Mares 1997, and Restall 1997). For a good overview of the recent literature on impossible worlds, see Berto 2013.[199]

Though a variety of reasons have been given for postulating such impossible worlds, not all of those reasons are cogent. For example, impossible worlds are often invoked to solve problems that arise when philosophers *represent* propositions as functions from possible worlds to truth-values (or as sets of possible worlds). Such representations, as is well known, *identify* propositions that are necessarily equivalent; if propositions p and q are just functions from worlds to truth-values, then they can't be distinguished when they have the same truth-value at every possible world. As a result, if one represents a belief as a relation between a person and a proposition so-conceived, then if x believes p, and p is necessarily equivalent to q, then x believes q. This just follows by the substitution of identicals and the identity of p and q. Such a result flies in the face of the data.[200] To solve this problem, it has been suggested that we can distinguish necessarily equivalent propositions if we consider their truth-values at impossible worlds. In effect, the suggestion is to represent propositions as functions from worlds generally, i.e., both possible and impossible worlds, to truth-values.

But, from the present perspective, we need not invoke impossible worlds to distinguish necessarily equivalent propositions. Our theory of propositions doesn't collapse necessarily equivalent propositions. Thus, one can use the present theory of propositions to represent beliefs without incurring the result that we believe everything that is necessarily equivalent to what we believe.

[199] See also Nolan 2013, Krakauer 2013, and Jago 2013, which also focus on impossible worlds. However, they don't consider the theory of impossible worlds developed here, which was first sketched in Zalta 1997a.

[200] Not only are there numerous examples of believing p without believing propositions necessarily equivalent to q, the problem of logical omniscience arises for this understanding of propositions. See Hintikka 1975 for a discussion of the problem and the suggestion that impossible worlds solve the problem.

Philosophers have also invoked impossible worlds to explain both impossibilities in fiction specifically and thoughts about impossible objects generally. But as the discussion in Chapters 16 will show, we can analyze such fictions and thoughts without invoking impossible worlds.

Given the present enterprise, the best case for impossible worlds comes from:

(A) the analysis of counterfactual and subjunctive conditionals with impossible antecedents, and

(B) the study of paraconsistent logic.

As to (A), consider the following sentences, which are clearly true:

- If Frege's system had been consistent, he would have died a happier man.
- If there were a set of all non-self-membered sets, it would be a member of itself iff not a member of itself.

The first example is a counterfactual conditional. On the standard analysis, such conditionals are true just in case the consequent is true at the closest possible world where the antecedent is true (Stalnaker 1968, Lewis 1973). But the antecedent of the example is true at no possible world, since one can derive a contradiction from Frege's axioms and thereby demonstrate his system's inconsistency. This inconsistency is not contingent; there is no possible world where the particular axioms of Frege's system are consistent. Hence, the antecedent of the above counterfactual conditional describes an impossibility and so the standard analysis implies that the sentence is false, thereby failing to preserve its truth-value.[201] It has been suggested that this problem might be solved if we amend the analysis so that if counterfactual conditionals are considered true just in case the consequent is true at the closest world (possible or impossible) where the antecedent is true.

The second case is a subjunctive conditional, and there are lots of similar examples, such as, "if four were prime, it would be divisible only by itself and one". Again, on the standard analysis of the truth conditions of such subjunctive conditionals, on which the consequent holds in the closest possible world where the antecedent holds, the sentence turns out to be false, contrary to intuition. But though there is no possible world where something is a set of all

[201] Of course, one could try to 'paraphrase away' the description "Frege's system", by interpreting "Frege's system" as the *non-rigid* description "the system Frege developed", so that the sentence in question implies: if Frege had developed a consistent system, he would have died a happier man. Here, the antecedent is possibly true; there are possible worlds where Frege developed a consistent system. But that is not what the original sentence implies. The paraphrase is not an accurate one; it just misrepresents what the original sentence means. If this is not convincing, just change the example to: if the system Frege in fact developed had been consistent, he would have died a happier man.

non-self-membered sets or where four is prime, it is claimed that there are impossible worlds where these propositions are true. Of course, these claims are typically just assumed rather than proved to be true. By amending the analysis so that the truth conditions become "the consequent holds at the closest world (possible or otherwise) where the antecedent holds", we seem to get correct truth conditions for the subjunctive conditional, assuming that the notion of an impossible world is sufficiently clear.

Concerning (B), the suggestion that paraconsistent logic governs impossible worlds is persuasive. Paraconsistent logic weakens classical logic so that an arbitrary proposition can't be derived from the contradiction $p \,\&\, \neg p$; thus, the principle *ex contradictione (sequitur) quodlibet* fails for such a logic. On the analysis developed below, we can see why one might think there is a connection between paraconsistent logic and impossible worlds: from the fact that $p \,\&\, \neg p$ is true at an impossible world, it doesn't follow that every proposition is true at that world.

Of course, once one accepts that impossible worlds help us to understand the data presented by (A) and (B), the question arises, what are impossible worlds exactly? Too frequently, the answer is given by switching to model theory and modeling impossible worlds as sets of propositions. Unfortunately, that is only a model, not a theory; a world, whether possible or impossible, is not a set of propositions. The propositions in a set don't characterize that set, whereas the propositions true in a world characterize the world.

In what follows, however, we develop a series of definitions and theorems that show impossible worlds to be abstract objects characterized by the propositions true at them. Moreover, the most important principles governing impossible worlds are derived as theorems, such as the fundamental theorem that for every way a world couldn't possibly be, there is a non-trivial impossible world that is that way; this is proved in (468) below. I don't know of any other *theory* of impossible worlds that yields similar consequences.[202]

The reader may wish to consult Zalta 1997a for a more detailed motivation of impossible worlds than the one just presented. The work below revises, corrects, and enhances the theorems and proofs first developed there.

(**459**) **Definition:** Impossible Worlds. In what follows, we continue to use the variables $s, s', s'', \ldots$ as restricted variables ranging over situations (363). Recalling the definition of $Maximal(s)$ (407) and the definition of $Possible(s)$ (395), we may say that a situation s is an *impossible world* just in case s is maximal and not possible:

$$ImpossibleWorld(s) \;=_{df}\; Maximal(s) \,\&\, \neg Possible(s)$$

[202] One way to question the arguments in Nolan 2013, Krakauer 2013, and Jago 2013, is to enquire whether their preferred theories of impossible worlds can yield the basic principles about impossible worlds as theorems, in the manner of Zalta 1997a and below.

Eliminating the restricted variable, this becomes:

$$\mathit{ImpossibleWorld}(x) =_{df} \mathit{Situation}(x) \,\&\, \mathit{Maximal}(x) \,\&\, \neg \mathit{Possible}(x)$$

Note here that we require maximality. We take it that only those situations that are maximal are correctly considered to be worlds. Situations that are maximal and possible are (provably) possible worlds (410), while situations that are maximal and impossible are impossible worlds by definition. A situation that isn't maximal has no legitimate claim to being called a 'world'.

(**460**) **Definition:** Truth at an Impossible World. Since $\mathit{ImpossibleWorld}(s)$ and $\mathit{ImpossibleWorld}(x)$ are well-defined conditions, we may introduce $i, i', i'', \ldots$ as restricted variables ranging over situations meeting the condition. Thus, at our convenience, we may take these variables to be either singly restricted or doubly restricted, as discussed in Remark (401) for the case of the variables $w, w', w'', \ldots$. As a result, notions defined on situations s may be applied to impossible worlds i without having to be redefined. An important example is the notion of *truth in* a situation, i.e., $s \models p$. By interpreting i as a doubly restricted variable, we may henceforth suppose that $i \models p$ is defined. We shall read $i \models p$ as 'p is true *at* i'. Thus, *truth at* an impossible world is simply a special case of the notion *truth in* a situation. Note that by definition (365), $i \models p$ is equivalent to $i\Sigma p$, and so by (216), both are equivalent to $i[\lambda y\, p]$. Hence, when a proposition p is true at i, the property $[\lambda y\, p]$ *characterizes* i by way of an encoding predication.

(**461**) **Theorem:** Identity of Impossible Worlds. Since impossible worlds are a species of situation, it follows that $i = i'$ iff all and only the propositions true at i are true at i':

$$i = i' \equiv \forall p(i \models p \equiv i' \models p)$$

(**462**) **Theorem:** $\boldsymbol{s_V}$ is an Impossible World. Recall that in (376.2), $\boldsymbol{s_V}$ was defined as $\iota x \mathit{TrivialSituation}(x)$, where $\mathit{TrivialSituation}(x)$ was defined in (374.2) as a situation x in which every proposition p is true. In (377.4), we gave a modally strict proof that $\mathit{TrivialSituation}(\boldsymbol{s_V})$. It is now provable as a modally strict theorem that $\boldsymbol{s_V}$ is an impossible world:

$$\mathit{ImpossibleWorld}(\boldsymbol{s_V})$$

Since this theorem implies that there are impossible worlds, we know the quantifiers $\forall i$ and $\exists i$ behave classically, in the sense that $\forall i \varphi \rightarrow \exists i \varphi$; cf. Remark (256).

(**463**) $\star$**Theorem:** The False is an Impossible World That Isn't Trivial. Recall the definition of The False ($\bot$) in (219.2). It follows from non-modally strict theorems that The False is a non-trivial impossible world:

$$ImpossibleWorld(\bot) \,\&\, \neg TrivialSituation(\bot)$$

Hence, we've established that there are non-trivial impossible worlds. Since $\bot$ is an impossible world at which every falsehood is true, we might affectionately think of it as *the worst of all impossible worlds*! By comparison, $\boldsymbol{s_V}$ has a saving grace, for though every false proposition is true there, every true proposition is true there as well.

(**464**) **Theorem:** There Are Non-Trivial Impossible Worlds. Though the previous theorem establishes that there are non-trivial impossible worlds, it did so by a non-modally strict proof. But there is modally strict proof that there are non-trivial impossible worlds:

$$\exists i(\neg TrivialSituation(i))$$

One way to prove this theorem is to show that $\bot$ (The False) is a non-trivial impossible world. But such a proof would fail to be modally strict, since it would rely on the necessitation-averse fact that $\forall F(\bot F \equiv \exists p(\neg p \,\&\, F = [\lambda y\, p]))$, derivable from (219.2) and (101.2)★. By contrast, the proof in the Appendix appeals to an instance of the Comprehension Principle for Situations which asserts $\exists s \forall F(sF \equiv \exists q(\neg q \& F = [\lambda y q]))$. In other words, the proof goes by way of a situation, like $\bot$, that is identified by the fact that it encodes all the falsehoods, but not identified via a description that ties its properties to the propositions that are in fact false. Rather, we pick an *arbitrary witness* to the instance of the Comprehension Principle for Situations and reason with respect to that witness. **Exercise:** Develop an alternative, modally strict proof by choosing an arbitrary possible world w and considering facts about The False at w ($\bot_w$), which was defined in (447.2). In other words, show that $\bot_w$ is a non-trivial impossible world.

(**465**) **Theorem:** Not All Impossible Worlds Are Modally Closed. It also follows that not all impossible worlds are modally closed, i.e., closed under necessary implication:

$$\neg \forall i ModallyClosed(i)$$

This theorem is modally strict. We can give a simpler, though non-modally strict proof of this result by pointing to $\bot$. It is a known impossible world and by appealing to non-modally strict theorems such as (227.4)★ and (227.5)★, one can easily establish that there are propositions p and q such that $p \Rightarrow q$, $\bot \models p$, and $\neg \bot \models q$.

(**466**) **Restricted Term Definition:** The p-Extension of Situation s. We canonically define *the p-extension of* situation s, written s^{+p}, as the situation that encodes not only the properties that s encodes but also the property $[\lambda y\, p]$:

$$s^{+p} =_{df} \imath s'\forall F(s'F \equiv (sF \vee F = [\lambda y\, p]))$$

Clearly, s^{+p} is a canonical situation (382), since when φ is $sF \vee F = [\lambda y\, p]$, it is easily provable that $\forall F(\varphi \rightarrow Propositional(F))$. Note that the expression s^{+p} is a binary functional term that involves two variables: the situation that s^{+p} denotes depends on the value of s and p. We may regard $\kappa^{+\Pi^0}$ as well-formed and logically proper for any 0-place relation term Π^0 and any individual term κ that is known, by hypothesis or by proof, to be such that $Situation(\kappa)$. For example, substituting $\perp$ for s and p_0 for p (where p_0 is $\forall x(E!x \rightarrow E!x)$) produces $\perp^{+p_0}$, which is provably a situation in which p_0 and every false proposition is true.

(**467**) **Lemmas:** The p-Extension of s is Strictly Canonical. Where φ is the formula $sF \vee F = [\lambda y\, p]$, it follows that (.1) every property such that φ is necessarily such that φ:

(.1) $\forall F(\varphi \rightarrow \Box\varphi)$

Hence s^{+p} is a strictly canonical situation and so it is a modally strict consequence of (466) and (189.2) that (.2) the p-extension of s encodes a property F if and only if either s encodes F or F just is $[\lambda y\, p]$:

(.2) $\forall F(s^{+p}F \equiv sF \vee F = [\lambda y\, p])$

Thus, it immediately follows that: (.3) if a proposition is true in s it is true in the p-extension of s, and (.4) p is true in the p-extension of s:

(.3) $s \models q \rightarrow s^{+p} \models q$

(.4) $s^{+p} \models p$

(**468**) **Theorem:** Fundamental Theorem of Impossible Worlds. The most important fact about impossible worlds is that if p isn't possibly true, then there is a non-trivial impossible world at which p is true:

$$\neg\Diamond p \rightarrow \exists i(\neg TrivialSituation(i)\ \&\ i \models p)$$

If we borrow a turn of phrase from Lewis (1986, 2), we might read the above as: every way a world couldn't possibly be is a way some non-trivial impossible world is. The proof of this theorem in the Appendix corrects an error in the proof of a corresponding theorem in Zalta 1997a.[203]

[203] In Zalta 1997a (647–8), we correctly asserted, but incorrectly proved, the theorem that $\neg\Diamond p \rightarrow \exists s(ImpossibleWorld(s)\ \&\ s \neq s_u\ \&\ s \models p)$. In this theorem, s_u is the *universal* situation (i.e., the trivial situation we're now calling $\boldsymbol{s_V}$, in which every proposition is true). The proof developed in 1997a correctly established that $\neg\Diamond p$ implies that there is a situation s which is maximal, not possible, and in which p is true. But the proof that s is distinct from s_u contained an assumption that ap-

Note that the proof in the Appendix assumes $\neg\Diamond p$, cites the fact that there is a situation that encodes all the truths, considers an arbitrary such situation, and then shows that its p-extension is a non-trivial impossible world where p is true. **Exercise:** Develop an alternative proof of this theorem that assumes $\neg\Diamond p$ and then shows, for an arbitrary possible world w, that ${\top_w}^{+p}$ is a non-trivial impossible world where p is true.

(**469**) **Theorems:** *Ex Contradictione Quodlibet* Fails for Impossible Worlds. Recall that we established, in (439.2), that the law *ex contradictione quodlibet* governs possible worlds. Formulated in the material mode, the law asserts: $w \models (p \,\&\, \neg p) \rightarrow w \models q$. But by reasoning from (464), we can show that this law fails for impossible worlds, i.e., we can show: (.1) there are impossible worlds i and propositions p and q such that $(p \,\&\, \neg p)$ is true at i but q fails to be true at i:

(.1) $\exists i \exists p \exists q(i \models (p \,\&\, \neg p) \,\&\, \neg i \models q)$

Since *ex contradictione quodlibet* fails for impossible worlds, it fails for situations. (As an exercise, show that there are situations other than impossible worlds for which the law fails.)

Moreover, we can also reason from the fundamental theorem for impossible worlds (468) to show that a variant version of *ex contradictione quodlibet* fails for impossible worlds, i.e., that (.2) there are impossible worlds i and propositions p and q such that both p and $\neg p$ are true at i but q fails to be true at i:

(.2) $\exists i \exists p \exists q(i \models p \,\&\, i \models \neg p \,\&\, \neg i \models q)$

Again, since this variant version of *ex contradictione quodlibet* fails for impossible worlds, it therefore fails for situations generally, though again, we leave it as an exercise to show that there are situations other than impossible worlds for which this variant fails. This theorem establishes that there are impossible worlds which can be used for the study of para-consistent logics in which *ex contradictione quodlibet* fails.

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peared correct but that isn't in fact provable, namely, that the impossible proposition p mentioned in the antecedent is distinct from the proposition $p \,\&\, \neg p$. The general claim that $q \neq (q \,\&\, \neg q)$ is certainly provable whenever q is a necessary, true, or even possible proposition. (For example, suppose $\Diamond q$, and assume for reductio that $q = (q \,\&\, \neg q)$. Then $\Diamond(q \,\&\, \neg q)$, which contradicts the fact that $\neg\Diamond(q \,\&\, \neg q)$.) However, when q is necessarily false, i.e., impossible, we can't prove the inequality $q \neq (q \,\&\, \neg q)$ from our axioms. Though our system requires that there be at least one impossible proposition (see (145.2)), it leaves open the question of whether there are multiple ones. Of course, it is consistent with the theory to assert that $q \neq (q \,\&\, \neg q)$ when q is necessarily false. If one does assert this, one can prove the existence of many new propositions. But when q is necessarily false, the identity $q = (q \,\&\, \neg q)$ is also consistent with the theory, and so one can't assume its negation. These facts were overlooked in the proof of this theorem in Zalta 1997a; by contrast, the proof of (468), which appears in the Appendix, has been amended accordingly. In the corrected proof, the impossible world where p is true is shown to be non-trivial by identifying a *contingently false proposition* that fails to be true in it.

(**470**) **Theorem:** Disjunctive Syllogism Fails for Impossible Worlds. Disjunctive syllogism is a valid rule of inference that we have often used to prove theorems. It justifies the inference to $\neg\psi$ from the premises $\varphi \vee \psi$ and $\neg\varphi$. We've expressed this in the formal mode as: $\varphi \vee \psi, \neg\varphi \vdash \neg\psi$ (64.4.b). A version of this principle governs possible worlds and propositions, for we established in (440.2) that the following principle governs possible worlds: $(w \models (p \vee q)\ \&\ (w \models \neg p)) \rightarrow w \models q$. However, disjunctive syllogism fails as a principle for impossible worlds. There are impossible worlds i and propositions p, q such that $p \vee q$ and $\neg p$ are true at i but q fails to be true at i:

$$\exists i \exists p \exists q [i \models (p \vee q)\ \&\ i \models \neg p\ \&\ \neg i \models q]$$

It follows immediately that disjunctive syllogism fails generally for situations, though impossible worlds weren't needed to show this general failure; there are situations other than impossible worlds for which the law fails as well. In any case, we have now shown that there are impossible worlds which can be used for the study of logics in which disjunctive syllogism fails.

(**471**) **Remark:** Consistent Impossible Worlds? The Fundamental Theorem for Impossible Worlds (468) implies that if p is necessarily false, then there is a non-trivial impossible world where p is true. Since we can not only prove that there are necessary falsehoods but identify particular ones as well (just take the negation of any tautology), it follows that there are non-trivial impossible worlds where those necessary falsehoods are true. But an interesting question arises: Are there *consistent* and *consistent** impossible worlds? That is, are there impossible worlds that are impossible for purely metaphysical reasons? To be maximally specific, our question can be put formally: can we prove that there are consistent and consistent* impossible worlds that are at which a metaphysical impossibility is true, i.e., prove:

$$\exists i (Consistent(i)\ \&\ Consistent^{*}(i)\ \&\ \exists p(\neg\Diamond p\ \&\ i \models p)) \qquad ?$$

To further address this question, we develop some definitions and then state a theorem.[204]

(**472**) **Restricted Term Definitions:** Distinguished Extensions and Restrictions of Situations. If given any situation s, let us define (.1) $s^{-\Rightarrow p}$ to be the situation that encodes all properties that s encodes *except* those properties constructed from propositions necessarily implied by p, and (.2) $s^{+\overline{\Rightarrow p}}$ to be the situation

[204] I am indebted to Uri Nodelman for prompting me to add the following sequence of items. In an earlier draft of this manuscript, these questions just discussed in the text were left open. While we were reading through the manuscript together, Uri suggested that they could be resolved. Though we worked out his initial idea together, he provided the impetus for ultimately reaching a proof of (473.1) below by finding a way to address every subtlety I raised in connection with the initial idea.

that encodes not only every property that s encodes but also those properties constructed from *the negations of* propositions necessarily implied by p; and :

(.1) $s^{-\Rightarrow p} \;=_{df}\; \imath s'(s'F \equiv (sF \,\&\, \neg\exists q((p \Rightarrow q) \,\&\, F = [\lambda y\, q])))$

(.2) $s^{+\overline{\Rightarrow p}} \;=_{df}\; \imath s'(s'F \equiv (sF \vee \exists q((p \Rightarrow q) \,\&\, F = [\lambda y\, \neg q])))$

Thus, we may regard terms of the form $\kappa^{-\Rightarrow\varphi^*}$ and $\kappa^{+\overline{\Rightarrow\varphi^*}}$ as well-defined and logically proper whenever κ is known to be a situation (either by proof or by hypothesis) and φ^* is any 0-place relation term. Think of $s^{-\Rightarrow p}$ as the *restriction* of s to those propositions that aren't necessary consequences of p, and think of $s^{+\overline{\Rightarrow p}}$ as the *extension* of s by all of the negations of the necessary consequences of p. We leave it as an exercise to show that $s^{-\Rightarrow p}$ and $s^{+\overline{\Rightarrow p}}$ are well-defined and strictly canonical situations. Note, though, that we can compose our two terms. For example, $(s^{-\Rightarrow p})^{+\overline{\Rightarrow p}}$ is the result of replacing the necessary consequences of p in s with their negations.

(**473**) **Theorem:** Consistent Impossible Worlds. We now prove that (.1) there are consistent and consistent* impossible worlds where some metaphysical impossibility is true; (.2) if p is possible and its negation is possible, then there is a consistent and consistent* impossible world at which p is necessary; and (.3) if p is possible and its negation is possible, then there is a consistent and consistent* impossible world at which p is impossible:

(.1) $\exists i(Consistent(i) \,\&\, Consistent^*(i) \,\&\, \exists p(\neg\Diamond p \,\&\, i \models p))$

(.2) $(\Diamond p \,\&\, \Diamond\neg p) \rightarrow \exists i(Consistent(i) \,\&\, Consistent^*(i) \,\&\, i \models \Box p))$

(.3) $(\Diamond p \,\&\, \Diamond\neg p) \rightarrow \exists i(Consistent(i) \,\&\, Consistent^*(i) \,\&\, i \models \neg\Diamond p))$

Note that we can't generalize these theorems further, to prove that for *any* necessary falsehood p, there is a consistent and consistent* impossible world where p is true. Clearly, when p is a contradiction of the form $\varphi^* \,\&\, \neg\varphi^*$, we can't show that there is an consistent and consistent* impossible world where p is true, for such a world is by definition inconsistent*. Nevertheless, by inspecting the proof of the above theorems, it should be clear that the key to finding consistent and consistent* impossible worlds is to start with a proposition that is both possibly true and possibly false. In the proof of (.1), we take a such a proposition (whose necessitation is clearly impossible) and consider a possible world where it is true. We then perform some operations on that possible world that turn it into an maximal and impossible situation at which the necessitation of p is true. That situation is provably consistent and consistent*.

(**474**) **Remark:** Consequences for Essentialism. The foregoing results have some important consequences for those who might wish to extend the present theory with some metaphysical axioms that imply that certain objects exemplify certain properties necessarily. For example, suppose that one were to adopt a form of essentialism and assert metaphysical axioms that imply both:

Necessarily, if Socrates is concrete, he fails to be a plant.
$\Box(E!s \rightarrow \neg Ps)$

Being a plant is a concreteness-entailing property.
$\Box\forall x(Px \rightarrow E!x)$

Then it would follow from these two metaphysical claims that Socrates couldn't have been a plant, i.e., $\neg\Diamond Ps$.[205] Consequently, the Fundamental Theorem for Impossible Worlds implies that there is a non-trivial impossible world where Socrates is a plant. But notice that by appealing to the Fundamental Theorem, the proof of this fact constructs an impossible world i, namely w^{+Ps}, at which (i) every true proposition is true (including the truth that $\neg Ps$) and (ii) it is true that Socrates is a plant. Clearly, then, i is inconsistent: there is a proposition, namely Ps such that both $i \models Ps$ and $i \models \neg Ps$.

So, once one adds metaphysical axioms that imply a proposition p is metaphysically impossible and that rule out the conjunction $\Diamond p$ & $\Diamond\neg p$, one can't build a consistent and consistent* impossible world where p is true. The Ps-extension of a possible world w doesn't constitute such an impossible world, since it is inconsistent. Nor can we extend $\bot$ or $\boldsymbol{s_V}$, for the former is inconsistent* and the latter is both inconsistent and inconsistent*.

On the other hand, if one rejects essentialist claims such as the above and allows both that possibly Socrates is a plant and possibly Socrates isn't a plant, then one can prove that there is a consistent and consistent* impossible world where the necessarily false claim, that necessarily Socrates fails to be a plant, is true, by (473.3).

12.5 Moments of Time and World-States

(**475**) **Remark:** An Outline. The definitions and theorems governing possible worlds that were developed in Section 12.2 can be adapted in a natural way

[205] To see why, first note that if the following derivation sequence holds:

(A) $E!s \rightarrow \neg Ps, \forall x(Px \rightarrow E!x) \vdash \neg Ps$

then it follows by Rule RN that:

(B) $\Box(E!s \rightarrow \neg Ps), \Box\forall x(Px \rightarrow E!x) \vdash \Box\neg Ps$

Now we can establish (A) as follows: assume both $E!s \rightarrow \neg Ps$ and $\forall x(Px \rightarrow E!x)$ but, for reductio, Ps. Then from the second assumption, it follows that $E!s$. So from the first assumption, we have $\neg Ps$. Contradiction. Since we have established (A), it follows that (B) by Rule RN. But the premises of (B) are just the two consequences implied by the metaphysical axioms assumed for this discussion. Hence, it follows that $\Box\neg Ps$, i.e., $\neg\Diamond Ps$.

to systematize *moments of time* and *world-states*. The simplest way to do this is to add to our language a primitive omnitemporality operator $\boxplus$ (*it is always the case that*, or more simply *always*) and stipulating that formulas of the form $\boxplus\varphi$ are well-formed. The dual operator, $\diamondplus$ (*it is sometimes the case that*, or more simply *sometimes*) can then be defined:

$$\diamondplus\varphi \;=_{df}\; \neg\boxplus\neg\varphi$$

Using this last definition, we can say that a situation s is a *moment of time* just in case sometimes, all and only true propositions are true in s:

$$\mathit{MomentOfTime}(s) \;=_{df}\; \diamondplus\forall p(s \models p \equiv p)$$

Similarly, a situation s is a *possible-world-state* just in case possibly, sometimes, all and only true propositions are true in s:

$$\mathit{PossibleWorldState}(s) \;=_{df}\; \Diamond\diamondplus\forall p(s \models p \equiv p)$$

Clearly, these definitions are analogous to the definition of *possible world*.

Alternatively, instead of a primitive omnitemporality operator, one could add the standard two tense operators of minimal tense logic to our language:

$\mathcal{H}$ ('it was always the case that')

$\mathcal{G}$ ('it will always be the case that')

The dual operators $\mathcal{P}\varphi$ ('it was once the case that φ') and $\mathcal{F}\varphi$ ('it will at some point be the case that φ') are then defined as follows:

$$\mathcal{P}\varphi \;=_{df}\; \neg\mathcal{H}\neg\varphi$$

$$\mathcal{F}\varphi \;=_{df}\; \neg\mathcal{G}\neg\varphi$$

Then one can either define $\boxplus\varphi$:

$$\boxplus\varphi \;=_{df}\; \mathcal{H}\varphi \,\&\, \varphi \,\&\, \mathcal{G}\varphi$$

and define $\diamondplus\varphi$ as we did above, or define $\diamondplus\varphi$ more directly, as:

$$\diamondplus\varphi \;=_{df}\; \mathcal{P}\varphi \vee \varphi \vee \mathcal{F}\varphi$$

No matter which of these ways we choose to define $\diamondplus\varphi$, we would be in a position to formulate the definitions of *moment of time* and *world-state* given above. However, in a system with the two tense operators $\mathcal{H}$ and $\mathcal{G}$, one can define past and future moments of time:

$$\mathit{PastMoment}(s) \;=_{df}\; \mathcal{P}\forall p(s \models p \equiv p)$$

$$\mathit{FutureMoment}(s) \;=_{df}\; \mathcal{F}\forall p(s \models p \equiv p)$$

Of course, the above developments would require some changes to certain defined notions of the theory. For example, one would have to redefine *ordinary* and *abstract* objects as:

$$O! =_{df} [\lambda x\, \Diamond \oplus E!x]$$

$$A! =_{df} [\lambda x\, \neg \Diamond \oplus E!x]$$

Thus, an abstract object would be defined as not the kind of thing that could ever be concrete.

One would also need to redefine identity for individuals and properties as follows:

$$=_E \;=_{df} [\lambda xy\, O!x \,\&\, O!y \,\&\, \Box\boxplus\forall F(Fx \equiv Fy)]$$

$$x = y \;=_{df}\; x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\boxplus\forall F(xF \equiv yF))$$

$$F^1 = G^1 \;=_{df}\; \Box\boxplus\forall x(xF^1 \equiv xG^1)$$

The rest of the definitions for the language of object theory can remain the same, though some may take on new significance in light of the above.

The point of revising our system in the above manner would be to derive interesting and important theorems that govern times and world-states. For example, one might want to derive the claims: (.1) sometimes p if and only if p is true at some moment of time; (.2) it is always the case that p if and only if p is true at every moment of time; (.3) possibly, sometimes p if and only if p is true at some world-state; and (.4) it is necessarily always the case that p if and only if p is true at every world-state:

(.1) $\oplus p \equiv \exists x(\mathit{MomentOfTime}(x) \,\&\, x \models p)$

(.2) $\boxplus p \equiv \forall x(\mathit{MomentOfTime}(x) \rightarrow x \models p)$

(.3) $\Diamond\oplus p \equiv \exists x(\mathit{PossibleWorldState}(x) \,\&\, x \models p)$

(.4) $\Box\boxplus p \equiv \forall x(\mathit{PossibleWorldState}(x) \rightarrow x \models p)$

There are, in addition, many other interesting theorems that one might want to derive, such as that there is a unique present moment, that moments of time and world-states are consistent, etc.

Of course, to derive theorems such as the above, our axioms and deductive system would have to be revised. Though the changes to the language are simple enough, the changes to the axioms and deductive system require much more care. One can't simply add the S5-type axioms to govern the omnitemporality operator $\boxplus$, or simply add the axioms of *minimal tense logic* govern $\mathcal{H}$ and $\mathcal{G}$. Other axioms are needed and a host of subtle issues arise and have to be addressed. A few moments' reflection about the issues involved may lead one to see that it is probably easier to accommodate the single omnitemporality operator $\boxplus$ than it is to accommodate the $\mathcal{H}$ and $\mathcal{G}$ operators. We conclude this chapter with a discussion of some of the more important points to consider.

In the following, I shall assume familiarity with the standard semantics for basic tense operators. The semantics we developed in Chapter 5 (Sections 5.2 – 5.6) would be revised to include a primitive domain $\boldsymbol{T}$ of *times* containing a distinguished time $\boldsymbol{t}_0$ representing the present moment. Moreover, we would need to revise the function that assigns an extension to every relation at each possible world, so that it becomes a function that assigns an extension to every relation at each world-time pair.[206] Then the definition of *truth* at $\boldsymbol{w}$ *with respect to* interpretation $\mathcal{I}$ and assignment f, i.e., $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$, would be revised to become a definition of *truth* at $\langle \boldsymbol{w}, \boldsymbol{t} \rangle$ *with respect to* interpretation $\mathcal{I}$ and assignment f, i.e., $\boldsymbol{w}, \boldsymbol{t} \models_{\mathcal{I},f} \varphi$. The clauses are, for the most part, straightforward. The interesting cases are the clauses for the necessity operator,[207] the actuality operator,[208] and any new tense operators. For example, if one adds only a primitive omnitemporal operator $\boxplus$, then the needed clause is:

T10t. if φ is a formula of the form $\boxplus\psi$, then $\boldsymbol{w}, \boldsymbol{t} \models_{\mathcal{I},f} \varphi$ if and only if for every moment of time $\boldsymbol{t}'$, $\boldsymbol{w}, \boldsymbol{t}' \models_{\mathcal{I},f} \psi$, i.e., iff $\forall \boldsymbol{t}'(\boldsymbol{w}, \boldsymbol{t}' \models_{\mathcal{I},f} \psi)$

Thus, the truth conditions for $\Box\varphi$ hold the time fixed and consider the truth conditions of φ at every primitive possible world, while the truth conditions for $\boxplus\varphi$ hold the world fixed and consider the truth conditions of φ at every primitive time. Note that the actuality operator $\mathcal{A}$ takes on new signficance when one adds tense operators to our system. A formula of the form $\mathcal{A}\varphi$ must be read as 'actually, presently φ' or 'it is actually now the case that φ'. Given the semantics of the actuality operator, the axiom (30)$\star$, i.e., $\mathcal{A}\varphi \equiv \varphi$, becomes both a necessitation-averse as well as a omnitemporalization-averse axiom.[209]

[206]Specifically, we would define a tertiary function, $\mathbf{ex}_{\boldsymbol{w},\boldsymbol{t}}(\boldsymbol{r}^n)$, indexed to its second and third arguments, that assigns each n-place relation $\boldsymbol{r}^n$ in $\mathbf{R}$ ($n \geq 0$) an *exemplification* extension at each world-time pair $\langle \boldsymbol{w}, \boldsymbol{t} \rangle$ as follows:

- for $n \geq 1$, $\mathbf{ex}_{\boldsymbol{w},\boldsymbol{t}}$ assigns, to each triple consisting of a relation $\boldsymbol{r}^n$ in $\mathbf{R}$, possible world $\boldsymbol{w}$ in $\mathbf{W}$, and moment of time $\boldsymbol{t}$ in $\boldsymbol{T}$, a set of n-tuples whose members are in $\mathbf{D}$; i.e., $\mathbf{ex}_{\boldsymbol{w},\boldsymbol{t}} : \mathbf{R}_n \times \mathbf{W} \times \boldsymbol{T} \to \wp(\mathbf{D}^n)$.
- for $n = 0$, assigns, to each triple consisting of a relation $\boldsymbol{r}^0$ in $\mathbf{R}$, possible world $\boldsymbol{w}$ in $\mathbf{W}$, and moment of time $\boldsymbol{t}$ in $\boldsymbol{T}$, one of the two truth-values $\mathbf{T}$ or $\mathbf{F}$; i.e., $\mathbf{ex}_{\boldsymbol{w},\boldsymbol{t}} : \mathbf{R}_0 \times \mathbf{W} \times \boldsymbol{T} \to \{\mathbf{T}, \mathbf{F}\}$.

[207] The clause for the necessity operator is:

T6t. if φ is a formula of the form $\Box\psi$, then $\boldsymbol{w}, \boldsymbol{t} \models_{\mathcal{I},f} \varphi$ if and only if for every possible world $\boldsymbol{w}'$, $\boldsymbol{w}', \boldsymbol{t} \models_{\mathcal{I},f} \psi$, i.e., iff $\forall \boldsymbol{w}'(\boldsymbol{w}', \boldsymbol{t} \models_{\mathcal{I},f} \psi)$

[208] The clause for the actuality operator is:

T7t. if φ is a formula of the form $\mathcal{A}\psi$, then $\boldsymbol{w}, \boldsymbol{t} \models_{\mathcal{I},f} \varphi$ if and only it $\boldsymbol{w}_0, \boldsymbol{t}_0 \models_{\mathcal{I},f} \psi$

[209]Similarly, the definite description operator $\imath$ has a new significance; the denotation of a definite

In order to assert axioms for the tensed system, one has to carefully define *temporal closures* and make sure that no temporal closures of axiom (30)★ are asserted as axioms. Moreover, a modified version of (32.4) has to be asserted, namely:

$$\Diamond\diamondplus\exists x(E!x \,\&\, \Diamond\diamondplus\neg E!x) \;\&\; \Diamond\diamondplus\neg\exists x(E!x \,\&\, \Diamond\diamondplus\neg E!x)$$

Clearly, we must now think of the *contingently concrete* individuals as those individuals that exemplify being concrete and that possibly, sometimes fail to exemplify being concrete. So the formula $\exists x(E!x \,\&\, \Diamond\diamondplus\neg E!x)$ asserts that contingently concrete individuals exist. Consequently the axiom displayed above asserts: possibly, sometimes there are contingently nonconcrete objects and possibly, sometimes there are no contingently concrete objects.

We already mentioned above that in the case of $\boxplus$, one would add the tensed S5 axioms, and in the case of $\mathcal{H}$ and $\mathcal{G}$, one would add the axioms of minimal tense logic. However, one must also introduce axioms that govern the interaction of the actuality operator and the tense operators, as well as axioms that govern the interaction of the tense operators and modal operators. For example, if $\boxplus$ is primitive, one might add:

$$\mathcal{A}\varphi \to \Box\boxplus\mathcal{A}\varphi$$

$$\Box\boxplus\varphi \equiv \mathcal{A}\Box\boxplus\varphi$$

$$\Box\boxplus\varphi \equiv \boxplus\Box\varphi$$

The first is the temporal analogue of (33.1). It asserts: if actually, presently φ, then it is necessarily, always the case that actually, presently φ.[210] The second is the temporal analogue of (33.2). It asserts: necessarily, always φ if and only if it is actually, presently the case that necessarily, always φ.[211] The third axiom tells us that the necessity operator and omnitemporality operator commute. Of course, different axioms will be needed if one starts with the two operators of minimal tense logic, $\mathcal{H}$ and $\mathcal{G}$.[212]

description such as $\imath x\varphi$ becomes the object, if there is one, that satisfies φ at the world-time pair $\langle \boldsymbol{w}_0, \boldsymbol{t}_0\rangle$. So we would need to read the description as 'the x now in fact such that φ'.

[210] The semantic validity of this axiom can be shown by appeal to clause T10t described above and clauses T6t and T7t defined in footnotes 207 and 208, respectively. For the proof, see Metatheorem ⟨12.1⟩ in the Appendix to this chapter. Moreover, the necessitations and omnitemporalizations of this axiom are valid (i.e., Metatheorems ⟨12.2⟩ and ⟨12.3⟩ are left as exercises).

[211] The validity of this claim is proved as Metatheorem ⟨12.4⟩. Its necessitations and omnitemporalizations are also valid, though the proofs (i.e., Metatheorems ⟨12.5⟩ and ⟨12.6⟩ are also left as exercises.)

[212] In any case, one must exert extreme caution here given the complexity of the issues. For example, $\Box\varphi \equiv \mathcal{A}\Box\varphi$ is valid and its necessitation is valid, but I suspect that its omnitemporalization, $\boxplus(\Box\varphi \equiv \mathcal{A}\Box\varphi)$, is not valid. (Remember, validity in our tensed system becomes defined, when one expands the definitions, as truth at distinguished world and present moment, i.e., at $\langle \boldsymbol{w}_0, \boldsymbol{t}_0\rangle$, with

Though the axioms governing complex terms won't need modification, the closures of the following revised axiom for the rigidity of encoding (37) are needed:

$$xF \to \Box\boxplus xF$$

Finally, the Comprehension Axiom for Abstract Objects (39) will remain the same.

The most interesting and subtle revisions that would be needed are those that concern the deductive system. One important notion that has to be introduced is that of *modally and temporally strict* derivation (or proof), i.e., one that doesn't depend on axiom (30)★. Moreover, one has to justify metarules of inference for the tense operators that correspond to the Rule of Necessitation. To state those metarules, one may need also need to define a *temporally strict* derivation (or proof). The fact is, the presence of necessitation- and omnitemporalization-averse axioms (and more generally, contingent premises) raises open questions about how to formulate, in the simplest way possible, a deductive system that both (a) accommodates the tense operators and (b) allows for reasoning with necessitation-averse axioms, omnitemporalization-averse axioms, and contingent premises. We leave those questions for another occasion. It should be clear, however, that the resulting system should yield proofs of the interesting and important theorems listed above and, moreover, ones that closely parallel the proofs of the analogous theorems governing possible worlds.

respect to every interpretation $\mathcal{I}$ and assignment f.) Similarly, whereas $\boxplus\varphi \equiv \mathcal{A}\boxplus\varphi$ is valid and its omnitemporalization is valid, its necessitation, $\Box(\boxplus\varphi \equiv \mathcal{A}\boxplus\varphi)$ may not be. I don't currently see a way of deriving the forms that are valid from the axioms suggested thus far. Hence, one may need to add, as axioms, not only $\Box\varphi \equiv \mathcal{A}\Box\varphi$ and its $\boxplus$-free closures, but also $\boxplus\varphi \equiv \mathcal{A}\boxplus\varphi$ and its $\Box$-free closures.

Chapter 13

Concepts

It is not always clear what is meant when philosophers talk about *concepts*. In this chapter, we define a notion of *concept* governed by a variety of interesting theorems, many of which represent principles that intuitively characterize this notion. Furthermore, some of the theorems that we derive look very similar to principles that Leibniz adopted in his work. Though many philosophers have supposed that Leibnizian concepts, like Plato's Forms, are to be analyzed as properties, the theorems below establish that Leibnizian principles about concepts fall out very naturally when the latter are analyzed as abstract individuals. Indeed, in (540) and (593) below, we explain difficulties that would arise if one were to analyze Leibnizian concepts as properties.

Thus, in what follows Leibnizian concepts are to be distinguished from Fregean concepts, which *are* taken to be properties in Chapter 14 (see the opening lines of Section 14.2, where this is justified). But whether or not one agrees that the following theorems constitute a good interpretation of Leibniz, we take it that they constitute an interesting and compelling philosophical theory of concepts in their own right.

(**476**) **Remark:** Leibnizian Concepts. Over the course of his life, Leibniz developed three different strands of his theory of concepts:

- a non-modal *calculus* of concepts,
- a *concept containment* theory of truth, and
- a *modal metaphysics* of complete, individual concepts.

We discuss these in turn.

Leibniz produced fragments of his non-modal calculus of concepts throughout his life, but only in his late works (1690a, 1690b) did he explicitly introduce a primitive operation symbol (which we shall write as $\oplus$) so that he could write

$A \oplus B$ to denote the *sum* of concepts A and B.[213] He also introduced the relations of concept *containment* and concept *inclusion*, so that he could say that the concept $A \oplus B$ contains both the concepts A and B, and that both A and B are included in $A \oplus B$. Some of the key axioms and theorems that emerge from this fragment of Leibniz's work are: that concept addition is idempotent, commutative, and associative; that concept containment and inclusion are reflexive, anti-symmetric, and transitive; that if concept A is included in concept B, then there is a concept C such that $A \oplus C = B$; and that if A is included in B, then $A \oplus B = B$.

Leibniz advocated a concept containment theory of truth throughout his life, though he often expressed it in terms of concept inclusion. Here is a classic statement, from his correspondence with Arnauld (June 1686, LA 63, G.ii 56):

> ... in every true affirmative proposition, necessary or contingent, universal or particular, the concept of the predicate is in a sense included in that of the subject; the predicate is present in the subject.

Stated in terms of containment, this becomes the claim that in a true subject-predicate statement, the concept of the subject contains the concept of the predicate. In what follows, we shall analyze the concept of a predicate in the material mode, as the concept of a *property*. Moreover, we shall define the concepts of such subjects as 'every person' and 'Alexander' as they occur in the sentences 'Every person is rational' and 'Alexander is a king'. We then show that from (the standard analysis of) the claim that every person is rational, one can derive (the Leibnizian analysis of) the claim that the concept *every person* contains the concept of *being rational*. Moreover, we show that from (the standard analysis of) the claim that Alexander is king, one can derive (the Leibnizian analysis) of the claim that the concept of Alexander contains the concept of *being king*.

Finally, in middle and late period works (*Discourse on Metaphysics* (1686), *Theodicy* (1709), and *The Monadology* (1714)), Leibniz developed a modal metaphysics of individual concepts that he used to analyze modal facts about individuals in terms of facts about various *individual concepts* that *appear at* other possible worlds. Some fundamental principles underlying Leibniz's view are:

- If an ordinary individual u is F but might not have been F, then (i) the individual concept of u contains the concept of F, and (ii) there is an individual concept that: (a) is a counterpart of the concept of u, (b) fails to contain the concept of F, and (c) appears at some other possible world.

- If an ordinary individual u isn't F but might have been F, then (i) the individual concept of u fails to contain the concept of F, and (ii) there is an

[213] Prior to those works, Leibniz indicated concept addition by concatenating the symbols for two concepts.

individual concept that: (a) is a counterpart of the concept of u, (b) contains the concept of F, and (c) appears at some other possible world.

Although Leibniz never actually states these principles explicitly, it seems clear that they are implicit in the views he expressed. As part of our goal of analyzing and unifying the various components of Leibniz's theory of concepts, we shall formalize and prove the above fundamental principles of Leibniz's modal metaphysics of concepts. The formalizations and derivations that we construct in analyzing Leibniz's theory realize, at least in part, his ideas of a *characteristica universalis* and *calculus ratiocinator*, though we won't argue for this here.

The developments that follow revise and enhance the work in Zalta 2000a in numerous ways. The discussion and procession of theorems has been revised, and the statement of the theorems and their proofs have been improved. Though a comparison of our work below with other work in the secondary literature on Leibniz would be useful, it is not attempted here. I shall, however, note that some commentators treat only Leibniz's nonmodal calculus of concepts and not the modal metaphysics of individual concepts,[214] while others treat only the modal metaphysics of individual concepts and not the nonmodal calculus.[215] Although Lenzen (1990) treats both, his work provides us with a *model* of Leibnizian concepts within set theory; it does not provide a *theory* of concepts. By contrast, the following reconstruction makes no set-theoretic assumptions, unlike most of the works in the secondary literature just cited. As mentioned previously, the work has significant philosophical interest in its own right, whether or not it constitutes a contribution to the secondary literature on Leibniz. That's why 'Leibnizian' is sometimes used within parentheses.

(**477**) **Definition:** (Leibnizian) Concepts. The key idea underlying our analysis of (Leibnizian) concepts is that they are the abstract individuals. Consequently, we identify the property *being a concept* with the property *being abstract*:

$$C! =_{df} A!$$

Thus, all of the previous theorems about abstract individuals become theorems about (Leibnizian) concepts. Note also that in light of theorem (153.2), $C!x \rightarrow \Box C!x$.

(**478**) **Theorems:** Immediate Equivalences and Identities. It is now a simple consequence of the previous definition that (.1) x is a concept that encodes exactly the properties such that φ if and only if x is an abstract object that encodes exactly the properties such that φ:

(.1) $(C!x \,\&\, \forall F(xF \equiv \varphi)) \equiv (A!x \,\&\, \forall F(xF \equiv \varphi))$

[214] See, for example, Rescher 1954; Kauppi 1960, 1967; Castañeda 1976, 1990; and Swoyer 1994, 1995.

[215] See Mates 1968, Mondadori 1973, and Fitch 1979.

We can further establish that (.2) the concept that encodes exactly the properties such that φ is identical to the abstract object that encodes exactly the properties such that φ:

(.2) $\imath x(C!x \,\&\, \forall F(xF \equiv \varphi)) = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$,
provided x doesn't occur free in φ

These are modally strict.

(**479**) **Theorems:** (Strong) Concept Comprehension and Canonical Concept Descriptions. From the preceding theorems, we can easily prove that (.1) there is a concept that encodes exactly the properties F such that φ; (.2) there is a unique concept that encodes exactly the properties F such that φ; and (.3) there is something which is *the* individual that is a concept encoding exactly the properties such that φ:

(.1) $\exists x(C!x \,\&\, \forall F(xF \equiv \varphi))$, provided x doesn't occur free in φ

(.2) $\exists !x(C!x \,\&\, \forall F(xF \equiv \varphi))$, provided x doesn't occur free in φ

(.3) $\exists y(y = \imath x(C!x \,\&\, \forall F(xF \equiv \varphi)))$, provided x, y don't occur free in φ

So by (.3) and the definition of $C!$, any appropriate formula φ can be used to formulate a logically proper *canonical concept description* of the form $\imath x(C!x \,\&\, \forall F(xF \equiv \varphi))$. Henceforth all of the machinery for, and discussions of, canonical and strictly canonical descriptions, in (181) – (190), can be repurposed for concepts. We'll see how on numerous occasions below.

(**480**) **Remark:** Restricted Variables and Canonical Concepts. Let us use the variables $c, d, e, f, \ldots$ as restricted variables ranging over concepts (thus, we are repurposing c from its use in Chapter 10 as a restricted variable ranging over classes). Let us also use $c_1, c_2, \ldots, d_1, d_2, \ldots$, etc., as restricted constants for concepts. Since it is clear that concepts exist, we may prove that $\forall c\varphi \rightarrow \exists c\varphi$ and hereafter assume it; cf. Remark (256).

Consequently, (479.1) – (479.3) can be expressed as follows:

$$\exists c \forall F(cF \equiv \varphi) \qquad (479.1)$$

$$\exists ! c \forall F(cF \equiv \varphi) \qquad (479.2)$$

$$\exists y(y = \imath c \forall F(cF \equiv \varphi)) \qquad (479.3)$$

Thus, the description $\imath c \forall F(cF \equiv \varphi)$ abbreviates the canonical concept description $\imath x(C!x \,\&\, \forall F(xF \equiv \varphi))$. Since the latter expands by definition to $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$, all of the theorems that govern canonical descriptions now also govern canonical concept descriptions.[216] We shall sometimes say that such descriptions describe *canonical* concepts.

(**481**) **Theorems:** Identity. In LLP 131–132 (G.vii 236), Propositions 1 and 3, Leibniz uses the substitution of identicals to derive that the notion of identity, as it applies to concepts, is symmetrical and transitive. However, '=' is defined in the present theory. Given theorems (71.1) – (71.3), it immediately follows that identity is reflexive, symmetrical, and transitive on the concepts. Hence, using our restricted variables, we have:

(.1) $c = c$

(.2) $c = d \rightarrow d = c$

(.3) $c = d \;\&\; d = e \rightarrow c = e$

Note that in Definition 1 of LLP 131 (G.vii 236), Leibniz uses '∞' as the identity symbol, and sometimes says that concepts that are the same are 'coincident'.

13.1 The Calculus of Concepts

13.1.1 Concept Addition

(**482**) **Definition:** Concept Addition (Summation). To define concept addition or summation, let us say that concept c is a *sum of* concepts *d and e* if and only if c encodes all and only the properties encoded by either d or e:

$$SumOf(c,d,e) =_{df} \forall F(cF \equiv dF \vee eF)$$

To produce an example of our definition:

- let c_1 be $\imath c \forall F(cF \equiv F = P)$
- let c_2 be $\imath c \forall F(cF \equiv F = P \vee F = Q)$
- let c_3 be $\imath c \forall F(cF \equiv F = P \vee F = Q \vee F = R)$

Since c_1 encodes just the property P, c_2 encodes just the properties Q and R, and c_3 encodes just the properties P, Q, and R, it is an easy exercise to show $SumOf(c_3, c_1, c_2)$.

(**483**) **Theorems:** The Sum of Concepts d and e Exists. In the usual manner, it follows that (.1) concepts d and e have a sum; (.2) concepts d and e have a unique sum; and (.3) the sum of concepts d and e exists:

[216] So, for example, it should be straightforward to show that a version of the Abstraction Principle (184)$\star$ applies to concepts, i.e.,

$$\imath c \forall F(cF \equiv \varphi))G \equiv \varphi^G_F, \text{ provided } c \text{ doesn't occur free in } \varphi \text{ and } G \text{ is substitutable for } F \text{ in } \varphi$$

This asserts: the concept encoding all and only the properties such that φ encodes G if and only if G is such that φ.

(.1) $\exists c SumOf(c,d,e)$

(.2) $\exists! c SumOf(c,d,e)$

(.3) $\exists y(y = \iota c SumOf(c,d,e))$

It it well to remember that if we eliminate the restricted variables, these become *conditional* existence claims. But since we know that concepts exist, we can derive unconditional existence claims from these theorems.

(**484**) **Restricted Term Definition:** The Sum of Concepts d and e. By our last theorem, we are entitled to introduce the following notation for *the* concept that is the sum of concepts d and e:

$$d \oplus e \quad =_{df} \quad \iota c SumOf(c,d,e)$$

Since there are free restricted variables occuring in the definition, an expression of the form $\kappa \oplus \kappa'$ is a binary functional term which we may regard as well-formed and logically proper only when both κ and κ' are known to be concepts, either by proof or by hypothesis. Moreover, by the definition of *SumOf*, we may regard $d \oplus e$ as (identical to) a canonical concept, for any concepts d and e. Finally, since concepts are just abstract individuals, it should be clear that concept addition applies to abstract individuals generally.

(**485**) **Lemmas:** Strict Canonicity of Sums. In light of Remark (446), in which we discussed strictly canonical terms with restricted variables, we compile the following theorems. If we let φ be the formula $dF \vee eF$, then we may use definition (188.1) to establish that (.1) φ is a rigid condition on properties, i.e.,

(.1) $\Box\forall F((dF \vee eF) \to \Box(dF \vee eF))$

Hence, $d \oplus e$ is a *strictly* canonical concept, by (188.2). By theorem (189.2) and the definition of $C!$, it follows that (.2) the sum of d and e is a concept that encodes all and only the properties F such that either d encodes F or e encodes F:

(.2) $C!d \oplus e \;\&\; \forall F(d \oplus eF \equiv dF \vee eF)$

Note that the first conjunct of (.2) is an exemplification formula consisting of a 1-place relation term $C!$ and a complex individual term $d \oplus e$. Moreover, it clearly follows from (.2) by definition (482) that (.3) $d \oplus e$ is a sum of d and e:

(.3) $SumOf(d \oplus e, d, e)$

(.1) – (.3) are modally strict theorems used frequently in the proofs of subsequent theorems.

(**486**) **Theorems:** Concept Addition Forms a Semi-Lattice. It follows immediately from the previous lemma that $\oplus$ is idempotent, commutative, and associative:

(.1) $c \oplus c = c$

(.2) $c \oplus d = d \oplus c$

(.3) $(c \oplus d) \oplus e = c \oplus (d \oplus e)$

In virtue of the last fact, we may leave off the parentheses in the expressions $(c \oplus d) \oplus e$ and $c \oplus (d \oplus e)$.

Thus, concept addition behaves in the manner that Leibniz prescribed. He took the first two of these theorems as *axioms* of his calculus, whereas we derive them as theorems.[217] Unfortunately, he omitted associativity from his list of axioms for $\oplus$; as Swoyer (1995, 1994) points out, it must be included for the proofs of certain theorems to go through.

(**487**) **Remark:** Concept Addition and Properties. We've analyzed concepts as abstract individuals and concept addition as a functional condition on concepts. Of course, Leibniz's texts use variables for concepts that suggest that he conceived of concepts as properties. For example, in 1690a and 1690b, he uses variables $A, B, \ldots, L, M, N$, etc., and he instantiates these variables with predicative expressions such as 'triangle', 'trilateral', 'rational', 'animal', etc. One may certainly try to reconstruct Leibnizian concepts as properties, but we now discuss an issue that arises for that analysis. On the property-theoretic analysis, concept addition is traditionally regarded as property conjunction, so that one would define:

$$F + G =_{df} [\lambda x\, Fx \,\&\, Gx]$$

So to derive (486.1) – (486.3) as theorems, one would have to derive:

$$[\lambda x\, Fx \,\&\, Fx] = F$$

$$[\lambda x\, Fx \,\&\, Gx] = [\lambda x\, Gx \,\&\, Fx]$$

$$[\lambda x\, [\lambda y\, Fy \,\&\, Gy]x \,\&\, Hx] = [\lambda x\, Fx \,\&\, [\lambda y\, Gy \,\&\, Hy]x]$$

If one were to take properties to be identical when materially (i.e., extensionally) equivalent, the above would be easy consequences, but it is well known that such a definition of property identity is incorrect. Moreover, one would also obtain the above consequences by defining property identity in terms of necessary equivalence, i.e., by defining $F = G$ as $\Box\forall x(Fx \equiv Gx)$. Again, the reasons why this definition is incorrect are well known, and the present theory doesn't endorse this definition. We have formulated our system so that

[217] See LLP 132 (G.vii 237), Axioms 2 and 1, respectively. Other idempotency assertions appear in LLP 40 (G.vii 222), LLP 56 (C 366), LLP 85 (C 396), LLP 90 (C 235), LLP 93 (C 421), and LLP 124 (G.vii 230). Swoyer (1995, footnote 5) also cites C 260 and C 262. Lenzen (1990) cites GI 171 for idempotency. Other commutativity assertions appear in LLP 40 (G.vii 222), LLP 90 (C 235), and LLP 93 (C 421).

one can consistently assert that there are properties F and G such that both $\Box\forall x(Fx \equiv Gx)$ and $F \neq G$.

If one allows for distinct properties that are necessarily equivalent with respect to exemplification, then it is not clear how to derive the above property identities in absence of further axioms governing property identity. Our policy has been to eschew such axioms. We have stated a comprehension principle (129.2) that asserts conditions under which properties exist, and defined precise conditions under which properties are identical (16.1). The latter tells us what exactly we are asserting or proving when we assert or prove that properties F and G are either identical or distinct. But, we take it to be an open question, to be decided as the case may demand, whether the above property identities are to be endorsed or not.

By understanding Leibnizian concepts as abstract individuals, one can derive (486.1) – (486.3) as theorems, but if one prefers the analysis of Leibnizian concepts as properties and takes properties seriously as intensional entities, then more work has to be done to derive the identities needed to show that property addition, as defined in this Remark, is idempotent, commutative, and associative. But it should also be noted that once *concepts* of properties are introduced in Section 13.2, where we define the concept of F and introduce the notation $\boldsymbol{c}_F$, then the following instances of (486.1) – (486.3) bring us even closer to Leibniz's texts:

$$\boldsymbol{c}_F \oplus \boldsymbol{c}_F = \boldsymbol{c}_F$$

$$\boldsymbol{c}_F \oplus \boldsymbol{c}_G = \boldsymbol{c}_G \oplus \boldsymbol{c}_F$$

$$(\boldsymbol{c}_F \oplus \boldsymbol{c}_G) \oplus \boldsymbol{c}_H = \boldsymbol{c}_F \oplus (\boldsymbol{c}_G \oplus \boldsymbol{c}_H)$$

These results at least provide an interpretation of the Leibnizian texts in which he asserts concept addition is idempotent and commutative.

(**488**) **Theorems:** Concept Addition and Identity. Leibniz proves two other theorems pertaining solely to concept addition and identity in LLP 133–4 (G.vii 238), Propositions 9 and 10:

(.1) $c = d \rightarrow c \oplus e = d \oplus e$

(.2) $c = d \;\&\; e = f \rightarrow c \oplus e = d \oplus f$

In the notes following Propositions 9 and 10 in LLP 133-134 (G.vii 238), Leibniz observes that counterexamples to the converses of these theorems can be produced. To produce a counterexample to the converse of (.1), first let P, Q, and R be any three distinct properties. We know there are such by (156.7). Then:

- let c_1 be $\imath c\forall F(cF \equiv F = P \vee F = Q \vee F = R)$

- let c_2 be $\imath c\forall F(cF \equiv F{=}P \vee F{=}Q)$
- let c_3 be $\imath c\forall F(cF \equiv F{=}R)$

Then it is straightforward to show that $c_1 \oplus c_3 = c_2 \oplus c_3$ and $c_1 \neq c_2$, contrary to the converse of (.1).

Similarly, to produce a counterexample to the converse of (.2), first let P, Q, R, and S be any four distinct properties. We know there are such by (156.7). Then:

- let c_1 be $\imath c\forall F(cF \equiv F{=}P \vee F{=}Q)$
- let c_2 be $\imath c\forall F(cF \equiv F{=}P)$
- let c_3 be $\imath c\forall F(cF \equiv F{=}R \vee F{=}S)$
- let c_4 be $\imath c\forall F(cF \equiv F{=}Q \vee F{=}R \vee F{=}S)$

It is then easy to show that $c_1 \oplus c_3 = c_2 \oplus c_4$ and $c_1 \neq c_2$ (and, indeed, $c_3 \neq c_4$), contrary to the converse of (.2).

13.1.2 Concept Inclusion and Containment

(**489**) **Definitions:** Inclusion and Containment. It is an algebraic fact that an idempotent, commutative, and associative operation on a domain induces a partial ordering on that domain. In the present case, concept addition induces the partial ordering of *concept inclusion* ($c \preceq d$) and a converse ordering of *concept containment* ($d \succeq c$). We'll prove these facts below. Leibniz was aware of this connection and derives these facts for the case of concept inclusion. In Definition 3 of LLP 132 (G.vii 237), Leibniz defined both $c \preceq d$ and $d \succeq c$ as $\exists e(c \oplus e = d)$.[218] Moreover, in Propositions 13 and 14 of LLP 135 (G.vii 239), Leibniz derives both directions of the equivalence $c \preceq d \equiv c \oplus d = d$ as theorems.

In object theory, however, a deeper level of analysis of concept inclusion and concept containment is available. Once that analysis is formulated, it follows that inclusion and containment partially order the concepts; this is item (490) below. Moreover, Leibniz's definition of inclusion in Definition 3 and the equivalence that he obtains via his Propositions 13 and 14 are both derivable from that analysis. These are theorems (493) and (494) below.

We begin by defining: *c is included in d* just in case d encodes every property c encodes. Formally:

(.1) $c \preceq d =_{df} \forall F(cF \rightarrow dF)$

[218] Strictly speaking, Leibniz didn't use the existential quantifier in his definition. But we make it clear in item (493) below that this is what he intended.

We shall see, in what follows, that this notion of concept inclusion is a generalization of the notion of *part-of*, which was defined on situations in (369).

Leibniz's notion of concept containment is now just the converse of inclusion. Let us say that d *contains* c just in case c is included in d:

(.2) $d \geq c \;=_{df}\; c \leq d$

Consequently, the theorems below are developed in pairs: one member of the pair governs concept inclusion and the other concept containment. However, in the Appendix, we prove the theorem only as it pertains to concept inclusion.

(**490**) **Theorems:** Concept Inclusion and Containment Are Partial Orders. It now follows that concept inclusion and containment are reflexive, anti-symmetric, and transitive. To show anti-symmetry, we use $c \not\leq d$ to abbreviate $\neg(c \leq d)$, and $c \not\geq d$ to abbreviate $\neg(c \geq d)$:

(.1) $c \leq c$
$c \geq c$

(.2) $c \leq d \;\rightarrow\; (c \neq d \rightarrow d \not\leq c)$
$c \geq d \;\rightarrow\; (c \neq d \rightarrow d \not\geq c)$

(.3) $c \leq d \;\&\; d \leq e \;\rightarrow\; c \leq e$
$c \geq d \;\&\; d \geq e \;\rightarrow\; c \geq e$

See LLP 133 (= G.vii 238), Proposition 7, for Leibniz's proof of the reflexivity of inclusion. See LLP 135 (= G.vii 240), Proposition 15, for Leibniz's proof of the transitivity of inclusion. See also LLP 33 (= G.vii 218) for the reflexivity of containment.

(**491**) **Theorems:** Inclusion, Containment, and Identity. Leibniz proves in LLP 136 (G.vii 240), Proposition 17, that when concepts c and d are included, or contained, in each other, they are identical. Hence we have the more general theorem:

(.1) $c = d \equiv c \leq d \;\&\; d \leq c$
$c = d \equiv c \geq d \;\&\; d \geq c$

Two interesting further consequences of concept inclusion and identity are that concepts c and d are identical whenever either of the following biconditionals hold: (.2) for all e, e is included in c iff e is included in d, and (.3) for all e, c is included in e iff d is included in e:

(.2) $c = d \equiv \forall e(e \leq c \equiv e \leq d)$
$c = d \equiv \forall e(c \geq e \equiv d \geq e)$

(.3) $c = d \equiv \forall e(c \leq e \equiv d \leq e)$
$c = d \equiv \forall e(e \geq c \equiv e \geq d)$

(**492**) **Theorems:** Inclusion and Addition. In LLP 33 (G.vii 218), Leibniz asserts '*ab* is *a*' and '*ab* is *b*'. Here it looks as if *ab* is to be interpreted as $a \oplus b$ and 'is' as containment. (This is an application of Leibniz's containment theory of truth, which will be discussed below.) Thus, in our system, these claims become: (.1) the sume of c and d contains c, and (.2) the sum of c and d contains d. In each case, we state the inclusion version first:

(.1) $c \preceq c \oplus d$
$c \oplus d \succeq c$

(.2) $d \preceq c \oplus d$
$c \oplus d \succeq d$

Leibniz also notes that if c is included in d, then the sum of e and c in included in the sum of e and d:

(.3) $c \preceq d \;\rightarrow\; e \oplus c \preceq e \oplus d$
$c \succeq d \;\rightarrow\; e \oplus c \succeq e \oplus d$

See LLP 134 (G.vii 239), Proposition 12. See also LLP 41 (G.vii 223), for the version governing containment. Note that there is a counterexample to the converse of (.3). If we let e_1 and c_1 both be $\imath c \forall F(cF \equiv F = P)$, and let d_1 be $\imath c \forall F(cF \equiv F = Q \vee F = R)$, where P, Q, R are all pairwise distinct, then it is easy to show that $e_1 \oplus c_1 \preceq e_1 \oplus d_1$ and $c_1 \not\preceq d_1$.

It also follows that $c \oplus d$ is included in e if and only if both c and d are included in e:

(.4) $c \oplus d \preceq e \;\equiv\; c \preceq e \,\&\, d \preceq e$
$e \succeq c \oplus d \;\equiv\; e \succeq c \,\&\, e \succeq d$

Leibniz notes a more economical form of the left-to-right direction of (.4). In LLP 136 (G.vii 240), Corollary to Proposition 15, he argues that if $A \oplus N$ is in B, then N is in B. However, he proves the right-to-left direction of (.4) at LLP 137 (G.vii 241), Proposition 18.

Finally, we may prove that if c is included in d and e is included in f, then $c \oplus e$ is included in $d \oplus f$:

(.5) $c \preceq d \,\&\, e \preceq f \;\rightarrow c \oplus e \preceq d \oplus f$
$c \succeq d \,\&\, e \succeq f \;\rightarrow c \oplus e \succeq d \oplus f$

See LLP 137 (= G.vii 241), Proposition 20. **Exercise:** Find a counterexample to the converse direction of (.5).

13.1.3 Concept Inclusion, Addition, and Identity

Now we show that our definitions of concept inclusion (containment), addition, and identity are all related in the appropriate way.

(**493**) **Theorem:** Leibnizian Definition of Inclusion. We prove Leibniz's definition of inclusion and containment as theorems:

$$c \preceq d \equiv \exists e(c \oplus e = d)$$
$$c \succeq d \equiv \exists e(c = d \oplus e)$$

Strictly speaking, while Leibniz's definition wasn't expressed an existential quantifier, it is clear that the above is what he intended. Compare the definiens in his Definition 3, referenced earlier:

> Definition 3. That *A* 'is in' *L*, or, that *L* 'contains' *A*, is the same as that *L* is assumed to be coincident with several terms taken together, among which is *A*.

This is the translation in LLP 132 of the passage in G.vii 237.

(**494**) **Theorem:** Leibniz's Equivalence. Our definition of $\preceq$ also validates the principal theorem governing Leibniz's non-modal calculus of concepts, namely, that c is included in d iff the sum of c and d is identical with d:

$$c \preceq d \equiv c \oplus d = d$$
$$c \succeq d \equiv c = c \oplus d$$

See LLP 135 (G.vii 239), Propositions 13 and 14. Though Leibniz apparently proves this theorem using our (493) as a definition, on our theory, no appeal to (493) needs to be made. So Leibniz's main principle governing the relationship between concept inclusion ($\preceq$), concept identity, and concept addition ($\oplus$) is derivable.

(**495**) **Theorem:** Leibniz's Proposition 23. In LLP 140, Proposition 23 is stated as "Given two disparate terms, A and B, to find a third term, C, different from them and such that A $\oplus$ B = A $\oplus$ C" (cf. G.vii 243). We can capture this as follows: (.1) If c is not included in d and d is not included in c, then there is a concept e such that (a) e is distinct from both c and d and (b) the sum of c and e is identical to the sum of c and d. Formally:

(.1) $(c \npreceq d \;\&\; d \npreceq c) \rightarrow \exists e(e \neq c \;\&\; e \neq d \;\&\; c \oplus e = c \oplus d)$

In the first steps of the proof, assume the antecedent and choose the witness to the existential claim to be $c \oplus d$; since c and d each encodes a property the other doesn't, $c \oplus d$ will be distinct from both c and d.

Note also the following consequence of our definitions, namely: (.2) c is included in d and d is not included in c if and only if some concept e not included in c is such that the sum of c and e is identical to d:

(.2) $(c \preceq d \;\&\; d \npreceq c) \equiv \exists e(e \npreceq c \;\&\; c \oplus e = d)$

This allows for the degenerate case in which where c encodes no properties and d encodes one or more properties.

13.1.4 The Algebra of Concepts

In this section we first investigate the algebraic principles derivable from our theory of concepts and then consider the extent to which this algebra constitutes a mereology. To anticipate a bit, we show that (Leibnizian) concepts are structured not only as a bounded lattice, but also as a complete Boolean algebra, including a null concept and a universal concept. We then examine the extent to which our algebra of concepts obeys the principles of mereology when our notion of concept inclusion, $x \preceq y$, is interpreted as: x is a part of y.

(**496**) **Definition:** Concept Multiplication (i.e., Concept Products). Recall that theorem (486) tells us that addition is idempotent, commutative, and associative on concepts and so forms a semi-lattice (following standard mathematical practice). Over the course of the next several items, we show that (Leibnizian) concepts form not just a semi-lattice but a *lattice*. We define concept multiplication by saying that c is a *product of* d and e just in case c encodes all and only the properties that d and e encode in common:

$$ProductOf(c,d,e) =_{df} \forall F(cF \equiv dF \,\&\, eF)$$

For example, where P, Q, R are three distinct properties:

let c_1 be $\iota c \forall F(cF \equiv F = P \vee F = Q)$

let c_2 be $\iota c \forall F(cF \equiv F = Q \vee F = R)$

let c_3 be $\iota c \forall F(cF \equiv F = Q)$

Since c_1 encodes just the properties P and Q, c_2 encodes just the properties Q and R, and c_3 encodes just the property Q, it is an easy exercise to show $ProductOf(c_3, c_1, c_2)$.

(**497**) **Theorems:** Existence of Products. In the usual way, we prove (.1) there exists a product of concepts d and e; (.2) there exists a unique product of concepts d and e; and (.3) the product of concepts d and e exists:

(.1) $\exists c ProductOf(c,d,e)$

(.2) $\exists! c ProductOf(c,d,e)$

(.3) $\exists y(y = \iota c ProductOf(c,d,e))$

These are, strictly speaking, conditional existence claims, given the free restricted variables, though we know that since concepts exist, unconditional existence claims can be derived from (.1) – (.3).

(**498**) **Restricted Term Definition:** Notation for the Product of Concepts d and e. By our last theorem, we are entitled to introduce notation for *the* product of concepts d and e, as follows:

$$d\otimes e \ =_{df} \ \imath c ProductOf(c,d,e)$$

As in the case of $\oplus$, expressions of the form $\kappa\otimes\kappa'$ may be considered well-formed and logically proper only when κ and κ' are known to be concepts, either by proof or by hypothesis. Finally, given the definition of *ProductOf*, it is clear that that $d\otimes e$ is (identical to) a canonical concept, for any concepts d and e.

(**499**) **Lemmas:** Strict Canonicity of Products. Where φ is the formula $dF \,\&\, eF$, it follows that (.1) φ is a rigid condition on properties:

(.1) $\Box\forall F((dF \,\&\, eF) \rightarrow \Box(dF \,\&\, eF))$

Thus $d\otimes e$ is a strictly canonical concept, by (188.2). So we can use (189.2) to establish the modally strict theorem that (.2) $d\otimes e$ is a concept that encodes just those properties F such that both dF and eF:

(.2) $C!d\otimes e \ \& \ \forall F(d\otimes eF \equiv dF \,\&\, eF)$

As in the case of $\oplus$, the first conjunct of (.2) is an exemplification formula consisting of a 1-place relation term $C!$ and a complex individual term $d\otimes e$. Finally, it follows that (.3) $d\otimes e$ is a product of d and e:

(.3) $ProductOf(d\otimes e, d, e)$

(.1) – (.3) are modally strict theorems used in the proofs of subsequent theorems.

(**500**) **Theorems:** Concept Multiplication Forms a Semi-Lattice. It follows immediately from the previous lemma that when we restrict our attention to concepts, $\otimes$ is (.1) idempotent, (.2) commutative, and (.3) associative:

(.1) $c\otimes c = c$

(.2) $c\otimes d = d\otimes c$

(.3) $(c\otimes d)\otimes e = c\otimes(d\otimes e)$

Thus concept multiplication, like concept addition, behaves like an algebraic operation.

(**501**) **Theorems:** The Laws of Absorption. We may now prove that laws of *absorption* hold with respect to $\oplus$ and $\otimes$. They are (.1) the sum of c and the product of c and d is identical to c; and (.2) the product of c and the sum of c and d is identical to c:

(.1) $c\oplus(c\otimes d) = c$

(.2) $c\otimes(c\oplus d) = c$

From theorems (486.1) – (486.3), (500.1) – (500.3), and (.1) – (.2) above, we have that (a) $\oplus$ and $\otimes$ both are idempotent, commutative, and associative, and (b) the absorption laws hold. Hence, an algebraist would say that the concepts are structured as a *lattice*, with $\oplus$ as the *join* and $\otimes$ as the *meet* for the lattice.

(**502**) **Theorems:** A Bounded Lattice of Concepts. Indeed, with just a little bit more work, one can show that (Leibnizian) concepts, and thus abstract individuals generally, are structured as a *bounded* lattice.

First, note that, by the definitions in (191), $Null(x)$ holds whenever x is an abstract object x that encodes no properties, and $Universal(x)$ holds whenever x is an abstract object that encodes every property. We showed that there is exactly one null object and exactly one universal object, and we introduced the notation $\boldsymbol{a}_{\varnothing}$ (193.1) for $\imath xNull(x)$ and the notation $\boldsymbol{a}_{\boldsymbol{V}}$ (193.2) for $\imath xUniversal(x)$. Since $C!$ is defined as $A!$, it follows that there is exactly one null concept and exactly one universal concept. So we may justifiably call $\boldsymbol{a}_{\varnothing}$ *the null concept* and call $\boldsymbol{a}_{\boldsymbol{V}}$ *the universal concept*.

Consequently the facts proved in (194) allow us to show (.1) the sum of a concept c and the null concept just is c; and (.2) the product of a concept c and the universal concept just is c:

(.1) $c \oplus \boldsymbol{a}_{\varnothing} = c$

(.2) $c \otimes \boldsymbol{a}_{\boldsymbol{V}} = c$

Thus, $\boldsymbol{a}_{\varnothing}$ is the identity element for concept addition, and $\boldsymbol{a}_{\boldsymbol{V}}$ is the identity element for concept multiplication. Indeed, $\boldsymbol{a}_{\varnothing}$ constitutes the minimal element and $\boldsymbol{a}_{\boldsymbol{V}}$ the maximal element in a bounded lattice of concepts. Since concepts just are abstract objects, we may observe that abstract objects generally are structured as a bounded lattice. Finally, note that (.3) the sum of c and the universal concept is just the universal concept, and (.4) the product of c and the null concept is just the null concept:

(.3) $c \oplus \boldsymbol{a}_{\boldsymbol{V}} = \boldsymbol{a}_{\boldsymbol{V}}$

(.4) $c \otimes \boldsymbol{a}_{\varnothing} = \boldsymbol{a}_{\varnothing}$

Thus, no concept survives addition with $\boldsymbol{a}_{\boldsymbol{V}}$ and no concept survives multiplication with $\boldsymbol{a}_{\varnothing}$.

(**503**) **Remark:** A Boolean Algebra. With just a few more definitions and theorems, we can show that (Leibnizian) concepts with concept addition and multiplication obey the principles of a Boolean algebra. This will occupy our attention over the course of the next few items. Since we already know, relative to the domain of concepts, that $\oplus$ and $\otimes$ are idempotent, commutative, and associative, that the absorption laws for $\oplus$ and $\otimes$ hold, that $\boldsymbol{a}_{\varnothing}$ is an identity element for $\oplus$, and that $\boldsymbol{a}_{\boldsymbol{V}}$ is an identity element for $\otimes$, it remains only to show that: (i) $\oplus$ distributes over $\otimes$, (ii) $\otimes$ distributes over $\oplus$, and (iii) concept complementation, $-c$, can be defined so that the complementation laws, $c \otimes -c = \boldsymbol{a}_{\varnothing}$ and $c \oplus -c = \boldsymbol{a}_{\boldsymbol{V}}$, both hold. The proofs that all of these fundamental axioms of Boolean algebra are theorems are completed below.

(**504**) **Theorems:** Distribution Laws. Since disjunction distributes over conjunction (63.7.b) and vice versa (63.7.a), it follows that (.1) $\oplus$ distributes over $\otimes$, and (.2) $\otimes$ distributes over $\oplus$:

(.1) $c \oplus (d \otimes e) = (c \oplus d) \otimes (c \oplus e)$

(.2) $c \otimes (d \oplus e) = (c \otimes d) \oplus (c \otimes e)$

(**505**) **Definition:** Complements. We say that c is a *complement of d* whenever c encodes exactly the properties that d fails to encode::

$$ComplementOf(c,d) =_{df} \forall F(cF \equiv \neg dF)$$

For example:

let c_1 be $\iota c\forall F(cF \equiv F = P)$

let c_2 be $\iota c\forall F(cF \equiv F \neq P)$

Since c_1 encodes just the property P and c_2 encodes all and only properties other than P, it follows that $ComplementOf(c_2, c_1)$.

(**506**) **Theorems:** Facts about Complementation. In the usual way, it follows that (.1) there exists a complement of d; (.2) there exists a unique complement of d; and (.3) there exists something that is the complement of d:

(.1) $\exists c ComplementOf(c,d)$

(.2) $\exists! c ComplementOf(c,d)$

(.3) $\exists y(y = \iota c ComplementOf(c,d))$

These existence claims are conditional on the fact that d is a concept, but since concepts exists, unconditional existence claims are derivable.

(**507**) **Restricted Term Definition:** The Complement of d. Given our last theorem, we are entitled to introduce notation for *the* concept that is a complement of d:

$$-d =_{df} \iota c ComplementOf(c,d)$$

Since d is a free restricted variable in this definition, an expression of the form $-\kappa$ may be regarded as well-formed and logically proper only when κ is known to be a concept, either by proof or by hypothesis.

(**508**) **Lemmas:** Strict Canonicity of Complements. It is a fact that (.1) the formula $\neg dF$ is a rigid condition on properties:

(.1) $\Box\forall F(\neg dF \to \Box\neg dF)$

Hence, $-d$ is a strictly canonical concept, by (188.2). So by (189.2), we can easily establish that (.2) $-d$ is a concept that encodes all and only the properties that d fails to encode:

(.2) $C!{-d}\ \&\ \forall F(-dF \equiv \neg dF)$

Finally, it follows that (.3) $-d$ is a complement of d:

(.3) $\mathit{ComplementOf}(-d, d)$

(.1) – (.3) are modally strict theorems used in the proofs of subsequent theorems.

(**509**) **Theorems:** Complementation Laws. The complementation laws are now theorems. They are (.1) the sum of c and $-c$ is the universal concept, and (.2) the product of c and $-c$ is the null concept:

(.1) $c \oplus -c = \boldsymbol{a_V}$

(.2) $c \otimes -c = \boldsymbol{a}_{\varnothing}$

Since we have now established the commutativity and associativity of $\oplus$ and $\otimes$, the distribution laws for $\oplus$ over $\otimes$ and for $\otimes$ over $\oplus$, the absorption laws, and the complementation laws, we have shown that concepts form a Boolean algebra. Of course, since our restricted variables for concepts can be considered as restricted variables for abstract individuals, given that $C!$ is defined as $A!$, we have shown that abstract individuals form a Boolean algebra.

(**510**) **Theorems:** Other Traditional Principles of Boolean Algebra. We close our discussion of the Boolean algebra of concepts by noting a few final theorems, namely, double complementation and the two De Morgan Laws. These are (.1) the complement of the complement of c just is c; (.2) the sum of $-c$ and $-d$ is identical to the complement of the product of c and d; and (.3) the product of $-c$ and $-d$ is identical to the complement of the sum of c and d:

(.1) $-(-c) = c$

(.2) $-c \oplus -d = -(c \otimes d)$

(.3) $-c \otimes -d = -(c \oplus d)$

(**511**) **Exercises:** Concept Difference and Overlap. In 1690a, Leibniz introduces concept subtraction (Theorems VIII – XII). He remarks, in a footnote, that "in the case of concepts, subtraction is one thing, negation another" (1690a, LLP 127). In Theorem IX, he introduces the notion of 'communicating' concepts, by which he seems to mean concepts that in some sense overlap. He then proves the theorem (Theorem X) that if N is the result of subtracting A from L, then A and N are uncommunicating (i.e., don't overlap). As an exercise, define:

(.1) $DifferenceOf(c, d, e) =_{df} \forall F(cF \equiv dF \,\&\, \neg eF)$

(.2) $Overlap(c, d) =_{df} \exists F(cF \,\&\, dF)$

Prove there is a unique concept that is the difference of d and e, that the difference concept of d and e exists, and introduce a restricted term, $d \ominus e$, for that concept:

(.3) $\exists! c DifferenceOf(c, d, e)$

(.4) $\exists y(y = \imath c DifferenceOf(c, d, e))$

(.5) $d \ominus e =_{df} \imath c DifferenceOf(c, d, e)$

Show that $d \ominus e$ is strictly canonical and prove, as a modally strict theorem, that $d \ominus e$ is a concept that encodes all and only the properties F such that dF and not eF:

(.6) $C!d \ominus e \,\&\, \forall F(d \ominus eF \equiv dF \,\&\, \neg eF)$

Then prove Theorem X in Leibniz 1690a:

(.7) $d \ominus e = c \rightarrow \neg Overlap(e, c)$

and show that it is equivalent to:

$$\neg Overlap(e, d \ominus e)$$

Show that the difference of d and e is identical to the product of d and $-e$, i.e., that:

(.8) $d \ominus e = d \otimes -e$

Formulate and prove other theorems governing concept difference and overlap, and determine whether other principles in Leibniz 1690a can be represented and derived. For example, while it is clear how to think about $d \ominus e$ when $e \preceq d$, what happens when e is not included in d? **Exercise 1:** Consider the scenario in which d and e don't overlap and, in particular, consider any three, pairwise distinct, properties, say P, Q, and R, and the following two objects:

$$d_1 = \imath c \forall F(cF \equiv F = P \vee F = Q)$$

$$e_1 = \imath c \forall F(cF \equiv F = R)$$

Give a systematic answer to the questions, what properties does $d_1 \ominus e_1$ encode, and what properties does $e_1 \ominus d_1$ encode? **Exercise 2:** Consider the scenario in which d is included in e but not identical to e and, in particular, consider:

$$d_1 = \imath c \forall F(cF \equiv F = P)$$

$$e_1 = \imath c \forall F(cF \equiv F = P \vee F = Q)$$

Identify $d_1 \ominus e_1$ and $e_1 \ominus d_1$ in terms of particular concepts we've already discussed.

Finally, prove:

(.9) $d = e \equiv (d \ominus e = e \ominus d)$

(.10) $Overlap(c,d) \equiv c \otimes d \neq \boldsymbol{a}_{\varnothing}$

(.11) $\neg Overlap(c,d) \equiv c \ominus d = c$

(.12) $\neg \exists e(c \oplus e = \boldsymbol{a}_{\varnothing})$

13.1.5 The Mereology of Concepts

(**512**) **Remark:** Mereology. Now that we have established that concepts form a Boolean algebra, we consider the ways in which they constitute a mereology, i.e., the ways in which the notions of *part* and *whole* can be defined and applied to concepts. Though some authors take mereology to apply primarily to the domain of concrete individuals (e.g., Simons 1987, 4),[219] others assume that mereology makes no assumptions about what kinds of entities have parts. Thus, Varzi (2015, §1) writes:

> ... it is worth stressing that mereology assumes no ontological restriction on the field of 'part'. In principle, the relata can be as different as material bodies, events, geometric entities, or spatio-temporal regions, ... as well as abstract entities such as properties, propositions, types, or kinds, ...As a formal theory ... mereology is simply an attempt to lay down the general principles underlying the relationships between an entity and its constituent parts, whatever the nature of the entity, just as set theory is an attempt to lay down the principles underlying the relationships between a set and its members. Unlike set theory, mereology is not committed to the existence of abstracta: the whole can be as concrete as the parts. But mereology carries no nominalistic commitment to concreta either: the parts can be as abstract as the whole.

Moreover, it is often thought that the entities of a mereological domain are structured algebraically. Simons notes that "the algebraic structure of a full classical mereology is that of a complete Boolean algebra with zero deleted"

[219]However, in 1987 (169–171), Simons does offer a few brief thoughts about mereology and abstract objects.

(1987, 25).[220] Consequently, since (a) mereological principles can be understood broadly as applying to abstract objects such as concepts, (b) concepts are abstract objects, and (c) the previous section shows that the domain of concepts is structured algebraically, it seems reasonable to investigate the extent to which concepts are provably governed by mereological principles.

Our discussion in the remainder of this section will be organized as follows. We begin by examining what happens when we interpret the mereological notion *part of* as the *inclusion* ($\preceq$) condition on concepts. That is, we confirm that core mereological principles are preserved when we both (a) define c is a *part of* d just in case d encodes every property c encodes and (b) define *proper part of* in the usual mereological way. Then we examine a variety of consequences of our definitions and consider whether they are acceptable as mereological principles. Finally, we examine what mereological principles provably apply to *non-null* concepts, i.e., concepts that encode at least one property. We discover that while some questionable mereological principles apply to concepts generally, they do not apply to non-null concepts.

In what follows, we assume familiarity with the basic notions and principles of mereology. Systems of mereology are typically, though not always, formulated in one of two ways. Some systems take *x is a part of y* ($x \preceq y$) as a primitive relation or as a relational condition that is, at a minimum, reflexive, anti-symmetric, and transitive. Others start with *x is a proper part of y* ($x \prec y$) as the primitive relation (or relational condition) that is, at a minimum, irreflexive, asymmetric, and transitive.[221] Little hangs on the choice of formulation since it is a well-known fact that to every non-strict partial order (e.g., one based on $\preceq$) that is reflexive, anti-symmetric and transitive, there corresponds a strict partial order (e.g., one based on $\prec$) that is provably irreflexive, asymmetric, and transitive. To obtain $\prec$ when starting with $\preceq$, one defines $x \prec y$ as $x \preceq y \mathbin{\&} x \neq y$. Then the irreflexivity, asymmetry, and transitivity of $\prec$ follow from facts about $\preceq$ and $\neq$. Alternatively, to obtain $\preceq$ when starting with $\prec$, one defines $x \preceq y$ as $x \prec y \vee x = y$. Then the reflexivity, anti-symmetry and transitivity of $\preceq$ follow from facts about $\prec$ and $=$. We begin by seeing how these ideas are confirmed in the present theory, under object-theoretic definitions of the notions involved.

(**513**) **Remark:** Part Of. Though mereology often takes *part of* as a primitive, we may define it object-theoretically and show that its core features are derivable. So let us *re*-introduce definition (489.1) but in such a way that the definiendum,

[220] A zero element is defined mereologically as an individual that is a part of every individual. It is often thought that there is no such zero element in a domain of concrete individuals.

[221] There are other ways of formulating a mereology, for example, by taking *x overlaps y* as a primitive (Goodman 1951), or by taking *x is disjoint from y* as a primitive (Leonard and Goodman 1940). But these variations need not distract us in what follows. See Simons 1987, p. 48ff.

$c \preceq d$, is to be read: concept c is a *part of* concept d. Thus, the following definition now systematizes a different pretheoretical notion:

$$c \preceq d \ =_{df} \ \forall F(cF \to dF) \tag{489.1}$$

Note that since the definiens has encoding subformulas, there is no guarantee that $\preceq$ defines a relation on concepts. Nevertheless, $c \preceq d$ is a well-defined 2-place condition on concepts and, indeed, on abstract objects generally.

Given the above definition, the theorems in (490) guarantee that *part of* ($\preceq$) is reflexive, anti-symmetric and transitive with respect to the concepts:

(.1) $c \preceq c$ (490.1)
A concept is a part of itself.

(.2) $c \preceq d \ \to \ (c \neq d \to d \npreceq c)$ (490.2)
If a concept c is a part of a distinct concept d, then d is not a part of c.

(.3) $c \preceq d \ \& \ d \preceq e \ \to \ c \preceq e$ (490.3)
If c is part of d and d is part of e, then c is part of e.

Thus, it is clear that the above definition of $\preceq$ captures the notion of *improper part of* since it allows for the case where both $c \preceq d$ and $d \preceq c$.

Varzi (2015) notes that principles such as these "represent a common starting point of all standard [mereological] theories" (2015, §2.2). However, he notes later that "[n]ot just any partial ordering qualifies as a part-whole relation, though, and establishing what further principles should be added ... is precisely the question a good mereological theory is meant to answer" (2015, §3). We shall return to this question below.

(**514**) **Definitions:** Proper Part Of. We define *proper part of* in the usual way, though relative to the domain of concepts. We say concept c is a *proper part* of concept d, written $c \prec d$, just in case c is a part of d and c is not equal to d:

$$c \prec d \ =_{df} \ c \preceq d \ \& \ c \neq d$$

Warning! In what follows, we sometimes cite Simons' classic text of 1987, which uses $<$ and $\ll$ for improper and proper parthood, respectively. Thus, our symbol $\preceq$ for improper parthood corresponds to his symbol $<$, and our symbol $\prec$ for proper parthood corresponds to his symbol $\ll$. So it is important not to confuse our symbol for proper parthood ($\prec$) with his symbol for improper parthood ($<$).

(**515**) **Theorems:** Principles of Proper Parthood. As expected, we can now derive that *proper parthood* is a strict partial ordering with respect to concepts. It immediately follows that (.1) c is not a proper part of itself; (.2) if c is a proper of d and d is a proper part of e, then c is a proper part of e; and (.3) if c is a proper part of d, then d is not a proper part of c:

(.1) $c \not\prec c$ (Irreflexivity)

(.2) $c \prec d \;\&\; d \prec e \rightarrow c \prec e$ (Transitivity)

(.3) $c \prec d \rightarrow d \not\prec c$ (Asymmetry)

Simons notes that these principles "fall well short of characterizing the [proper] part-relation; there are many [strict] partial orderings which we should never call part-whole systems" (1987, 26). In what follows, we examine the extent to which our theorems conform to the accepted principles of part-whole systems, by considering the broader picture of how *traditional* mereological notions fare in the current setting.

(**516**) **Definitions:** Bottom Element, Concepts With a Single Proper Part, and Atoms. Three of the most basic mereological issues are: (a) whether there exists an individual that is a part of every individual (i.e., whether there exists a 'bottom' or 'zero' element), (b) whether there are any individuals that have a single proper part (i.e., exactly one proper part), and (c) whether there exist any individuals that have no proper parts (i.e., whether there exist any 'atoms').

The traditional mereology of concrete individuals eschews the existence of bottom elements and individuals with exactly one proper part, though the existence of atoms is permitted. For example, Simons writes (1987, 13):

> . . . in normal set theory even two disjoint sets have an intersection, namely the null set, whereas disjoint individuals precisely lack any common part. Most mereological theories have no truck with the fiction of a null individual which is part of all individuals, although it neatens up the algebra somewhat.

Clearly, the preference Simons is describing seems reasonable when one supposes, as he does, that mereology is restricted to the study of the part-whole relation on concrete individuals. But that is not the case in the present context.

Similarly, traditional mereology eschews concrete wholes having a single proper part. Simons rhetorically asks and then asserts, "How could an individual have a *single* proper part? That goes against what we mean by 'part'" (1987, 26). Again, this view seems reasonable when mereology is limited to the field of concrete objects, for if there is no bottom individual, then whenever a concrete individual has one proper part, it would seem that there must be a second proper part, disjoint from the first, that supplements the first so as to form the whole.[222]

By contrast, in our algebra of abstract individuals, it follows that there is a unique bottom concept and that there are concepts having a single proper

[222] It is worth noting that Varzi (2015, §3.1) compiles a list of objects with a single proper part that have been postulated by philosophers.

part. Moreover, there are conceptual atoms. To see that these are facts, let us begin by saying that (.1) c is a *bottom* concept just in case c is a part of every concept; and (.2) c is an *atom* just in case c has no concepts as proper parts:

(.1) $Bottom(c) \ =_{df} \ \forall d(c \preceq d)$

(.2) $Atom(c) \ =_{df} \ \neg\exists d(d \prec c)$

In these definitions, then, standard mereological notions have been adapted to the present context (Simons 1987, 16; Varzi 2015, §3.4).

(**517**) **Theorems:** Facts about Bottom Concepts, Concepts With a Single Proper Part, and Atoms. Recall that we relabeled $\boldsymbol{a}_{\varnothing}$ as *the null concept* (502). Since modally-strict theorem (194.3), i.e., $Null(\boldsymbol{a}_{\varnothing})$, implies that the null concept encodes no properties, we can now prove: (.1) the null concept is a bottom concept; (.2) there is a unique bottom concept; (.3) the null concept is a proper part of the thin Form of G; and (.4) the thin Form of G has exactly one proper part; (.5) bottom concepts are atoms; (.6) the null concept is an atom; and (.7) there is exactly one atom:

(.1) $Bottom(\boldsymbol{a}_{\varnothing})$

(.2) $\exists! c Bottom(c)$

(.3) $\boldsymbol{a}_{\varnothing} \prec \boldsymbol{a}_G$

(.4) $\exists! d(d \prec \boldsymbol{a}_G)$

(.5) $Bottom(c) \rightarrow Atom(c)$

(.6) $Atom(\boldsymbol{a}_{\varnothing})$

(.7) $\exists! c Atom(c)$

Later, we shall return to the discussion of bottom concepts, concepts with exactly one proper part, and atoms; if we study how *part of* behaves when restricted to *non-null concepts* (i.e., concepts that encode at least one property), then there are related notions of bottoms and atoms that do not yield the above consequences. To give a hint of what is to come, $\boldsymbol{a}_{\varnothing}$ fails to be a *non-null bottom* (i.e., a non-null concept that is a part of every non-null concept) since it fails to be a non-null concept; indeed, there is no non-null bottom element. Moreover, $\boldsymbol{a}_{\varnothing}$ fails to be a *non-null atom* (i.e., a non-null concept that has no proper parts), though as we shall see, there nevertheless are non-null atoms. But we shall discuss these ideas in more detail below, starting with item (524).

(**518**) **Definitions:** Mereological Overlap. In Exercise (511), we suggested the following definition as part of an exercise: *c overlaps d* whenever there is a property F that both c and d encode:

(.1) $Overlap(c,d) =_{df} \exists F(cF \,\&\, dF)$

Clearly, this is a natural understanding of overlap in our object-theoretic setting. However, readers familiar with texts on mereology will recognize that the standard definition of *mereological overlap* is different from (.1). The standard mereological definition is that individuals x and y overlap whenever they have a common part (Simons 1987, 11, 28; Varzi 2015, §2.2). Let us formulate this latter definition in the present context by saying that *c overlaps** *d* just in case there is a concept e that is a part of both concepts c and d:

(.2) $Overlap^*(c,d) =_{df} \exists e(e \leq c \,\&\, e \leq d)$

These two notions of overlap have some interesting consequences.

(**519**) **Theorems:** Overlap vs. Overlap*. We first observe that the notions of overlap and overlap* are not equivalent; (.1) overlap implies overlap*, but (.2) overlap* doesn't imply overlap:

(.1) $Overlap(c,d) \to Overlap^*(c,d)$

(.2) $\exists c \exists d(Overlap^*(c,d) \,\&\, \neg Overlap(c,d))$

Clearly, since the null concept $\boldsymbol{a}_{\varnothing}$ encodes no properties, it overlaps with nothing and so doesn't overlap with itself. Hence it follows that (.3) overlap is not reflexive:

(.3) $\neg \forall c Overlap(c,c)$

But, trivially, since every concept is a part of itself, (.4) overlap* is reflexive:

(.4) $Overlap^*(c,c)$

By contrast, both overlap and overlap* are symmetric:

(.5) $Overlap(c,d) \to Overlap(d,c)$

(.6) $Overlap^*(c,d) \to Overlap^*(d,c)$

More conclusively, though, whereas (.7) not all concepts overlap each other, (.8) all concepts overlap* each other:

(.7) $\neg \forall c \forall d Overlap(c,d)$

(.8) $Overlap^*(c,d)$

(.8) is a consequence of the definition of overlap* and the fact that the null concept, $\boldsymbol{a}_{\varnothing}$, is a bottom element: since $\boldsymbol{a}_{\varnothing}$ is a part of every concept, it is a common part of every two concepts. Given the mereological commitment to the definition of *overlap* as *having a common part*, a consequence such as (.8) makes it clear why traditional mereologists eschew a *bottom* element. Varzi thus notes that "[I]n general ... mereologists tend to side with traditional wisdom and steer clear of (P.10) [which asserts the existence of a bottom element] altogether" (2015, §3.4). We shall see, however, that mereological overlap again becomes a useful condition when we turn our attention to the non-null concepts – mereological overlap doesn't hold universally on that subdomain.

Let's return, then, to the object-theoretic definition of overlap, whereby concepts overlap if they encode a common property. Then note that for arbitrary concepts c and d, (.9) there is a concept e that is a part of d but which doesn't overlap with c:

(.9) $\exists e(e \preceq d \;\&\; \neg Overlap(e,c))$

(The reason for ordering the free variables in this way will become apparent when we discuss supplementation principles, in the next item.) While (.9) is made trivially true when $\boldsymbol{a}_\varnothing$ is taken as a witness, the theorem also holds when $d \ominus c$ is selected as the witness; see the proof in the Appendix. Furthermore, the theorem holds even in the degenerate case where c and d take $\boldsymbol{a}_\varnothing$ as their value; in this case, the witness is still $\boldsymbol{a}_\varnothing$ (which is then identical to $d \ominus c$), since $\boldsymbol{a}_\varnothing$ is a part of itself and fails to overlap with itself.

Finally, note that the principle which results when we replace $\preceq$ in (.9) with $\prec$ doesn't hold for arbitrary concepts c and d. We have:

(.10) $\neg\forall c \forall d \exists e(e \prec d \;\&\; \neg Overlap(e,c))$

For suppose $\boldsymbol{a}_\varnothing$ is a substitution instance for d and any concept you please is taken as a substitution instance for c. Then since we know $\boldsymbol{a}_\varnothing$ is an atom, it has no proper parts, i.e., $\neg\exists e(e \prec \boldsymbol{a}_\varnothing)$. *A fortiori*, $\neg\exists e(e \prec \boldsymbol{a}_\varnothing \;\&\; \neg Overlap(e,c))$.

(**520**) **Theorems:** Supplementation Principles. Both weak and strong supplementation principles of mereology trivially follow from (519.9). The *weak supplementation* principle of mereology is that if an individual x is a proper part of individual y, then there exists an individual z that is both a part of y and fails to overlap x (Varzi 2015, §3.1, item P.4). The *strong supplementation* principle is that if y fails to be a part of x, then there is a z such that z both is a part of y and fails to overlap x (Varzi 2015, §3.2, item P.5; Simons 1987, 29, SA5/SSP). When we consider the domain of concepts, these become the theorems that (.1) if c is a proper part of d, then there is a concept e such that e is a part of d and e fails to overlap c, and (.2) if d fails to be a part of c, then there is a concept e such that e is a part of d and fails to overlap c:

(.1) $c \prec d \rightarrow \exists e(e \preceq d \;\&\; \neg Overlap(e,c))$

(.2) $d \npreceq c \rightarrow \exists e(e \preceq d \;\&\; \neg Overlap(e,c))$

Since the consequents of both of these conditionals are just instances of (519.9), both theorems follow trivially. Given that (.1) and (.2) are theorems independent of the truth of their antecedents, a mereologist will no doubt regard this as a reason to question the definition of overlap. This is a fair point and we shall address this issue when we examine notions of overlap as they apply to non-null concepts.

However, note that the following variant of weak supplementation is a non-trivial theorem, namely, (.3) that if c is a proper part of d, then there is a concept e that is a proper part of d that fails to overlap with c:

(.3) $c \prec d \rightarrow \exists e(e \prec d \;\&\; \neg \mathit{Overlap}(e,c))$

We saw in (519.10) that the consequent of the above claim doesn't hold for arbitrary concepts c and d. So (.3) isn't a theorem merely in virtue of the truth of the consequent. Note, however, that one can't take the witness to the consequent to be $d \ominus c$. For in the case where c is the null concept $\boldsymbol{a}_{\varnothing}$, it is not the case that $d \ominus c \prec d$, since $d \ominus c$ just is d. So the proof of (.3) requires that the witness to the consequent be $\boldsymbol{a}_{\varnothing}$. Hence, one can reasonably argue that (.3) is not a true supplementation principle, since it is reasonable to suppose that $\boldsymbol{a}_{\varnothing}$ doesn't supplement anything. This becomes a non-issue, however, when we study supplementation principles with respect to the non-null concepts below.

(**521**) **Definition:** Underlap and Superconcepts. In traditional mereology, two individuals underlap just in case there is an individual of which they are both a part. The object-theoretic notion of underlap is no different; in the theory of concepts, we may say (.1) c *underlaps* d just in case there is a concept e such that both c and d are a part of e:

(.1) $\mathit{Underlap}(c,d) =_{df} \exists e(c \preceq e \;\&\; d \preceq e)$

Moreover, let us define (.2) a *superconcept* to be any concept of which every concept is a part:

(.2) $\mathit{Superconcept}(c) =_{df} \forall d(d \preceq c)$

(**522**) **Theorem:** Superconcepts and Underlap. It is a well-known fact of mereology that if there exists a universal entity, of which every individual is a part, then every individual underlaps every individual. This is borne out by the present theory. We have that (.1) $\boldsymbol{a}_{\boldsymbol{V}}$ is a superconcept:

(.1) $\mathit{Superconcept}(\boldsymbol{a}_{\boldsymbol{V}})$

It now follows that (.2) underlap holds universally among concepts:

(.2) $\mathit{Underlap}(c,d)$

Note, by the way, that definition (521.1) and (.2) above are excellent examples of how the elimination of free restricted variables may require distinct methods depending on the context.[223]

(**523**) **Remark:** Dodging Mereological Controversies. We've now seen a variety of new theorems, with $\leq$ considered as *improper parthood*. Some of those theorems, so understood, preserve key mereological principles, while others may be thought doubtful as such. The main concerns are the consequences of the fact that the null concept $\boldsymbol{a}_{\varnothing}$ is a bottom concept that is a part of every concept, and the fact that the definition of overlap, which is stated in terms of encoding rather than in terms of the mereological notion of parthood, yields uninteresting supplementation principles.

But as mentioned previously, one can sidestep these concerns by finding a way to exclude the null concept as a part (improper or proper) of any concept. There are at least two ways in which this can be done. Varzi suggests (2015, §3.4) that one "treat the null item as a mere algebraic fiction" and revise the definition of parthood as follows:

Genuine Parthood
$GP(x,y) \ =_{df} \ x \leq y \,\&\, \exists z \neg(x \leq z)$

To understand the suggestion better, note that the second conjunct of the definiens is equivalent to $\neg\forall z(x \leq z)$. Given that $Bottom(x,z)$ is defined as $\forall z(x \leq z)$, the above definition becomes: x is a *genuine part* of y iff x is a part of y and not a bottom element. Now in the present theory of concepts, we know that there is a unique bottom concept, namely, $\boldsymbol{a}_{\varnothing}$. So in the context of our theory, Varzi's suggestion could be implemented by defining: c is a *genuine part* of d just in case c is a part of d other than $\boldsymbol{a}_{\varnothing}$ (or just in case c is part of d and c encodes some property). Varzi then suggests a definition of 'genuine overlap' in terms of genuine parthood; in object theory, his definition would be implemented as follows: *c genuinely overlaps d* if and only if there is a concept e that is a genuine part of both.

While this is a perfectly legitimate course to follow, in the present theory, it makes better sense to hold the notions of parthood and proper parthood fixed and consider how they behave on a subdomain, i.e., approach the matter by investigating how the defined notions of overlap, bottom, atom, etc., behave

[223]The free restricted variables in the expression $Underlap(c,d)$ are to be eliminated in different ways depending on the context. They are to be eliminated one way when used in a definition and another way when they appear in a theorem. In particular, definition (521) is shorthand for:

$$Underlap(x,y) \ =_{df} \ C!x \,\&\, C!y \,\&\, \exists z(C!z \,\&\, x \leq z \,\&\, y \leq z)$$

By contrast, (.2) is shorthand for:

$$(C!x \,\&\, C!y) \rightarrow Underlap(x,y)$$

We thus have a nice example of one of the points raised in Chapter 10, Section 10.4.

when restricted to the concepts that encode at least one property, i.e., with respect to *non-null* concepts. We now turn to a development of this idea.

(**524**) **Definition:** Non-null Concepts. Let us say that x is a *non-null concept*, written $Concept^{+}(x)$, just in case x is a concept and x encodes some property:

$$Concept^{+}(x) =_{df} C!x \,\&\, \exists FxF$$

Though we could just as well said that a non-null concept is any concept other than the null concept $\boldsymbol{a}_{\varnothing}$, the above definition avoids the additional layer of definition that would be contributed by using a notion of (non-)identity in the definiens.

Clearly, $Concept^{+}(x)$ is a well-defined condition and there are abstract objects that satisfy the condition. Since it is also clear that concepts^{+} are concepts, we may introduce well-behaved (doubly) restricted variables $\underline{c}, \underline{d}, \underline{e}, \ldots$ to range over concepts^{+}. These are doubly restricted because we may eliminate the variables in one of two ways, as discussed in (401). For example, we may regard $\forall \underline{c}\varphi$ either as $\forall x(Concept^{+}(x) \to \varphi)$ or as $\forall c(Concept^{+}(c) \to \varphi)$.

(**525**) **Definitions:** Non-null Bottoms. Let us say that a non-null concept $\underline{c}$ is a *non-null bottom*, written $Bottom^{+}(\underline{c})$, just in case $\underline{c}$ is a part of every non-null concept:

$$Bottom^{+}(\underline{c}) =_{df} \forall \underline{d}(\underline{c} \preceq \underline{d})$$

(**526**) **Theorems:** Facts about Non-null Bottoms. The foregoing definition has the following consequences: (.1) The null concept is not a non-null bottom; and (.2) no non-null concept is a non-null bottom:

(.1) $\neg Bottom^{+}(\boldsymbol{a}_{\varnothing})$

(.2) $\neg\exists \underline{c} Bottom^{+}(\underline{c})$

So the mereologist's expectation that there be no bottom element is met when we restrict our attention to non-null concepts (cf. Simons 1987, 13, 25; Varzi 2015, §3.4).

(**527**) **Definition:** Non-null Atoms. Let us say that a non-null concept $\underline{c}$ is a *non-null atom*, written $Atom^{+}(\underline{c})$, just in case $\underline{c}$ has no non-null concepts as proper parts:

$$Atom^{+}(\underline{c}) =_{df} \neg\exists \underline{d}(\underline{d} \prec \underline{c})$$

(**528**) **Theorems:** Facts about Non-null Atoms and Proper Parthood for Non-null Concepts. The foregoing definition yields the following consequences: (.1) the Thin Form of G is a non-null atom; (.2) no non-null concepts have a single (non-null concept as a) proper part; (.3) non-null atoms encode at most one property; and (.4) a non-null concept is a non-null atom if and only if it is a thin form:

(.1) $Atom^{+}(\boldsymbol{a}_G)$

(.2) $\neg\exists\underline{c}\exists!\underline{d}(d \prec c)$

(.3) $Atom^{+}(\underline{c}) \rightarrow \forall F\forall G((\underline{c}F \,\&\, \underline{c}G) \rightarrow F = G)$

(.4) $Atom^{+}(\underline{c}) \equiv ThinForm(\underline{c})$

Note that (.2) preserves Simon's intuition, mentioned earlier (1987, 26), that no mereological individual has a single proper part.

(**529**) **Theorem:** Overlap on Non-null Concepts. By restricting our attention to non-null concepts, it emerges that non-null concepts $\underline{c}$ and $\underline{d}$ overlap just in case there is a non-null concept $\underline{e}$ that is a part of both:

$$Overlap(\underline{c},\underline{d}) \equiv \exists\underline{e}(\underline{e} \preceq \underline{c} \,\&\, \underline{e} \preceq \underline{d})$$

This consequence merits a brief discussion.

(**530**) **Remark:** Object-Theoretic and Mereological Overlap on Non-null Concepts. The previous theorem shows that the object-theoretic definition of *overlap* is equivalent to the traditional mereological definition of overlap when we restrict our attention to non-null concepts. That is, suppose we revise the definition of overlap* in (518.2) to read: non-null concepts $\underline{c}$ and $\underline{d}$ *mereologically overlap*, written $Overlap^{+}(\underline{c},\underline{d})$, just in case there is a non-null concept $\underline{e}$ that is a part of both, i.e.,

$$Overlap^{+}(\underline{c},\underline{d}) =_{df} \exists\underline{e}(\underline{e} \preceq \underline{c} \,\&\, \underline{e} \preceq \underline{d})$$

Then (529) tells us that object-theoretic overlap, defined in (518.1), is *equivalent* to mereological overlap. Henceforth, then, we'll suppose that object-theoretic overlap is a *bona fide* mereological notion.

Of course, lots of questions now arise. For example: (i) which of the suspicious theorems about overlapping concepts can be preserved as *bona fide* principles about overlapping non-null concepts? (ii) what mereological principles of supplementation hold with respect to non-null concepts? These and other questions will be answered below.

(**531**) **Theorems:** Facts about Overlap and Non-null Concepts. (.1) Not all non-null concepts overlap; (.2) overlap is reflexive on the non-null concepts; (.3) overlap is symmetric on the non-null concepts; (.4) overlap is not transitive on the non-null concepts; (.5) it is not the case that for any non-null concepts c and d, there is non-null concept e that is a part of d but which doesn't overlap c; and (.6) it is not the case that for any non-null concepts c and d, there is non-null concept e that is a proper part of d but which doesn't overlap c:

(.1) $\neg\forall\underline{c}\forall\underline{d}Overlap(\underline{c},\underline{d})$

(.2) $Overlap(\underline{c},\underline{c})$

(.3) $Overlap(\underline{c},\underline{d}) \rightarrow Overlap(\underline{d},\underline{c})$

(.4) $\neg\forall\underline{c}\forall\underline{d}\forall\underline{e}(Overlap(\underline{c},\underline{d})\ \&\ Overlap(\underline{d},\underline{e}) \rightarrow Overlap(\underline{c},\underline{e}))$

(.5) $\neg\forall\underline{c}\forall\underline{d}\exists\underline{e}(\underline{e} \leq \underline{d}\ \&\ \neg Overlap(\underline{e},\underline{c}))$

(.6) $\neg\forall\underline{c}\forall\underline{d}\exists\underline{e}(\underline{e} < \underline{d}\ \&\ \neg Overlap(\underline{e},\underline{c}))$

Contrast (.1) with (519.8). The latter showed that the mereological notion of overlap defined on concepts generally (518.2) trivially holds for every two concepts, while the former, in light of fact (529) and Remark (530), shows that mereological overlap defined on the non-null concepts is not trivial.

(.5) is of interest because it contrasts with (519.9). The latter implies that for any two concepts c and d, there is a concept e that is a part of d but which doesn't overlap with c. But the former asserts that this fails to hold for non-null concepts; there are non-null concepts $\underline{c}$ and $\underline{d}$ such that every non-null part of $\underline{d}$ mereologically overlaps with $\underline{c}$. Since a principle like (519.9) doesn't hold for arbitrary non-null concepts $\underline{c}$ and $\underline{d}$, the way is now clear to prove *non-trivial* supplementation principles.

Note that (.6) shows that (519.10) remains a theorem when restricted to non-null concepts, albeit for a different reason. If, in (.6), we take any concept you please as an instance of c and take the Thin Form of G, $\boldsymbol{a}_G$ (which is a non-null concept, for any property G), as a substitution instance for d, then since we know $\boldsymbol{a}_G$ is a non-null atom (528.1), it has no non-null concepts as proper parts, i.e., $\neg\exists\underline{e}(\underline{e} < \boldsymbol{a}_G)$. *A fortiori*, $\neg\exists\underline{e}(\underline{e} < \boldsymbol{a}_\varnothing\ \&\ \neg Overlap(\underline{e},\underline{c}))$.

(**532**) **Theorems:** Non-Trivial Supplementation Principles on Non-null Concepts. The following weak and strong supplementation principles, formulated so that they apply only to non-null concepts, are not trivial: (.1) if $\underline{c}$ is a proper part of $\underline{d}$, then some non-null part of $\underline{d}$ fails to overlap $\underline{c}$; (.2) if $\underline{d}$ fails to be a part of $\underline{c}$, then some non-null part of $\underline{e}$ fails to overlap $\underline{c}$:

(.1) $\underline{c} < \underline{d} \rightarrow \exists\underline{e}(\underline{e} \leq \underline{d}\ \&\ \neg Overlap^{+}(\underline{e},\underline{c}))$ (Weak Supplementation)

(.2) $\underline{d} \not\leq \underline{c} \rightarrow \exists\underline{e}(\underline{e} \leq \underline{d}\ \&\ \neg Overlap^{+}(\underline{e},\underline{c}))$ (Strong Supplementation)

Cf. Varzi 2015, §3.1 (P.4), and §3.2 (P.5). Finally, note that when we restrict (520.3) to non-null concepts, we obtain the following counterpart, which is slightly stronger than (.1) above:

(.3) $\underline{c} < \underline{d} \rightarrow \exists\underline{e}(\underline{e} < \underline{d}\ \&\ \neg Overlap(\underline{e},\underline{c}))$

Varzi (2015, §3.1 (P.4′)) calls this last result *proper supplementation*. It is immune to the concern raised about (520.3); it is a true supplementation principle since the witness to the quantifier in the consequent is not the null concept.

(**533**) **Exercises:** Consider the following questions (ranging from easy to hard), all of which arise by considering how previous definitions fare when (adjusted and) applied to non-null concepts:

- Is there a genuine superconcept$^+$, i.e., a concept$^+$ such that $\forall \underline{d}(\underline{d} \preceq \underline{c})$?

- Consider the following mereological definition of concept summation (Simons 1987, 32, cf. SD7; Varzi 2015, §4.2, 39_3):

  $$Sum^+(\underline{c}, \underline{d}, \underline{e}) \ =_{df} \ \forall \underline{f}(Overlap(\underline{c}, \underline{f}) \equiv (Overlap(\underline{f}, \underline{d}) \vee Overlap(\underline{f}, \underline{e}))$$

  How does this mereological notion of concept summation relate to the object-theoretic notion we defined in (482) when the latter is restricted to non-null concepts? Are the two notions equivalent? If not, what distinctive theorems do the non-equivalent notions give rise to? For example, we established that *every* two concepts have a unique sum (483.2); does this hold for non-null concepts on the alternative definition of concept summation?

- What lattice-theoretic and algebraic facts are preserved when we restrict our attention to non-null concepts? What facts of this kind fail to hold with respect to non-null concepts? Are the non-null concepts structured in the way Simons says a 'full classical mereology' is structured, namely, as a "complete Boolean algebra with zero deleted" (1987, 25)?

- Finally, does the following *complementation* principle hold:

  $$\underline{c} \not\preceq \underline{d} \rightarrow \exists \underline{e} \forall \underline{f}(\underline{f} \preceq \underline{e} \equiv (\underline{f} \preceq \underline{c} \ \& \ \neg Overlap(\underline{f}, \underline{d})))$$

  (cf. Varzi 2015, §3.3, P.6)?

We leave the above as open questions for the interested reader to pursue.

13.2 Concepts of Properties and Individuals

In the subsections that follow, we describe: (a) concepts of properties, such as the concept of *being a king*, the concept of *being red*, etc., (b) concepts of ordinary individuals, such as the concept of Adam, the concept of Alexander, etc., and (c) generalized concepts, such as the concept of *every human*, the concept of *something red*, etc.

13.2.1 Concepts of Properties

(**534**) **Definitions:** Concepts of Properties. In (339) we said, *G necessarily implies F*, written $G \Rightarrow F$, just in case $\Box\forall x(Gx \rightarrow Fx)$. Let us now say that c is a *concept of G* just in case c encodes exactly the properties necessarily implied by G:

$$ConceptOf(c,G) \ =_{df} \ \forall F(cF \equiv G \Rightarrow F)$$

For example, c is a concept of *being human* just in case c encodes exactly the properties necessarily implied by *being human*. Clearly, then, we are distinguishing the property *being human* from the concept of that property. Remember that we may eliminate the restricted variable in the above definition, so that the notion $ConceptOf(x,G)$ becomes defined as $C!x \,\&\, \forall F(xF \equiv G \Rightarrow F)$.

(**535**) **Theorems:** Existence Conditions for Concepts of Properties. In the usual way, it follows that (.1) there is a concept of G, (.2) there is a unique concept of G, and (.3) there is something that is *the* concept of G:

(.1) $\exists c ConceptOf(c,G)$

(.2) $\exists! c ConceptOf(c,G)$

(.3) $\exists y(y = \imath c ConceptOf(c,G))$

These are almost immediate consequences of the previous definition (534), comprehension for concepts (479), and our conventions for restricted concept variables (480).

(**536**) **Term Definition:** Notation for The Concept of G. We may now introduce the notation $\boldsymbol{c}_G$ for *the* concept of G:

$$\boldsymbol{c}_G \ =_{df} \ \imath c ConceptOf(c,G)$$

By our previous theorems, we know $\boldsymbol{c}_G$ is logically proper, for every property G. Moreover, it is clear that $\boldsymbol{c}_G$ is (identical to) a canonical concept. Note here that the boldface symbol '$\boldsymbol{c}$' in the expression '$\boldsymbol{c}_G$' is *not* a variable ranging over concepts and, hence, not a restricted variable. The only variable in the expression '$\boldsymbol{c}_G$' is the symbol 'G'. So we have an unrestricted term definition that introduces a unary functional term.

(**537**) **Lemmas:** Facts about the Concept of G. In (345) we established that necessarily, for any property F, if G necessarily implies F, then it is necessary that G necessarily implies F, i.e., that $\Box\forall F(G \Rightarrow F \rightarrow \Box G \Rightarrow F)$. Thus, where φ is $G \Rightarrow F$, this establishes that φ is a rigid condition on properties, by (188.1). So $\boldsymbol{c}_G$ is a strictly canonical concept, by (188.2). Hence, by theorem (189.2), it follows that there is a modally strict proof of (.1) for any property F, $\boldsymbol{c}_G$ encodes F if and only if G necessarily implies F:

(.1) $\forall F(\boldsymbol{c}_G F \equiv G \Rightarrow F)$

Moreover, since we can establish $G \Rightarrow G$ by modal predicate logic alone, it follows from (.1) that:

(.2) $\boldsymbol{c}_G G$

(**538**) **Theorem:** Identity of Sums. If we remember the definition of $\oplus$, then it is a simple consequence of the foregoing definitions that the sum of the concept of G and the concept of H is identical to the concept that encodes just the properties implied by G or implied by H:

(.1) $\boldsymbol{c}_G \oplus \boldsymbol{c}_H = \iota c \forall F(cF \equiv G \Rightarrow F \vee H \Rightarrow F)$

Moreover, (.2) no matter how one reorders the (finite) sum $c_{G_1}, \ldots, c_{G_n}$, the original sum and the reordered sum are identical:

(.2) $\boldsymbol{c}_{G_1} \oplus \ldots \oplus \boldsymbol{c}_{G_i} \oplus \ldots \oplus \boldsymbol{c}_{G_j} \oplus \ldots \oplus \boldsymbol{c}_{G_n} \;=\; \boldsymbol{c}_{G_1} \oplus \ldots \oplus \boldsymbol{c}_{G_j} \oplus \ldots \oplus \boldsymbol{c}_{G_i} \oplus \ldots \oplus \boldsymbol{c}_{G_n}$
$(1 \leq i \leq \ldots \leq j \leq n)$

(**539**) **Theorems:** Inclusion Chain. A further consequence of the foregoing is that for any properties $G_1, \ldots, G_n$, the concept of G_1 is included in the sum of the concepts of G_1 and G_2, which in turn is included in the sum of the concepts of G_1, G_2, and G_3, ..., which in turn is included in the sum of the concepts G_1, G_2, ..., G_n; and the reverse is true for concept containment:

$$\boldsymbol{c}_{G_1} \preceq \boldsymbol{c}_{G_1} \oplus \boldsymbol{c}_{G_2} \preceq \ldots \preceq \boldsymbol{c}_{G_1} \oplus \ldots \oplus \boldsymbol{c}_{G_n}$$
$$\boldsymbol{c}_{G_1} \oplus \ldots \oplus \boldsymbol{c}_{G_n} \succeq \ldots \succeq \boldsymbol{c}_{G_1} \oplus \boldsymbol{c}_{G_2} \succeq \boldsymbol{c}_{G_1}$$

(**540**) **Remark:** Concepts and Properties. Earlier, in (494), we proved Leibniz's Equivalence in a completely general form, without reference to concepts of properties, for both concept inclusion and concept containment:

$$c \preceq d \equiv c \oplus d = d$$

$$c \succeq d \equiv c = c \oplus d$$

But if we instantiate these to concepts of properties, we obtain the following corollaries, one for concept inclusion and one for concept containment:

$$\boldsymbol{c}_F \preceq \boldsymbol{c}_G \;\equiv\; \boldsymbol{c}_F \oplus \boldsymbol{c}_G = \boldsymbol{c}_G$$

(ϑ) $\boldsymbol{c}_F \succeq \boldsymbol{c}_G \;\equiv\; \boldsymbol{c}_F = \boldsymbol{c}_F \oplus \boldsymbol{c}_G$

These are of interest because they more closely resemble Leibniz's texts. In Leibniz 1690b (LLP 131–132, G.vii 239), we find the following (in which the variables L and B have been replaced by G and F, respectively):

Proposition 13:
If $G + F = G$, then F will be in G

Proposition 14:
If F is in G, then $G + F = G$

Clearly, if we conjoin these and substitute $F + G$ for $G + F$ (which we can do since concept addition is commutative), then we obtain:

F is in G if and only if $F + G = G$

And the containment version is:

(ζ) F contains G if and only if $F = F + G$

On the present analysis of Leibnizian concepts, we've captured (ζ) as (ϑ).

(ζ), however, raises an issue for the standard view that Leibnizian concepts are to be interpreted as properties instead of as abstract individuals. In Remark (487), we considered interpreting the variables in principles like (ζ) as ranging over properties. Now suppose further that one gives the following property-theoretic analysis of concept summation and concept containment:[224]

$F + G =_{df} [\lambda x\, Fx \,\&\, Gx]$.

F contains G $=_{df}$ $F \Rightarrow G$

Notice that, on such an analysis, the present theory doesn't guarantee $F + G$ is identical to $G + F$, since our property theory does not imply (at least not without further axioms) that $[\lambda x\, Fx \,\&\, Gx]$ is identical to $[\lambda x\, Gx \,\&\, Fx]$. But let's put aside the fact that this analysis of concept summation in terms of property conjunction doesn't guarantee the commutativity of the former.

On this analysis of Leibnizian *containment* and *addition* just given, (ζ) becomes:

(ξ) $F \Rightarrow G \equiv F = [\lambda x\, Fx \,\&\, Gx]$

(ξ), however, is inconsistent with reasonable principles about properties that one might wish to adopt. From the fact that property F implies property G, it does *not* follow that F just is identical to the conjunctive property $[\lambda x\, Fx \,\&\, Gx]$. One may consistently extend our theory with the claim:

$\exists F \exists G(F \Rightarrow G \;\&\; F \neq [\lambda x\, Fx \,\&\, Gx])$

[224]The analysis that follows in the text would allow one to conclude that since *being a brother* necessarily implies *being male*, the property *being a brother* contains the property *being male*. So the analysis would provide us with a way of understanding the Leibnizian claim that the concept *brother* contains the concept *male*.

For example, one may reasonably claim both (a) that the property *being a brother* necessarily implies the property *being male* yet (b) that *being a brother* is not identical to *being a brother and male*. Or if we let K be the necessary property $[\lambda x\, Fx \vee \neg Fx]$ (where F is any property), then one might reasonably argue that, for any property G distinct from K, both $G \Rightarrow K$ (indeed, this is provable) and $G \neq [\lambda y\, Gy \,\&\, Ky]$. Indeed, some philosophers might wish to go further and assert that the right condition of (ξ) fails universally, i.e., that no (distinct) properties F and G are such that F is identical to $[\lambda y\, Fy \,\&\, Gy]$. In any case, we only need the existence of a single counterexample to show that (ξ) is unacceptable.

Of course, one could try to reinterpret the identity claim in (ζ) in terms of some weaker notion. Castañeda (1990, 17) suggests that Leibniz's relation of coincidence is indeed a weaker relation than identity on concepts, but Ishiguro (1990, Chapter 2) argues that this isn't really consistent with Leibniz's texts. Among other things, it conflicts with Leibniz's reading of the symbol '∞' as 'the same' in Definition 1 of LLP 131 (G.vii 236).

By contrast, we've seen that when the concept F is analyzed as $\boldsymbol{c}_F$, concept addition as $\oplus$, and concept containment as $\succeq$, then (ζ) becomes analyzed as (ϑ). The latter has the following instance:

$$\boldsymbol{c}_B \succeq \boldsymbol{c}_M \;\equiv\; \boldsymbol{c}_B = \boldsymbol{c}_B \oplus \boldsymbol{c}_M$$

> The concept of *being a brother* contains the concept of *being male* if and only if the concept of *being a brother* is identical with the sum of the concepts of *being a brother* and *being male*.

Indeed, if one adds to our system the property-theoretic postulate that *being a brother* necessarily implies *being male* ($B \Rightarrow M$), one may derive either side of above biconditional.[225]

We conclude the present remark on concepts and properties by considering whether the following constitutes counterevidence to our analysis of concepts as abstract individuals. Some philosophers, including Leibniz, might assert that the following is true:

(A) The concept *brother* is identical to the sum of the concept *male* and the concept *sibling*.

[225] To see this, assume $B \Rightarrow M$, i.e., $\Box\forall x(Bx \rightarrow Mx)$. Now to derive the left-side of the biconditional, i.e., $\boldsymbol{c}_B \succeq \boldsymbol{c}_M$, we have to show $\boldsymbol{c}_M \preceq \boldsymbol{c}_B$, by definition (489.2). So by definition (489.1), we have to show $\forall F(\boldsymbol{c}_M F \rightarrow \boldsymbol{c}_B F)$. By GEN, it suffices to show $\boldsymbol{c}_M F \rightarrow \boldsymbol{c}_B F$. So assume $\boldsymbol{c}_M F$. Then by (537.1), $M \Rightarrow F$, i.e., $\Box\forall x(Mx \rightarrow Fx)$. But it follows from our assumption and this last fact that $\Box\forall x(Bx \rightarrow Fx)$, i.e., $B \Rightarrow F$. So by (537.1), $\boldsymbol{c}_B F$, completing our conditional proof. Now to derive the right-side of the biconditional from $B \Rightarrow M$, we could just appeal to the derivation we just completed and the instance of (ϑ) above in the text. Independently, however, we can easily show $\boldsymbol{c}_B = \boldsymbol{c}_B \oplus \boldsymbol{c}_M$ by showing they encode the same properties. ($\rightarrow$) Assume $\boldsymbol{c}_B F$. Then $\boldsymbol{c}_B F \vee \boldsymbol{c}_M F$. Hence $\boldsymbol{c}_B \oplus \boldsymbol{c}_M F$. ($\leftarrow$) Assume $\boldsymbol{c}_B \oplus \boldsymbol{c}_M F$. Then by (538.1), $(B \Rightarrow F) \vee (M \Rightarrow F)$. If $B \Rightarrow F$, then $\boldsymbol{c}_B F$, by (537.1). If $M \Rightarrow F$, then given our assumption $B \Rightarrow M$, it follows that $B \Rightarrow F$. Hence, again, $\boldsymbol{c}_B F$.

Now in certain pre-theoretical, ordinary language contexts, in which *concept* talk is just loose talk about *properties* and concept summation is intuitively understood as property conjunction, (A) is no doubt best interpreted as asserting that the property *being a brother* ('B') is identical to the property $[\lambda x\, Mx \,\&\, Sx]$, which conjoins the property of *being male* ('M') and the property of *being a sibling* ('S'). This would yield a true interpretation of (A). But we have suggested in this subsection that Leibniz's talk of 'the concept brother' should be analyzed as a reference to *the concept of being a brother*, i.e., $\boldsymbol{c}_B$, and similarly for his talk of 'the concept male' and 'the concept sibling'. So (A), on the present analysis, would be formalized as follows:

(B) $\boldsymbol{c}_B = \boldsymbol{c}_M \oplus \boldsymbol{c}_S$

But (B) is provably false if given, as minimal facts about these properties, that the property *being a brother* necessarily implies the conjunctive property *being a male sibling* (i.e., that $B \Rightarrow [\lambda x\, Mx \,\&\, Sx]$), and that neither M nor S necessarily imply this conjunctive property. For, given such facts, it follows by definition of $\boldsymbol{c}_B$ and $\boldsymbol{c}_M \oplus \boldsymbol{c}_S$ that $\boldsymbol{c}_B$ encodes $[\lambda x\, Mx \,\&\, Sx]$ and that $\boldsymbol{c}_M \oplus \boldsymbol{c}_S$ does not. Since $\boldsymbol{c}_B$ encodes a property that $\boldsymbol{c}_M \oplus \boldsymbol{c}_S$ fails to encode, they are distinct, contradicting (B).

It may be tempting to conclude that this result shows that the property-theoretic analysis of the Leibnizian notion of concepts is preferable to the present analysis, since it avoids attributing to him a provable falsehood. As noted above, if we take the concepts discussed in (A) to be properties, and take the sum of the concepts *male* and *sibling* to be $[\lambda x\, Mx \,\&\, Sx]$, then (C) becomes an interpretation of (A):

(C) $B = [\lambda x\, Mx \,\&\, Sx]$

(C) is in fact true. The property *being a brother* is identical to the property *being a male sibling*. Moreover, on the basis of (C), it follows that the concept of *being a brother* is identical to the concept of the conjunctive property *being a male sibling*:

(D) $\boldsymbol{c}_B = \boldsymbol{c}_{[\lambda x\, Mx \,\&\, Sx]}$

Once (C) is added to our system (say, as part of a theory of brotherhood), then (D) becomes an easy theorem.

So I think the conclusion to be drawn here is that either (A) is to be interpreted as (C), in which case Leibniz has asserted a property identity in the guise of concept identities, or (A) is to be interpreted as (D). If one doesn't rigorously distinguish properties and concepts of properties, one might well say (A) when intending (C) or (D). Leibniz did in fact distinguish between properties and their concepts, but it doesn't look like he did so rigorously. There are passages where he distinguishes an accident and its notion, and passages where he distinguishes a predicate and its concept. Consider the following passage from Article 8 in the *Discourse on Metaphysics* (PW 19, G.iv 433):

> ... an accident is a being whose notion does not include all that can be attributed to the subject to which this notion is attributed. Take, for example, the quality of being a king, ...

In this passage, I take it, Leibniz is distinguishing the property (i.e., the accident or the quality) from the concept or notion of that property. And recall the following passage from his correspondence with Arnauld (June 1686, LA 63, G.ii 56):

> ... in every true affirmative proposition, necessary or contingent, universal or particular, the concept of the predicate is in a sense included in that of the subject; the predicate is present in the subject.

These passages show Leibniz did distinguish properties and their concepts. But he didn't regiment the distinction and systematically adhere to it in his logic of concepts and containment theory of truth. So it shouldn't be surprising if sometimes Leibniz asserted concept identities like (A) when he meant to assert either something like (C) or (D). This conclusion may address the concern that (A) constitutes counterevidence to the present analysis.

Notice also that from (C), we can also derive the property-theoretic postulate discussed earlier, namely, (a) $B \Rightarrow M$, and so derive that (b) the concept of *being a brother* contains the concept of *being male*, i.e., that $\boldsymbol{c}_B \succeq \boldsymbol{c}_M$. For (a), note that $[\lambda x\, Mx \,\&\, Sx] \Rightarrow M$ (exercise) and so by (C), it follows that $B \Rightarrow M$. Hence, for (b), the reasoning we used in footnote 225 now yields the conclusion that $\boldsymbol{c}_B \succeq \boldsymbol{c}_M$.

In conclusion, on the present theory, there is a subtle and important difference between properties and concepts, between the sum concept $\boldsymbol{c}_M \oplus \boldsymbol{c}_S$ and the conjunctive property $[\lambda x\, Mx \,\&\, Sx]$, and hence between $\boldsymbol{c}_M \oplus \boldsymbol{c}_S$ and $\boldsymbol{c}_{[\lambda x\, Mx \,\&\, Sx]}$. In general, it is important to remember that adding concepts $\boldsymbol{c}_F$ and $\boldsymbol{c}_G$ to obtain $\boldsymbol{c}_F \oplus \boldsymbol{c}_G$ is not the same as conjoining properties F and G to obtain $[\lambda x\, Fx \,\&\, Gx]$. The conjunction $[\lambda x\, Fx \,\&\, Gx]$ is a property whose extension (in the technical sense of Chapter 10, Section 10.3) is the intersection of the extensions of F and G. The sum of $\boldsymbol{c}_F$ and $\boldsymbol{c}_G$ is a concept that encodes the union of the properties implied by F and those implied by G.

(**541**) **Theorem:** Identity of The Concept of G and The (Thick) Form of G. It should come as no surprise to those who have worked through the chapter on Forms that the concept of G is identical to the (thickly-conceived) Form of G:

$$\boldsymbol{c}_G = \Phi_G$$

This establishes an interesting link between the work of Plato and Leibniz. As an exercise, consider what the resulting theorems say when we substitute $\boldsymbol{c}_G$ for Φ_G in the theorems about Forms, and substitute Φ_G for $\boldsymbol{c}_G$ in the theorems about concepts of properties.

13.2.2 Concepts of Individuals

In this subsection and for the remainder of this chapter, we use u, v, w as variables ranging over ordinary individuals.

(**542**) **Definition:** Concepts of Ordinary Individuals. Let us say that c is a *concept of* u just in case c encodes exactly the properties that u exemplifies:

$$ConceptOf(c,u) \ =_{df} \ \forall F(cF \equiv Fu)$$

To take an example, if Socrates ('s') is an ordinary object and a concept c encodes exactly the properties Socrates exemplifies, then $ConceptOf(c,s)$. Note that, strictly speaking, we have defined:

$$ConceptOf(x,y) \ =_{df} \ O!y \,\&\, C!x \,\&\, \forall F(xF \equiv Fy)$$

Nevertheless the first definition with restricted variables still implies the expected equivalence: $ConceptOf(c,u) \equiv \forall F(cF \equiv Fu)$.[226]

(**543**) **Theorems:** Conditional Existence Conditions for Concepts of Ordinary Individuals. It follows that (.1) there is a a concept of u; (.2) there is a unique concept of u; and (.3) there is something that is *the* concept of u:

(.1) $\exists c ConceptOf(c,u)$

(.2) $\exists! c ConceptOf(c,u)$

(.3) $\exists z(z = \iota c ConceptOf(c,u))$

These are straightforward consequences of the definition (542) and comprehension for concepts (479).

Of course, in these theorems, u is a free restricted variable, and so strictly speaking, all of the above are conditional. But we can derive *unconditional*

[226]To see why, note that our discussion of free restricted variables in definitions, in (253), is generalizable for two free restricted variables, as follows. If we instantiate the (restricted) variable c for x and the (restricted) variable u for y in the second definition, we obtain:

$$ConceptOf(c,u) \equiv O!u \,\&\, C!c \,\&\, \forall F(cF \equiv Fu)$$

But, since the restrictions on c and u entail that $C!c$ and $O!u$ are in effect axioms, the above equivalence reduces to:

$$ConceptOf(c,u) \equiv \forall F(cF \equiv Fu)$$

So the first definition converts to an equivalence, as is to be expected.

existence claims from them since it follows from (174) and the T schema that ordinary individuals exist.

(**544**) **Restricted Term Definition:** Notation for the Concept of u. We henceforth use the notation $\boldsymbol{c}_u$ to denote the concept of u:

$$\boldsymbol{c}_u =_{df} \imath c ConceptOf(c, u)$$

This introduces $\boldsymbol{c}_u$ as a unary functional term with the free restricted variable u. Hence, we shall use expressions of the form $\boldsymbol{c}_\kappa$ only when the singular term κ is known to be an ordinary object, either by proof or by hypothesis, for only then is $\boldsymbol{c}_\kappa$ both well-formed and logically proper. Clearly, by definitions (542) and (477), $\boldsymbol{c}_u$ is (identical to) $\imath x(A!x \,\&\, \forall F(xF \equiv Fu))$. So $\boldsymbol{c}_u$ is canonical.

(**545**) **Remark:** The Concept of u is Not Strictly Canonical. Intuitively, ordinary individuals exemplify some of their properties contingently. This means that when we apply our system by extending it with facts about the world, we should be prepared for the addition of facts of the form $Fu \,\&\, \Diamond\neg Fu$. It is important, therefore, to make sure that the concept of u, $\boldsymbol{c}_u$, isn't strictly canonical, for if it were, every property exemplified by an ordinary individual would be a necessary property, as explained below.

If we let φ be the formula Fu, then we can develop an indirect argument that φ is not a rigid condition on properties, as this is defined in (188.1). This argument is given below but note first several important consequences implied by the conclusion, namely, that (a) the description $\imath x(A!x \,\&\, \forall F(xF \equiv Fu))$ fails to be strictly canonical, (b) $\boldsymbol{c}_u$ consequently fails to be strictly canonical, and (c) $\boldsymbol{c}_u$ isn't subject to theorem (189.2), which means one has to rely on (101.2)$\star$ to derive general facts about the properties that $\boldsymbol{c}_u$ encodes.

The argument concluding that φ is not a rigid condition on properties rests on the following fact and the demands of consistency. It is easy to show that there is a proof that some ordinary individual u is such that it is not necessary that every property u exemplifies is necessarily exemplified, i.e.,

$$(\vartheta) \ \vdash \exists u \neg\Box\forall F(Fu \rightarrow \Box Fu)$$

From this fact, established below, it clearly follows that $\vdash \neg\forall u\Box\forall F(Fu \rightarrow \Box Fu)$. But if φ were a rigid condition, then it would follow from definition (188.1) that $\vdash \forall u\Box\forall F(Fu \rightarrow \Box Fu)$, by GEN. So if we can establish (ϑ), φ isn't a rigid condition, on pain of inconsistency.

Now we can produce the proof witness to (ϑ) with the help of axiom (32.4). The first conjunct of that axiom asserts that $\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x)$. By BF$\Diamond$, it follows that $\exists x\Diamond(E!x \,\&\, \Diamond\neg E!x)$. Suppose b is such an object, so that we know:

$$(\xi) \ \Diamond(E!b \,\&\, \Diamond\neg E!b)$$

But by standard modal reasoning, this implies:[227]

(ζ) $\neg\Box\forall F(Fb \to \Box Fb)$

Note also that (ξ) also implies $\Diamond E!b$, which by β-Conversion, yields $[\lambda x \Diamond E!x]b$, i.e., $O!b$. Since b is ordinary, it follows from (ζ) that $\exists u \neg\Box\forall F(Fu \to \Box Fu)$. Thus we have a proof that is a witness to (ϑ).

So φ isn't a rigid condition on properties. It then easily follows that $\boldsymbol{c}_u$ is not strictly canonical. This is, of course, as it should be. As noted above, ordinary individuals exemplify some of their properties contingently and though our system doesn't assert, of any particular ordinary individual u and property F, that $Fu \,\&\, \Diamond\neg Fu$, axiom (32.4) does guarantee that some ordinary individual might be such that it is concrete but possibly not. And, of course, when it is time to apply our system by representing our modal beliefs, we might certainly extend our theory with facts like the ones discussed in (185), i.e., Pb and $\Diamond\neg Pb$, where b denotes some particular individual and P denotes the property of *being a philosopher*. In (185), we saw that in the presence of such added facts, the following claim is not a necessary truth:

$$\imath x(A!x \,\&\, \forall F(xF \equiv Fb))P \equiv Pb$$

As an instance of our Abstraction Principle (184)★, the above is a necessitation-averse theorem, for the reasons discussed in (185). Since $\boldsymbol{c}_u$ is identical to $\imath x(A!x \,\&\, \forall F(xF \equiv Fu))$, we should not expect general theorems about the properties $\boldsymbol{c}_u$ encodes to be modally strict, though as we saw in item (187), it can be shown by modally strict means that $\boldsymbol{c}_u$ encodes *necessary* properties. But the general claim that $\boldsymbol{c}_u G \equiv Gu$ is necessitation-averse and cannot be proved by modally strict means. Nevertheless, it is a non-modally strict theorem, as we shall now see.

(**546**) ★**Lemmas:** Facts about Concepts of Ordinary Objects. It is an immediate consequence of the foregoing that (.1) the concept of u encodes G iff u exemplifies G, and (.2) the concept of u encodes G iff the concept of u contains the concept of G:

(.1) $\boldsymbol{c}_u G \equiv Gu$

(.2) $\boldsymbol{c}_u G \equiv \boldsymbol{c}_u \succeq \boldsymbol{c}_G$

[227] Here is the proof:

1.	$\Diamond(E!b \,\&\, \Diamond\neg E!b)$	Assumption
2.	$\Diamond(E!b \,\&\, \neg\Box E!b)$	from 1, by (117.2) and the Rule of Substitution
3.	$\Diamond\neg(E!b \to \Box E!b)$	from 2, by (63.5.b) and the Rule of Substitution
4.	$\neg\Box(E!b \to \Box E!b)$	from 3, by (117.2) and the Rule of Substitution
5.	$\exists F\neg\Box(Fb \to \Box Fb)$	from 4, by ∃I
6.	$\neg\forall F\Box(Fb \to \Box Fb)$	from 5, by (86.2)
7.	$\neg\Box\forall F(Fb \to \Box Fb)$	from 6, by BF (32.4) and Rule of Substitution

These theorems are not modally strict.

(**547**) **Definition:** Completeness of Concepts. Recall that $\overline{F}$ was defined in (136.1) to be $[\lambda y\, \neg Fy]$. We now say that a concept c is *complete* just in case for every property F, either c encodes F or c encodes $\overline{F}$:

$$Complete(c) \;=_{df}\; \forall F(xF \vee x\overline{F})$$

(**548**) ⋆**Theorem:** The Concept of Individual u Is Complete.

(.1) $Complete(\boldsymbol{c}_u)$

This theorem and theorem (546)⋆ fully capture Leibniz's suggestion, in Article 8 of the *Discourse on Metaphysics* (PW 18–19, G.iv 433), that:

> ...it is in the nature of an individual substance, or complete being, to have a notion so complete that it is sufficient to contain and render deducible from itself, all the predicates of the subject to which this notion is attributed.

Theorem (546)⋆ ensures that all of the properties that u exemplifies are derivable from $\boldsymbol{c}_u$, and the present theorem ensures that $\boldsymbol{c}_u$ is provably complete.

While the present theorem isn't modally strict, there is a modally strict theorem close by that makes use of definition (542), namely, $ConceptOf(c,u) \rightarrow Complete(c)$.[228] It is also noteworthy that although we can't use the Rule of Necessitation to derive that the present theorem is necessary, (.1) is nevertheless a necessary truth; one can prove, as a non-modally strict theorem, that necessarily the concept of u is complete:

(.2) $\Box Complete(\boldsymbol{c}_u)$.

(**549**) **Remark:** The Focus on Concepts of Ordinary Individuals. One might wonder why we are focusing on concepts of ordinary individuals and not on concepts of individuals generally. There are both theoretical reasons and practical reasons. One theoretical reason is that, strictly speaking, we don't need a distinction between an abstract individual and a concept of that abstract individual. An abstract individual x is, by its very nature, something distinguished

[228] Assume $ConceptOf(c,u)$. Then by definition (542), $\forall F(cF \equiv Fu)$. So by $\forall$E, we know both:

$$cF \equiv Fu$$
$$c\overline{F} \equiv \overline{F}u$$

But since it follows from the tautology $Fu \vee \neg Fu$, an appropriate instance of β-Conversion, and the Rule of Substitution that $Fu \vee \overline{F}u$, we may conclude by disjunctive syllogism that $cF \vee c\overline{F}$. Since the variable F was not free in our assumption, we may use GEN to conclude: $\forall F(cF \vee c\overline{F})$. So, by definition and modally strict reasoning, it follows that $Complete(c)$.

and identified by the properties it encodes. So any concept that encodes exactly the distinguishing and identifying properties of x just *is* x.

Moreover, we already know that the properties that an abstract object exemplifies do not necessarily distinguish it from other abstract objects. Theorem (197) tells us that there are abstract objects, say a and b, such that $a \neq b$ and $\forall F(Fa \equiv Fb)$. That is, there are abstract objects a and b that are distinct (i.e., encode different properties) but which are indiscernible (i.e., exemplify the same properties). Thus, if we had extended the definition of *ConceptOf* to allow for concepts of abstract individuals, $\boldsymbol{c}_a$ and $\boldsymbol{c}_b$ would be identical concepts for such a and b (exercise). Thus, concepts of distinct but indiscernible abstract individuals would collapse. This gives us a reason to think that it would be a mistake to suppose that a *concept of* an abstract individual x encodes exactly what x exemplifies. We take this to be an insight, not a limitation.

Such theoretical reasons do not apply to concepts of ordinary individuals. Ordinary individuals are distinguished and identified by the properties they exemplify (170), and since the concepts of such individuals encode the latter's exemplified properties, it makes sense to distinguish ordinary individuals and concepts of them. Moreover, whenever ordinary individuals a and b are distinct, their concepts $\boldsymbol{c}_a$ and $\boldsymbol{c}_b$ are distinct (exercise).

A practical reason for restricting our focus is that concepts of ordinary individuals suffice for representing the examples Leibniz and others have used in discussing the metaphysics of individual concepts. When expounding his calculus of concepts, containment theory of truth, and modal metaphysics, Leibniz always used concepts of ordinary individuals as examples. Moreover, as we shall see, the study of concepts of ordinary individuals yields some very nice results in the modal metaphysics of concepts, discussed in Section 13.4 below.

13.2.3 Concepts of Generalizations

In what follows, we introduce concepts of *generalizations*, such as the concept *everything that exemplifies G* and the concept *something that exemplifies G*. In our metalanguage, we take the liberty of rendering these expressions more simply as the concept *every G* and the concept *some G*, and we introduce and define notation like $\boldsymbol{c}_{\forall G}$ and $\boldsymbol{c}_{\exists G}$, respectively, to represent them. To further simplify the discussion, these formal expressions will be defined directly in terms of canonical descriptions.

(**550**) **Term Definitions:** Concepts of Generalizations. Let us say (.1) the concept *everything that exemplifies G* (i.e., the concept *every G*) is the concept that encodes just the properties F such that every G exemplifies F; and (.2) the concept *something that exemplifies G* (i.e., the concept *some G*) is the concept that encodes just the properties F such that some G exemplifies F:

(.1) $\boldsymbol{c}_{\forall G} =_{df} \imath c\forall F(cF \equiv \forall x(Gx \to Fx))$

(.2) $\boldsymbol{c}_{\exists G} =_{df} \imath c\forall F(cF \equiv \exists y(Gy \,\&\, Fy))$

In these definitions, the variables x, y are standard and range over everything whatsoever.

Thus, $\boldsymbol{c}_{\forall G}$ and $\boldsymbol{c}_{\exists G}$ are canonically defined. It follows from (180), (477), and our conventions for eliminating the restricted variable c ranging over concepts that these new expressions are well-formed and logically proper for every value of G. Hence they are unary functional terms.

We leave it as an exercise to show that $\boldsymbol{c}_{\forall G}$ and $\boldsymbol{c}_{\exists G}$ are not strictly canonical, i.e., that when φ is either $\forall x(Gx \to Fx)$ or $\exists x(Gx \,\&\, Fx)$, then φ is *not* a rigid condition on properties. [Hint: Consider the argument in the latter half of Remark (241).]

(**551**) ⋆**Lemmas:** Facts about Concepts of Generalizations. It is an immediate consequence of the foregoing that (.1) the concept *every G* encodes F iff every G exemplifies F, and (.2) the concept *some G* encodes F iff some G exemplifies F:

(.1) $\boldsymbol{c}_{\forall G}F \equiv \forall x(Gx \to Fx)$

(.2) $\boldsymbol{c}_{\exists G}F \equiv \exists x(Gx \,\&\, Fx)$

It also follows that (.3) the concept *every G* encodes the property F iff the concept *every G* contains the concept of F, and (.4) the concept *some G* encodes the property F iff the concept *some G* contains the concept of F:

(.3) $\boldsymbol{c}_{\forall G}F \equiv \boldsymbol{c}_{\forall G} \succeq \boldsymbol{c}_F$

(.4) $\boldsymbol{c}_{\exists G}F \equiv \boldsymbol{c}_{\exists G} \succeq \boldsymbol{c}_F$

These are not modally strict theorems.

(**552**) **Remark:** Generalized Quantifiers. In the next section, we examine Leibniz's containment theory of truth. The reader might wish to keep the following question in mind when reading that material: did Leibniz anticipate the idea of a generalized quantifier or Montague's (1974) subject-predicate analysis of basic sentences of natural language? Montague gave a uniform subject-predicate analysis of a fundamental class of English sentences by treating such noun phrases as 'John' and 'every person' as sets of properties. He supposed that the proper name 'John' denotes the set of properties that John exemplifies and supposed that the noun phrase 'every person' denoted the set of properties that every person exemplifies. Then, on Montague's theory, English sentences such as 'John is rational' and 'Every person is rational' could be given a uniform subject-predicate analysis: such sentences are true iff the property denoted by the predicate 'is rational' is a member of the set of properties denoted by the subject term. This, as we shall see, is captured by Leibniz's containment theory of truth, which offers a similar unification of the analysis of singular and general predications. Moreover, one could posit other concepts of generalizations, such as the concepts *most things that exemplify F*, *many things that exemplify F*, *few things that exemplify F*, etc., once the relevant quantifiers are introduced into our language and either axiomatized or defined.

13.3 The Containment Theory of Truth

(**553**) ⋆**Theorems:** Exemplification and Containment. It is a consequence of the preceding definitions and lemmas that exemplification predication is equivalent to the Leibnizian analysis of predication for ordinary individuals, for it follows that (.1) u exemplifies G if and only if the concept of u contains the concept of G. Furthermore, simple universal and existentially quantified statements are equivalent to the Leibnizian analysis involving concepts of generalization, given the following theorems: (.2) everything exemplifying G exemplifies F if and only if the concept *everything that exemplifies* G contains the concept of F; and (.3) something exemplifies both G and F if and only if the concept *something that exemplifies* G contains the concept of F:

(.1) $Gu \equiv \boldsymbol{c}_u \succeq \boldsymbol{c}_G$

(.2) $\forall x(Gx \rightarrow Fx) \equiv \boldsymbol{c}_{\forall G} \succeq \boldsymbol{c}_F$

(.3) $\exists x(Gx \,\&\, Fx) \equiv \boldsymbol{c}_{\exists G} \succeq \boldsymbol{c}_F$

These are easy consequences of previous theorems.

(**554**) **Remark:** The Containment Theory of Truth. To see how Leibniz's containment theory of truth is preserved in these theorems, first recall the passage quoted at the outset, from the correspondence with Arnauld (June 1686, LA 63, G.ii 56):

> . . . in every true affirmative proposition, necessary or contingent, universal or particular, the concept of the predicate is in a sense included in that of the subject; the predicate is present in the subject.

Leibniz also produced an early statement of his containment theory of truth in a work subsequently titled *Elements of a Calculus*, where he spoke of universal propositions and wrote (1679, 18–19; source C 51):

> . . . every true universal affirmative categorical proposition simply shows some connection between predicate and subject (a *direct* connection, which is what is always meant here). This connection is, that the predicate is said to be in the subject, or to be contained in the subject; either absolutely and

> regarded in itself, or at any rate, in some instance; i.e., that the subject is said to contain the predicate in a stated fashion. This is to say that the concept of the subject, either in itself or with some addition, involves the concept of the predicate....

In Article 8 of the *Discourse on Metaphysics* (1686, Bennett's translation of G.iv 433) we find:[229]

> So the [notion of the] subject term must always include [that of] the predicate, so that anyone who understood the subject notion perfectly would also judge that the predicate belongs to it.

Now to see how these passages are validated by our theorems, let a stand for Alexander the Great and let us assume Alexander is an ordinary object. Then the concept of Alexander, $\boldsymbol{c}_a$, is well-formed and logically proper, by (544) and (543.3). Moreover, let K stand for the property *being a king*. Then the concept of *being a king*, $\boldsymbol{c}_K$, is well-formed and logically proper, by (536) and (535.3). Now, given the above quotations, Leibniz analyzes the ordinary language predication:

(.1) Alexander is a king

in the following terms:

(.2) The concept of Alexander contains the concept of *being a king*

On our representation of Leibniz's analysis, this becomes:

(.3) $\boldsymbol{c}_a \succeq \boldsymbol{c}_K$

Whereas the modern analysis of (.1) in the predicate calculus is:

(.4) Ka

So by (553.1)★, our representation (.3) of Leibniz's analysis (.2) of the ordinary claim (.1) is equivalent to the modern analysis (.4) of (.1).

As a second example, consider the ordinary language predication:

(.5) Every person is rational.

Leibniz's analysis would be:

(.6) The concept *every person* contains the concept of *being rational*

We represented the Leibnizian concept *every person* as the concept *everything that exemplifies being a person*, i.e., $\boldsymbol{c}_{\forall P}$, which is defined in (550.1). By (180), the definition of $C!$ and our conventions for eliminating the restricted variable c, $\boldsymbol{c}_{\forall P}$ is well-formed and logically proper. So our representation of Leibniz's analysis (.6) becomes:

[229] The words in brackets were interpolated by Bennett, so as to clarify Leibniz's intention.

(.7) $\boldsymbol{c}_{\forall P} \succeq \boldsymbol{c}_R$

Whereas the modern analysis of (.5) in the predicate calculus is:

(.8) $\forall x(Px \rightarrow Rx)$

So by (553.2)★, our representation (.7) of Leibniz's analysis (.6) of the ordinary claim (.5) is equivalent to the modern analysis (.8) of (.6). And analogously for the ordinary language predication 'Some person is rational',

(**555**) **Remark:** Hypothetical Necessity. Leibniz's containment theory of truth gives rise to an interesting objection. Leibniz anticipated the objection in Article 13 of the *Discourse on Metaphysics*. In that Article, Leibniz reiterates that "the notion of an individual substance involves, once and for all, everything that can ever happen to it" (PW 23, G.iv 436), and then says:

> But it seems this will destroy the difference between contingent and necessary truths, that human freedom will no longer hold, and that an absolute fatality will rule over all our actions as well as over the rest of what happens in the world.

Leibniz's contemporary, Antoine Arnauld, took this criticism in a theistic direction, suggesting that the view not only implies that everything that happens to a person happens by necessity but also that it places constraints on God's freedom to shape what happens to the history of the human race (letter to Count Ernst von Hessen-Rheinfels, March 13, 1686, LA 9, G.ii 15).

Let us put aside Arnauld's theistic turn. Given Leibniz's phrasing, the objection charges that the containment theory of truth somehow collapses contingency and necessity; at best his analysis *represents* contingent truths as necessary truths and at worst implies that the actual world exhibits no contingency. If concept containment is not relative to any circumstance, then Leibniz has analyzed the contingent statement 'Alexander is a king' in terms of a claim that appears to be a necessary truth, namely, the concept of Alexander contains the concept of *being a king*. A similar worry arises about the analysis of the contingent general claims 'Every person is rational' and 'Some person is rational', which if true, would also appear to be rendered as necessary truths given Leibniz's analysis; but we'll put these two other examples aside and focus just on the first.

Leibniz's response to the objection is somewhat intriguing. In the next passage of the *Discourse*, he continues (PW 23-24, G.iv 437):

> To give a satisfactory answer to it, I assert that connexion or sequence is of two kinds. The one is absolutely necessary, whose contrary implies a contradiction; this kind of deduction holds in the case of eternal truths, such as those of geometry. The other is only necessary by hypothesis (*ex*

> *hypothesi*), and so to speak by accident; it is contingent in itself, since its contrary does not imply a contradiction.

So Leibniz defends himself against the objection by appealing to a distinction between absolute necessity and hypothetical necessity. The question is, what is hypothetical necessity?

Before we answer this question, note that when Jonathan Bennett translates the passage from the *Discourse* just quoted, he interpolates some text that he thinks helps us to understand what Leibniz is saying. Bennett's translation is:

> To that end, I remark that there are two kinds of connection or following from. One is absolutely necessary, and its contrary implies a contradiction; such deduction pertains to eternal truths, such as those of geometry. The other is necessary [not absolutely, but] only *ex hypothesi*, and, so to speak, accidentally. [It doesn't bring us to *It is necessary that P*, but only to *Given Q, it follows necessarily that P*.] Something that is necessary only ex hypothesi is contingent in itself, and its contrary doesn't imply a contradiction.

I'm not sure why Bennett interpolated the text in square brackets; it doesn't appear in the original at G.iv 437. But his explanation of the distinction between absolute and hypothetical necessity seems on point. A hypothetical necessity is a claim that either (a) is *logically required* given some contingent hypothesis, or (b) is metaphysically necessary despite being derived from a contingent hypothesis. We can make these two options more precise as follows.

On the one hand, hypothetical necessities might simply be non-modally strict *theorems*. Speaking in the formal mode (i.e., our metalanguage), these theorems are necessary in the sense that they are logically necessary consequences of our axioms. They are hypothetical in the sense of being *derived from* some claim that is necessitation-averse. Leibniz could then argue that the equivalence $Ka \equiv \boldsymbol{c}_a \succeq \boldsymbol{c}_K$, which is an instance of (553.1)★, is hypothetically necessary in this sense. Moreover, if we derive $\boldsymbol{c}_a \succeq \boldsymbol{c}_K$ from this equivalence by appealing to the contingent truth Ka, then $\boldsymbol{c}_a \succeq \boldsymbol{c}_K$ becomes another example of a hypothetical necessity. In other words, it would be hypothetically necessary that the concept of Alexander contains the concept of *being a king*.

On the other hand, hypothetical necessities might be metaphysically necessary truths that are established by non-modally strict means. An example of this is the claim that necessarily, the concept of Alexander encodes *being a king*, i.e., $\Box\boldsymbol{c}_a K$. If Ka is assumed as a contingent truth, then we can derive the fact that the concept of Alexander encodes *being a king*, i.e., $\boldsymbol{c}_a K$, from (546.1)★. Then, by axiom (37), we can conclude $\Box\boldsymbol{c}_a K$. So the claim $\Box\boldsymbol{c}_a K$ is provably a *necessary truth*, though the proof depends on a contingent truth.

On neither way understanding hypothetical necessities does it follow that the Leibnizian containment theory of truth banishes contingency from the actual world. But one might still wonder whether Leibniz and Arnauld might have made the objection stronger by recasting it as follows: the equivalence of an analysandum and its analysans should be a necessary truth but the equivalence of the analysandum 'Alexander is a king' (Ka) and the analysans $\boldsymbol{c}_a \succeq \boldsymbol{c}_K$ is not. For the equivalence, $Ka \equiv \boldsymbol{c}_a \succeq \boldsymbol{c}_K$, is an instance of theorem (553.1)$\star$ and so necessitation-averse. This is a fair point, though it is not absolutely clear why we should accept that the analysans should be necessarily equivalent to the analysandum in a system in which all terms rigidly designate and abstractions can be defined on the basis of contingencies. In the present system, $\boldsymbol{c}_a$ is defined on the basis of the properties Alexander in fact exemplifies, some of which are contingently exemplified. It is therefore inevitable that claims about the properties $\boldsymbol{c}_a$ (rigidly) encodes become provably, but not necessarily, equivalent to claims about the properties Alexander in fact exemplifies. Thus, it may be a correct analysis requires that the equivalence of the analysans and analysandum be provable *a priori* rather necessary.

There are two final points to make about this concern that Leibniz's containment theory of truth turns contingencies into necessities. For the first, reconsider the example claim that Alexander is a king (ignoring, as we've been doing, the issue of tense). If we acknowledge from the outset that this claim is true in fact, by asserting that it is *actually* the case that Alexander is a king, i.e., $\mathcal{A}Ka$, then the analysandum is no longer a mere contingency. $\mathcal{A}Ka$ implies $\Box\mathcal{A}Ka$, by axiom (33.1). We can therefore formulate and prove a modally strict variant of (553.1)$\star$, namely, that $\mathcal{A}Gu \equiv \boldsymbol{c}_u \succeq \boldsymbol{c}_G$. The proof, which is left to a footnote, goes by way of Actualized Abstraction (186).[230] The biconditional $\mathcal{A}Ka \equiv \boldsymbol{c}_a \succeq \boldsymbol{c}_K$ is an instance of this modally strict theorem. So it may be that Leibniz's hypothetical necessity is the *hypothesis* that it is actually the case that Alexander is king.

The second point is not unrelated. In the next section (Section 13.4.4), we shall define the concept of an ordinary individual u at world w ($\boldsymbol{c}_u^w$) as the concept that encodes exactly the properties F such that that Fu is true at w. It will then be established as a modally strict theorem (590.2), that u exemplifies G at a world w if and only if $\boldsymbol{c}_u^w$ contains the concept of G, i.e., $w \models Gu \equiv \boldsymbol{c}_u^w \succeq \boldsymbol{c}_G$. This is a world-relativized version of the containment theory of truth! As an instance, we have that Alexander exemplifies *being a king* at the actual world

[230]We prove both directions: ($\rightarrow$) Assume $\mathcal{A}Gu$. To show $\boldsymbol{c}_u \succeq \boldsymbol{c}_G$, we have to show $\boldsymbol{c}_G \preceq \boldsymbol{c}_u$ and so show $\forall F(\boldsymbol{c}_G F \rightarrow \boldsymbol{c}_u F)$. So, by GEN, assume $\boldsymbol{c}_G F$. Then, by facts about $\boldsymbol{c}_G$ (537.2), we know $G \Rightarrow F$, i.e., $\Box\forall x(Gx \rightarrow Fx)$. Then $\mathcal{A}\forall x(Gx \rightarrow Fx)$, by (89). So by axiom (31.3), $\forall x\mathcal{A}(Gx \rightarrow Fx)$. Hence $\mathcal{A}(Gu \rightarrow Fu)$. But our assumption is $\mathcal{A}Gu$. So by axiom (31.2), it follows that $\mathcal{A}Fu$. But then by an alphabetic variant of Actualized Abstraction (186) to avoid clash of variables, it follows that $\imath x(A!x \,\&\, \forall H(xH \equiv Hu))F$. Hence, by applying definitions (544) and (542), $\boldsymbol{c}_u F$. ($\leftarrow$) Assume $\boldsymbol{c}_u \succeq \boldsymbol{c}_G$. Then by previous reasoning, we know $\forall F(\boldsymbol{c}_G F \rightarrow \boldsymbol{c}_u F)$. Instantiating to G yields $\boldsymbol{c}_G G \rightarrow \boldsymbol{c}_u G$. Since the antecedent is a theorem, we have $\boldsymbol{c}_u G$. So by applying the definitions (544) and (542), it then follows that $\imath x(A!x \,\&\, \forall F(xF \equiv Fu))G$. Hence by Actualized Abstraction, $\mathcal{A}Gu$.

(i.e., $\boldsymbol{w}_\alpha \models Ka$) if and only if the concept of Alexander at the actual world contains the concept of *being a king* (i.e., $\boldsymbol{c}_a^{\boldsymbol{w}_\alpha} \succeq \boldsymbol{c}_K$). Though this biconditional is a necessary truth, we can derive the right side only if we assume, as a hypothesis, that Alexander is a king at the actual world. Since this latter hypothesis, i.e., $\boldsymbol{w}_\alpha \models Ka$, is an encoding claim, it is a necessary truth if true. As such, it yields an understanding of hypothetical necessities without abandoning the idea that an analysans and its analysandum are necessarily equivalent.

13.4 The Modal Metaphysics of Concepts

(**556**) **Remark:**. Primitive vs. Defined Counterparts. The theorems described in this section articulate a modal metaphysics of concepts inspired by Leibniz's work and we may introduce these theorems via an issue in Leibniz scholarship. Some Leibniz scholars have suggested that the best way to reconstruct Leibniz's modal metaphysics of concepts is to suppose that the *counterpart* relation partially systematized in Lewis 1968 should be applied to individual concepts. This general view is adopted here, but instead of taking *counterpart* to be a primitive, as in Lewis 1968 and in the work of other Leibniz scholars, we shall define a notion of *individual concept* and define the conditions under which one individual concept is a *counterpart of* another. Since our system axiomatizes a fixed domain of individuals and presupposes that every individual exemplifies properties in every possible world, it turns out that the modal metaphysics will preserve elements of Leibnizian, Lewisian, and Kripkean metaphysics.

The view adopted by various Leibniz scholars, that one should use counterpart theory to reconstruct Leibniz's metaphysics, traces back to work of Mondadori (1973, 1975), who notes that the natural reading of certain passages in the Leibnizian corpus are suggestive of that theory (cf. Ishiguro 1972, 123–134). Here is a passage from the *Theodicy* (T 371, G.vi 363) which Mondadori cites:

> I will now show you some [worlds], wherein shall be found, not absolutely the same Sextus as you have seen (that is not possible, he carries with him always that which he shall be) but several Sextuses resembling him, possessing all that you know already of the true Sextus, but not that is already in him imperceptibly, nor in consequence all that shall yet happen to him. You will find in one world a very happy and noble Sextus, in another a Sextus content with a mediocre state, . . .

Mondadori also cites the letter to Count Ernst von Hessen-Rheinfels of April 12, 1686, where Leibniz talks about the different possible Adams, all of which differ from each other (PW 51, G.ii 20):

> For by the individual notion of Adam I undoubtedly mean a perfect representation of a particular Adam, with given individual conditions and distinguished thereby from an infinity of other possible persons very much like him, but yet different from him... There is one possible Adam whose posterity is such and such, and an infinity of others whose posterity would be different; is it not the case that these possible Adams (if I may so speak of them) are different from one another, and that God has chosen only one of them, who is exactly our Adam?

When Mondadori suggests using counterpart theory to model Leibniz's views, he notes that whereas for Lewis the counterpart relation is a relation on individuals, "in Leibniz's case, it is best regarded as being a relation between (complete) concepts" (1973, 248). This suggestion is explicitly built into the Leibnizian system described in Fitch 1979, and has been adopted by other commentators as well.[231]

According to this view, in a Leibnizian modal metaphysics, a possible world is not a locus of compossible individuals but rather a locus of *compossible individual concepts*. A reconstruction of such metaphysics would involve: (1) a compossibility partition of individual concepts so as to induce a correspondence: for each possible world, there is a group of compossible individual concepts that appear at that world, and (2) a notion of *counterpart* that connects each individual concept of a given possible world to various other individual concepts in other possible worlds. Such a reconstruction has to ensure that an ordinary claim such as:

> Alexander is a king but might not have been.

becomes equivalent to the following claim:

> The concept of Alexander contains the concept of *being a king*, but there is an individual concept c such that: (i) c is a counterpart of the concept of Alexander, (ii) c doesn't contain the concept of *being a king*, and (iii) c appears at some other possible world.

In our reconstruction of these ideas below, this equivalence is preserved as a fundamental theorem of Leibniz's modal metaphysics, as item (598.1)★.

Now when Leibniz talks about the 'many possible Adams' and 'several Sextuses' that are all distinct from one another, the above-mentioned commentators take him to be talking about different concepts of the same individual

[231] See Wilson 1979 and Vailati 1986. Lloyd (1978) also accepts that Leibniz 'resorts to counterparts' (p. 379), though she discovers some Leibnizian features in a Kripkean semantics of rigid designators, which assumes that the same individual can appear in other possible worlds. Interestingly, Kripke notes that "Many have pointed out to me that the father of counterpart theory is probably Leibnitz [sic]" (1980, 45, n. 13).

rather than different possible individuals. In the modal metaphysics developed below, this suggestion is preserved, but we do *not* stipulate that the different concepts of Adam that appear at the various possible worlds stand in a primitive counterpart relation. Instead, we begin with the Kripkean view that Adam himself exemplifies different properties at different possible worlds. We then examine the different concepts of Adam that these exemplification facts *induce* on the domain of abstract objects. On the basis of this structure, we can define the sense in which the various concepts of Adam constitute counterparts of one another.

It is also important to remember again that our definitions will be cast within the context of the simplest quantified modal logic, in which there is a single, fixed domain of individuals. Thus, not only do the properties that Adam exemplifies at one possible world differ from the properties he exemplifies at other possible worlds, the ordinary individual Adam exemplifies properties at *every* possible world. For example, although Adam is concrete at our world and at certain other possible worlds, there are possible worlds where he fails to be concrete. In many of the possible worlds where Adam is concrete, his 'posterity is different'. At possible worlds where Adam is not concrete, he has no posterity.

Now although the same ordinary individual Adam exemplifies properties at every other possible world, a Leibnizian metaphysics of 'world-bound' abstract individuals emerges once we consider, for each possible world w, the concept that encodes exactly the properties that Adam exemplifies at w. At each possible world, Adam realizes a *different* concept, since concepts differ whenever they encode distinct properties. The concept that encodes all and only the properties Adam exemplifies at one possible world is distinct from the concept that encodes all and only the properties Adam exemplifies at a different possible world, though all of the different concepts of Adam will be counterparts of one another. Of course, we will define only one of these concepts to be *the concept of Adam*, namely, the concept that encodes just what Adam exemplifies at the actual world. In other words, once we relativize concepts of Adam to a world w, then *the* concept of Adam will be identified with the concept of Adam at $\boldsymbol{w}_\alpha$. When Leibniz talks about 'possible Adams', we may take him to be talking about different Adam-at-w concepts. We'll explain this in more detail once the definitions and theorems have been presented.

What is more interesting is the fact that these individual concepts have certain other Leibnizian features. We shall not just *stipulate* that compossibility is an equivalence condition on individual concepts, but rather *define* compossibility and *prove* that it is such a condition. Moreover, we shall not define possible worlds as sets of compossible individual concepts, but rather prove that there is a one-to-one correspondence between each group of compossible concepts and the possible worlds. We shall also show that there is a sense of *mirrors* for which it is provable that each member of a group of compossible individual concepts is a perfect mirror of its corresponding possible world.

So, although we do not use counterpart-theoretic primitives in our reconstruction, we nevertheless *recover* a modal metaphysics in which complete individual concepts are *world-bound*. These individual concepts will provide an interpretation for much of what Leibniz says about necessity, contingency, completeness, mirroring, etc. and, in the process, offer a way to reconcile Kripke's and Lewis's modal metaphysics.

13.4.1 Realization, Appearance, Mirroring

In this subsection, we continue to use the variables u, v to range over ordinary objects.

(**557**) **Definition:** Realization at a World. Let us say that an ordinary object *u realizes* a concept *c at* possible world w just in case for all properties F, the proposition that u-exemplifies-F is true at w if and only if c encodes F:

$$\mathit{RealizesAt}(u,c,w) =_{df} \forall F(w \models Fu \equiv cF)$$

In other words, u realizes c at w just in case u exemplifies at w exactly the properties c encodes:

(**558**) **Remark:** In what follows, we shall be studying notions definable, and theorems expressible, in terms of *RealizesAt*. This will be an investigation in the abstract; we won't prove any particular claims of the form $\mathit{RealizesAt}(u,c,w)$. Indeed, we can't do so, and the reason is that our system, which hasn't yet been applied, doesn't identify any particular ordinary objects, though we know that there are some.

To be maximally specific here, note that the following is provable, though by non-modally strict means:

$$\exists u \exists c \exists w(\mathit{RealizesAt}(u,c,w))$$

Proof. By (174.2) and the T-schema, $\exists x O!x$. Let a be such an object, so that we know $O!a$. Now consider $\boldsymbol{c}_a$. By (546)★, we know $\boldsymbol{c}_a F \equiv Fa$. Now consider $\boldsymbol{w}_\alpha$. By (426)★, we know $Fa \equiv \boldsymbol{w}_\alpha \models Fa$. Hence, $\boldsymbol{c}_a F \equiv \boldsymbol{w}_\alpha \models Fa$ and, by the symmetry of the bicondtional, $\boldsymbol{w}_\alpha \models Fa \equiv \boldsymbol{c}_a F$. So by GEN, $\forall F(\boldsymbol{w}_\alpha \models Fa \equiv \boldsymbol{c}_a F)$. Hence, by ∃I, $\exists u \exists c \exists w \forall F(w \models Fu \equiv cF)$, i.e., $\exists u \exists c \exists w(\mathit{RealizesAt}(u,c,w))$. This conclusion remains once we discharge our assumption about a by ∃E.

But though we can prove this claim, we can't prove, for any particular ordinary individual, that it realizes a particular concept at a world. Our unapplied theory doesn't identify or individuate any particular ordinary individual. So, the reader should recognize that though the theorems in what follows do govern ordinary objects, concepts, and worlds, we shall in some sense be *projecting* the claims being established without proving particular instances of them. Of course, on occasion, it will be helpful to illustrate these claims with examples drawn from the actual world. But such examples presuppose that the theory has been appropriately extended with new (and sometimes contingent) facts.

(**559**) **Theorem:** Facts about Realization. The definition of realization at a world has the following consequences. (.1) if u realizes c at w and u realizes d at w, then c and d are identical; (.2) if u realizes c at w and v realizes c at w, then u is identical to v; and (.3) if u realizes c at w and u realizes c at w', then w is identical to w':

(.1) $RealizesAt(u,c,w) \,\&\, RealizesAt(u,d,w) \rightarrow c = d$

(.2) $RealizesAt(u,c,w) \,\&\, RealizesAt(v,c,w) \rightarrow u = v$

(.3) $RealizesAt(u,c,w) \,\&\, RealizesAt(u,c,w') \rightarrow w = w'$

(**560**) **Definition:** Appearance at a World. We say that a concept c *appears at* a possible world w just in case some ordinary object realizes c at w:

$$AppearsAt(c,w) \ =_{df} \ \exists u RealizesAt(u,c,w)$$

(**561**) **Theorem:** Fact About Appearance. In light of the foregoing facts about realization at a world, we also have the following fact about appearance at a world: if a concept c appears at possible world w, then a unique ordinary object realizes c at w:

$$AppearsAt(c,w) \rightarrow \exists! u(RealizesAt(u,c,w))$$

(**562**) **Theorem:** Appearance and Being Ordinary. It proves useful to remember that if a concept c appears at a world w, then c encodes the property of *being ordinary*:

$$AppearsAt(c,w) \rightarrow cO!$$

(**563**) **Definition:** Mirroring. Recall that in (216) we stipulated that x encodes a proposition p, written $x\Sigma p$, just in case x is abstract and encodes $[\lambda y\, p]$. Since concepts are abstract objects (477), $c\Sigma p$ is defined and it follows that $c\Sigma p \equiv c[\lambda y\, p]$. So we now say that a concept c *mirrors* a possible world w just in case for any proposition p, c encodes p if and only if p is true at w:

$$Mirrors(c,w) \ =_{df} \ \forall p(c\Sigma p \ \equiv \ w \models p)$$

Also, as noted in (402), $w \models p$ is equivalent to $w\Sigma p$. So the above definition entails the following equivalence:

$$Mirrors(c,w) \equiv \forall p(c\Sigma p \equiv w\Sigma p)$$

This better reveals why the definiens introduces a notion of mirroring: concept c mirrors a possible world w just in case c and w encode the same propositions.

(**564**) **Theorem:** Appearance and Mirroring. It now follows that if a concept appears at a possible world, it mirrors that world:

$$AppearsAt(c,w) \to Mirrors(c,w)$$

It is important to understand why the right-to-left direction fails, i.e., why it is provable that $\exists c \exists w(Mirrors(c,w) \,\&\, \neg AppearsAt(c,w))$. To find witnesses to this claim, consider any possible world, say w_1 (we know that there are are at least two, by (435.4)). Since possible worlds are abstract objects, they are also concepts. So w_1 is both a possible world and a concept. Then, clearly, $Mirrors(w_1,w_1)$. However, $\neg AppearsAt(w_1,w_1)$. For suppose otherwise, i.e., suppose that $AppearsAt(w_1,w_1)$. Then, by our useful fact (562), it follows that $w_1 O!$. But since w_1 is, by hypothesis, a possible world, it is a situation (400). Hence every property w_1 encodes is a propositional property (363). It follows that $Propositional(O!)$. But this contradicts (202.4.c).

(**565**) **Theorem:** New Fact About Appearance. If a concept c appears at possible worlds w and w', then $w=w'$:

$$AppearsAt(c,w) \,\&\, AppearsAt(c,w') \to w=w'$$

(**566**) **Theorem:** Appearance At is Rigid. A concept c appears at a possible world w if and only if it necessarily does so:

$$AppearsAt(c,w) \equiv \Box AppearsAt(c,w)$$

(**567**) **Theorem:** New Fact About Realization. We can now easily prove that if u realizes c at w and v realizes c at w', then both w is identical to w' and u is identical to v.

$$RealizesAt(u,c,w) \,\&\, RealizesAt(v,c,w') \to (w=w' \,\&\, u=v)$$

Intuitively, this theorem tells us that the concept alone fixes the other parameters of the condition $RealizesAt(u,c,w)$.

(**568**) ⋆**Lemma:** Concepts of Ordinary Individuals, Realization, Appearance, and Mirroring. (.1) u realizes the concept of u at the actual world; (.2) the concept of u appears at the actual world; (.3) the concept of u mirrors the actual world:

(.1) $RealizesAt(u, \boldsymbol{c}_u, \boldsymbol{w}_\alpha)$

(.2) $AppearsAt(\boldsymbol{c}_u, \boldsymbol{w}_\alpha)$

(.3) $Mirrors(\boldsymbol{c}_u, \boldsymbol{w}_\alpha)$

None of these theorems are modally strict.

13.4.2 Individual Concepts

(**569**) **Definition:** Individual Concepts. We now introduce a notion of *individual concept*, as distinct from concepts of individuals. An individual concept is any concept that appears at some possible world:

$$IndividualConcept(c) =_{df} \exists w AppearsAt(c, w)$$

(**570**) **Theorem:** Concepts of Ordinary Individuals are Individual Concepts. Recalling the definition of a concept of an ordinary individual (542), it follows that if c is a concept of an ordinary individual u, then c is an individual concept:

$$\exists u(ConceptOf(c, u)) \rightarrow IndividualConcept(c)$$

There is an easy, but non-modally strict proof that appeals to theorem (426)★, i.e., $p \equiv \boldsymbol{w}_\alpha \models p$.[232] Yet with some work, this theorem can be proved by modally strict means.

(**571**) ★**Theorem:** The Concept of u is an Individual Concept.

$$IndividualConcept(\boldsymbol{c}_u)$$

This theorem fails to be modally strict because the properties $\boldsymbol{c}_u$ encodes depend on the properties u (in fact) exemplifies.

(**572**) **Theorem:** Individual Concepts Exist. Since we know, by (174) and the T schema, that ordinary individuals exist, it is relatively easy to establish that individual concepts exist as well:

$$\exists c IndividualConcept(c)$$

[232] Assume $\exists u ConceptOf(c, u)$, and suppose a be such an ordinary object, so that we know $ConceptOf(c, a)$. If we can show $\forall F(\boldsymbol{w}_\alpha \models Fa \equiv cF)$, then by existentially generalizing on a and $\boldsymbol{w}_\alpha$ and applying definitions (557), (560), and (569), we're done. But by definition (542), $ConceptOf(c, a)$ implies $\forall F(cF \equiv Fa)$. Moreover, as an instance of (426)★, we know $Fa \equiv \boldsymbol{w}_\alpha \models Fa$, which by GEN yields $\forall F(Fa \equiv \boldsymbol{w}_\alpha \models Fa)$. So by the laws of quantified biconditionals, $\forall F(cF \equiv \boldsymbol{w}_\alpha \models Fa)$, i.e., $\forall F(\boldsymbol{w}_\alpha \models Fa \equiv cF)$.

Since the notion of an individual concept is well-defined, we may introduce restricted variables to range over them. Let us use the circumflexed, lower-case italic letters $\hat{c}, \hat{d}, \hat{e}, \ldots$ as restricted variables ranging over individual concepts. Note that the above theorem implies that the quantifiers $\forall\hat{c}$ and $\exists\hat{c}$ behave classically in the sense that $\forall\hat{c}\varphi \to \exists\hat{c}\varphi$; cf. Remark (256). Note also that our remarks about doubly restricted variables in (401) apply to our restricted variables for individual concepts.

(**573**) **Theorem:** Appearance at a Unique Possible World. We may now establish (.1) there is a unique possible world at which an individual concept appears; and (.2) that there is a unique world at which an individual concept necessarily appears:

(.1) $\exists!w AppearsAt(\hat{c}, w)$

(.2) $\exists!w \Box AppearsAt(\hat{c}, w)$

Clearly, (.1) is sufficient to establish (.3) there is something which is the world at which individual concept $\hat{c}$ appears:

(.3) $\exists x(x = \imath w AppearsAt(\hat{c}, w))$

(**574**) **Restricted Term Definition:** The Possible World At Which an Individual Concept Appears. The previous theorem allows us to introduce the notation $\boldsymbol{w}_{\hat{c}}$ to refer to the possible world at which individual concept $\hat{c}$ appears:

$$\boldsymbol{w}_{\hat{c}} =_{df} \imath w(AppearsAt(\hat{c}, w))$$

Thus, $\boldsymbol{w}_{\kappa}$ is a functional term that we may regard as well-formed and logically proper whenever κ is an individual term that signifies something known to be an individual concept.

(**575**) **Theorem:** Modally Strict Fact About $\boldsymbol{w}_{\hat{c}}$. It is now a modally strict fact that an individual concept appears at the world where it appears:

$$AppearsAt(\hat{c}, \boldsymbol{w}_{\hat{c}})$$

This theorem sounds trivial and there is a non-modally strict proof that *is* trivial. But the modally strict proof involves (573.2), theorem (108.1), and definition (574).

(**576**) **Theorems:** Individual Concepts, Mirroring, and Containment. It now follows (.1) that an individual concept $\hat{c}$ mirrors the possible world where it appears:

(.1) $Mirrors(\hat{c}, \boldsymbol{w}_{\hat{c}})$

Cf. Leibniz 1714, Section 56, where we find (PW 187, G.vi 616):

> Now this *connexion* or adaptation of all created things with each, and of each with all the rest, means that each simple substance has relations which express all the others, and that consequently, it is a perpetual living mirror of the universe.

Here, we have to interpret Leibniz's talk of simple substances in terms of the individual concepts of ordinary individuals. Moreover, only those ordinary individuals that are living have individual concepts that 'are' alive, in the sense that they encode *being alive*.

Furthermore, from previous theorems we may conclude that (.2) an individual concept $\hat{c}$ contains the possible world where it appears:

(.2) $\hat{c} \succeq \boldsymbol{w}_{\hat{c}}$

Cf. Leibniz 1686, Article 9, where we find (PW 19–20, G.iv 434):

> Further, every substance is like an entire world and like a mirror of God, or of the whole universe, which each one expresses in its own way, very much as one and the same town is variously represented in accordance with different positions of the observer. Thus, the universe is in a way multiplied as many times as there are substances, ...

So the metaphor of mirroring was used early on in Leibniz's work. Once we interpret Leibniz to be talking about the individual concepts of ordinary individuals, we see that the world where an individual concept appears ($\boldsymbol{w}_{\hat{c}}$) is indeed 'multiplied as many times as there are substances' since each individual concept $\hat{c}$ in each group of 'compossible' concepts (see the next section) has $\boldsymbol{w}_{\hat{c}}$ as a part.

(**577**) **Theorem:** Individual Concepts Contain the Concepts of Encoded Properties. It is also a consequence of the foregoing that an individual concept encodes a property if and only if it contains the concept of that property:

$$\hat{c}G \equiv \hat{c} \succeq \boldsymbol{c}_G$$

(**578**) **Theorems:** Individual Concepts and Property Negation. Facts about property negation become reflected in individual concepts as follows: (.1) an individual concept encodes G if and only if it fails to encode the negation of G; (.2) an individual concept encodes the negation of G if and only if it fails to encode G; (.3) an individual concept contains the concept of G if and only if it doesn't contain the concept of the negation of G; and (.4) an individual concept doesn't contain the concept of G if and only if it contains the concept of the negation of G:

(.1) $\hat{c}G \equiv \neg\hat{c}\overline{G}$

(.2) $\hat{c}\overline{G} \equiv \neg\hat{c}G$

(.3) $\hat{c} \succeq \boldsymbol{c}_G \equiv \hat{c} \not\succeq \boldsymbol{c}_{\overline{G}}$

(.4) $\hat{c} \not\succeq \boldsymbol{c}_G \equiv \hat{c} \succeq \boldsymbol{c}_{\overline{G}}$

(**579**) **Theorem:** Individual Concepts and Completeness. Recall that in (547) we defined a sense in which concepts are complete. It straightforwardly follows that an individual concept is complete:

$$\mathit{Complete}(\hat{c})$$

This provides further confirmation of Article 8 of the *Discourse on Metaphysics*, quoted earlier, where Leibniz says that "it is in the nature of an individual substance, or complete being, to have a notion so complete that it is sufficient to contain and render deducible from itself, all the predicates of the subject to which this notion is attributed" (PW 18–19, G.iv 433).

13.4.3 Compossibility

(**580**) **Definition:** Compossibility. We say that two individual concepts are compossible just in case they appear at the same possible world:

$$\mathit{Compossible}(\hat{c},\hat{e}) =_{df} \exists w(\mathit{AppearsAt}(\hat{c},w) \,\&\, \mathit{AppearsAt}(\hat{e},w))$$

(**581**) **Lemma:** Common Possible World of Compossible Individual Concepts. It follows from the previous definition that if individual concepts $\hat{c}$ and $\hat{e}$ are compossible, then the possible worlds where they appear are identical:

$$\mathit{Compossible}(\hat{c},\hat{e}) \equiv \boldsymbol{w}_{\hat{c}} = \boldsymbol{w}_{\hat{e}}$$

(**582**) **Theorems:** Compossibility Partitions the Individual Concepts. It follows from the previous lemma that compossibility is reflexive, symmetric, and transitive with respect to individual concepts:

(.1) $\mathit{Compossible}(\hat{c},\hat{c})$

(.2) $\mathit{Compossible}(\hat{c},\hat{e}) \to \mathit{Compossible}(\hat{e},\hat{c})$

(.3) $\mathit{Compossible}(\hat{c},\hat{d}) \,\&\, \mathit{Compossible}(\hat{d},\hat{e}) \to \mathit{Compossible}(\hat{c},\hat{e})$

Since compossibility is reflexive, symmetrical, and transitive with respect to individual concepts, we know that the latter are partitioned. In light of lemma (581), all of the individual concepts that are compossible with one another appear at a common possible world.

13.4.4 World-Relative Concepts of Individuals

(**583**) **Definition:** World-Relative Concepts of Ordinary Individuals. Let us say that c is a *concept of* u *at* w just in case c encodes exactly the properties that u exemplifies at w:

$$ConceptOfAt(c,u,w) \ =_{df} \ \forall F(cF \equiv w \models Fu)$$

Note that we could have used $RealizesAt(u,c,w)$ as the definiens for the definiendum $ConceptOfAt(c,u,w)$, since the definiens of $RealizesAt(u,c,w)$ is equivalent to the one above; they differ only by commuting the (quantified) biconditional. Though the linguistic notion of *focus* doesn't apply to the formulas in our system, it nevertheless has some application to the regimented natural language we use to read those formulas. When we read the above definiens, the focus is on the concept c and the properties it encodes, whereas when we read the definiens of $RealizesAt(u,c,w)$, the focus is on u and the properties it exemplifies at w.

(**584**) **Theorems:** Conditional Existence Conditions for World-Relative Concepts of Individuals. It follows that (.1) there is a concept of u at w; (.2) there is a unique concept of u at w; and (.3) there is something that is *the* concept of u at w:

(.1) $\exists c ConceptOfAt(c,u,w)$

(.2) $\exists! c ConceptOfAt(c,u,w)$

(.3) $\exists z(z = \imath c ConceptOfAt(c,u,w))$

Strictly speaking, these are conditional existence claims, given the presence of the restricted variables u and w. But since we know that ordinary individuals exist (by (174) and the T schema) and that there are at least two possible worlds (435.4), we can derive *unconditional* existence claims.

(**585**) **Restricted Term Definition:** Notation for the Concept of u at w. We henceforth use the notation $\boldsymbol{c}_u^w$ to denote the concept of u at w:

$$\boldsymbol{c}_u^w \ =_{df} \ \imath c ConceptOfAt(c,u,w)$$

This introduces $\boldsymbol{c}_u^w$ as a binary functional term with the free restricted variables u and w. Hence, we shall use expressions of the form $\boldsymbol{c}_{\kappa_2}^{\kappa_1}$ only when the singular terms κ_1 and κ_2 are known, respectively, to be a possible world and an ordinary object, either by proof or by hypothesis, for only then is $\boldsymbol{c}_{\kappa_2}^{\kappa_1}$ both well-formed and logically proper. Clearly, it follows that $\boldsymbol{c}_u^w$ is (identical to) a canonical concept, since it easily follows that $\boldsymbol{c}_u^w$ equals $\imath c \forall F(cF \equiv w \models Fu)$.

(**586**) **Theorems:** $\boldsymbol{c}_u^w$ is Strictly Canonical. If we let φ be the formula $w \models Fu$, then it follows that φ is a rigid condition on properties, i.e., that (.1) necessarily, every property such that φ is necessarily such that φ:

(.1) $\Box\forall F(w \models Fu \rightarrow \Box w \models Fu)$

So $\boldsymbol{c}_u^w$ is (identical to) a strictly canonical concept, by (188.2). It therefore follows by (189.2) that (.2) $\boldsymbol{c}_u^w$ is a concept that encodes exactly those properties F that u exemplifies at w:

(.2) $C!\boldsymbol{c}_u^w \,\&\, \forall F(\boldsymbol{c}_u^w F \equiv w \models Fu)$

(**587**) **Lemma:** Basic Facts about World-Relative Concepts of Ordinary Individuals. (.1) u realizes the concept of u at w at w [*sic*]; (.2) the concept of u at w appears at w; (.3) the concept of u at w is an individual concept; (.4) the concept of u at w mirrors w; and (.5) the concept of u at w is complete:

(.1) $\mathit{RealizesAt}(u, \boldsymbol{c}_u^w, w)$

(.2) $\mathit{AppearsAt}(\boldsymbol{c}_u^w, w)$

(.3) $\mathit{IndividualConcept}(\boldsymbol{c}_u^w)$

(.4) $\mathit{Mirrors}(\boldsymbol{c}_u^w, w)$

(.5) $\mathit{Complete}(\boldsymbol{c}_u^w)$

These are all modally strict.

(**588**) **Theorem:** Equivalence of Individual Concepts and World-Relative Concepts of Individuals. From the definitions of individual concept and world-relative concepts of individuals, we may prove that (.1) a concept c is an individual concept iff there is some ordinary object u and possible world w such that c is a concept of u at w; and (.2) a concept c is an individual concept iff there is some and ordinary object u and possible world w such that c is identical to the concept of u at w:

(.1) $\mathit{IndividualConcept}(c) \equiv \exists u \exists w \mathit{ConceptOfAt}(c, u, w)$

(.2) $\mathit{IndividualConcept}(c) \equiv \exists u \exists w(c = \boldsymbol{c}_u^w)$

(**589**) **Lemma:** The Concept of an Individual at the Actual World. It is a basic fact about the notions that we've defined so far that the concept of u is identical to the concept of u at the actual world $\boldsymbol{w}_\alpha$:

$$\boldsymbol{c}_u^{\boldsymbol{w}_\alpha} = \boldsymbol{c}_u$$

While it is easy to give a non-modally strict proof of this theorem, the interest is in the fact that it is capable of a modally strict proof.

(**590**) **Lemmas:** Further Facts about World-Relative Concepts of Individuals. The following 5 lemmas are also immediately forthcoming: (.1) the concept of

u at w encodes a property G iff it contains $\boldsymbol{c}_G$; (.2) u exemplifies G at w iff the concept of u at w contains $\boldsymbol{c}_G$; (.3) if the concept of u at w is identical to the concept of v at w, then u and v are identical; (.4) if the concept of u at w is identical to the concept of u at w', then $w=w'$; and (.5) if the concept of u at w is identical to the concept of v at w', then $w=w'$ and $u=v$:

(.1) $\boldsymbol{c}_u^w G \equiv \boldsymbol{c}_u^w \succeq \boldsymbol{c}_G$

(.2) $w \models Gu \equiv \boldsymbol{c}_u^w \succeq \boldsymbol{c}_G$

(.3) $\boldsymbol{c}_u^w = \boldsymbol{c}_v^w \rightarrow u=v$

(.4) $\boldsymbol{c}_u^w = \boldsymbol{c}_u^{w'} \rightarrow w=w'$

(.5) $\boldsymbol{c}_u^w = \boldsymbol{c}_v^{w'} \rightarrow (w=w' \;\&\; u=v)$

(**591**) **Theorem:** Compossible World-Relative Concepts of Individuals. It also follows that the concept of u at w and the concept of v at w are compossible:

$$Compossible(\boldsymbol{c}_u^w, \boldsymbol{c}_v^w)$$

(**592**) **Remark:** Interpreting Leibniz's Theodicy. Though the system of individual concepts and possible worlds (i.e., the group of theorems that pertain to individual concepts and possible worlds) are completely silent about the existence of God, they nevertheless preserve an element of Leibniz's theodicy, namely, his conception of the work that God had to do to 'create' the actual world. Our results suggest that in order to evaluate all the possible worlds, in the effort to determine which one should be actualized, all God had to do was to inspect an arbitrarily chosen individual concept from each group of compossible individual concepts. That one inspection alone would reveals all the facts about *the* world where that individual concept appears, since every individual concept of the group mirrors that world. Note that our metaphysics doesn't tell us which world is the actual world other than by the principle that the actual world is the one that encodes all and only the true propositions. So God could actualize a world w (having deciding it was the best) by making it the case that every proposition encoded in w is true. This is what God had to do to create the actual world. Indeed, when inspecting the possible worlds by examining an arbitrary individual concept from each group of compossible concepts, God could just actualize the individual concept c that represented the possible world which turned out to be the best. To do so, God would have made it the case that there is in fact an ordinary object which exemplifies all the properties that c encodes. In doing *that*, God would as a consequence actualize $\boldsymbol{w}_{\hat{c}}$; when God makes it the case that there is an ordinary object that exemplifies the properties c encodes, God ends up making it the case that all of the propositions true in $\boldsymbol{w}_{\hat{c}}$ are true, since c mirrors $\boldsymbol{w}_{\hat{c}}$ (576.1).

(**593**) **Remark:** Another Reason Why Concepts Are Not Properties. It is worth mentioning here that another reason not to identify concepts as properties is that such a view gets the Leibnizian *metaphysics* of individual concepts wrong. It is central to Leibniz's view of individual concepts that for each ordinary individual u and possible world w, a unique individual concept corresponds to u at w. But if concepts are analyzed as properties, and individual concepts become a special kind of property, then uniqueness will fail unless the property theorist *requires* that properties be identical when necessarily equivalent, i.e., that $\Box\forall x(Fx \equiv Gx) \to F = G$.

To see why, suppose one were to define:

> F is an individual concept of u at w if and only if both (a) F necessarily implies all of the properties that u exemplifies at w, and (b) u uniquely exemplifies F at w.

To see the problem with this definition, consider the concept of Adam. If the necessary equivalence of properties doesn't imply their identity, then one can't prove that for every world w, there is a unique individual concept of Adam at w, as defined above. For suppose property P is Adam's complete individual concept at w. Now suppose Q is a property necessarily equivalent to, but distinct from, P. Then Q is an individual concept of Adam, by the following reasoning:

> Q satisfies clause (a) of the definition. Since P and Q are necessarily equivalent, then they necessarily imply the same properties by (340.3). So if P necessarily implies all the properties Adam exemplifies at w, then so does Q.
>
> Q satisfies clause (b) of the definition. Since Adam uniquely exemplifies P at w, and P and Q are necessarily equivalent, then Adam uniquely exemplifies Q at w (we leave the proof as an exercise).

So if necessary equivalence doesn't imply identity, and there are properties like Q that are necessarily equivalent to P, then P isn't a unique individual concept of Adam at w. Thus, the property theorist faces a dilemma: either require that necessarily equivalent properties are identical and derive the existence of a unique concept of Adam at w or omit the requirement that necessarily equivalent properties are identical and give up the claim that there is a unique concept of Adam at w. Such a dilemma is not faced on the present theory.

In the next section, we interpret Leibniz's talk of 'the several Sextuses' and 'the many possible Adams' as a reference to the variety of individual concepts connected with Sextus and Adam, in a sense to be precisely defined. Even though our subsystem of individual concepts was carved out without appeal to counterpart theory, we can, nevertheless, still speak with the counterpart theorists!

13.4.5 Counterpart Theory

(**594**) **Definition:** Counterpart Of. We say that individual concept $\hat{e}$ is a *counterpart of* individual concept $\hat{c}$ just in case there is an ordinary individual u and there are possible worlds w and w' such that u realizes $\hat{c}$ at w and u realizes $\hat{e}$ at w':

$$\mathit{CounterpartOf}(\hat{e},\hat{c}) =_{df} \exists u\exists w\exists w'(\mathit{RealizesAt}(u,\hat{c},w) \,\&\, \mathit{RealizesAt}(u,\hat{e},w'))$$

For example, if Alexander is an ordinary object that realizes an individual concept $\hat{c}$ at the actual world and realizes an individual concept $\hat{e}$ at some non-actual possible world, then $\hat{e}$ is a counterpart of $\hat{c}$.

(**595**) **Theorem:** Counterpart Of is an Equivalence Condition. We now have (.1) an individual concept is a counterpart of itself; (.2) if $\hat{e}$ is a counterpart of $\hat{c}$, then $\hat{c}$ is a counterpart of $\hat{e}$; and (.3) if $\hat{e}$ is a counterpart of $\hat{d}$, and $\hat{d}$ is a counterpart of $\hat{c}$, then $\hat{e}$ is a counterpart of $\hat{c}$:

(.1) $\mathit{CounterpartOf}(\hat{c},\hat{c})$

(.2) $\mathit{CounterpartOf}(\hat{e},\hat{c}) \rightarrow \mathit{CounterpartOf}(\hat{c},\hat{e})$

(.3) $\mathit{CounterpartsOf}(\hat{e},\hat{d}) \,\&\, \mathit{CounterpartOf}(\hat{d},\hat{c}) \rightarrow \mathit{CounterpartOf}(\hat{e},\hat{c})$

(**596**) **Theorem:** Counterparts and Realization. It follows that if $\hat{e}$ is a counterpart of $\hat{c}$, then there is a unique ordinary individual that realizes $\hat{c}$ at some world w and that realizes $\hat{e}$ at some world w':

$$\mathit{CounterpartOf}(\hat{e},\hat{c}) \equiv \exists!u\exists w\exists w'(\mathit{RealizesAt}(u,\hat{c},w) \,\&\, \mathit{RealizesAt}(u,\hat{e},w'))$$

(**597**) **Theorem:** World-Relative Concepts and Counterparts. It follows that (.1) the concept of u at w' is a counterpart of the concept of u-at-w [hyphens have been added here, and will be added elsewhere, to rule out the interpretation: the concept of u at w' is a counterpart at w of the concept of u — *counterpart of* is not a condition relativized to possible worlds]; and (.2) $\hat{e}$ is a counterpart of $\hat{c}$ if and only if there is an ordinary individual u and there are worlds w and w' such that $\hat{e}$ is the concept of u at w and $\hat{c}$ is the concept of u at w':

(.1) $\mathit{CounterpartOf}(\boldsymbol{c}_u^{w'},\boldsymbol{c}_u^{w})$

(.2) $\mathit{CounterpartOf}(\hat{e},\hat{c}) \equiv \exists u\exists w\exists w'(\hat{c} = \boldsymbol{c}_u^{w} \,\&\, \hat{e} = \boldsymbol{c}_u^{w'})$

The first of these plays an important role in a fundamental theorem of Leibnizian modal metaphysics.

13.4.6 Fundamental Theorems

(**598**) ⋆**Theorem:** A Fundamental Theorem of Leibnizian Modal Metaphysics. It seems reasonable to suggest that Leibniz's modal metaphysics is driven by the following conditionals (although Leibniz never explicitly formulates them as such), namely, (.1) if an ordinary individual u exemplifies F but might not have, then both (a) the concept of u contains the concept of F and (b) there is an individual concept $\hat{c}$ such that $\hat{c}$ is a counterpart of the concept of u, $\hat{c}$ doesn't contain the concept of F, and $\hat{c}$ appears at a possible world distinct from the actual world; and (.2) if an ordinary individual u doesn't exemplify F but might have, then both (a) the concept of u doesn't contain the concept of F and (b) there is an individual concept $\hat{c}$ such that $\hat{c}$ is a counterpart of the concept of u, $\hat{c}$ does contain the concept of F, and $\hat{c}$ appears at a possible world distinct from the actual world:

(.1) $Fu \,\&\, \Diamond\neg Fu \rightarrow$
$$\boldsymbol{c}_u \succeq \boldsymbol{c}_F \,\&\, \exists\hat{c}(CounterpartOf(\hat{c},\boldsymbol{c}_u) \,\&\, \hat{c} \not\succeq \boldsymbol{c}_F \,\&\, \exists w(w \neq \boldsymbol{w}_\alpha \,\&\, AppearsAt(\hat{c},w)))$$

(.2) $\neg Fu \,\&\, \Diamond Fu \rightarrow$
$$\boldsymbol{c}_u \not\succeq \boldsymbol{c}_F \,\&\, \exists\hat{c}(CounterpartOf(\hat{c},\boldsymbol{c}_u) \,\&\, \hat{c} \succeq \boldsymbol{c}_F \,\&\, \exists w(w \neq \boldsymbol{w}_\alpha \,\&\, AppearsAt(\hat{c},w)))$$

If we suppose our theory has been applied, then as an example of (.1), we have: if Alexander is king but might not have been, then the concept of Alexander contains the concept of *being a king* and there is an individual concept that is a counterpart of the concept of Alexander, that doesn't contain the concept of *being a king*, and that appears at some world other than the actual world. As an example of (.2), we have: if Alexander fails to be a philosopher but might have been, then the concept of Alexander fails to contain the concept of *being a philosopher* and there is an individual concept that is a counterpart of the concept of Alexander, that does contain the concept of *being a philosopher*, and that appears at some world other than the actual world.

(**599**) ⋆**Theorems:** Biconditional Fundamental Theorems. Though we've labeled the preceding theorem a fundamental theorem of Leibnizian modal metaphysics, it is reasonable to suggest that, strictly speaking, a fundamental theorem so-called should be a biconditional. Though the converse of the preceding theorem is indeed derivable, a study of the matter reveals that the converse is stronger than it needs to be. As we shall see below, one can derive $Fu \,\&\, \Diamond\neg Fu$ from the simpler conjunction $\boldsymbol{c}_u \succeq \boldsymbol{c}_F \,\&\, \exists\hat{c}(CounterpartOf(\hat{c},\boldsymbol{c}_u) \,\&\, \hat{c} \not\succeq \boldsymbol{c}_F)$. This shows how much information is packed into the notions of concept containment and counterparts.

Consequently, we can formulate biconditional fundamental theorems as follows: (.1) an ordinary individual u exemplifies F but might not have if and only if both (a) the concept of u contains the concept of F and (b) there is an individual concept $\hat{c}$ such that $\hat{c}$ is a counterpart of the concept of u and $\hat{c}$ doesn't contain the concept of F; and (.2) an ordinary individual u doesn't exemplify F but might have if and only if both (a) the concept of u doesn't contain the concept of F and (b) there is an individual concept $\hat{c}$ such that $\hat{c}$ is a counterpart of the concept of u and $\hat{c}$ does contain the concept of F:

(.1) $Fu \;\&\; \Diamond\neg Fu \equiv \boldsymbol{c}_u \succeq \boldsymbol{c}_F \;\&\; \exists\hat{c}(\mathit{CounterpartOf}(\hat{c}, \boldsymbol{c}_u) \;\&\; \hat{c} \not\succeq \boldsymbol{c}_F)$

(.2) $\neg Fu \;\&\; \Diamond Fu \equiv \boldsymbol{c}_u \not\succeq \boldsymbol{c}_F \;\&\; \exists\hat{c}(\mathit{CounterpartOf}(\hat{c}, \boldsymbol{c}_u) \;\&\; \hat{c} \succeq \boldsymbol{c}_F)$

These theorems constitute *Leibnizian truth conditions* for the most fundamental statements of contingency.

(**600**) **Theorems:** Even More Fundamental Theorems. It is interesting to note that there are several ways of adjusting the formulation of the fundamental theorems so as to produce modally strict versions of these theorems. The most basic of these ways is to relativize both sides of the conditionals in (598)★, and both sides of the biconditional in (599)★, to a possible world. This yields the facts that (.1) if it is true at possible world w that an ordinary individual u exemplifies F but might not have, then (a) the concept of u at w contains the concept of F and (b) there is an individual concept $\hat{c}$ such that $\hat{c}$ is a counterpart of the concept of u-at-w, $\hat{c}$ doesn't contain the concept of F, and $\hat{c}$ appears at a possible world distinct from w; and (.2) if it is true, at possible world w, that an ordinary individual u doesn't exemplify F but might have, then both (a) the concept of u at w doesn't contain the concept of F and (b) there is an individual concept $\hat{c}$ such that $\hat{c}$ is a counterpart of the concept of u-at-w, $\hat{c}$ does contain the concept of F, and $\hat{c}$ appears at a possible world distinct from w; (.3) it is true, at possible world w, that an ordinary individual u exemplifies F but might not have if and only if (a) the concept of u at w contains the concept of F and (b) there is an individual concept $\hat{c}$ such that $\hat{c}$ is a counterpart of the concept of u-at-w and $\hat{c}$ doesn't contain the concept of F; and (.4) it is true, at possible world w, that an ordinary individual u doesn't exemplify F but might have if and only if both (a) the concept of u at w doesn't contain the concept of F and (b) there is an individual concept $\hat{c}$ such that $\hat{c}$ is a counterpart of the concept of u-at-w and $\hat{c}$ does contain the concept of F:

(.1) $w \models (Fu \;\&\; \Diamond\neg Fu) \rightarrow$
$\boldsymbol{c}_u^w \succeq \boldsymbol{c}_F \;\&\; \exists\hat{c}(\mathit{CounterpartOf}(\hat{c}, \boldsymbol{c}_u^w) \;\&\; \hat{c} \not\succeq \boldsymbol{c}_F \;\&\; \exists w'(w' \neq w \;\&\; \mathit{AppearsAt}(\hat{c}, w')))$

(.2) $w \models (\neg Fu \;\&\; \Diamond Fu) \rightarrow$
$\boldsymbol{c}_u^w \not\succeq \boldsymbol{c}_F \;\&\; \exists\hat{c}(\mathit{CounterpartOf}(\hat{c}, \boldsymbol{c}_u^w) \;\&\; \hat{c} \succeq \boldsymbol{c}_F \;\&\; \exists w'(w' \neq w \;\&\; \mathit{AppearsAt}(\hat{c}, w')))$

(.3) $w \models (Fu \;\&\; \Diamond\neg Fu) \equiv \boldsymbol{c}_u^w \succeq \boldsymbol{c}_F \;\&\; \exists\hat{c}(\mathit{CounterpartOf}(\hat{c}, \boldsymbol{c}_u^w) \;\&\; \hat{c} \not\succeq \boldsymbol{c}_F)$

(.4) $w \models (\neg Fu \,\&\, \Diamond Fu) \equiv \boldsymbol{c}_u^w \not\succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^w) \,\&\, \hat{c} \succeq \boldsymbol{c}_F)$

We now turn to the development of a precise account of mathematical objects and relations, beginning with natural cardinals and natural numbers.

(**601**) **Exercises:** Show that the following versions of (598.1)★ and (598.2)★ are modally strict:

(.1) $\Diamond(Fu \,\&\, \Diamond\neg Fu) \to \exists w(\boldsymbol{c}_u^w \succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^w) \,\&\, \hat{c} \not\succeq \boldsymbol{c}_F \,\&$
$\exists w'(w' \neq w \,\&\, AppearsAt(\hat{c}, w'))))$

(.2) $\Diamond(\neg Fu \,\&\, \Diamond Fu) \to \exists w(\boldsymbol{c}_u^w \not\succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^w) \,\&\, \hat{c} \succeq \boldsymbol{c}_F \,\&$
$\exists w'(w' \neq w \,\&\, AppearsAt(\hat{c}, w'))))$

Also, show that the following versions of (598.1)★ and (598.2)★ are modally strict:

(.3) $\mathcal{A}(Fu \,\&\, \Diamond\neg Fu) \to \boldsymbol{c}_u^{w_\alpha} \succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^{w_\alpha}) \,\&\, \hat{c} \not\succeq \boldsymbol{c}_F \,\&$
$\exists w'(w' \neq w_\alpha \,\&\, AppearsAt(\hat{c}, w')))$

(.4) $\mathcal{A}(\neg Fu \,\&\, \Diamond Fu) \to \boldsymbol{c}_u^{w_\alpha} \not\succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^{w_\alpha}) \,\&\, \hat{c} \succeq \boldsymbol{c}_F \,\&$
$\exists w'(w' \neq w_\alpha \,\&\, AppearsAt(\hat{c}, w')))$

Now show that the following versions of (599.1)★ and (599.2)★ are modally strict:

(.5) $\Diamond(Fu \,\&\, \Diamond\neg Fu) \equiv \exists w(\boldsymbol{c}_u^w \succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^w) \,\&\, \hat{c} \not\succeq \boldsymbol{c}_F))$

(.6) $\Diamond(\neg Fu \,\&\, \Diamond Fu) \equiv \exists w(\boldsymbol{c}_u^w \not\succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^w) \,\&\, \hat{c} \succeq \boldsymbol{c}_F))$

Also, show that the following versions of (599.1)★ and (599.2)★ are modally strict:

(.7) $\mathcal{A}(Fu \,\&\, \Diamond\neg Fu) \equiv \boldsymbol{c}_u^{\boldsymbol{w}_\alpha} \succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^{\boldsymbol{w}_\alpha}) \,\&\, \hat{c} \not\succeq \boldsymbol{c}_F)$

(.8) $\mathcal{A}(\neg Fu \,\&\, \Diamond Fu) \equiv \boldsymbol{c}_u^{\boldsymbol{w}_\alpha} \not\succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u^{\boldsymbol{w}_\alpha}) \,\&\, \hat{c} \succeq \boldsymbol{c}_F)$

(**602**) **Remark:** The importance of the theorems in this final section shouldn't be understated. An unanalyzed truth to the effect that an ordinary object has a property but might not have (or that an ordinary object doesn't have a property but might have), which is represented and regimented in terms of our modern notions of exemplification (Fx), negation ($\neg$) and possibility ($\Diamond$), implies a complex web of facts in Leibniz's modal metaphysics, involving the notions of: concepts ($C!$), concept containment ($\succeq$), possible worlds (w), identity ($=$), concepts of individuals ($\boldsymbol{c}_u$), individual concepts ($\hat{c}$) concepts of properties ($\boldsymbol{c}_F$), appearance, and counterpart of, not to mention the notions in terms of which these notions are defined. The most important of these latter are: encoding (xF), abstract objects ($A!$), situations, truth in a situation ($\models$), and propositional properties ($[\lambda y\, p]$). Moreover, these theorems have powerful consequences when the theory is applied. Once we start extending our theory with familiar, uncontroversial truths about the properties that ordinary objects exemplify (or fail to exemplify) contingently, an elaborate network of truths involving primitive and defined notions emerges and describes an elegant and precise metaphysical picture that articulates both the structural aspects of Leibniz's view of the mind of God as well as some (though not all) of Lewis's views about counterparts, even while preserving Kripkean intuitions that ground and anchor the structure by means of the properties each given ordinary individual exemplifies at every possible world.

Chapter 14

Natural Numbers

14.1 Setting the Stage

What are the natural numbers? Can the natural numbers number absolutely anything, as Frege assumed, or do they have a more limited range of application? Is there an infinity of natural numbers and, if so, can this be established in some way other than by stipulation? What primitive notions and axioms do we need to prove the basic postulates of number theory? Must we assume primitive mathematical notions and mathematical axioms for the proof of these postulates, or can we derive them from more general principles?

We try to answer these questions in the present chapter. We begin by recalling Remark (232), in which we distinguished *natural* from *theoretical* mathematics. In that Remark, it was noted that ordinary statements of number ("there are eight planets", etc.) constitute pretheoretical claims that we make independent of any explicit mathematical theory of numbers. These statements of number are therefore part of *natural* mathematics. They are to be distinguished from the statements mathematicians make when either asserting the axioms or theorems of some number theory, such as Dedekind-Peano number theory, real number theory, etc. These latter are part of *theoretical* mathematics.

In this chapter, we analyze numbers in natural mathematics by adapting techniques Frege used to define them and derive the principles that govern them. Whereas Frege thought that the natural numbers are cardinal numbers that count *all* the objects that exemplify a property, our analysis takes them to have a more limited range of application; for us, the natural numbers are natural cardinals that count the *ordinary* (i.e., possibly concrete) objects that exemplify a given property. Despite this difference, when we *extend* the present theory of abstract objects with a few intuitive, non-mathematical axioms that philosophers and logicians should accept, the resulting theory is capable of (a) defining the natural numbers as abstract objects, rather than taking them as primitive, and (b) deriving the most important number-theoretic postulates as theorems rather than taking them as axioms. Consequently, in this chapter, the primitive notions of Dedekind-Peano number theory will be defined and the postulates of that theory will be derived as theorems.

The work in this chapter, then, is to be contrasted with that in the next (Chapter 15), where we use a somewhat different method to analyze the language, axioms, and theorems of theoretical mathematics. Theoretical mathematics includes the various number theories, set theories, algebras and group theories, etc., and so includes any theory that assumes mathematical primitives and axioms. Thus, the Dedekind-Peano number theory will make an appearance in the next chapter as well as in this one, though in the next chapter, the primitive notions will *not* be analyzed using Frege's methods, but rather using techniques that apply to any theoretical mathematical theory.

Taken together, then, the views developed in this and the next chapter together bear some similarity to the famous quip attributed to Leopold Kronecker by his students, namely, that "God made the whole numbers; all the others are the work of men".[233] But instead of appealing to God as the source of the natural numbers, we find their origins in the principles governing abstract individuals.[234]

Answers will be developed to the questions posed at the outset. Moreover, we shall show that the existence of an infinite cardinal can be proved without appealing to *any* mathematical primitives or mathematical axioms. Thus, we shall be able to address what Heck (2011, 152) calls the fundamental epistemological question of the philosophy of arithmetic, namely, "What is the basis of our knowledge of the infinity of the series of natural numbers?" Our answer

[233]This is reported by H. Weber and attributed to Kronecker in Weber 1893 (15). The passage in Weber attributing the quote to Kronecker is:

> Manche von Ihnen werden sich des Ausspruchs erinnern, den er in einem Vortrag bei der Berliner Naturforscher-Versammlung im Jahre 1886 that [sic] "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk".

As far as I've been able to discover, Kronecker never published this particular statement.

[234]Similarly, we avoid the appeal to God (as well as to the primitive concepts of a *unit* and *adjunction*) found in Errett Bishop's work of 1967, where he says (2):

> The primary concern of mathematics is number, and this means the positive integers. ... The positive integers and their arithmetic are presupposed by the very nature of our intelligence and, we are tempted to believe, by the very nature of intelligence in general. The development of the positive integers from the primitive concept of the unit, the concept of adjoining a unit, and the process of mathematical induction carries complete conviction. In the words of Kronecker, the positive integers were created by God.

will be that such knowledge can be derived from principles governing abstract objects generally; no mathematics has to be assumed.

(**603**) **Remark:** The Dedekind-Peano Postulates for Number Theory. One of the goals of this chapter is to derive the Dedekind-Peano postulates for number theory as *theorems*. Both Dedekind 1888 and Peano 1889 contain statements of these basic postulates. We formulate them below, though the presentation doesn't exactly match that of either author.[235] To assert the postulates in a logically perspicuous way, three primitives are needed. They are:

- the individual *Zero*, denoted by the constant '0'
- the property *being a number*, denoted by the 1-place relation constant N
- the 2-place relation *succeeds* or its converse *precedes*, denoted respectively by the 2-place relation constants S and P

We may understand formulas of the form Nx as asserting that x exemplifies *being a number*, formulas of the form Sxy as asserting that x *immediately succeeds* y, and formulas of the form Pxy as asserting that x *immediately precedes* y.

We can state the Dedekind-Peano postulates in terms of the above primitives as follows, where m, n, o are restricted variables ranging over the assumed domain of numbers; in some of the axioms below, we give an alternative statement using *precedes* after the statement using *succeeds*:

1. Zero is a number.
   $N0$
2. Zero doesn't succeed any number.
   $\neg\exists n S0n$

   No number precedes Zero.
   $\neg\exists n Pn0$
3. If a number o succeeds numbers n and m, then $n = m$.
   $\forall n\forall m\forall o(Son \,\&\, Som \to n = m)$

   If numbers n and m precede a number o, then $n = m$.
   $\forall n\forall m\forall o(Pno \,\&\, Pmo \to n = m)$

[235] Dedekind's work in 1888 doesn't initially appear to constitute axioms for number theory. He basically stipulates what must obtain for a set N to be 'simply infinite' or inductive, namely, N must contain an element 1 and be a subset of some set S for which there is a function f on S such that (a) f maps N into N, (b) N is the minimal closure of the unit set $\{1\}$ in S under f, (c) 1 is not the value of f for any member of N, and (d) f is a one-to-one function. See Reck 2016 (Section 2.2), where he notes "it is not hard to show that these Dedekindian conditions are a notational variant of Peano's axioms for the natural numbers."

4. Every number is succeeded by some number.
   $\forall n \exists m Smn$

   Every number precedes some number.
   $\forall n \exists m Pnm$

5. Mathematical Induction: If (a) Zero exemplifies F and (b) Fm implies Fn whenever n succeeds m, then every number exemplifies F.
   $F0 \,\&\, \forall n \forall m(Snm \rightarrow (Fm \rightarrow Fn)) \rightarrow \forall n Fn$

   If (a) Zero exemplifies F and (b) Fn implies Fm whenever n precedes m, then every number exemplifies F.
   $F0 \,\&\, \forall n \forall m(Pnm \rightarrow (Fn \rightarrow Fm)) \rightarrow \forall n Fn$

In addition, Boolos (1995, 293; 1996, 275), and Heck (2011, 288) include the following among the Dedekind-Peano postulates:

6. If x succeeds n, x is a number.
   $\forall n \forall x(Sxn \rightarrow Nx)$

   If n precedes x, x is a number.
   $\forall n \forall x(Pnx \rightarrow Nx)$

7. If numbers n and m succeed a number o, then $n = m$.
   $\forall n \forall m \forall o(Sno \,\&\, Smo \rightarrow n = m)$

   If a number o precedes numbers n and m, then $n = m$.
   $\forall n \forall m \forall o(Pon \,\&\, Pom \rightarrow n = m)$

All seven postulates will be derived in what follows. Note that we shall sometimes refer to *succeeds* as the *successor* relation and *precedes* as the *predecessor* relation.

Our derivation of these postulates will borrow heavily from some of the methods used in Frege's Theorem, in which the Dedekind-Peano postulates are derived from Hume's Principle in second-order logic. This is explained in detail below, but before we turn to that discussion, it will be useful to have some definitions and a theorem before us.

(**604**) **Definition:** One-to-One Correspondence. Let us say that a 2-place relation R is a *one-to-one correspondence between F and G*, written $R : F \overset{1\text{-}1}{\longleftrightarrow} G$, just in case (a) every object exemplifying F is R-related to a unique object exemplifying G, and (b) every object exemplifying G is such that a unique object exemplifying F bears R to it:

$$R : F \overset{1\text{-}1}{\longleftrightarrow} G \ =_{df} \ \forall x(Fx \rightarrow \exists! y(Gy \,\&\, Rxy)) \,\&\, \forall x(Gx \rightarrow \exists! y(Fy \,\&\, Ryx))$$

(**605**) **Definitions:** Functions from F to G. Let us now say: (.1) R is a *function from F to G* just in case every object exemplifying F is related to a unique object exemplifying G:

(.1) $R: F \longrightarrow G =_{df} \forall x(Fx \rightarrow \exists! y(Gy \,\&\, Rxy))$

Moreover, we say (.2) R is a *one-to-one function from F to G* just in case R is a function from F to G and for every x, y, and z such that Fx, Fy, and Gz, if Rxz and Ryz, then $x = y$:

(.2) $R:F \xrightarrow{1\text{-}1} G =_{df} R:F \longrightarrow G \,\&\, \forall x \forall y \forall z((Fx \,\&\, Fy \,\&\, Gz) \rightarrow (Rxz \,\&\, Ryz \rightarrow x = y))$

Finally, we say (.3) R is a function from F *onto* G just in case R is a function from F to G and every G is such that some F bears R to it:

(.3) $R: F \xrightarrow[\text{onto}]{} G =_{df} R: F \longrightarrow G \,\&\, \forall x(Gx \rightarrow \exists y(Fy \,\&\, Ryx))$

(**606**) **Theorems:** One-to-One Correspondences and Functions. The following well-known fact is now derivable from the definitions in (604) and (605), namely, R is a one-to-one correspondence between F and G just in case R is a one-to-one function from F onto G. If we combine the notation for one-to-one and onto functions from F to G, we may write this theorem as follows:

$$R: F \xleftrightarrow{1\text{-}1} G \equiv R: F \xrightarrow[\text{onto}]{1\text{-}1} G$$

In light of this result, logicians often say that R is a *bijection* from F to G whenever it is a one-to-one function from F onto G.

(**607**) **Remark:** Frege's Theorem. Frege's system of 1893, despite being inconsistent, contained one of the most astonishing intellectual achievements in logic and philosophy. This is now known as Frege's Theorem, though Frege himself never formulated the result explicitly as a theorem, nor even thought of it as a *result*. Nevertheless, the theorem Frege proved, despite taking himself to be doing something else, is that the Dedekind-Peano axioms of number theory can be validly derived using only the resources of second-order logic supplemented by a single principle, namely, Hume's Principle.[236]

To state Hume's Principle, two notions are needed. The first is expressed by the definite description *the number of F*s. Frege thought that this description is logically proper because he believed that for every property F, there is a unique individual x that *numbered F*. The other notion needed for Hume's Principle is the *equinumerosity* of F and G. Without a definition of equinumerosity, Hume's Principle has an air of triviality to those encountering it for the first time, for it asserts:

[236] See Wright 1983, Heck 1993, and Zalta 2015, for details.

> The number of Fs is equal to the number of Gs if and only if F and G are equinumerous.

But let us say that F and G are *equinumerous* just in case there exists a one-to-one correspondence between F and G, i.e.,

(.1) $F \approx G =_{df} \exists R[R : F \xleftrightarrow{\text{1-1}} G]$

Then, if we abbreviate *the number of F*s as $\#F$, Hume's Principle may be formally represented as the following non-trivial claim:

(.2) **Hume's Principle**
$\#F = \#G \equiv F \approx G$

From Hume's Principle, (.1), and theorem (606), it follows that $\#F = \#G$ if and only if there is a relation that is a one-to-one function from F onto G, i.e., if and only if there is a relation that is a bijection from F to G.

With this understanding of Hume's Principle we are in a position to summarize how Frege derived the Dedekind-Peano postulates in second-order logic from that principle. Before we do, however, it is worth digressing briefly to say more about how Frege both attempted to define $\#F$ and *derive* Hume's Principle from his theory of extensions. We can simplify the discussion of how Frege thought this could be done by (a) representing Frege's notion *the extension of F* as ϵF and (b) supposing that ϵF is axiomatized by Basic Law V, which for present purposes, may be written as: $\epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx)$. It is well known that the addition of Basic Law V to second-order logic (with *unrestricted* second-order comprehension for properties) is subject to Russell's paradox. But Frege was initially unaware of the paradox and so defined $\#F$ as the extension of the property *being an extension of a property equinumerous to F*. We can easily represent this property using λ-notation as:

$$[\lambda x \, \exists G(x = \epsilon G \,\&\, G \approx F)]$$

and then represent Frege's definition of the number of Fs as:[237]

$$\#F =_{df} \epsilon[\lambda x \, \exists G(x = \epsilon G \,\&\, G \approx F)]$$

Frege's next step was then to derive Hume's Principle, $\#F = \#G \equiv F \approx G$, from Basic Law V. Once he had Hume's Principle, he then made no further *essential*

[237] Note that the λ-expression used here isn't well-formed in the present theory, since the identity symbol = not only appears in the first conjunct ($x = \epsilon F$) of the quantified matrix, but also is buried in the definition of the second conjunct ($F \approx G$). Interestingly, the presence of the term ϵF in the λ-expression is allowed in our system, even if we define ϵF as we did in Section 10.3. That's because none of the encoding formulas appearing as subformulas of the matrix of the description used to define ϵF themselves occur as subformulas of the matrix of the λ-expression $[\lambda x \exists F(x = \epsilon F \& F \approx G)]$.

appeal to Basic Law V when deriving the Dedekind-Peano postulates as theorems; his derivations of these postulates appealed only to Hume's Principle and the theorems of second-order logic (Heck 1993).

So, if one starts with $\#F$ instead of ϵF as a primitive and adds Hume's Principle instead of Basic Law V to second-order logic, the resulting system is provably consistent.[238] Frege's Theorem proceeds by constructing the following definitions, in which P stands for the *predecessor* relation and N stands for *being a natural* (or *finite*) *number*:

$$Pxy =_{df} \exists F \exists z(Fz \;\&\; y = \#F \;\&\; x = \#[\lambda w\, Fw \;\&\; w \neq z])$$

$$0 =_{df} \#[\lambda x\, x \neq x]$$

$$P^*xy =_{df} \forall F[\forall z(Pxz \rightarrow Fz) \;\&\; \forall x' \forall y'(Px'y' \rightarrow (Fx' \rightarrow Fy')) \rightarrow Fy] \quad \text{(The ancestral of } P\text{)}$$

$$P^+xy =_{df} P^*xy \vee x = y \quad \text{(The weak ancestral of } P\text{)}$$

$$Nx =_{df} P^+0x$$

From these definitions of the primitive notions used in the Dedekind-Peano postulates, Frege derived the latter as theorems of second-order logic supplemented by Hume's Principle.

Frege's Theorem, astonishing as it is, doesn't accomplish Frege's goals. After all, Hume's Principle requires a primitive mathematical notion, $\#F$, and the principle itself is clearly a mathematical axiom. So one of Frege's goals, namely, that of defining the numbers and deriving their governing principles without appeal to mathematical notions and axioms, isn't achieved if we take $\#F$ as primitive and Hume's Principle as an axiom. By contrast, in what follows, we derive the Dedekind-Peano postulates without mathematical primitives and axioms.

Moreover, there are a number of concerns about the methodology of adding principles such as Hume's Principle to second-order logic, some of which were raised by Frege himself. Though there is now a substantial literature on this topic and it would take us too far afield to delve into the details, it is worth noting only one concern that Frege raised, namely, the Julius Caesar problem (1884, §55). If the only principle we know about $\#F$ is Hume's Principle, i.e., if Hume's Principle is the sole axiom governing *the number of* Fs, then the open formula $\#F = x$ is not defined and the theory of numbers derived from Hume's Principle provides no necessary and sufficient conditions for establishing whether the number of planets, say, is identical to Julius Caesar. Though

[238] See Hodes 1984 (138), Burgess 1984 (639), and Hazen 1985 (252). Geach 1976 (446–7) develops the model that the others describe, but doesn't specifically identify it as a model of second-order logic plus Hume's Principle.

much has been written about this problem, I think the deeper issue here is that modern mathematicians almost *never* introduce a new kind of mathematical object by way of a single principle that implies both existence and identity claims in the manner of Hume's Principle and Basic Law V.[239] Instead they typically formulate their theories using separate existence principles and identity principles. The classical axioms for Zermelo-Fraenkel set theory, for example, include existence principles (e.g., Null Set axiom, Pair Set axiom, Unions, Power Set, Infinity axiom, Separation, and Replacement) and a distinct identity principle (Extensionality). We shall follow this modern mathematical practice even though our implementation of it will be informed by other aspects of Frege's methodology.

Frege's methods, however, can be adapted and applied in object theory only so far, and we'll see why below in (608). Nevertheless, his goals can be achieved by extending object theory in interesting and justifiable ways. We plan to show that $\#F$, *Precedes*, and 0 can all be defined in object-theoretic terms, and that:

- Dedekind-Peano postulates 1, 2, 3, 5, 6, and 7 can be derived in object theory by extending it with the claim that *Precedes* and its weak ancestral are relations,[240] and
- Dedekind-Peano postulate 4 (every number has a successor) can be derived in object theory by extending it with a claim that makes explicit an intuitively true modal belief that we already implicitly accept.

After this is done and the usual definitions for natural arithmetic are given, we conclude with an interesting result, namely, that the existence of an infinite cardinal and an infinite set can be derived without any mathematical primitives or axioms.

14.2 Equinumerosity w.r.t. Ordinary Objects

Throughout the remainder of this chapter, we use u and v as restricted variables ranging over ordinary objects. We continue to use x, y, and z as variables ranging over all individuals. Note also that when we compare one of the theorems proved below to one of the theorems found in Frege's work, we assume that our notion of a *property* (i.e., a 1-place relation) corresponds to Frege's notion of a *concept* (i.e., a function that maps objects to truth values). The

[239] For example, Hume's Principle clearly offers conditions for the identity $\#F = \#G$. And by the reflexivity of equinumerosity, $\forall F(F \approx F)$, Hume's Principle implies $\#F = \#F$, which in turn implies $\exists x(x = \#F)$, for every F. So a single principle implies existence and identity conditions for numbers. Similar reasoning shows Basic Law V also implies existence and identity principles.

[240] Strictly speaking, we don't need the axiom that Precedes is a relation to prove postulates 1, 2, and 3, though the axiom does simplify the presentation. See the discussion in Remark (657).

justification for this comes from Frege himself, who said (1892, 51) that the concepts under which an object falls are its properties.

(**608**) **Remark:** Classical Equinumerosity Isn't an Equivalence Condition. One of the keys to Frege's theorem that we haven't discussed is the fact that the equinumerosity of F and G, as defined in (607.1), is an equivalence condition in the classical second-order predicate calculus. We leave it as an exercise for the reader to show that in classical second-order logic (without encoding), $F \approx G$ is reflexive, symmetric, and transitive. Frege relied on this fact, which intuitively partitions the domain of properties into equivalence classes of equinumerous properties, to introduce a new object, $\#F$, to represent the class of all properties equinumerous to F.

However, in the present system, classical equinumerosity, as defined in (607.1), provably fails to be an equivalence condition on properties, and this is the first obstacle we must surmount if we are to adapt Frege's methods to object theory. To see why the equinumerosity of F and G fails to be an equivalence condition, recall that it was established in (197) that there are distinct abstract objects that exemplify the same properties:

$$\exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, \forall F(Fx \equiv Fy)) \qquad (197)$$

From this it follows that $A!$ is not equinumerous to any property, i.e.,

(.1) $\forall G(A! \not\approx G)$

Proof. Given (197), assume a and b are such objects, so that we know $A!a$, $A!b$, $a \neq b$, and $\forall F(Fa \equiv Fb)$. Suppose, for reductio, that $\exists G(A! \approx G)$. Let Q be such a property so that we know $A! \approx Q$. Then, by definition of $\approx$ (607.1), there is a relation, say R, such that $R : F \xleftrightarrow{\text{1-1}} G$. So by (606), R is a one-to-one function from $A!$ onto Q and, *a fortiori*, a function from $A!$ to Q. The latter fact and our assumption $A!a$ jointly imply that there is an object, say c, such that both Qc and Rac. So by β-Conversion, it follows that $[\lambda z\, Rzc]a$. But, since a and b, by hypothesis, exemplify the same properties, $[\lambda z\, Rzc]b$. So by β-Conversion, Rbc. But this contradicts the fact R is a one-to-one function from $A!$ to Q, for we now have $A!a$, $A!b$, Rac and Rbc, which by (605.2) implies $a = b$, contrary to assumption.

In particular, then, it follows that $A! \not\approx A!$. So we have established that equinumerosity is not a reflexive condition:

(.2) $\exists F(F \not\approx F)$

Since equinumerosity is not reflexive, it is not an equivalence condition. Intuitively, then, equinumerosity doesn't partition the domain of properties into mutually exclusive and jointly exhaustive cells of equinumerous properties.

Since the existence of such a partition is essential to Frege's method of abstracting out a distinguished object, $\#G$, that numbers all and only the properties equinumerous to G, it will not do us much good to define $\#G$ as $\imath x(A!x \& \forall F(xF \equiv F \approx G))$. Fortunately, there is a notion of equinumerosity in the neighborhood which does much, though not all, of what Frege's method requires.

Recall that the relation $=_E$ was shown to be an equivalence relation on the ordinary objects (168). We also established that ordinary objects are classically behaved in the sense that distinct ordinary objects have distinct haecceities: from (171) it follows that $(O!x \& O!z \& x \neq z) \rightarrow [\lambda y\, y =_E x] \neq [\lambda y\, y =_E z]$. These facts allow us to formulate a notion of equinumerosity that holds whenever there is a one-to-one correspondence between the ordinary objects that exemplify F and those that exemplify G.

(**609**) **Definitions:** Equinumerosity with respect to the Ordinary Objects. Using our restricted variables u and v to range over ordinary objects, let us first introduce a special unique existence quantifier for ordinary objects. We say that (.1) *there is a unique ordinary object such that* φ if and only if (a) there is an ordinary object such that φ, and (b) every ordinary object such that φ is identical$_E$ to it:

(.1) $\exists! u\varphi =_{df} \exists u(\varphi \,\&\, \forall v(\varphi^v_u \rightarrow v =_E u))$

This unique existence quantifier is defined only for quantification over ordinary objects; by comparing it to definition (87.1) when the latter is similarly restricted, we see that the only difference is that identity$_E$ replaces identity in the definiens. Clearly, nothing is lost when we do so since we know by (169) that for ordinary objects u and v, $u =_E v$ is equivalent to $u = v$. So in what follows, it is important to remember that the unique existence of an ordinary object such that φ is defined, in part, to be a matter of identity$_E$. The virtue of using $=_E$ in the definition of $\exists! u\varphi$ is that the definiens and, hence, the definiendum, are propositional formulas.

In any case, we may now use our special unique existence quantifier to say that (.2) a 2-place relation R is a *one-to-one correspondence*$_E$ between (the ordinary objects exemplifying) F and (the ordinary objects exemplifying) G, written $R : F \xleftrightarrow{\text{1-1}}_E G$, just in case (a) every ordinary object exemplifying F bears R to a unique ordinary object exemplifying G, and (b) every ordinary object exemplifying G is such that a unique ordinary object exemplifying F bears R to it:

(.2) $R : F \xleftrightarrow{\text{1-1}}_E G =_{df} \forall u(Fu \rightarrow \exists! v(Gv \,\&\, Ruv)) \,\&\, \forall u(Gu \rightarrow \exists! v(Fv \,\&\, Rvu))$

We then say that (.3) properties F and G are *equinumerous with respect to the ordinary objects*, or *equinumerous*$_E$ (written $F \approx_E G$), just in case some relation R is a one-to-one correspondence$_E$ between F and G:

(.3) $F \approx_E G =_{df} \exists R(R : F \xleftrightarrow{\text{1-1}}_E G)$

In the proofs of the theorems that follow, we say that a relation R is a *witness* to the equinumerosity$_E$ of F and G whenever $R : F \xleftrightarrow{\text{1-1}}_E G$. Also, we sometimes say that F is equinumerous$_E$ *to* G when F and G are equinumerous$_E$.

(**610**) **Theorems:** Equinumerosity$_E$ Partitions the Domain of Properties. It follows that: equinumerosity$_E$ is (.1) reflexive; (.2) symmetric; and (.3) transitive:

(.1) $F \approx_E F$

(.2) $F \approx_E G \rightarrow G \approx_E F$

(.3) $(F \approx_E G \,\&\, G \approx_E H) \rightarrow F \approx_E H$

It also follows that (.4) if F and G are equinumerous$_E$, then for every property H, H and F are equinumerous$_E$ if and only if H and G are equinumerous$_E$::

(.4) $F \approx_E G \equiv \forall H(H \approx_E F \equiv H \approx_E G)$

(.4) is based on Theorem 25 in Frege 1893 (§61). Frege, of course, used rather different notation to state this theorem. Though we shall frequently indicate in what follows when a theorem proved here is in some way analogous to one that Frege proved in 1893, it goes beyond the scope of the present monograph to explain Frege's notation.

(**611**) **Remark:** One-to-one Correspondences$_E$ and Functions$_E$. Clearly, the notion of one-to-one correspondence$_E$ defined in (609.2) is adapted from the notion of one-to-one correspondence defined in (604). We can also adapt the notions defined in (605) as follows: (.1) R is a *function$_E$ from F to G* just in case every ordinary object exemplifying F is related to a unique ordinary object exemplifying G:

(.1) $R : F \longrightarrow_E G =_{df} \forall u(Fu \rightarrow \exists! v(Gv \,\&\, Ruv))$

Exercise. Given the notion of a function defined in (605.1), show that the following is equivalent to the above:

$$R : F \longrightarrow_E G =_{df} R : [\lambda x\, O!x \,\&\, Fx] \longrightarrow [\lambda x\, O!x \,\&\, Gx]$$

Similarly, we may say (.2) R is a *one-to-one function$_E$ from F to G* just in case R is a function$_E$ from F to G and for all ordinary objects t, u, and v such that Ft, Fu, and Gv, if Rtv and Ruv, then $t =_E u$:

(.2) $R : F \xrightarrow{\text{1-1}}_E G =_{df}$
$R : F \longrightarrow_E G \,\&\, \forall t \forall u \forall v((Ft \,\&\, Fu \,\&\, Gv) \rightarrow (Rtv \,\&\, Ruv \rightarrow t =_E u))$

Exercise. Given the notion of a one-to-one function defined in (605.2), show that the following is equivalent to the above::

$$R\!:\!F \xrightarrow{\text{1-1}}_E G \;=_{df}\; R\!:\![\lambda x\, O!x \,\&\, Fx] \xrightarrow{\text{1-1}} [\lambda x\, O!x \,\&\, Gx]$$

Moreover, we say (.3) R is a *function$_E$ from F onto G* just in case R is a function$_E$ from F to G and every ordinary object exemplifying G is such that some ordinary object exemplifying F bears R to it:

(.3) $R : F \underset{\text{onto}}{\longrightarrow}_E G \;=_{df}\; R : F \longrightarrow_E G \;\&\; \forall u(Gu \rightarrow \exists v(Fv \,\&\, Rvu))$

Exercise. Given the notion of an onto function defined in (605.3), show that the following is equivalent to the above:

$$R : F \underset{\text{onto}}{\longrightarrow}_E G \;=_{df}\; R : [\lambda x\, O!x \,\&\, Fx] \underset{\text{onto}}{\longrightarrow} [\lambda x\, O!x \,\&\, Gx]$$

Finally, we leave the following theorems as exercises. (.4) R is a one-to-one correspondence$_E$ between F and G if and only if R is a one-to-one function$_E$ from F onto G, and (.5) R is a one-to-one correspondence$_E$ between F and G if and only if R is a one-to-one correspondence between $[\lambda x\, O!x \,\&\, Fx]$ and $[\lambda x\, O!x \,\&\, Gx]$:

(.4) $R : F \xleftrightarrow{\text{1-1}}_E G \;\equiv\; R : F \underset{\text{onto}}{\xrightarrow{\text{1-1}}}_E G$

(.5) $R : F \xleftrightarrow{\text{1-1}}_E G \;\equiv\; R : [\lambda x\, O!x \,\&\, Fx] \xleftrightarrow{\text{1-1}} [\lambda x\, O!x \,\&\, Gx]$

These exercises are designed, in part, to show that our defined notions of 1-1 correspondence$_E$ and function$_E$ reduce to familiar notions of 1-1 correspondence and functions, but with restrictions on the properties involved.

(**612**) **Definition:** Material Equivalence with Respect to the Ordinary Objects. We say that properties F and G are *materially equivalent with respect to the ordinary objects*, written $F \equiv_E G$, if and only if F and G are exemplified by the same ordinary objects:

$$F \equiv_E G \;=_{df}\; \forall u(Fu \equiv Gu)$$

(**613**) **Lemmas:** Equinumerous$_E$ and Equivalent$_E$ Properties. The following consequences concerning equinumerous$_E$ and materially equivalent$_E$ properties are easily provable: (.1) if F and G are materially equivalent$_E$, then they are equinumerous$_E$; and (.2) if F is equinumerous$_E$ to G and G is materially equivalent$_E$ to H, then F is equinumerous$_E$ to H:

(.1) $F \equiv_E G \rightarrow F \approx_E G$

(.2) $(F \approx_E G \,\&\, G \equiv_E H) \rightarrow F \approx_E H$

(**614**) **Theorems:** Equinumerosity$_E$ and Empty Properties. To show that $\#G$ is not strictly canonical, we shall need to appeal to the following modally-strict facts about the equinumerosity$_E$ of empty properties: (.1) if no ordinary objects exemplify F and none exemplify H, then F and H are equinumerous with respect to the ordinary objects; and (.2) if some ordinary object exemplifies exemplifies F and no ordinary object exemplifies H, then F and H aren't equinumerous with respect to the ordinary objects:

(.1) $(\neg\exists uFu \,\&\, \neg\exists vHv) \rightarrow F \approx_E H$

(.2) $(\exists uFu \,\&\, \neg\exists vHv) \rightarrow \neg(F \approx_E H)$

(**615**) **Term Definition:** Being F But Not Identical$_E$ to u. We introduce the notation F^{-u}, where u ranges over ordinary individuals, to denote *being F but not identical*$_E$ *to u*:

$$F^{-u} \;=_{df}\; [\lambda z\, Fz \,\&\, z \neq_E u]$$

(**616**) **Lemma:** An Equinumerosity$_E$ Lemma. If F and G are equinumerous$_E$, u exemplifies F, and v exemplifies G, then F^{-u} and G^{-v} are equinumerous$_E$:

$$F \approx_E G \,\&\, Fu \,\&\, Gv \;\rightarrow\; F^{-u} \approx_E G^{-v}$$

Compare this theorem with the result marked ϑ just prior to Theorem 87 in Frege 1893 (§95).[241]

(**617**) **Lemma:** Another Equinumerosity$_E$ Lemma. If F^{-u} is equinumerous$_E$ with G^{-v}, u exemplifies F, and v exemplifies G, then F and G are equinumerous$_E$:

$$F^{-u} \approx_E G^{-v} \,\&\, Fu \,\&\, Gv \;\rightarrow\; F \approx_E G$$

Compare Frege 1893, Theorem 66.[242]

14.3 Natural Cardinals and The Number of *F*s

(**618**) **Definitions:** Numbering a Property. We may appeal to our definition of equinumerosity$_E$ to say when an object numbers a property: *x numbers* (the ordinary objects exemplifying) G iff x is an abstract object that encodes just the properties equinumerous$_E$ with G:

[241] There are several differences. First, on the basis of Hume's Principle, Frege uses numeral identities instead of equinumerosity claims in the antecedent and consequent, and we substitute these into the present theorem, we obtain $\#F = \#G \,\&\, Fu \,\&\, Gv \rightarrow \#F^{-u} = \#G^{-v}$. Second, Frege proves the contrapositive, switches the order of the antecedents, and puts everything into conditional form. Thus, he proves: $Fu \rightarrow (Gv \rightarrow (\#F^{-u} \neq \#G^{-v} \rightarrow \#F \neq \#G))$. Note also that Frege uses c and b, respectively, where we use u and v, and he uses v and u, respectively, where we use $\#F$ and $\#G$.

[242] The present theorem differs from Frege's Theorem 66 only by two applications of Hume's Principle: in Frege's Theorem, $\#F^{-u} = \#G^{-v}$ is substituted for $F^{-u} \approx_E G^{-v}$ in the antecedent, and $\#F = \#G$ is substituted for $F \approx_E G$ in the consequent.

$$Numbers(x,G) =_{df} A!x \,\&\, \forall F(xF \equiv F \approx_E G)$$

(**619**) **Theorem:** Equinumerosity$_E$, Material Equivalence$_E$, and Numbering. (.1) if G and H are equinumerous$_E$, then x numbers G if and only if x numbers H; (.2) if x numbers both G and H, then G and H are equinumerous$_E$; and (.3) if G is materially equivalent to H with respect to the ordinary objects, then x numbers G if and only if y numbers H:

(.1) $G \approx_E H \to (Numbers(x,G) \equiv Numbers(x,H))$

(.2) $(Numbers(x,G) \,\&\, Numbers(x,H)) \to G \approx_E H$

(.3) $G \equiv_E H \to (Numbers(x,G) \equiv Numbers(y,H))$

(**620**) **Theorem:** Modally Strict Fact Underlying Hume's Principle. It is a modally strict consequence of the foregoing that if x numbers G and y numbers H, then x is identical to y if and only if G and H are equinumerous with respect to the ordinary objects:

$$(Numbers(x,G) \,\&\, Numbers(y,H)) \to (x = y \equiv G \approx_E H)$$

(**621**) **Remark:** Numbered Properties and Material Equivalence$_E$. One may well wonder about the status of a claim analogous to (619.2) but stated with respect to $\equiv_E$, namely, if x numbers both G and H, then G and H are materially equivalent$_E$:

$$(Numbers(x,G) \,\&\, Numbers(x,H)) \to G \equiv_E H$$

Intuitively, this claim is false: the fact that x numbers both G and H shouldn't imply that G and H are materially equivalent$_E$. For example, if one and the same abstract object numbers *being a planet* and *being a human artifact on my desk*, it doesn't follow that all and only the ordinary objects that exemplify *being a planet* exemplify *being a human artifact on my desk*.

But one might ask: can one prove the negation of the claim displayed above *within* our system without additional assumptions, i.e., can we prove, for some object x and properties G and H, that both $Numbers(x,G)$ and $Numbers(x,H)$ but G is not equivalent$_E$ to H:

$$\exists x \exists G \exists H(Numbers(x,G) \,\&\, Numbers(x,H) \,\&\, \neg G \equiv_E H)$$

As we shall see, it doesn't take much to prove this, but we do need the assumption that there are at least two ordinary objects. Given that condition, we have the following theorem.

(**622**) **Theorem:** Numbered Properties Not Materially Equivalent$_E$. If there are at least two ordinary objects, then there is an abstract object x and properties G and H such that x numbers both G and H but G and H are not materially equivalent$_E$:

$$\exists u \exists v(u \neq v) \to \exists x \exists G \exists H(Numbers(x,G) \,\&\, Numbers(x,H) \,\&\, \neg G \equiv_E H)$$

Note, however, that nothing at present guarantees that there are at least two ordinary objects.

(**623**) **Definition:** Natural Cardinals. Intuitively, natural cardinals are things that answer the question, "How many ordinary *F*s are there?". To define them, we say that *x* is a *natural cardinal* iff *x* numbers some property:

$$NaturalCardinal(x) \;=_{df}\; \exists G(Numbers(x,G))$$

The reader may wish to consider how this departs from Frege's definition of *Anzahl* in 1884, §72, and in 1893, §42. We'll say a bit more about this in (626) below.

(**624**) **Theorem:** Natural Cardinals Encode What They Number. It is now provable that a natural cardinal encodes all and only the properties that it numbers:

$$NaturalCardinal(x) \to \forall F(xF \equiv Numbers(x,F))$$

(**625**) **Theorems:** Existence of (Unique) Numberers. In the usual way, it now follows that (.1) something numbers *G*; (.2) there is a unique individual that numbers *G*:

(.1) $\exists x Numbers(x,G)$

(.2) $\exists! x Numbers(x,G)$

Note here that by RN, (.1) and (.2) are both necessarily true. Intuitively, at every possible world, there is a unique *x* that numbers *G* there, i.e., that encodes all and only the properties *F* equinumerous$_E$ to *G* there. However, for the next few sections, we shall be studying, for each *G*, the *x* that in fact numbers *G*, since we also know that (.3) the *x* that (in fact) numbers *G* exists:

(.3) $\exists y(y = \imath x Numbers(x,G))$

(**626**) **Term Definition:** The Number of (Ordinary) *G*s. Since it follows that the description $\imath x Numbers(x,G)$ is logically proper for every *G*, we may use it to introduce the notation $\#G$ to rigidly refer to *the number of* (ordinary) *G*s:

$$\#G \;=_{df}\; \imath x Numbers(x,G)$$

In what follows, we often read $\#G$ as *the number of G*s.

(**627**) **Theorem:** Equinumerosity$_E$ and Contingency. Of course, our intuitions tell us that there are properties *F* and *G* that are equinumerous$_E$ in some worlds but not in others. For example, even if *being a planet* and *being a human artifact on my desk* are equinumerous$_E$, things might have been different, so that these two properties wouldn't have been *equinumerous*$_E$. But we now show that without any additional assumptions, we can find two properties that have this modal behavior.

Consider the impossible property $\overline{L}$, i.e., the negation of L, where L was defined as $[\lambda x\, E!x \rightarrow E!x]$ (140). Now consider the property $[\lambda x\, E!x \,\&\, \diamond\neg E!x]$, i.e., *being contingently concrete*. Then it is a modally strict theorem that possibly: $\overline{L}$ and $[\lambda x\, E!x \,\&\, \diamond\neg E!x]$ are equinumerous$_E$ and possibly they aren't:

$$\diamond(\overline{L} \approx_E [\lambda x\, E!x \,\&\, \diamond\neg E!x] \,\&\, \diamond\neg\overline{L} \approx_E [\lambda x\, E!x \,\&\, \diamond\neg E!x])$$

This fact will help us to establish that $\#G$ is not strictly canonical. Note that by (119.12), it is equivalent to $\diamond\overline{L} \approx_E [\lambda x\, E!x \,\&\, \diamond\neg E!x] \,\&\, \diamond\neg\overline{L} \approx_E [\lambda x\, E!x \,\&\, \diamond\neg E!x]$, which by definition (144.2), tells us that the equinumerosity$_E$ of $\overline{L}$ and $[\lambda x E!x \,\&\, \diamond\neg E!x]$ is contingent.

(**628**) **Remark:** $\#G$ is Not Strictly Canonical. Clearly, by definitions (626) and (618):

$$\#G = \imath x(A!x \,\&\, \forall F(xF \equiv F \approx_E G))$$

So by (181), $\#G$ is (identical to) a canonical individual. However, in (188.2), we stipulated that a canonical description $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ is *strictly* canonical just in case φ is a rigid condition on properties, i.e., by (188.1), just in case $\vdash \Box\forall F(\varphi \rightarrow \Box\varphi)$. But we can use (627) to show that when φ is $F \approx_E G$, then φ fails to be a rigid condition on properties and, hence, that $\#G$ fails to be (identical to) a strictly canonical individual. The argument is an indirect one: we find a property G for which $\vdash \neg\Box\forall F(\varphi \rightarrow \Box\varphi)$ and conclude that φ fails to be rigid, on pain of inconsistency.

Given the equivalence of $\neg\Box\forall F(\varphi \rightarrow \Box\varphi)$ and $\diamond\exists F(\varphi \,\&\, \neg\Box\varphi)$, it suffices to show that there is a proof of the latter, for some property G. But when we choose G to be $[\lambda x\, E!x \,\&\, \diamond\neg E!x]$, then by (627), we know:

$$\diamond(\overline{L} \approx_E G \,\&\, \diamond\neg\overline{L} \approx_E G)$$

So by $\exists$I,

$$\exists F \diamond(F \approx_E G \,\&\, \diamond\neg F \approx_E G)$$

Hence, by CBF$\diamond$:

$$\diamond\exists F(F \approx_E G \,\&\, \diamond\neg F \approx_E G)$$

So, by standard modal reasoning:

$$\diamond\exists F(F \approx_E G \,\&\, \neg\Box F \approx_E G)$$

Since we've found a value for the free variable in $\Diamond\exists F(\varphi \mathbin{\&} \neg\Box\varphi)$ for which this claim is derivable, it follows that φ fails to be rigid and that $\#G$ isn't strictly canonical, on pain of inconsistency.

(**629**) **Theorem**. Some Numbers Contingently Number. If we use some of the reasoning in the previous remark, it is relatively straightforward to show that there is a property G and an object x such that x numbers G but doesn't necessarily number G:

$$\exists x\exists G(Numbers(x,G) \mathbin{\&} \neg\Box Numbers(x,G))$$

Clearly, this theorem and the Remark that precedes it *intuitively* tell us that the Frege conception of numbers, as abstractions over equivalence classes of equinumerous properties, yields *different* numbers at other possible worlds.

(**630**) ⋆**Lemmas:** Equinumerosity$_E$ and The Number of Gs. It now follows, by non-modally strict proofs, that: (.1) x numbers G iff x is the number of Gs; (.2) the number of Gs numbers G; (.3) the number of Gs encodes F iff F is equinumerous$_E$ with G; and (.4) the number of Gs encodes G:

(.1) $Numbers(x,G) \equiv x = \#G$

(.2) $Numbers(\#G,G)$

(.3) $\forall F(\#GF \equiv F \approx_E G)$

(.4) $\#GG$

In these formulas, the expression $\#GF$ asserts that the number of *Gs* encodes F; we put the G immediately prefaced by the octothorpe in a slightly smaller font size solely for ease of readability.

(**631**) **Remark:** Digression on the Definition of Natural Cardinal. In (623), we defined $NaturalCardinal(x)$ as $\exists G(Numbers(x,G))$. But in a previous work (1999, 630), $NaturalCardinal(x)$ was defined as $\exists G(x = \#G)$. The 1999 definition more closely follows Frege's definition of *Anzahl* in 1884, §72, and in 1893, §42. But Frege wasn't working in a modal context. Since $\#G$ is naturally defined by a rigid definite description, the Fregean notion of *natural cardinal* is less flexible from a modal point of view; it is more difficult to prove modally strict theorems about natural cardinals given such a definition.

Of course, non-modally strict theorems are fine as theorems. But whereas a simple claim like:

$$NaturalCardinal(x) \to \exists G\forall F(xF \equiv F \approx_E G)$$

can be proved as a modally strict claim from definition (623), it can't be so proved from the Fregean definition. To see this, note that given (623), the above claim follows trivially by applying definitions and simple predicate logic. But given the Fregean definition, we would have to reason as follows:

> Assume $NaturalCardinal(x)$. Then $\exists G(x = \#G)$. Suppose P is such a property, so that we know $x = \#P$. Then by (630.3)★, we know $\forall F(\#PF \equiv F \approx_E P)$. Hence, $\forall F(xF \equiv F \approx_E P)$. So, by $\exists$I, $\exists G \forall F(xF \equiv F \approx_E G)$.

Here the appeal to (630.3)★ would turn this into a non-modally strict theorem.

(**632**) ⋆**Theorem:** Hume's Principle. It is a non-modally strict theorem that the number of Fs is identical to the number of Gs if and only if F and G are equinumerous$_E$:

$$\#F = \#G \equiv F \approx_E G$$

This principle encapsulates the work Frege does in 1884 (§63–§72), though relativized to our context of numbering the ordinary objects. If we expand the above by definitions by (609.3) and (609.2), and remember exercise (611.4), then Hume's Principle implies that the number of Fs is identical to the number of Gs if and only if there is a relation that is a one-to-one function$_E$ from F onto G.

In 1893, Hume's Principle is not proved as a biconditional. Tennant (2004, 108–9) was first to observe that Frege proves each direction as a separate theorem. The right-to-left direction is proved in §65, Theorem 32, and the contraposed version of the left-to-right direction is proved in §69, Theorem 49. See May & Heck forthcoming, for further discussion. Finally, readers who compare the above with Frege's original should note that instead of the octothorpe symbol, Frege uses the symbol $\mathfrak{P}$, which operates on the extension of the concept being numbered. Thus, $\mathfrak{P}u$ designates the number of the concept for which u is the extension. See 1893, §40, Definition Z.

(**633**) ⋆**Theorem:** Corollary to Hume's Principle. If F and G are materially equivalent with respect to the ordinary objects, then the number of Fs is identical to the number of Gs:

$$F \equiv_E G \rightarrow \#F = \#G$$

(**634**) ⋆**Theorem:** Encoding and The Number of Fs. A natural cardinal encodes a property F just in case it is identical to the number of Fs:

$$NaturalCardinal(x) \rightarrow \forall F(xF \equiv x = \#F)$$

Clearly, once we apply the theory, then it follows from this theorem that if x is a natural cardinal and is equal to the number of planets, then x encodes *being a planet*.

(**635**) **Lemma:** Fact about Non-Identity$_E$. Since theorem (168.1) tells us that $O!x \rightarrow x =_E x$, it follows that nothing whatsoever exemplifies the property of *being an ordinary object that fails to be self-identical*$_E$:

$$\neg\exists y[\lambda x\, O!x \,\&\, \neg(x =_E x)]y$$

(**636**) **Definition:** Zero. We therefore define Zero to be the number of the property *being an ordinary object that fails to be self-identical$_E$*. Using u as a restricted variable ranging over ordinary objects and infix notation for the negated identity$_E$ claim, we can write this definition as follows:

$$0 \;=_{df}\; \#[\lambda u\; u \neq_E u]$$

Cf. Frege 1884, §74, and 1893, §41. We've kept the definition of Zero as close to Frege's definition as possible. But we could have used any other property that no ordinary object in fact exemplifies, such as $\overline{L}$ (i.e., the negation of L, where L is the property $[\lambda x\, E!x \to E!x]$).

(**637**) ⋆**Theorem:** Zero is a Natural Cardinal.

$$\mathit{NaturalCardinal}(0)$$

(**638**) ⋆**Theorem:** Zero Encodes the Properties Unexemplified by Ordinary Objects. It is now a theorem that Zero encodes all and only the properties that no ordinary object exemplifies:

$$0F \equiv \neg\exists u Fu$$

(**639**) ⋆**Corollary:** Zero Numbers Empty Properties. As simple consequences of the previous theorems and definitions we have (.1) F fails to be exemplified by ordinary objects iff Zero numbers F; and (.2) F fails to be exemplified by ordinary objects iff the number of Fs is Zero:

(.1) $\neg\exists u Fu \equiv \mathit{Numbers}(0, F)$

(.2) $\neg\exists u Fu \equiv \#F = 0$

Cf. Frege 1884 (§75) and 1893 (§99, Theorem 97) with (.2).

14.4 Predecessor

(**640**) **Definition:** Predecessor. We now define: *x immediately precedes y*, written $\mathit{Precedes}(x, y)$, if and only if there is a property F and ordinary object u such that (a) u exemplifies F, (b) y numbers F, and (c) x numbers *being-F-but-not-identical$_E$-to-u*:

$$\mathit{Precedes}(x, y) \;=_{df}\; \exists F\exists u(Fu \,\&\, \mathit{Numbers}(y, F) \,\&\, \mathit{Numbers}(x, F^{-u}))$$

Note that if we replace the notions $Numbers(y, F)$ and $Numbers(x, F^{-u})$ in the definiens with their respective definientia according to (618), then the definiens clearly contains encoding subformulas. So Comprehension for Relations (129.1) does not guarantee that the condition $Precedes(x, y)$ defines a relation. Later, when the need arises, we shall assert, as an axiom, that it does.

(**641**) **Remark:** Departure from Frege's Definition. In 1884 (§76) and 1893 (§43), Frege defined:

$$Succeeds(y, x) =_{df} \exists F \exists u(Fu \;\&\; y = \#F \;\&\; x = \#F^{-u})$$

If we put aside the fact that we're reconstructing Frege's conception of numbers in terms of equinumerosity$_E$ instead of equinumerosity, then our definition (640) departs from Frege's definition in two ways. First, we used $Precedes(x, y)$ as the definiendum instead of $Succeeds(y, x)$. Second, in the definiens, we substituted the conditions $Numbers(y, F)$ and $Numbers(x, F^{-u})$, respectively, for the conditions $y = \#F$ and $x = \#F^{-u}$. Neither departure is significant, for the following reasons.

As we saw in Section 14.1, the basic principles of number theory can be formulated with either predecessor or successor; they are converses of one another. But we prefer to work with the notion of predecessor for a simple reason: unlike $Succeeds(y, x)$, the order of the variables in the expression $Precedes(x, y)$ matches the numerical order of any corresponding cardinal numbers that satisfy the condition. This may slightly reduce one's cognitive load when attempting to parse and understand complex claims, especially concerning the strong and weak ancestral of a relation.

The second departure is also insignificant since by (630.1)$\star$, which asserts $Numbers(x, G) \equiv x = \#G$, the definiens of (640) is materially equivalent to the definiens Frege uses. The conditions $Numbers(y, F)$ and $Numbers(x, F^{-u})$, however, are not only several steps closer to primitive notation than the conditions $y = \#F$ and $x = \#F^{-u}$, but more importantly, they don't contain terms defined by a (rigid) definite description. This means our definiens is more flexible from a modal point of view: whereas the condition $y = \#F$ identifies y as the object that in fact numbers F, the condition $Numbers(y, F)$ identifies y as something that, in any modal context, numbers F relative to that context.

(**642**) **Theorem:** Predecessor is a One-to-One Condition. It now follows that if x and y precede z, then $x = y$:

$$Precedes(x, z) \;\&\; Precedes(y, z) \rightarrow x = y$$

Cf. Frege 1884, §78; and 1893, Theorem 89. Intuitively, this theorem establishes that *predecessor* is a one-to-one 2-place condition on objects.

(**643**) **Theorem:** Predecessor is a Functional Condition. If x precedes both y and z, then y is z:

$$Precedes(x,y) \,\&\, Precedes(x,z) \rightarrow y = z$$

Cf. Frege 1893, Theorem 71. Recall that this principle was listed as one of the basic postulates of number theory; see postulate 7 of Remark (603).

14.5 The Strong and Weak Ancestrals of R

(**644**) **Definition:** Properties Hereditary w.r.t. a Relation. Let us say that a property F *is hereditary with respect to* a relation R if and only if every pair of R-related objects are such that if the first exemplifies F then so does the second:

$$Hereditary(F,R) \;=_{df}\; \forall x \forall y(Rxy \rightarrow (Fx \rightarrow Fy))$$

In what follows, we sometimes say F is R-hereditary instead of F is hereditary w.r.t. R.

(**645**) **Definition:** The (Strong) Ancestral of a Relation R. Let us say that x is an *R-ancestor of y*, written $R^*(x,y)$, if and only if y exemplifies every property F that is both (a) exemplified by all the objects to which x is R-related and (b) hereditary with respect to R:

$$R^*(x,y) \;=_{df}\; \forall F[(\forall z(Rxz \rightarrow Fz) \,\&\, Hereditary(F,R)) \rightarrow Fy]$$

In 1884 (§79) and 1893 (§45), Frege regarded the above as a definition of *y follows x in the R-series*.

It should be noted here that $R^*(x,y)$ never holds merely vacuously. To see why, note that for every relation R, there exists a relational property definable in terms of R, namely $[\lambda y' \,\exists x' Rx'y']$. As we shall see in the proof of (647.5) below, this relational property is guaranteed to be such that (a) $\forall z(Rxz \rightarrow Fz)$ and (b) $Hereditary(F,R)$. Without the guarantee that there are properties that satisfy (a) and (b), there would be degenerate cases in which $R^*(x,y)$ holds because of a failure of the antecedent. The quantified conditional that serves as the definiens of (645) asserts that $R^*(x,y)$ holds whenever y exemplifies *every* property F such that both (a) and (b) hold. So if (a) or (b) fails, then the antecedent of the conditional fails, and the definiens of $R^*(x,y)$ becomes vacuously true. To see how the relational property $[\lambda y' \,\exists x' Rx'y']$ is guaranteed to satisfy (a) and (b), see the proof of (.5) in the Appendix.

(**646**) **Remark:** Explanation of Notation. Since the definiens of $R^*(x,y)$ expands by definition to a propositional formula, $R^*(x,y)$ defines a relation, by Comprehension for Relations (129.2). This means we could have introduced R^* as a constant by way of the following term definition:

$$R^* \;=_{df}\; [\lambda xy \,\forall F((\forall z(Rxz \rightarrow Fz) \,\&\, Hereditary(F,R)) \rightarrow Fy)]$$

But given both that (a) the notion of a *weak* R-ancestor (see below) doesn't automatically define a relation, and (b) we want the expressions for the strong and weak ancestral of a relation to look somewhat similar, there is no harm in resting with the formula definition of $R^*(x,y)$.

(**647**) **Lemmas:** Facts about the Ancestral of R. The following are immediate consequences of the two previous definitions: (.1) if x bears R to y, then x is an R-ancestor of y; (.2) if (a) x is an R-ancestor of y, (b) F is exemplified by every object to which x bears R, and (c) F is R-hereditary, then y exemplifies F; (.3) if (a) x exemplifies F, (b) x is an R-ancestor of y, and (c) F is R-hereditary, then y exemplifies F; (.4) if (a) x bears R to y and y is an R-ancestor of z, then x is an R-ancestor of z; and (.5) if x is an R-ancestor of y, then something bears R to y:

(.1) $Rxy \rightarrow R^*(x,y)$

(.2) $(R^*(x,y) \,\&\, \forall z(Rxz \rightarrow Fz) \,\&\, \mathit{Hereditary}(F,R)) \rightarrow Fy$

(.3) $(Fx \,\&\, R^*(x,y) \,\&\, \mathit{Hereditary}(F,R)) \rightarrow Fy$

(.4) $(Rxy \,\&\, R^*(y,z)) \rightarrow R^*(x,z)$

(.5) $R^*(x,y) \rightarrow \exists z Rzy$

Cf. (.2) with Frege 1893, Theorem 123 (beware Frege's use of italic *a* and Fraktur $\mathfrak{a}$ in the same formula), (.3) with Theorem 128, (.4) with Theorem 129, and (.5) with Theorem 124.

(**648**) **Definition:** The Weak Ancestral of R or y is a Member of the R-Series Beginning With x. Let us say that x is a *weak R-ancestor* of y, or y is a *member of the R-series beginning with x*, written $R^+(x,y)$, just in case either x is an R-ancestor of y or $x = y$:

$$R^+(x,y) \;=_{df}\; R^*(x,y) \vee x = y$$

We shall often use Frege's reading of the definiendum as *y is a member of the R-series beginning with x* (1884, §81; 1893, §46). Though there is increased cognitive load given that the variable for the object occurring later in the R-series occurs earlier in the expression, the effect is mitigated by the phrase *beginning with x*.

Note that the definiens for $R^+(x,y)$ involves the identity sign, which is defined in terms of encoding subformulas. So the condition $R^+(x,y)$ isn't guaranteed to define a relation, but clearly, every R, x, and y such that $R^*(x,y)$ are such that $R^+(x,y)$.

(**649**) **Lemmas:** Facts about the Weak Ancestral of R. The following are immediate consequences of our definitions: (.1) if x bears R to y, then x is a weak R-ancestor of y; (.2) if (a) x exemplifies F, (b) x is a weak R-ancestor of y, and

(c) F is R-hereditary, then y exemplifies F; (.3) if x is a weak R-ancestor of y and y bears R to z, then x is an R-ancestor of z; (.4) if x is an R-ancestor of y and y bears R to z, then x is a weak R-ancestor of z; (.5) if x bears R to y, and y is a weak R-ancestor of z, then x is an R-ancestor of z; and (.6) if x is an R-ancestor of y, then x is a weak R-ancestor of something that bears R to y:

(.1) $Rxy \rightarrow R^{+}(x,y)$

(.2) $(Fx \,\&\, R^{+}(x,y) \,\&\, \mathit{Hereditary}(F,R)) \rightarrow Fy$

(.3) $(R^{+}(x,y) \,\&\, Ryz) \rightarrow R^{*}(x,z)$

(.4) $(R^{*}(x,y) \,\&\, Ryz) \rightarrow R^{+}(x,z)$

(.5) $(Rxy \,\&\, R^{+}(y,z)) \rightarrow R^{*}(x,z)$

(.6) $R^{*}(x,y) \rightarrow \exists z(R^{+}(x,z) \,\&\, Rzy)$

Cf. Frege 1893, Theorem 144 with (.2) above; Theorem 134 with (.3); Theorem 132 with (.5), and Theorem 141 with (.6).

14.6 Natural Numbers

(**650**) **Axioms:** Predecessor and Its (Weak) Ancestral Are Relations. We now assert, as an axiom, that (.1) there is a relation R such that necessarily, R is exemplified by all and only those objects x and y such that x precedes y:

(.1) $\exists R \Box \forall x \forall y (Rxy \equiv \mathit{Precedes}(x,y))$

Given this axiom, we stipulate that the defined formula $\mathit{Precedes}(x,y)$, but not its definiens in (640), is propositional. That is, we're now *extending* the definition of *propositional formula* to include $\mathit{Precedes}(x,y)$ but not its definiens. (We discuss this further in Remark (651) below.) So, in what follows, we may regard $[\lambda xy\, \mathit{Precedes}(x,y)]$ as well-formed. If we abbreviate this expression as *Precedes*, then we may treat $\mathit{Precedes}(x,y)$ as an exemplification formula of the form Rxy. Consequently, we may also regard the λ-expression $[\lambda xy\, \mathit{Precedes}^{*}(x,y)]$ as well-formed, since its matrix $\mathit{Precedes}^{*}(x,y)$ expands by definitions (645) and (644) to the propositional formula:

$$\forall F[(\forall z(\mathit{Precedes}(x,z) \rightarrow Fz) \,\&\, \forall x' \forall y'(\mathit{Precedes}(x',y') \rightarrow (Fx' \rightarrow Fy'))) \rightarrow Fy]$$

Again, for simplicity, we henceforth abbreviate $[\lambda xy\, \mathit{Precedes}^{*}(x,y)]$ as $\mathit{Precedes}^{*}$ and treat $\mathit{Precedes}^{*}(x,y)$ as an exemplification formula of the form Rxy.

Thus, the following is an instance of definition (648):

$$\mathit{Precedes}^{+}(x,y) \;=_{df}\; \mathit{Precedes}^{*}(x,y) \vee x = y$$

Given this definition, we now assert, as an axiom, that there is a relation R such that, necessarily, R is exemplified by all and only those objects x and y such that x is a weak ancestor of y with respect to *Precedes*:

(.2) $\exists R\Box\forall x\forall y(Rxy \equiv \mathit{Precedes}^{+}(x,y))$

Again, given this axiom, we stipulate that the defined formula $\mathit{Precedes}^{+}(x,y)$ (but not its definiens) is propositional. So $[\lambda xy\, \mathit{Precedes}^{+}(x,y)]$ is well-formed. If we abbreviate this λ-expression as $\mathit{Precedes}^{+}$, then we may treat $\mathit{Precedes}^{+}(x,y)$ as an exemplification formula of the form Rxy.[243]

(**651**) **Remark:** Consequence for the Theory of Definition. The preceding stipulations require us to note some exceptions to our understanding of definitions. In (19) and (208), we set the policy of regarding the definiendum in a formula definition as non-propositional if the definiens is non-propositional. Given axioms (650.1) and (650.2) and the surrounding discussion, the definitions of $\mathit{Precedes}(x,y)$ and $\mathit{Precedes}^{+}(x,y)$ in (640) and (648), respectively, now have to be regarded as exceptions to this policy. Moreover, in (208.5), we established, with respect to formula definitions, that a definiens may be substituted for a definiendum in any context whatsover. Given the exception we've just made, however, this metatheoretic fact no longer holds; $[\lambda xy\mathit{Precedes}(x,y)]$ and $[\lambda xy\mathit{Precedes}^{+}(x,y)]$ are exceptions. Though the matrices of these λ-expressions are propositional formulas, the definientia of these matrices are not. So the exceptions to (208.5) are that the definientia for $\mathit{Precedes}(x,y)$ and $\mathit{Precedes}^{+}(x,y)$ may not be substituted for these definienda when the latter occur inside a λ-expression. This applies to any λ-expression whose matrix explicitly contains $\mathit{Precedes}(x,y)$ or $\mathit{Precedes}^{+}(x,y)$ as a subformula.

(**652**) ⋆**Lemma:** Nothing Precedes or Ancestrally Precedes Zero. It is a consequent of our definitions that nothing precedes Zero:

(.1) $\neg\exists x\mathit{Precedes}(x,0)$

[243] There is an alternative procedure for asserting axioms corresponding to (.1) and (.2), namely, stipulate that the following λ-expressions, when the defined expressions are appropriately expanded, are to be considered well-formed and hence exceptions to definition (3.5.a):

$$[\lambda xy\, \exists F\exists u(Fu\ \&\ \mathit{Numbers}(y,F)\ \&\ \mathit{Numbers}(x,F^{-u}))]$$

$$[\lambda xy\, \mathit{Precedes}^{*}(x,y) \vee x = y]$$

Abbreviate the first as *Precedes* and the second as $\mathit{Precedes}^{+}$. If these are stipulated to be well-formed 2-place relation terms, then we have immediately, as consequences, that:

$$\exists R(R = \mathit{Precedes})$$

$$\exists R(R = \mathit{Precedes}^{+})$$

We've decided not to proceed this way since allowing encoding subformulas into λ-expressions seems more drastic than adjusting adding new propositional formulas.

Cf. Frege 1893, Theorem 108. Since this theorem asserts that nothing whatsoever precedes Zero, it follows that no natural cardinal precedes Zero.

Moreover, it also follows that (.2) nothing is a predecessor ancestor of Zero, and (.3) Zero is not a predecessor ancestor of itself:

(.2) $\neg\exists x Precedes^{*}(x, 0)$

(.3) $\neg Precedes^{*}(0, 0)$

Cf. Frege 1893, Theorem 126, with (.2).

(**653**) **Definition:** Natural Numbers. We now say that x is a *natural number* just in case Zero is a weak Predecessor-ancestor of x:

$$\mathbb{N}x =_{df} Precedes^{+}(0, x)$$

Thus, a natural number is, in Frege's terminology, any member of the Predecessor-series beginning with Zero.

Note that since $Precedes^{+}(x, y)$ is, per our recent discussion, a propositional formula, we could have introduced $\mathbb{N}$ as a 1-place relation constant by way of the following term definition:

$$\mathbb{N} =_{df} [\lambda x\, Precedes^{+}(0, x)]$$

But since it makes no practical difference in what follows, we may rest with the formula definition. The important thing to remember is that the expression $\mathbb{N}x$ may appear in a well-formed λ-expression.

(**654**) **Theorem:** Zero is a Natural Number.

$$\mathbb{N}0$$

Interestingly, Frege (1893) doesn't seem to prove this claim as a theorem, possibly because it is a trivial consequence of definitions and facts about identity. Instead, he proves only the general theorem $R^{+}xx$, i.e., that x is a weak R-ancestor of itself (1893, Theorem 140), though he doesn't label the instance $Precedes^{+}(0, 0)$ as a separate theorem.

With this theorem, however, we have derived the first Dedekind-Peano axiom. In what follows, let us use 'm, n, o' as restricted variables ranging over natural numbers. Since we know that there are natural numbers, claims of the form $\forall n\varphi$ imply $\exists n\varphi$.

(**655**) ★**Theorems:** Zero Is Not the Successor of Any Natural Number. It also follows that no natural number precedes Zero.[244]

$$\neg\exists n(Precedes(n, 0))$$

[244] cf. Frege, 1893, Theorem 126.

With this theorem, we have derived the second Dedekind-Peano axiom.

(**656**) **Theorems:** No Two Natural Numbers Have the Same Successor. From (642), it follows that no two natural numbers have the same successor:

$$\forall n \forall m \forall o(Precedes(n,o) \,\&\, Precedes(m,o) \rightarrow n = m)$$

With (656), we have derived the third Dedekind-Peano axiom.

(**657**) **Remark and Exercise:** Digression on When Axiom (650) Is Needed. We began this section by asserting as an axiom that *immediately precedes* and its weak ancestral are relations, in (650). We then proved a variety of theorems in terms of these relations. But it is important to recognize that, strictly speaking, the axioms in (650) are not needed for the proof of (652.1)★, (654), (655)★, and (656), and so not needed for the proof of the first three Dedekind-Peano axioms. In this remark, we explain why these axioms are not needed for these theorems and then explain why we nevertheless asserted them before stating theorems in question.

To see why the axiom isn't needed, remember that, in item (640), the definiens for $Precedes(x,y)$ is a 2-place condition stated in terms of a non-propositional formula (i.e., a formula having encoding subformulas). But analogues of the notions of *hereditary property*, *strong ancestral*, and *weak ancestral* are definable relative to such a 2-place condition. For example, in the case of *Precedes*, we could have defined, respectively:

$$Hereditary(F, Precedes) =_{df} \ \forall x \forall y(Precedes(x,y) \rightarrow (Fx \rightarrow Fy))$$

$$Precedes^*(x,y) \ =_{df} \\ \quad \forall F[(\forall z(Precedes(x,z) \rightarrow Fz) \,\&\, Hereditary(F, Precedes)) \rightarrow Fy]$$

$$Precedes^+(x,y) \ =_{df} \ Precedes^*(x,y) \vee x = y$$

These generalize to any 2-place condition, independent of whether that condition defines a relation. Given the above definitions, we could have derived analogues of most of the theorems governing the strong and weak ancestral. In particular, analogues of (647.1) – (647.4) and (649.1) – (649.5) are derivable using the above definitions, without requiring that $Precedes(x,y)$ defines a relation. We leave the proofs as exercises. They are simple adaptions of the proofs in the Appendix.

In turn, these analogues of the theorems governing ancestrals are sufficient to prove most of the theorems in this section *without* asserting that $Precedes(x,y)$ defines a relation. That is, from the definitions and the analogues of the theorems mentioned in the previous paragraph, one can prove:

$$\neg \exists x Precedes(x,0) \qquad (652.1)\bigstar$$

$$\mathbb{N}0 \tag{654}$$

$$\neg\exists n(\mathit{Precedes}(n,0)) \qquad (655)\star$$

$$\forall n \forall m \forall o(\mathit{Precedes}(n,o) \,\&\, \mathit{Precedes}(m,o) \rightarrow m = n) \tag{656}$$

Again, the proofs are left as exercises. But note that the last three establish that the first three Dedekind-Peano axioms do not require the axioms in (650).

Interestingly, the defined ancestrals of 2-place non-propositional conditions aren't strong enough to yield analogues of (647.5) and (649.6). (647.5) asserts $R^*(x,y) \rightarrow \exists z Rzy$, and (649.6) asserts $R^*(x,y) \rightarrow \exists z(R^+(x,z) \,\&\, Rzy)$. The proof of (647.5) requires that we consider the property $[\lambda y' \, \exists x' Rx'y']$, which is definable in terms of relation R, and the proof of (649.6) requires that we consider the property $[\lambda y \, \exists x(R^+(a,x) \,\&\, Rxy)]$, again something definable from relation R. If $\varphi(x,y)$ is a 2-place condition that has encoding subformulas, we can't formulate the properties $[\lambda y \, \exists x \varphi(x,y)]$ and $[\lambda y \, \exists x(\varphi^+(a,x) \,\&\, \varphi(x,y))]$. (647.5) and (649.6) are used in the proof of:

$$\neg\exists x \mathit{Precedes}^*(x,0) \qquad (652.2)\star$$

$$\neg \mathit{Precedes}^*(0,0) \qquad (652.3)\star$$

These are the only two theorems proved in this section that make use of the new axioms in (650).

By contrast, the theorems in the next two sections do make use of the axioms in (650). The reason we asserted the axioms that predecessor and its weak ancestral are relations at the outset of this section is that it (a) allows us to apply *all* the theorems about strong and weak ancestrals developed in the preceding section, and instantiate *Precedes*, $\mathit{Precedes}^*$, and $\mathit{Precedes}^+$, for the free variables R, R^* and R^+, respectively in those theorems, and (b) allows us to derive the first three Dedekind-Peano axioms as theorems about the *relation* of predecessor and its ancestrals.[245]

[245] The observations in this Remark establish that the oversight in Zalta 1999, in which $\mathit{Precedes}(x,y)$ was instantiated for the relation variable R without first asserting it defined a relation, was not an egregious one. In §4 of Zalta 1999, items (29) and (31) defined the strong and weak ancestral of a relation R, respectively. Then, in §5, item (33), we defined a natural number to be any object to which 0 bears the weak ancestral of the predecessor relation. But at this point, we hadn't yet asserted that predecessor is a relation in item (42). Fortunately, to define the notion of a natural number, we don't strictly need the axiom that predecessor is a relation. It suffices that predecessor is a 2-place condition governed by certain theorems.

Moreover, in Zalta 1999, we also instantiated the condition $\mathit{Precedes}(x,y)$ for a relation variable R in Lemma (30.5), which asserted that if x bears the strong ancestral of R to y, then something bears R to y. Lemma (30.5) was used in the proof that natural numbers are natural cardinals (34), which in turn was used in the proof that every number has a successor (41). In what follows, we correct the order of presentation by stating the corresponding theorems only after we assert, as an axiom, that $\mathit{Precedes}(x,y)$ is a relation.

(**658**) **Remark:** A Potential But Unfounded Worry. In light of the theorems in (196), the new axioms asserted in (650) have a consequence that raises some questions, though fortunately ones that can be put to rest. Recall that (196.1) and (196.2) assert:

$$\forall R \exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda z\, Rzx] = [\lambda z\, Rzy]) \tag{196.1}$$

$$\forall R \exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda z\, Rxz] = [\lambda z\, Ryz]) \tag{196.2}$$

These imply, respectively, that:

$$\exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda z\, Precedes(z,x)] = [\lambda z\, Precedes(z,y)])$$

$$\exists x \exists y (A!x \,\&\, A!y \,\&\, x \neq y \,\&\, [\lambda z\, Precedes(x,z)] = [\lambda z\, Precedes(y,z)])$$

Let a and b be witnesses to the first existential claim, and c and d be witnesses to the second. That is suppose:

(ϑ) $A!a \,\&\, A!b \,\&\, a \neq b \,\&\, [\lambda z\, Precedes(z,a)] = [\lambda z\, Precedes(z,b)]$

(ζ) $A!c \,\&\, A!d \,\&\, c \neq d \,\&\, [\lambda z\, Precedes(c,z)] = [\lambda z\, Precedes(d,z)]$

These consequences give rise to the following questions: does anything precede either a or b, and does c or d precede anything? If the answer to the first is yes, then we could derive a contradiction from (ϑ) and the fact that *Predecessor* is a function:

> Suppose $\exists y Precedes(y,a)$. Let e be such an object, so that $Precedes(e,a)$. Then by β-Conversion, $[\lambda z\, Precedes(z,a)]e$. So by ($\vartheta$), $[\lambda z\, Precedes(z,b)]e$. By β-Conversion, $Precedes(e,b)$. But since (ϑ) also implies $a \neq b$, we now we have $Precedes(e,a)$, $Precedes(e,b)$ and $a \neq b$. This contradicts (643), whose significance in light of axiom (650.1) is that *Precedes* is a function (i.e., a relation R such that $Rxy \,\&\, Rxz \rightarrow y = z$).

Similarly, if c or d precede anything, then then we could derive a contradiction from (ζ) and the fact that *Predecessor* is a one-to-one function (exercise).

Fortunately, we can regard these derivations as *reductio* arguments to the conclusions that nothing precedes a and b, and that c and d don't precede anything. More generally, we can conclude that a and b (alternatively, c and d) can't both be natural numbers. An extended Aczel model of object theory is available that shows that object theory is consistent with the additional two axioms if consistent without them. The extended model is developed in the appendix to this chapter. The model described there improves on the model described in Zalta 1999, which omits some important details.

14.7 Natural Numbers Have Successors

We now work our way towards a proof that every natural number has a unique successor.

(**659**) **Lemma:** Successors of Natural Numbers are Natural Numbers. If a natural number precedes an object x, then x is itself a natural number:

$$Precedes(n,x) \to \mathbb{N}x$$

Recall that this principle was listed as one of the basic postulates of number theory; see postulate 6 of Remark (603).

(**660**) ★**Lemma:** Natural Numbers are Natural Cardinals. It is a consequence of our definitions that natural numbers are natural cardinals:

$$\mathbb{N}x \to NaturalCardinal(x)$$

This lemma and the preceding one play a role in the proof that every natural number has a successor.

(**661**) **Axioms:** Richness of Possible Objects. The closures of the following modal claim are asserted *a priori*, as axioms: if it is actually the case that some natural number numbers G, then there might have been a concrete object distinct$_E$ from every ordinary object that *actually* exemplifies G. We may formalize this *a priori* truth as follows:

$$\mathcal{A}\exists x(\mathbb{N}x \,\&\, Numbers(x,G)) \to \Diamond\exists y(E!y \,\&\, \forall u(\mathcal{A}Gu \to u \neq_E y))$$

We'll justify this axiom in Remark (664) below.[246]

(**662**) **Lemma:** Distinctness of Possible Objects. The following is a consequence of the modal logic of actuality and the modal logic of identity$_E$: if it is possible that ordinary object v is distinct$_E$ from every ordinary object that actually exemplifies G, then v is distinct$_E$ from every ordinary object that actually exemplifies G:

$$\Diamond\forall u(\mathcal{A}Gu \to u \neq_E v) \to \forall u(\mathcal{A}Gu \to u \neq_E v)$$

[246] In a draft of this material prior to the publication of Zalta 1999, I had used a modal axiom schema of the form 'There might have been at least n concrete objects' (where each 'n' is eliminated in terms of numerical quantifiers). Karl-Georg Niebergall noted that the derivation that every number has a successor therefore required an appeal to an ω-rule and suggested that, if possible, I should try to do without such a rule. In the attempt to find a proof that did not appeal to the ω-rule, I sought to replace the original modal axiom schema with a simpler one. After helpful discussions with Niebergall about general number systems, it occurred to me to that by strategically employing the actuality operator, we could prove that every number has a successor without an ω-rule.

(**663**) ★**Theorem:** Natural Numbers Have a Unique Successor. It now follows from (661) and (662) that a natural number n precedes a unique natural number:

$$\exists! m Precedes(n, m)$$

With this theorem, we have derived the fourth Dedekind-Peano Axiom.

(**664**) **Remark:** Justification of Axiom (661) on the Richness of Possible Objects. Since the closures of (661) are axioms, its necessitation in particular is an axiom. So (661) is not contingently true. Moreover, (661) does not imply the existence of concrete objects and is assertible *a priori*. (661) conditionalizes the possible existence of concrete objects on the truth of an actuality statement: if it is actually the case that G is numbered by a natural number, then there might have been a concrete object distinct$_E$ from all the ordinary objects actually G.

Clearly, (661) is one way of capturing the metaphysical intuition that there might have been different concrete objects than there in fact are. But it also grounds a metaphysical claim that is sometimes expressed semantically (namely, that the domain of objects *might* be of any size) when logicians defend the view that logic should imply nothing about the size of the domain. For example, in Boolos 1987 (18; 1998, 199), we find:

> In logic, we ban the empty domain as a concession to technical convenience but draw the line there: We firmly believe that the existence of even two objects, let alone infinitely many, cannot be guaranteed by logic alone. ... Since *there might be fewer than two items* [emphasis added] that we happen to be talking about, we cannot take even $\exists x \exists y(x \neq y)$ to be valid.

The attitude expressed here is a common one, namely, that logical principles shouldn't imply the existence of contingent or concrete objects. I take it Boolos would be willing to generalize the emphasized claim in the final sentence to "there might have been fewer than n ordinary objects", for $n \geq 2$, to be true *a priori*. So I take it that claims of the form "there might have been more than n ordinary objects", for any n, are true *a priori* as well. Our modal axiom (661), together with the rest of our system, validates this latter intuition.[247]

Not only do many other philosophical logicians justify their attitude about the ontological neutrality of logic by appeal to modal intuitions, but some philosophical logicians now accept the role that modality plays in mathematics. For example, Hodes 1984, write (149):

[247] I have not spent time commenting on the opening sentence of the quote from Boolos. The present theory doesn't ban the empty domain as a concession to technical convenience. It asserts well-justified logico-metaphysical principles, e.g., axioms (32.4) and (39), that ensure the domain is non-empty.

> For a long time it has been thought that modality played not role in mathematics, since purely mathematical truths were uniformly necessary. ... But modality really permeates the terms in which we learn and discuss mathematics. For example, if we ask a bright child what it means to say that there are infinitely many numbers, the anwer we want is something like "No matter how high I were to count, I could go on and count higher."

Note that theorem (663)★, that every number has a unique successor, does not imply that there are an infinite number of concrete objects. Rather, as we shall see in Section 14.9, it implies only that there are an infinite number of possibly concrete, i.e., ordinary, objects. But (661) is not like the axiom of infinity asserted in Whitehead and Russell 1910–1913 or in Zermelo-Fraenkel set theory. Without the Barcan Formula (122.1), the logic of actuality, and many other components of our system, (661) by itself doesn't yield a successor for every natural number.

(**665**) **Remark:** Digression on Frege's Proof That Every Number Has a Successor. Frege proved that every number has a successor from Hume's Principle without any additional axioms or rules. If we had tried to follow Frege's procedure, we would have run into the following obstacle when attempting to prove that every number has a successor. The claim to be proved may be represented as:

(a) $\exists m(Precedes(n, m))$

Recall that to prove this claim, it suffices to prove the following in virtue of theorem (659):

(b) $\exists x(Precedes(n, x))$

To prove (b), Frege's strategy was to prove by induction that every number n immediately precedes the number of members in the *Predecessor* series ending with n; intuitively, that n immediately precedes the number of natural numbers less than or equal to n. So Frege's method was to show:

(c) $Precedes(n, \#[\lambda x\, Precedes^{+}(x, n)])$

Cf. Frege 1893, Theorem 155. However, in our system, (c) is provably false; natural numbers and natural cardinals only number the ordinary objects that fall under a property, no number n is the number of the property *being a member of the predecessor series ending with n*.

To see exactly where the failure occurs, suppose for reductio (c) that is true, for some n. Then, by the definition of *Predecessor*, (c) implies that there is a property P and ordinary object a such that:

(d) $Pa \,\&\, Numbers(\#[\lambda x\, Precedes^{+}(x, n)], P) \,\&\, Numbers(n, P^{-a})$

From the second conjunct of (d), it follows by (630.1)★ that $\#[\lambda x\, Precedes^{+}(x,n)]$ is identical to $\#P$. So by Hume's Principle (632)★:

(e) $[\lambda x\, Precedes^{+}(x,n)] \approx_E P$

By the definition of $\approx_E$, then, there is a relation, say R, that is a one-to-one correspondence$_E$ from the ordinary objects exemplifying $[\lambda x\, Precedes^{+}(x,n)]$ to the ordinary objects exemplifying P. Since (d) implies Pa, there is an ordinary object, say b, that exemplifies $[\lambda x\, Precedes^{+}(x,n)]$ and such that Rba. But if b exemplifies $[\lambda x\, Precedes^{+}(x,n)]$, then by β-Conversion, $Precedes(b,n)$. But, then, b must be abstract, by the definitions of *Predecessor* and $Numbers(x,F)$. Contradiction. So (c) is provably false in the present system.

So, again, (c) fails because our natural numbers and natural cardinals only number the ordinary (i.e., possibly concrete) objects that exemplify a property. They do not count abstract objects. But since it is provable that no object in the exemplification extension of the property $[\lambda x\, Precedes^{+}(x,n)]$ is ordinary, the abstract object that numbers this property is Zero. Moreover, our definitions can't straightforwardly be extended so that natural cardinals and numbers count both the ordinary and the abstract objects in the exemplification extension of a property. The theorems in (196) and (197) guarantee that there are just too many abstract objects to count by means of the classical notions of exemplification and equinumerosity. But as we shall see, we can still use *general numerical quantifiers* to assert the existence of exactly n objects, ordinary or abstract, that fall under a property or that satisfy a condition φ. See definition (681).

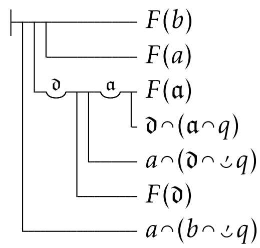
14.8 Mathematical Induction

(**666**) **Theorem:** Generalized Induction. Let us say "F is hereditary with respect to the members of the R-series beginning with z" whenever: for any elements x and y of the R-series starting with z, if x bears R to y, then if x exemplifies F, then y exemplifies F, i.e., $\forall x \forall y((R^{+}(z,x) \,\&\, R^{+}(z,y)) \rightarrow (Rxy \rightarrow (Fx \rightarrow Fy)))$. Then we may state the Generalized Principle of Induction as follows: if both (i) z exemplifies F and (ii) F is hereditary with respect to the members of the R-series beginning with z, then every member of the R-series beginning with z exemplifies F:

$$[Fz \,\&\, \forall x \forall y((R^{+}(z,x) \,\&\, R^{+}(z,y)) \rightarrow (Rxy \rightarrow (Fx \rightarrow Fy)))] \rightarrow \forall x(R^{+}(z,x) \rightarrow Fx)$$

This is a variant of Frege 1893, Theorem 152. Those interested in the differences between the above version and Frege's may find the following Remark useful.

(**667**) **Remark:** Changes to Frege's Version of Generalized Induction. (666) differs from Frege's Theorem 152 in two ways: (a) we formulate the consequent of the principle as a universal claim, and (b) we add, for reasons to be explained below, an additional, though strictly unnecessary, conjunct to the second conjunct of the antecedent. In this Remark, we explain the differences in more detail and justify both changes. Consider Frege's Theorem 152:



This can be rewritten in our notation for predication, conditionals, and quantification as follows, where a and b are preserved as individual variables, the Gothic letters $\mathfrak{d}$ and $\mathfrak{a}$ are replaced by the variables x and y, respectively, q is replaced by R, $\smile q$ is replaced by R^+:

$$[R^+(a,b) \to \forall x(Fx \to (R^+(a,x) \to \forall y(Rxy \to Fy)))] \to (Fa \to Fb)$$

If we validly swap $R^+(a,b)$ in the antecedent of the antecedent with Fa in the antecedent of the consequent and apply Importation (63.8.b) in a couple of places, this starts to look more familiar:

$$[Fa \,\&\, \forall x(Fx \,\&\, R^+(a,x) \to \forall y(Rxy \to Fy))] \to (R^+(a,b) \to Fb)$$

In other words, if a exemplifies F and every F-thing in the R-series beginning with a passes F on to everything to which it is R-related, then if b is in the R-series beginning with a, b exemplifies F.

Now if we rearrange the second conjunct of the antecedent a bit more using (83.7), symmetry of &, and Exportation (63.8.a), the above is equivalent to:

$$[Fa \,\&\, \forall x\forall y(R^+(a,x) \to (Rxy \to (Fx \to Fy))] \to (R^+(a,b) \to Fb)$$

Since b is being used here as a free variable, GEN tells us this holds for every b (if it holds) and so by (79.2):

$$[Fa \,\&\, \forall x\forall y(R^+(a,x) \to (Rxy \to (Fx \to Fy))] \to \forall b(R^+(a,b) \to Fb)$$

The consequent is now a universal claim, and so the only remaining difference with (666) is that the above version drops $R^+(a,y)$ from the second conjunct of the antecedent of (666). Frege, of course, recognized that $R^+(a,y)$ isn't needed in the statement of the theorem, for it is provable that if x is in the R-series beginning with z and x bears R to y, then y is in the R-series beginning with z, i.e., it is a theorem that $(R^+(z,x) \,\&\, Rxy) \to R^+(z,y)$.[248] I leave it as an exercise to show that the General Principle of Induction can be proved in any of the above forms.

It will soon become clear, however, why our formulation of (666) departs from Frege's in the ways described above: once (666) is instantiated in the manner described below, an immediate rewrite using restricted variables yields the simple, classical statement of the Principle of Mathematical Induction. See the proof of the next item.

(**668**) **Corollary:** Mathematical Induction. The Principle of Mathematical Induction falls out immediately as a special case of (666), by GEN and our conventions for restricted variables. For every property F, if Zero exemplifies F and Fn implies Fm whenever n and m are any two successive natural numbers, then every natural number exemplifies F:

$$\forall F[F0 \,\&\, \forall n \forall m(Precedes(n,m) \to (Fn \to Fm)) \to \forall n Fn]$$

With this theorem, we have derived the fifth and final Dedekind-Peano axiom.

14.9 An Infinite Natural Cardinal and Class

(**669**) **Definition:** Finite and Infinite Cardinals. Let us use κ as a restricted variable ranging over natural cardinals, as these were defined in (623). Then we may say (.1) κ is *finite* if and only if κ is a natural number, and (.2) κ is *infinite* if and only if κ is not finite:

(.1) $Finite(\kappa) =_{df} \mathbb{N}\kappa$

(.2) $Infinite(\kappa) =_{df} \neg Finite(\kappa)$

We justify these definitions in the following Remark.

(**670**) **Remark:** On Frege's Definition of Finite and Infinite Cardinals. In (653), we defined $\mathbb{N}x$ as $Precedes^+(0,x)$. But in 1884 (§83) and 1893 (§§46, 108, 122, 158), Frege indicated that $Precedes^+(0,x)$ is the definiens for the notion x is a *finite cardinal number*.

In 1884, §83 is titled 'Definition of Finite Number' and in this section, Frege writes:

> ...I define as follows:
>
> > the proposition "n is a member of the series of natural numbers beginning with 0"
>
> is to mean the same as:

[248] Assume $R^+(z,x) \,\&\, Rxy$. Then by (649.3), $R^*(z,y)$. Hence, by $\vee$I and definition (648), $R^+(z,y)$.

"n is a finite number"

In 1893, §46, he writes:[249]

> Accordingly, $\Delta \frown (\Theta \frown \smile\Upsilon)$ is the truth-value of: Θ belongs to the Υ-series starting with Δ. Thus, $0 \frown (\Theta \frown \smile\mathfrak{f})$ is the truth-value of: Θ belongs to the cardinal number series starting with 0, for which I can also say Θ is a *finite* cardinal number.

The statement $0 \frown (\Theta \frown \smile\mathfrak{f})$ would be written in our notation as $Precedes^{+}(0, x)$, which is the definiens for $\mathbb{N}x$ in (653).

Furthermore, in 1893, the main heading of §108 is:

$$\vdash\!\!\top\; b \frown (b \frown \underline{\smile}\mathfrak{f}) \\ \llcorner\; 0 \frown (b \frown \smile\mathfrak{f})$$

Frege then writes:

> The proposition mentioned in the main heading states that no object belonging to the cardinal number series starting with Zero follows after itself in the cardinal number series. Instead, we could also say: "No *finite* cardinal number follows after itself in the cardinal number series".

In our notation, Frege's main heading would be written:

$$Precedes^{+}(0, x) \to \neg Precedes^{*}(x, x)$$

and in (685.1)★, we will prove this in the simpler form $n \nless n$.

Again, in 1893, the main heading of §122 is:

$$\vdash\!\!\top\; 0 \frown (\infty \frown \smile\mathfrak{f})$$

Frege then writes:

> There are cardinal numbers that do not belong to the cardinal numbers series beginning with 0, or, as we shall also say, that are not finite, that are infinite. One such cardinal number is that of the concept *finite cardinal number*; I propose to call it *Endlos* and designate it with '∞'. I define it thus:
>
> $$\Vdash \mathfrak{p}(0 \frown \grave{\varepsilon} \smile\mathfrak{f}) = \infty \qquad \text{(M}$$
>
> For $0 \frown \grave{\varepsilon} \smile\mathfrak{f}$ is the extension of the concept *finite cardinal number*. The proposition mentioned in the heading says that the cardinal number Endlos is not a finite cardinal number.

[249] In the following quote, I've corrected a known one-character transcription error in the first edition of the Ebert and Rossberg 2013 translation, p. 60. In the second sentence of the following quotation, the character $\mathfrak{f}$ has been substituted for Υ in the formula. This correction is based on the original (1893, p. 60) and is included in the revised, paperback edition of the translation (2016).

Note that in what follows, we shall prove that a particular abstract object is an infinite cardinal (671.2)★ and then introduce ∞ to name it (672). We then note, in the subsequent discussion, that this results in a proof of the claim in Frege's main heading of §122. In our notation, this claim becomes $\neg Precedes^{+}(0,\infty)$.

Finally, in 1893, the main heading of §156 is:

$$\begin{array}{l} \vdash\!\!\!\top\!\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\; \emptyset \frown (\mathfrak{p}u \frown \dot{\smile}\mathfrak{f}) \\ \quad\llcorner\!\!\top\!\!-\!\overset{\mathfrak{A}}{\smile}\!-\!\overset{\mathfrak{q}}{\smile}\!-\!\!\top\; u = \mathfrak{A}\,\underline{\Lambda}\,\mathfrak{q} \end{array}$$

Frege then writes:

> For finite cardinal numbers we can prove a proposition similar to the last, namely that the cardinal number of a concept is finite if the objects falling under it can be ordered into a *simple* (non-branching, non-looping back into itself) series starting with a certain object and ending with a certain object.

Frege's main heading in §156 is a conditional and in the consequent of this conditional, he continues to use $\emptyset \frown (\mathfrak{p}u \frown \dot{\smile}\mathfrak{f})$, i.e., $\mathfrak{p}u$ is a member of the predecessor series starting with $\emptyset$, to express that the number of u ($\mathfrak{p}u$) is finite.

Given these examples, it is clear that Frege regards a cardinal number κ as *finite* just in case Zero is a weak predecessor ancestor of κ, i.e., just in case κ is a natural number. And just as clearly, he suggests that a cardinal is *infinite* just in case it fails to be finite. So our definitions in (669) conform with Frege's usage, though as we shall see below, the existence of an infinite cardinal is not established by considering the number of the concept *finite cardinal* (since for us, the *natural* cardinal of that concept is Zero).

(**671**) ★**Theorem:** There Exists an Infinite Cardinal. It is a straightforward consequence of previous definitions and theorems that: (.1) the number of *being ordinary* is a natural cardinal; (.2) the number of *being ordinary* is infinite; and (.3) there exists an infinite cardinal:

(.1) $NaturalCardinal(\#O!)$

(.2) $Infinite(\#O!)$

(.3) $\exists\kappa Infinite(\kappa)$

Hence, we have established the existence of an infinite cardinal from no mathematical primitives! We don't need any primitive notions, or assert any axioms, from set theory or number theory to identify an infinite number.

(**672**) **Definition:** Infinity. Since we've shown that $\#O!$ is infinite, we may define *Infinity*, written ∞, to be the number of *being ordinary*:

$$\infty =_{df} \#O!$$

(**673**) ⋆**Theorems:** The preceding definition yields the following immediate consequences. (.1) Infinity is a natural cardinal; (.2) Infinity is infinite; (.3) Infinity is not finite; (.4) Infinity is not a natural number; and (.5) Infinity isn't a member of the predecessor series beginning wth Zero:

(.1) $\mathit{NaturalCardinal}(\infty)$

(.2) $\mathit{Infinite}(\infty)$.

(.3) $\neg \mathit{Finite}(\infty)$,

(.4) $\neg \mathbb{N}\infty$

(.5) $\neg \mathit{Precedes}^{+}(0, \infty)$

Note that (.5) is just Theorem 167 (Frege 1893, §122), i.e., $\vdash\!\!\!\top\ \emptyset \frown (\varpi \frown \acute{\smile}\mathfrak{f})$. This theorem was anticipated in Remark (670).

(**674**) **Definition:** Infinite Class. We now stipulate that x is an *infinite class* if and only if x is a class of some property that an infinite cardinal numbers:

$$\mathit{InfiniteClass}(x) =_{df} \exists G(\mathit{ClassOf}(x, G) \,\&\, \exists \kappa(\mathit{Infinite}(\kappa) \,\&\, \mathit{Numbers}(\kappa, G)))$$

(**675**) ⋆**Theorem:** Existence of An Infinite Class. We can now prove that (.1) the extension of *being ordinary* is an infinite class; and (.2) there exists an infinite class:

(.1) $\mathit{InfiniteClass}(\epsilon O!)$

(.2) $\exists x \mathit{InfiniteClass}(x)$

So, we've established the existence of an infinite class without appealing to any mathematical primitives or asserting any mathematical axioms.

14.10 Natural Arithmetic and Other Applications

(**676**) **Definition:** Notation for Successors. By theorem (663)⋆ we know that every natural number has a unique successor. Hence, we may introduce the notation n' to abbreviate the definite description *the natural number that n precedes*:

$$n' =_{df} \imath m \mathit{Precedes}(n, m)$$

We henceforth refer to n' as the successor of n. By introducing the prime notation for successor, the prime symbol becomes a term-forming operator on terms that denote natural numbers. So we must henceforth refrain from using primes on our restricted variables n, m, and o as a means of forming new variables. However, we may continue to use prime notation on other variables — x and x' may be used as distinct general variables, and u and u' as distinct variables for ordinary objects.

(**677**) **Definitions:** Introduction of the Numerals. Since every natural number has a unique successor, we may introduce the (base 10) numerals '1', '2', '3', … , as abbreviations, respectively, for the descriptions *the successor of 0*, *the successor of 1*, *the successor of 2*, etc.

$$1 =_{df} 0'$$

$$2 =_{df} 1'$$

$$3 =_{df} 2'$$

$$\vdots$$

The ellipsis is to be continued by a sequence of definitions with analogous definienda and definientia, ordered according to the base 10 representation of the natural numbers. Note that the terms being introduced here are all logically proper. In English, the new names may be read: One, Two, Three, etc.

(**678**) **Definitions:** The Exact Numerical Quantifiers for Ordinary Objects. We now inductively define the *exact numerical quantifiers*, or *cardinality quantifiers*, for ordinary objects. Let us say (.1) *there are exactly* Zero ordinary objects that exemplify F, written $\exists!_0 uFu$, just in case no ordinary objects exemplify F; and (.2) *there are exactly n'* ordinary objects that exemplify F, written $\exists!_{n'} uFu$, just in case there is an ordinary object u that exemplifies F and there are exactly n ordinary objects that exemplify F^{-u}:

(.1) $\exists!_0 uFu \ =_{df} \ \neg\exists uFu$

(.2) $\exists!_{n'} uFu \ =_{df} \ \exists u(Fu \,\&\, \exists!_n vF^{-u}v)$

Note that the definition of $\exists!_n uFu$ is a condition with the variables n and F free.

(**679**) **Theorem:** Proof of Correctness. If we set n to Zero in the preceding definition, then since $0'$ is 1, it is straightforward to establish (.1) there is exactly One ordinary object that exemplifies F if and only if there is exactly one ordinary object that exemplifies F:

(.1) $\exists!_1 uFu \equiv \exists!uFu$

Furthermore, we can prove generally, for $n \geq 2$, that there are exactly n ordinary F-exemplifiers if and only if there are ordinary objects $u_1,\ldots,u_n$ such that (i) $u_1,\ldots,u_n$ all exemplify F, (ii) $u_1,\ldots,u_n$ are pairwise distinct, and (iii) every ordinary object that exemplifies F is identical$_E$ to one of $u_1,\ldots,u_n$:

(.2) $\exists!_n u Fu \equiv \exists u_1 \ldots \exists u_n (Fu_1 \,\&\, \ldots \,\&\, Fu_n \,\&\, u_1 \neq_E u_2 \,\&\, \ldots \,\&\, u_1 \neq_E u_n \,\&$
$u_2 \neq_E u_3 \,\&\, \ldots \,\&\, u_2 \neq_E u_n \,\&\, \ldots \,\&\, u_{n-1} \neq_E u_n \,\&\, \forall v(Fv \rightarrow (v =_E u_1 \vee \ldots \vee v =_E u_n)))$

These theorems establish that (678) correctly predicts truth conditions for the exact numerical quantifiers for ordinary objects.

(**680**) ⋆**Theorem:** Natural Numbers and Numerical Quantifiers. It is now provable that (.1) n is the abstract object that encodes just the properties F such that there are exactly n ordinary objects exemplifying F; (.2) there are exactly n ordinary objects exemplifying G if and only if n encodes G; and (.3) there are exactly n ordinary objects exemplifying G if and only if n is the number of Gs:

(.1) $n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n u Fu))$

(.2) $\exists!_n u Gu \equiv nG$

(.3) $\exists!_n u Gu \equiv n = \#G$

Note how (.1) validates an idea put forward by Hodes in papers from 1984 and 1990. He summarizes his 'coding fictionalism' as the view that numbers are "fictions created to encode cardinality quantifiers, thereby clothing a certain higher-order logic in the attractive garments of lower-order logic" (1990, 350). By 'higher-order logic', Hodes means third-order logic.[250] If we replace the word 'fictions' by 'logical patterns', then even Hodes might accept that (.1) implies that natural numbers are logical patterns (abstracted from exemplification facts about ordinary objects and equinumerosity facts about properties of individuals) that literally encode the ordinary object quantifiers. Though one might suggest that Hodes's view has greater generality given that the view holds for the numerical object-quantifiers generally (and not just numerical ordinary object quantifiers), but then, he has to accept far richer assumptions to

[250] His position was laid out earlier in 1984 where we find, on p. 143:

> In making what appears to be a statement about numbers one is really making a statement primarily about cardinality object-quantifiers; what appears to be a first-order theory about objects of a distinctive sort really is an encoding of a fragment of third-order logic.

And on p. 144:

> The mathematical-object picture may be described in two equivalent ways. . . . or we may see it as a pretense of positing objects that intrinsically represent type 2 entities. This second description makes mathematical discourse, when carried on within the mathematical-object picture, a special sort of fictional discourse: numbers are fictions "created" with a special purpose, to encode numerical object-quantifiers and thereby enable us to "pull down" a fragment of third-order logic, dressing it in first-order clothing.

In our system, the Comprehension Principle for Abstract Objects that grounds the way we abstract the numbers from these higher-order facts.

even state the view, namely, the assumptions of third-order logic, model theory, applied set theory, etc. By contrast, object theory can express and derive a version of the view without even requiring full second-order logic.

(**681**) **Remark:** Exact Numerical Quantifiers for *Any* Object or Property and *Any* Condition. We can now make use of the natural numbers to inductively define exact numerical quantifiers for any objects or properties that meet any condition φ. These quantifiers will be unlike the numerical quantifiers for ordinary objects: we won't be able to transform assertions involving them into facts about the number indexing the quantifier. Nevertheless, the following definitions are precise and allow us to assert *there are exactly n objects such that* φ and *there are exactly n m-place relations such that* φ, as the case may be:

$$\exists!_0\alpha\varphi \;=_{df}\; \neg\exists\alpha\varphi$$
$$\exists!_{n'}\alpha\varphi \;=_{df}\; \exists\alpha(\varphi \,\&\, \exists!_n\beta(\varphi^{\beta}_{\alpha} \,\&\, \beta\neq\alpha))$$

As the simplest (non-vacuous) case of this definition, let α be x and φ be Fx, so that we have:

$$\exists!_0 xFx \;=_{df}\; \neg\exists xFx$$
$$\exists!_{n'} xFx \;=_{df}\; \exists x(Fx \,\&\, \exists!_n y(Fy \,\&\, y\neq x))$$

Similarly, for the case of m-place relations ($m \geq 0$), our definition yields:

$$\exists!_0 F^m\varphi \;=_{df}\; \neg\exists F^m\varphi$$
$$\exists!_{n'} F^m\varphi \;=_{df}\; \exists F^m(\varphi \,\&\, \exists!_n G^m(\varphi^{G^m}_{F^m} \,\&\, G^m\neq F^m))$$

Clearly, we can't use these definitions to derive theorems analogous to (680.2)★ and (680.3)★. In particular, we can't show $\exists!_n xFx$ is equivalent to either nF or $n = \#F$. Similarly, we can't derive facts about what n encodes or about the number of any property from $\exists!_n F\varphi$. Nevertheless, the definienda provide nominal answers the questions "How many individuals are such that φ?" and "How many Fs are such that φ?". The answers make use of the natural numbers merely as informative indices — once the answers are completely expanded by the definitions, they make no reference to the natural numbers.

(**682**) **Definition:** One-to-One Relations. Recall that in (642), we intuitively established that *Precedes* is a one-to-one condition by showing: if *Precedes*(x,z) and *Precedes*(y,z), then x is identical to y. Since it is now axiomatic that *Precedes* is a relation, (642) asserts that it is a one-to-one relation. Let us say generally that R is a *one-to-one* relation, written *1-1*(R), just in case Rxz and Ryz imply $x = y$:

$$\mathit{1\text{-}1}(R) \;=_{df}\; \forall x\forall y\forall z(Rxz \,\&\, Ryz \rightarrow x=y)$$

(**683**) **Lemmas:** One-to-One Relations and Their Ancestrals. We now have the following facts about one-to-one relations and their ancestrals: (.1) if R is one-to-one, x bears R to y and z is an R-ancestor of y, then z is a weak R-ancestor of x; (.2) if R is one-to-one, then for any R-related objects x and y, if x fails to be an R-ancestor of x, y fails to be an R-ancestor of y; (.3) if both R is one-to-one and x fails to be an R-ancestor of x, then if x is a weak R-ancestor of y, y fails to be a weak R-ancestor of y:

(.1) $\mathit{1\text{-}1}(R) \,\&\, Rxy \,\&\, R^*(z,y) \rightarrow R^+(z,x)$

(.2) $\mathit{1\text{-}1}(R) \rightarrow \forall x \forall y[Rxy \rightarrow (\neg R^*(x,x) \rightarrow \neg R^*(y,y))]$

(.3) $(\mathit{1\text{-}1}(R) \,\&\, \neg R^*(x,x)) \rightarrow (R^+(x,y) \rightarrow \neg R^*(y,y))$

These facts are used in the proofs that no natural number is less than itself or precedes itself.

(**684**) **Definitions:** Less Than, Less Than Or Equal To, Greater Than, Greater Than Or Equal To. We now define:

(.1) $n < m \;=_{df}\; \mathit{Precedes}^*(n,m)$

(.2) $n \leq m \;=_{df}\; \mathit{Precedes}^+(n,m)$

(.3) $n > m \;=_{df}\; m < n$

(.4) $n \geq m \;=_{df}\; m \leq n$

The reader should convince herself that these are all *relations*, given that the strong and weak ancestral of *Precedes* are relations.

(**685**) ⋆**Theorems:** Facts About Natural Numbers. It is a consequence of the previous lemmas that (.1) no natural number is less than itself; (.2) no natural number precedes itself; and (.3) no natural number is identical to its own successor:

(.1) $n \nless n$

(.2) $\neg \mathit{Precedes}(n,n)$

(.3) $n \neq n'$

(**686**) **Definition:** Positive Integers. A positive integer is any natural number not equal to Zero:

$$\mathit{PositiveInteger}(x) \;=_{df}\; \mathbb{N}x \,\&\, x \neq 0$$

We henceforth use i as a restricted variable for positive integers.

(**687**) **Lemma:** Predecessors of Positive Integers are Natural Numbers. If an object x precedes a positive integer i, then x is itself a natural number:

$$Precedes(x,i) \rightarrow \mathbb{N}x$$

(**688**) ⋆**Theorem:** Every Positive Integer is Preceded by a Unique Natural Number. Where i is a restricted variable ranging over positive integers, our theorem may be written:

$$\exists! n Precedes(n,i)$$

Cf. Frege 1884, §78, in which he asserts that every cardinal number other than Zero has a predecessor.

Part III

Metaphilosophy

Part IV

Technical Appendices, Bibliography, Index

Appendix: Proofs of Theorems and Metarules

NOTE: The items below numbered as (n), $(n.m)$, or $(n.m.a)$ refer to numbered items in Part II. So, for example, references to item (9.1) do *not* refer to Chapter 9.1, but rather to item (9.1), which occurs in Part II, Chapter 7.

(45.1) (Exercise)

(45.2) (Exercise)

(46.1) We establish the rule only for $\vdash$. If φ is an element of $\mathbf{\Lambda}$, then the one element sequence φ is a proof of φ, by (42.2). ⋈

(46.2) We establish the rule only for $\vdash$. If φ is an element of Γ, then the one element sequence φ is a derivation of φ from Γ, by (42.1). ⋈

(46.3) We establish the rule only for $\vdash$. If $\vdash \varphi$, then by definition (42.2), there is a sequence of formulas every element of which is either a member of $\mathbf{\Lambda}$ or a direct consequence of some of the preceding members of the sequence by virtue of MP. Since $\mathbf{\Lambda} \subseteq \mathbf{\Lambda} \cup \Gamma$, there is a sequence of formulas every element of which is either a member of $\mathbf{\Lambda} \cup \Gamma$ or a direct consequence from some of the preceding members of the sequence by virtue of MP. Hence, by definition (42.1), $\Gamma \vdash \varphi$. ⋈

(46.4) (Exercise)

(46.5) We establish the rule only for $\vdash$. Assume $\Gamma \vdash \varphi$ and $\Gamma \vdash (\varphi \to \psi)$. Then there is a sequence $\chi_1, \ldots, \chi_{n-1}, \varphi$ (= S_1) that is a derivation of φ from Γ and there is a sequence $\theta_1, \ldots, \theta_{m-1}, \varphi \to \psi$ (= S_2) that is a derivation of $\varphi \to \psi$ from Γ. So the consider the sequence:

$$\chi_1, \ldots, \chi_{n-1}, \varphi, \theta_1, \ldots, \theta_{m-1}, \varphi \to \psi, \psi \qquad (S_3)$$

Since every element of S_1 and S_2 is either an element of $\mathbf{\Lambda} \cup \Gamma$ or follows from preceding members by MP, the same holds for every member of the initial segment of S_3 up to and including $\varphi \to \psi$. Since the last member of S_3 follows from previous members by MP, we know that every element of S_3 is either an element of $\mathbf{\Lambda} \cup \Gamma$ or follows from preceding members by MP. Hence, S_3 is a derivation of ψ from Γ. ⋈

(46.6) (Exercise)

(46.7) We establish the rule only for $\vdash$. Assume $\Gamma \vdash \varphi$ and that $\Gamma \subseteq \Delta$. Then by the former, there is a sequence S ending in φ such that every member of the sequence is either in $\mathbf{\Lambda} \cup \Gamma$ or follows from previous members by MP. But since $\Gamma \subseteq \Delta$, it follows that every member of S is either in $\mathbf{\Lambda} \cup \Delta$ or follows from previous members by MP, i.e., $\Delta \vdash \varphi$. ⋈

(46.8) We establish the rule only for $\vdash$. Assume $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$. By the former, there is a sequence $S_1 = \chi_1, \ldots, \chi_{n-1}, \varphi$ such that every member of the sequence is either in $\mathbf{\Lambda} \cup \Gamma$ or follows from previous members by MP. By the latter, there is a sequence $S_2 = \theta_1, \ldots, \theta_{m-1}, \psi$ such that every member of the sequence is either in $\mathbf{\Lambda} \cup \{\varphi\}$ or follows from previous members by MP. So consider, then, the following sequence:

$$\chi_1, \ldots, \chi_{n-1}, \theta_1, \ldots, \theta_{m-1}, \psi \tag{S_3}$$

This sequence is the concatenation of the first $n-1$ members of S_1 with the entire sequence S_2. Since S_1 and S_2 are derivations, we know that all the χ_i $(1 \le i \le n-1)$ and θ_j $(1 \le j \le m-1)$ are either elements of $\mathbf{\Lambda} \cup \Gamma$ or follow from two of the preceding members of the sequence by MP. The only potential exceptions are possible occurrences of φ among the θ_js. But note that φ follows by MP from two members of $\chi_1, \ldots, \chi_{n-1}$. Thus S_3 is a derivation of ψ from Γ and, hence, $\Gamma \vdash \psi$. ⋈

(46.9) We establish the rule only for $\vdash$. Suppose $\Gamma \vdash \varphi$. Since the instances of (21.1) are axioms, we know by (46.1) that $\vdash \varphi \to (\psi \to \varphi)$, where ψ is any formula. So by (46.3), we have $\Gamma \vdash \varphi \to (\psi \to \varphi)$. From our initial hypothesis and this last result, it follows by an instance of (46.5) (i.e., an instance in which we set ψ in (46.5) to $\psi \to \varphi$) that $\Gamma \vdash (\psi \to \varphi)$, where ψ is any formula. ⋈

(46.10) We establish the rule only for $\vdash$. Suppose $\Gamma \vdash (\varphi \to \psi)$. Since $\Gamma \subseteq \Gamma \cup \{\varphi\}$, it follows from (46.7) that:

(ϑ) $\Gamma \cup \{\varphi\} \vdash \varphi \to \psi$

But since $\varphi \in \Gamma \cup \{\varphi\}$, it follows by (46.2) that:

(ξ) $\Gamma \cup \{\varphi\} \vdash \varphi$

So from (ϑ) and (ξ), it follows by (46.5) that $\Gamma \cup \{\varphi\} \vdash \psi$, i.e., $\Gamma, \varphi \vdash \psi$. ⋈

(49) Suppose (a) $\Gamma \vdash \varphi$, and (b) α doesn't occur free in any formula in Γ. We show by induction on the length of the derivation of φ from Γ that $\Gamma \vdash \forall\alpha\varphi$.

Base case. The derivation of φ from Γ is a one-element sequence, in which case the sequence must be φ itself since a derivation of φ from Γ must end with φ. Then by the definition of *derivation from*, (42.1), $\varphi \in \mathbf{\Lambda} \cup \Gamma$. So we have two cases: (A) φ is an element of $\mathbf{\Lambda}$, i.e., φ is one of the axioms asserted in Chapter 8, or (B) φ is an element of Γ.

Case A. $\varphi \in \mathbf{\Lambda}$. Then $\forall\alpha\varphi \in \mathbf{\Lambda}$, since the generalizations of all axioms are axioms.[268] So, $\forall\alpha\varphi \in \mathbf{\Lambda} \cup \Gamma$, and so by (46.4), it follows that $\Gamma \vdash \forall\alpha\varphi$.

Case B. $\varphi \in \Gamma$. Then, by hypothesis, α doesn't occur free in φ. Consequently, $\varphi \to \forall\alpha\varphi$ is an instance of axiom (29.4) meeting the condition that α doesn't occur free in φ. So by (42.1), the sequence $\varphi, \varphi \to \forall\alpha\varphi, \forall\alpha\varphi$ is a witness to $\Gamma \vdash \forall\alpha\varphi$, since every member of the sequence is either a member of $\mathbf{\Lambda} \cup \Gamma$ or is a direct consequence of two previous members by MP.

Inductive Case. Suppose that the derivation of φ from Γ is a sequence of length n, where $n > 1$. Then either $\varphi \in \mathbf{\Lambda} \cup \Gamma$ or φ follows from two previous members of the sequence, namely, $\psi \to \varphi$ and ψ, by MP. If $\varphi \in \mathbf{\Lambda} \cup \Gamma$, then using the same reasoning as in the base case, $\Gamma \vdash \forall\alpha\varphi$. If φ follows from previous members $\psi \to \varphi$ and ψ by MP, then by the definition of derivation, we know that $\Gamma \vdash \psi \to \varphi$ and $\Gamma \vdash \psi$, where these are derivations of length less than n. Since our IH is that the theorem holds for all derivations of formulas from Γ of length less than n, it follows that $\Gamma \vdash \forall\alpha(\psi \to \varphi)$ and $\Gamma \vdash \forall\alpha\psi$. So there is a sequence $S_1 = \chi_1, \ldots, \chi_i$, where $\chi_i = \forall\alpha(\psi \to \varphi)$, that is a witness to the former and a sequence $S_2 = \theta_1, \ldots, \theta_j$, where $\theta_j = \forall\alpha\psi$, that is a witness to the latter. Now by using an instance of axiom (29.3), we may construct the following sequence:

$$\chi_1, \ldots, \chi_i, \theta_1, \ldots, \theta_j, \forall\alpha(\psi \to \varphi) \to (\forall\alpha\psi \to \forall\alpha\varphi), \forall\alpha\psi \to \forall\alpha\varphi, \forall\alpha\varphi \qquad (S_3)$$

The antepenultimate member of S_3 is an instance of axiom (29.3), and so an element of $\mathbf{\Lambda}$ and hence of $\mathbf{\Lambda} \cup \Gamma$. The penultimate member of S_3 follows from previous members (namely, the antepenultimate member and χ_i) by MP, and the last member of S_3 follows from previous members (namely, the penultimate member and θ_j) by MP. Hence, every element of S_3 is either in $\mathbf{\Lambda} \cup \Gamma$ or follows from previous members by MP. So $\Gamma \vdash \forall\alpha\varphi$. ⋈

(51) Suppose $\Gamma \vdash_{\Box} \varphi$, i.e., that there is a modally-strict derivation of φ from Γ. We show by induction on the length of the derivation that $\Box\Gamma \vdash \Box\varphi$, i.e., that there is a derivation of $\Box\varphi$ from $\Box\Gamma$.

Base Case. If $n = 1$, then the modally-strict derivation of φ from Γ consists of a single formula, namely, φ itself. So by the definition of $\Gamma \vdash_{\Box} \varphi$, φ must

[268] We did not, however, assert the *necessitations* and *actualizations* of all the axiomatic formulas as axioms; in particular, we did not assert the necessitations or actualizations of axiom (30)$\star$. But we did assert their universal closures as axioms. So, indeed, if $\varphi \in \mathbf{\Lambda}$, then $\forall\alpha\varphi \in \mathbf{\Lambda}$.

be in $\mathbf{\Lambda}_\Box \cup \Gamma$. So either (a) φ is in $\mathbf{\Lambda}_\Box$ or (b) φ is in Γ. If (a), then φ must be a necessary axiom and so its necessitation $\Box\varphi$ is an axiom. So $\vdash \Box\varphi$ by (46.1) and $\Box\Gamma \vdash \Box\varphi$ by (46.3).[269] If (b), then $\Box\varphi$ is in $\Box\Gamma$, by the definition of $\Box\Gamma$. Hence by (46.2), it follows that $\Box\Gamma \vdash_\Box \Box\varphi$. But then by (45.1), it follows that $\Box\Gamma \vdash \Box\varphi$.

Inductive Case. Suppose that the modally-strict derivation of φ from Γ is a sequence S of length n, where $n > 1$. Then either $\varphi \in \mathbf{\Lambda}_\Box \cup \Gamma$ or φ follows by MP from two previous members of the sequence, namely, $\psi \to \varphi$ and ψ. If $\varphi \in \mathbf{\Lambda}_\Box \cup \Gamma$, then using the reasoning in the base case, it follows that $\Box\Gamma \vdash \Box\varphi$. If φ follows from previous members $\psi \to \varphi$ and ψ by MP, then by the definition of a modally-strict derivation, we know both that $\Gamma \vdash_\Box \psi \to \varphi$ and $\Gamma \vdash_\Box \psi$. Consequently, since our IH is that the theorem holds for all such derivations of length less than n, it implies:

(a) $\Box\Gamma \vdash \Box(\psi \to \varphi)$

(b) $\Box\Gamma \vdash \Box\psi$

Now since instances of the K schema (32.1) are axioms, we know:

$$\vdash \Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$$

So by (46.3), it follows that:

$$\Box\Gamma \vdash \Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$$

So by (46.5), it follows from this and (a) that:

$$\Box\Gamma \vdash \Box\psi \to \Box\varphi$$

And again by (46.5), it follows from this and (b) that:

$$\Box\Gamma \vdash \Box\varphi$$ ⋈

(53) Axiom (21.2) asserts:

$$\varphi \to (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

If we let φ in the above be φ, let ψ in the above be $(\varphi \to \varphi)$, and let χ in the above be φ, then we obtain the following instance of (21.2):

$$(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$$

But the following is an instance of (21.1):

$$\varphi \to ((\varphi \to \varphi) \to \varphi)$$

[269] Note that (46.3) says that if $\vdash \varphi$, then $\Gamma \vdash \varphi$, for *any* Γ. So, in this case, we've substituted $\Box\Gamma$ for Γ in (46.3). The clash of variables is not an egregious one.

Since this latter is the antecedent of the former, we may apply MP to obtain:

$$(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$$

But now the following is also an instance of (21.1):

$$\varphi \to (\varphi \to \varphi)$$

By applying MP to our last two results we obtain:

$$\varphi \to \varphi$$ ⋈

(54) Suppose $\Gamma, \varphi \vdash \psi$. We show by induction on the length of a derivation of ψ from $\Gamma \cup \{\varphi\}$ that $\Gamma \vdash (\varphi \to \psi)$.

Base case. The derivation of ψ from $\Gamma \cup \{\varphi\}$ is a one-element sequence, namely, ψ itself. Then by the definition of *derivation from*, (42.1), $\psi \in \mathbf{\Lambda} \cup \Gamma \cup \{\varphi\}$. So we have two cases: (A) ψ is an element of $\mathbf{\Lambda} \cup \Gamma$, i.e., ψ is one of the axioms asserted in Chapter 8 or an element of Γ, or (B) $\psi = \varphi$.

Case A. $\psi \in \mathbf{\Lambda} \cup \Gamma$. Then by (42.1), $\Gamma \vdash \psi$. Since the instances of (21.1) are axioms governing conditionals, we know $\vdash (\psi \to (\varphi \to \psi))$, by (46.1). So, by (46.3), it follows that $\Gamma \vdash (\psi \to (\varphi \to \psi))$. Hence by (46.5), it follows that $\Gamma \vdash (\varphi \to \psi)$.

Case B. $\psi = \varphi$. Then by (53), we know $\vdash (\psi \to \psi)$. So, $\vdash (\varphi \to \psi)$, and hence, by (46.3), it follows that $\Gamma \vdash (\varphi \to \psi)$.

Inductive Case. The derivation of ψ from $\Gamma \cup \{\varphi\}$ is a sequence of length n, where $n > 1$. Then either $\psi \in \mathbf{\Lambda} \cup \Gamma \cup \{\varphi\}$ or ψ follows from two previous members of the sequence, namely, $\chi \to \psi$ and χ, by MP. If $\psi \in \mathbf{\Lambda} \cup \Gamma \cup \{\varphi\}$, then using the same reasoning as in the base case, $\Gamma \vdash (\varphi \to \psi)$. If ψ follows from previous members $\chi \to \psi$ and χ by MP, then since our IH is that the theorem holds for all derivations of formulas from Γ of length less than n, it implies both:

(a) $\Gamma \vdash (\varphi \to \chi)$

(b) $\Gamma \vdash (\varphi \to (\chi \to \psi))$

Now since the instances of (21.2) are axioms governing conditionals, we know, by (46.1):

$$\vdash (\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$$

So, by (46.3), it follows that:

$$\Gamma \vdash (\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$$

From this and (b), it follows by (46.5) that:

$$\Gamma \vdash (\varphi \to \chi) \to (\varphi \to \psi)$$

And from this last conclusion and (a), it follows that:

$$\Gamma \vdash (\varphi \to \psi) \qquad \bowtie$$

(55.1) Assume:

(a) $\Gamma_1 \vdash \varphi \to \psi$

(b) $\Gamma_2 \vdash \psi \to \chi$

So, by definition (42.1), there is a sequence, say S_1, that is a witness to (a) and a sequence, say S_2, that is a witness to (b). Then consider the sequence S_3 consisting of the members of S_1, followed by the members of S_2, followed by $\varphi \to \psi$, $\psi \to \chi$, φ, ψ, and ending in χ. It is not hard to show that this is a witness to:

(ϑ) $\Gamma_1, \Gamma_2, \varphi \to \psi, \psi \to \chi, \varphi \vdash \chi$

since every element of S_3 either: (a) is an element of Γ_1, or (b) is an element of Γ_2, or (c) is just the formula $\varphi \to \psi$, $\psi \to \chi$, or φ, or (d) follows from previous members of the sequence by MP. By an application of the Deduction Theorem to (ϑ), it follows that:

$$\Gamma_1, \Gamma_2, \varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$$

By an application of the Deduction Theorem to the above, and another application to the result, we obtain:

(ξ) $\Gamma_1, \Gamma_2, \vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

But from (a) and (b), respectively, it follows by (46.7) that:

(c) $\Gamma_1, \Gamma_2 \vdash \varphi \to \psi$

(d) $\Gamma_1, \Gamma_2 \vdash \psi \to \chi$

So from (ξ) and (c) it follows by (46.5) that:

(ζ) $\Gamma_1, \Gamma_2, \vdash (\psi \to \chi) \to (\varphi \to \chi)$

And from (ζ) and (d) it again follows by (46.5) that $\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$. $\bowtie$

(55.2) (Exercise)

(55.3) Consider the premise set $\Gamma = \{\varphi \to \psi, \psi \to \chi, \varphi\}$. From the first and third members of Γ, we obtain ψ by MP. From ψ and the second member of Γ, we obtain χ by MP. Hence, the sequence consisting of the members of Γ

followed by ψ and χ constitute a witness to $\varphi \to \psi$, $\psi \to \chi$, $\varphi \vdash \chi$. So by the Deduction Theorem (54), $\varphi \to \psi$, $\psi \to \chi \vdash \varphi \to \chi$. ⋈

(55.4) Consider the premise set $\Gamma = \{\varphi \to (\psi \to \chi), \psi, \varphi\}$. Then from the first and third members of Γ, we obtain $\psi \to \chi$ by MP, and from this and the second member of Γ we obtain χ by MP. Hence the sequence consisting of the members of Γ followed by $\psi \to \chi$ and χ constitute a witness to $\varphi \to (\psi \to \chi)$, ψ, $\varphi \vdash \chi$. So by the Deduction Theorem (54), it follows that $\varphi \to (\psi \to \chi)$, $\psi \vdash \varphi \to \chi$. ⋈

(58.1) As an instance of (21.3), we have: $(\neg\varphi \to \neg\neg\varphi) \to ((\neg\varphi \to \neg\varphi) \to \varphi)$. Moreover, by (53), we have $\neg\varphi \to \neg\varphi$. Then by (56.2), it follows that $(\neg\varphi \to \neg\neg\varphi) \to \varphi$. But $(\neg\neg\varphi \to (\neg\varphi \to \neg\neg\varphi))$ is an instance of (21.1). So it follows that $\neg\neg\varphi \to \varphi$, by (55.1). ⋈

(58.2) As an instance of (21.3), we have: $(\neg\neg\neg\varphi \to \neg\varphi) \to ((\neg\neg\neg\varphi \to \varphi) \to \neg\neg\varphi)$. Moreover, as an instance of (58.1), we know: $\neg\neg\neg\varphi \to \neg\varphi$. So by MP, it follows that $(\neg\neg\neg\varphi \to \varphi) \to \neg\neg\varphi$. But as an instance of (21.1), we know: $\varphi \to (\neg\neg\neg\varphi \to \varphi)$. So by (55.1), it follows that $\varphi \to \neg\neg\varphi$. ⋈

(58.3) Assume $\neg\varphi$ for conditional proof. Now assume φ for a conditional proof nested within our conditional proof. Then since $\varphi \to (\neg\psi \to \varphi)$ is an instance of axiom (21.1), it follows from this and our second assumption that:

(a) $\neg\psi \to \varphi$

Moroever, since $\neg\varphi \to (\neg\psi \to \neg\varphi)$ is an instance of axiom (21.1), it follows from this and our first assumption that:

(b) $\neg\psi \to \neg\varphi$

But as an instance of axiom (21.3), we know:

(c) $(\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$

From (b) and (c), it follows that $(\neg\psi \to \varphi) \to \psi$. And from this and (a), it follows that ψ. So, discharging the premise of our nested conditional proof, it follows that $\varphi \to \psi$. Hence, discharging the premise of our original conditional proof, it follows that $\neg\varphi \to (\varphi \to \psi)$. ⋈

(58.4) We establish $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$ by conditional proof. Assume $\neg\psi \to \neg\varphi$. Then as an instance of (21.3), we know: $(\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$. So it follows that $(\neg\psi \to \varphi) \to \psi$. But as an instance of (21.1), we know: $\varphi \to (\neg\psi \to \varphi)$. So by hypothetical syllogism (55.1) from from our last two results, it follows that $\varphi \to \psi$. So, by conditional proof (CP), it follows that $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$. ⋈

(58.5) We establish $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ by conditional proof. Assume $\varphi \to \psi$. We know by (58.1) that $\neg\neg\varphi \to \varphi$. So it follows by hypothetical syllogism (56.1) that:

(a) $\neg\neg\varphi \to \psi$

But by (58.2), we know:

(b) $\psi \to \neg\neg\psi$

So it follows from (a) and (b) by hypothetical syllogism (56.1) that $\neg\neg\varphi \to \neg\neg\psi$. But as an instance of (58.4), we know: $(\neg\neg\varphi \to \neg\neg\psi) \to (\neg\psi \to \neg\varphi)$. Hence it follows that $\neg\psi \to \neg\varphi$. So, by conditional proof (CP), it follows that $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$. ⋈

(58.6) Assume $\varphi \to \neg\psi$, to show $\psi \to \neg\varphi$ by conditional proof. Now assume ψ for a conditional proof nested within our conditional proof. From ψ it follows by (58.1) that $\neg\neg\psi$. Then from $\varphi \to \neg\psi$ and $\neg\neg\psi$, it follows by Modus Tollens that $\neg\varphi$. So discharging the premise of our nested conditional proof, we have $\psi \to \neg\varphi$. And discharging the premise of our original conditional proof, it follows that $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$. ⋈

(58.7) (Exercise)

(58.8) – **(58.9)** Follow the proofs in Mendelson 1997, Lemma 1.11(f) – (g), pp. 39–40. ⋈

(58.10) (Exercise)

(59.1) Assume $\Gamma_1 \vdash (\varphi \to \psi)$ and $\Gamma_2 \vdash \neg\psi$. Since $\Gamma_1 \subseteq \Gamma_1 \cup \Gamma_2$, it follows from the first assumption by (46.7) that:

(a) $\Gamma_1, \Gamma_2 \vdash (\varphi \to \psi)$

Since $\Gamma_2 \subseteq \Gamma_1 \cup \Gamma_2$, it follows from the second assumption by (46.7) that:

(b) $\Gamma_1, \Gamma_2 \vdash \neg\psi$

Now as an instance of (58.5), we know:

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

and hence by (46.3) that:

$$\Gamma_1, \Gamma_2 \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

So by applying (46.5) to this last result and (a), we have: $\Gamma_1, \Gamma_2 \vdash \neg\psi \to \neg\varphi$. And by applying (46.5) to this result and (b), we have $\Gamma_1, \Gamma_2 \vdash \neg\varphi$. ⋈

(59.2) (Exercise)

(60.1) ($\to$) Assume:

(ϑ) $\Gamma \vdash \varphi \to \psi$

But given (58.5), we know:

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

and hence by (46.3) that:

$$\Gamma \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

Hence by applying (46.5) to this last result and (ϑ), we obtain $\Gamma \vdash (\neg\psi \to \neg\varphi)$. ($\leftarrow$) By symmetrical reasoning, but using (58.4). $\bowtie$

(60.2) ($\to$) Assume:

(ϑ) $\Gamma \vdash \varphi \to \neg\psi$

But given (58.6), we know:

$$\vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$$

and hence by (46.3) that:

$$\Gamma \vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$$

Hence by applying (46.5) to this last result and (ϑ), we obtain $\Gamma \vdash (\psi \to \neg\varphi)$. ($\leftarrow$) By symmetrical reasoning, but using (58.7) $\bowtie$

(61.1) Assume $\Gamma_1, \neg\varphi \vdash \neg\psi$ and $\Gamma_2, \neg\varphi \vdash \psi$. By analogy with the first step of the reasoning in (59.1), it follows by (46.7) that both:

(a) $\Gamma_1, \Gamma_2, \neg\varphi \vdash \neg\psi$

(b) $\Gamma_1, \Gamma_2, \neg\varphi \vdash \psi$

Now, by the Deduction Theorem (54), it follows from (a) and (b), respectively, that:

(ϑ) $\Gamma_1, \Gamma_2 \vdash (\neg\varphi \to \neg\psi)$

(ζ) $\Gamma_1, \Gamma_2 \vdash (\neg\varphi \to \psi)$

But the instances of (21.3) are axioms and hence theorems, by (46.1). So we know:

$$\vdash (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$$

From this result it follows by (46.3) that:

(ξ) $\Gamma_1, \Gamma_2 \vdash (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

But by apply (46.5) to (ϑ) and (ξ), and to the result and (ζ), apply (46.5) again. It follows that $\Gamma_1, \Gamma_2 \vdash \varphi$. ⋈

(61.2) (Exercise)

(62.1) (Exercise)

(62.2) (Exercise)

(63.1.a) – **(63.10.e)** (Exercises)

(64.1) – **(64.8)** (Exercises)

(66.1) If we let $\tau = \alpha$, then as an instance of axiom (29.2), we have $\exists\beta(\beta = \alpha)$ (clearly, the variable β is not free in the variable α, and so the condition of the axiom is met). Since this is a theorem, we may apply GEN to obtain $\forall\alpha\exists\beta(\beta = \alpha)$. ⋈

(66.2) If we let $\tau = \alpha$, then as a *bona fide* instance of axiom (29.2), we have $\exists\beta(\beta = \alpha)$. Since this is a $\Box$-theorem (recall that in item (47)★ we defined $\Box$-theorems as those theorems having modally-strict proofs), it follows by RN (51) that $\Box\exists\beta(\beta = \alpha)$. ⋈

(66.3) Since (66.1) is a $\Box$-theorem, we may apply RN (51) to obtain $\Box\forall\alpha\exists\beta(\beta = \alpha)$. ⋈

(66.4) From (66.2), by GEN. ⋈

(66.5) From the $\Box$-theorem (66.4), by RN (51). ⋈

(67.1) As an instance of (63.4.a), we have $xF^1 \equiv xF^1$. Since this is a theorem, we may apply GEN (49) to obtain $\forall x(xF^1 \equiv xF^1)$. Since this is a $\Box$-theorem, it follows by RN (51) that $\Box\forall x(xF^1 \equiv xF^1)$. Thus, by the definition of property identity (16.1), $F^1 = F^1$. ⋈

(67.2) Let φ be $F^1 = F^1$ and φ' be $G^1 = F^1$. Since we've taken the closures of the substitution of identicals (25) as axioms, the following universal generalization is an axiom:

(ϑ) $\forall F^1 \forall G^1(F^1 = G^1 \to (F^1 = F^1 \to G^1 = F^1))$

Since $\exists H^1(H^1 = F^1)$ and $\exists H^1(H^1 = G^1)$ are *bona fide* instances of axiom (29.2), we may formulate the relevant instances of axiom (29.1) that allow us to instantiate F^1 for $\forall F^1$ and G^1 for $\forall G^1$ in (ϑ) so as to conclude:

$$F^1 = G^1 \to (F^1 = F^1 \to G^1 = F^1)$$

Now assume $F^1 = G^1$, for conditional proof. Then it follows by MP that $F^1 = F^1 \to G^1 = F^1$. And from this last result and the theorem that $F^1 = F^1$ (67.1), it follows by MP that $G^1 = F^1$. So by conditional proof, $F^1 = G^1 \to G^1 = F^1$. ⋈

(67.3) By (67.2), we know that:

(a) $F^1 = G^1 \rightarrow G^1 = F^1$

Now let φ be $G^1 = H^1$ and let φ' be $F^1 = H^1$. Then, instantiating appropriately into a universal generalization of (25), we also know:

(b) $G^1 = F^1 \rightarrow (G^1 = H^1 \rightarrow F^1 = H^1)$

From (a) and (b), it now follows by hypothetical syllogism (56.1) that:

$$F^1 = G^1 \rightarrow (G^1 = H^1 \rightarrow F^1 = H^1)$$

So by Importation (63.8.b), it follows that:

$$F^1 = G^1 \;\&\; G^1 = H^1 \rightarrow F^1 = H^1$$ ⋈

(67.4) Let F^n be an arbitrary n-place relation, $n \geq 2$. So, as instances of (67.1), we have the following property identities of the form $G^1 = G^1$:

- $[\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, F^n y x_1 \ldots x_{n-1}]$
- $[\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}]$

  $\vdots$

- $[\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, F^n x_1 \ldots x_{n-1} y]$

Hence, by &I (64.1), we may conjoin all of these to form the claim:

$$\begin{aligned}&[\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, F^n y x_1 \ldots x_{n-1}] \;\& \\ &\quad [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] \;\& \ldots \& \\ &\quad [\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, F^n x_1 \ldots x_{n-1} y]\end{aligned}$$

Since this is a theorem, we may apply GEN (49) $n-1$ times, universally generalizing on each of the variables $x_1, \ldots, x_{n-1}$, to obtain:

$$\begin{aligned}&\forall x_1 \ldots \forall x_{n-1}([\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, F^n y x_1 \ldots x_{n-1}] \;\& \\ &\quad [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] \;\& \ldots \& \\ &\quad [\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, F^n x_1 \ldots x_{n-1} y])\end{aligned}$$

Hence, by the definition of relation identity (16.2), it follows that $F^n = F^n$. ⋈

(67.5) From (67.4) using reasoning analogous to (67.2). ⋈

(67.6) From (67.5) using reasoning analogous to (67.3). ⋈

(67.7) As an instance of (67.1), we have $[\lambda y\, p] = [\lambda y\, p]$. So $p = p$, by (16.3). ⋈

(67.8) From (67.7), using reasoning analogous to (67.2). ⋈

(67.9) From (67.8), using reasoning analogous to (67.3). ⋈

(68) Let (a) τ be any complex n-place relation term ($n \geq 0$), (b) τ' be an alphabetic variant of τ, (c) α be the variable F with same arity as τ and τ', (c) τ and τ' both be substitutable for F in φ, and (d) φ' be the result of replacing zero or more occurrences of τ in φ^{τ}_{F} with occurrences of τ'. Note independently that the n-place relation case of the axiom for the substitution of identicals (25) has the following universal generalization with respect to φ, where the superscripts on F^n and G^n are suppressed:

(ϑ) $\forall F \forall G(F = G \rightarrow (\varphi \rightarrow \varphi''))$, where φ'' is the result of replacing zero or more occurrences of F in φ with occurrences of G.

Since τ and τ' are both n-place relation terms, we know that for some variable β that doesn't occur free in τ, the following are instances of axiom (29.2):

$$\exists \beta(\beta = \tau)$$
$$\exists \beta(\beta = \tau')$$

So by an appropriate instance of axiom (29.1), we may first instantiate τ for $\forall F$ in (ϑ) and then instantiate τ' for $\forall G$ in the result, to obtain:

$$\tau = \tau' \rightarrow (\varphi^{\tau}_{F} \rightarrow \varphi'),$$

where φ' is defined by hypothesis (d), i.e., as the result of replacing zero or more occurrences of τ in φ^{τ}_{F} with occurrences of τ'. Now we know independently that the following is an instance of α-Conversion (36.1), given that τ and τ' are alphabetically-variant n-place relation terms:

$$\tau = \tau'$$

It follows from our last two displayed results by MP that:

$$\varphi^{\tau}_{F} \rightarrow \varphi'$$

Hence, by (46.10), we have established:

(ξ) $\varphi^{\tau}_{F} \vdash \varphi'$

Now assume that $\Gamma \vdash \varphi^{\tau}_{F}$. Then from this and ($\xi$) it follows by (46.8) that $\Gamma \vdash \varphi'$. ⋈

(69.1) Since the universal generalizations of β-Conversion (36.2) are axioms, the following is an axiom:

$$\forall x_1 \forall x_2([\lambda y_1 y_2\, O!y_1 \,\&\, O!y_2 \,\&\, \Box\forall F(Fy_1 \equiv Fy_1)]x_1 x_2 \equiv (O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F(Fx_1 \equiv Fx_2)))$$

Since $[\lambda xy\, O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]$ is an alphabetic variant of $[\lambda y_1 y_2\, O!y_1 \,\&\, O!y_2 \,\&\, \Box\forall F(Fy_1 \equiv Fy_1)]$, it follows by rule (68) that:

(ϑ) $\forall x_1 \forall x_2([\lambda xy\, O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]x_1 x_2 \equiv (O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F(Fx_1 \equiv Fx_2)))$

By definition (12), (ϑ) becomes:

$$\forall x_1 \forall x_2 (=_E x_1 x_2 \equiv (O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F(Fx_1 \equiv Fx_2)))$$

and by definition (13), this last formula becomes:

(ξ) $\forall x_1 \forall x_2 (x_1 =_E x_2 \equiv (O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F(Fx_1 \equiv Fx_2)))$

Now since $\exists z(z = x)$ and $\exists z(z = y)$ are *bona fide* instances of axiom (29.2), an appropriate instance of (29.1) allows us to instantiate x for $\forall x_1$ in (ϑ) and then instantiate y for $\forall x_2$ in the result, to conclude:

$$x =_E y \equiv (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \qquad \bowtie$$

(69.2) Assume $x =_E y$, for conditional proof. Then it follows by $\vee$Introduction (64.3.a) that:

$$x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

But this is, by definition (15), $x = y$. $\bowtie$

(69.3) By $\equiv$I (64.5), it suffices to show both directions of the biconditional. ($\rightarrow$) Assume $x = y$, for conditional proof. Then, by definition (15), this is equivalent to:

(a) $x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$

But by theorem (69.1), we also know the following about the left disjunct of (a):

(b) $x =_E y \equiv (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$

So by appealing to a disjunctive syllogism of the form (64.4.e), it follows from (a), (b) and the tautology $\psi \equiv \psi$, where ψ is the right disjunct of (a), that:

$$(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

($\leftarrow$) Reverse the reasoning, but first apply the tautology (63.3.g), i.e., $(\varphi \equiv \psi) \equiv (\psi \equiv \varphi)$ to (69.1) to obtain the commuted form of (b). $\bowtie$

(70.1) By theorem (69.3) and biconditional syllogism (64.6.b), it suffices to show:

(ϑ) $(O!x \,\&\, O!x \,\&\, \Box\forall F(Fx \equiv Fx)) \vee (A!x \,\&\, A!x \,\&\, \Box\forall F(xF \equiv xF))$

Now to establish (ϑ), we shall be reasoning by a disjunctive syllogism (64.4.d), as follows:

From:

(a) $\Diamond E!x \vee \neg\Diamond E!x$

(b) $\Diamond E!x \rightarrow (O!x \,\&\, O!x \,\&\, \Box\forall F(Fx \equiv Fx))$

(c) $\neg\Diamond E!x \rightarrow (A!x \,\&\, A!x \,\&\, \Box\forall F(xF \equiv xF))$

we may conclude (ϑ).

So if we can establish (a), (b), and (c), we're done. Now (a) is trivial, as it is an instance of Excluded Middle (63.2). We now show (b) and (c) in turn:

(b) Assume $\Diamond E!x$, for CP. Note that $[\lambda y\, \Diamond E!y]x \equiv \Diamond E!x$ is an instance of β-Conversion (36.2). So, by a biconditional syllogism $\equiv$E (64.6.b) from this instance and our assumption, it follows that $[\lambda y\, \Diamond E!y]x$. By our Rule of Substitution of Alphabetically-Variant Relation Terms (68), it follows that $[\lambda x\, \Diamond E!x]x$. Hence, by definition of $O!$ (11.1), it follows that $O!x$. By the idempotency of & (63.3.a), it follows that $O!x \,\&\, O!x$. Note, independently of our conditional proof, that as an instance of (63.4.a), we have $Fx \equiv Fx$. Since this is a theorem, we may apply GEN (49) to obtain $\forall x(Fx \equiv Fx)$. Since this is a $\Box$-theorem, it follows by RN (51) that $\Box\forall x(Fx \equiv Fx)$. Returning to our conditional proof, we may conjoin this last result by &I (64.1) with what we have established so far, to conclude $O!x \,\&\, O!x \,\&\, \Box\forall x(Fx \equiv Fx)$. Hence, by conditional proof, $\Diamond E!x \rightarrow (O!x \,\&\, O!x \,\&\, \Box\forall x(Fx \equiv Fx))$.

(c) Assume $\neg\Diamond E!x$, for CP. Note that $[\lambda y\, \neg\Diamond E!y]x \equiv \neg\Diamond E!x$ is an instance of β-Conversion (36.2). So, by a biconditional syllogism $\equiv$E (64.6.b) from this instance and our assumption, it follows that $[\lambda y\, \neg\Diamond E!y]x$. By our Rule of Substitution of Alphabetically-Variant Relation Terms (68), it follows that $[\lambda x\, \neg\Diamond E!x]x$. Hence, by definition of $A!$ (11.2), it follows that $A!x$. By the idempotency of & (63.3.a), it follows that $A!x \,\&\, A!x$. Note, independently of our conditional proof, that as an instance of (63.4.a), we have $xF \equiv xF$. Since this is a theorem, we may apply GEN to obtain $\forall x(xF \equiv xF)$. Since this is a $\Box$-theorem, it follows by RN that $\Box\forall x(xF \equiv xF)$. Returning to our conditional proof, we may conjoin this last result by &I (64.1) with what we have established so far, to conclude $A!x \,\&\, A!x \,\&\, \Box\forall x(xF \equiv xF)$. By conditional proof, $\neg\Diamond E!x \rightarrow (A!x \,\&\, A!x \,\&\, \Box\forall x(xF \equiv xF))$. $\bowtie$

(70.2) From (70.1), using reasoning analogous to (67.2). $\bowtie$

(70.3) From (70.2), using reasoning analogous to (67.3). $\bowtie$

(71.1) By cases (67.1), (67.4), (67.7) and (70.1). $\bowtie$

(71.2) By cases (67.2), (67.5), (67.8) and (70.2). $\bowtie$

(71.3) By cases (67.3), (67.6), (67.9) and (70.3). ⋈

(72.1) Apply GEN to theorem (71.1). Since the result is a $\Box$-theorem, apply RN. ⋈

(72.2) Since (71.1) is a $\Box$-theorem, we may apply RN to obtain $\Box(\alpha = \alpha)$ as a theorem. Hence, by GEN, we have: $\forall\alpha\Box(\alpha = \alpha)$. ⋈

(73.1) Proof by cases, where the cases are: (A) τ is any term other than a description, and (B) τ is a definite description.

Case A. If τ is any term other than a description, then by axiom (29.2), we know $\exists\beta(\beta = \tau)$, where β is a variable of the same type as τ that doesn't occur free in τ. Hence, by axiom (21.1), it follows that $\tau = \tau' \to \exists\beta(\beta = \tau)$.

Case B. Suppose τ is the description $\imath x\chi$, and for conditional proof, assume $\tau = \tau'$, to show $\exists\nu(\nu = \tau)$, where ν is an individual variable not free in τ. Then by definition (15), it follows that:

$$\tau =_E \tau' \vee (A!\tau \,\&\, A!\tau' \,\&\, \Box\forall F(\tau F \equiv \tau' F))$$

Our proof strategy is to reason by cases (64.4.a) to establish $\exists\nu(\nu = \tau)$. If we let φ be $\tau =_E \tau'$ and let ψ be $A!\tau \,\&\, A!\tau' \,\&\, \Box\forall F(\tau F \equiv \tau' F))$, then we can reach the conclusion (d) by the reasoning:

(a) $\varphi \vee \psi$
(b) $\varphi \to \exists\nu(\nu = \tau)$
(c) $\psi \to \exists\nu(\nu = \tau)$
(d) $\exists\nu(\nu = \tau)$

We have already established (a).

To show (b), assume φ, i.e., $\tau =_E \tau'$. Then by definition (13), it follows that $=_E \tau\tau'$, which is an exemplification formula in which τ occurs as one of the individual terms. Now consider the formula $=_E z\tau'$, where z doesn't appear free in τ'. Call this formula ψ, so that $=_E \tau\tau'$ is $\psi_z^{\imath x\chi}$. Then where ν is an individual variable that doesn't occur free in τ, the conditions of (29.5) are met and we know $\psi_z^{\imath x\chi} \to \exists\nu(\nu = \tau)$ is an axiom. Since we've established the antecedent, it follows that $\exists\nu(\nu = \tau)$.

To show (c), assume ψ, i.e., $A!\tau \,\&\, A!\tau' \,\&\, \Box\forall F(\tau F \equiv \tau' F)$. Then by an application of &E (64.2.a), it follows that:

$$A!\tau$$

Since this is exemplification formula in which τ is a description appearing as an argument, the conditions of (29.5) are met and so the following is an instance of that axiom:

$A!\tau \to \exists\nu(\nu = \tau)$, where ν doesn't occur free in τ

Hence, from our last two displayed results, we have, by MP, $\exists\nu(\nu = \tau)$.

Hence by (a), (b), and (c), it follows that (d). ⋈

(73.2) By reasoning analogous to (73.1). ⋈

(74.1) Since (71.1) is a theorem, we know:

$$\vdash \alpha = \alpha$$

Hence by GEN, we know:

(a) $\vdash \forall\alpha(\alpha = \alpha)$

Since we know that the instances of (29.1) are axioms, we also know, where β is some variable that doesn't occur free in τ:

(b) $\vdash \forall\alpha(\alpha = \alpha) \to (\exists\beta(\beta = \tau) \to \tau = \tau)$

So from (a) and (b), it follows by (46.6) that:

(c) $\vdash \exists\beta(\beta = \tau) \to \tau = \tau$

By hypothesis, τ is any term other than a description and so the instances of (29.2), given our choice of β, are axioms, yielding:

(d) $\vdash \exists\beta(\beta = \tau)$

Hence, from (c) and (d) it follows by (46.6) that $\vdash \tau = \tau$. ⋈

(74.2) Assume $\Gamma_1 \vdash \varphi^{\tau}_{\alpha}$ and $\Gamma_2 \vdash \tau = \tau'$. So by (46.7), where $\Delta = \Gamma_1 \cup \Gamma_2$, we know:

(ξ) $\Gamma_1, \Gamma_2 \vdash \varphi^{\tau}_{\alpha}$

(ϑ) $\Gamma_1, \Gamma_2 \vdash \tau = \tau'$

We have to show $\Gamma_1, \Gamma_2 \vdash \varphi'$. Now by applying (46.10) to theorems (73.1) and (73.2), we obtain, for some variable γ that doesn't occur free in τ:

(a) $\tau = \tau' \vdash \exists\gamma(\gamma = \tau)$

(b) $\tau = \tau' \vdash \exists\gamma(\gamma = \tau')$

Hence by applying (46.8) first to the pair (ϑ) and (a) and then to the pair (ϑ) and (b), we obtain, respectively:

(c) $\Gamma_1, \Gamma_2 \vdash \exists\gamma(\gamma = \tau)$

(d) $\Gamma_1, \Gamma_2 \vdash \exists\gamma(\gamma = \tau')$

Since generalizations of the axiom schema (25) for the substitution of identicals are axioms, we know by (46.1) that:

$\vdash \forall\alpha\forall\beta(\alpha = \beta \rightarrow (\varphi \rightarrow \varphi'))$, whenever β is substitutable for α in φ and φ' is the result of replacing zero or more occurrences of α in φ with occurrences of β.

It follows from this last result by (46.3) that:

(e) $\Gamma_1, \Gamma_2 \vdash \forall\alpha\forall\beta(\alpha = \beta \rightarrow (\varphi \rightarrow \varphi'))$, whenever β is substitutable for α in φ and φ' is the result of replacing zero or more occurrences of α in φ with occurrences of β.

Now let ψ be $\forall\beta(\alpha = \beta \rightarrow (\varphi \rightarrow \varphi'))$ so that we may abbreviate (e) as:

(e) $\Gamma_1, \Gamma_2 \vdash \forall\alpha\psi$

Since instances of (29.1) are axioms, we also know, for some variable γ not free in τ, that $\vdash \forall\alpha\psi \rightarrow (\exists\gamma(\gamma = \tau) \rightarrow \psi^{\tau}_{\alpha})$, by (46.1). By (46.3) this yields:

(f) $\Gamma_1, \Gamma_2 \vdash \forall\alpha\psi \rightarrow (\exists\gamma(\gamma = \tau) \rightarrow \psi^{\tau}_{\alpha})$

From (e) and (f) it follows by (46.5) that:

(g) $\Gamma_1, \Gamma_2 \vdash \exists\gamma(\gamma = \tau) \rightarrow \psi^{\tau}_{\alpha}$

From (g) and (c) it follows by (46.5) that $\Gamma_1, \Gamma_2 \vdash \psi^{\tau}_{\alpha}$, i.e.,

(h) $\Gamma_1, \Gamma_2 \vdash \forall\beta(\tau = \beta \rightarrow (\varphi^{\tau}_{\alpha} \rightarrow \varphi'))$, where φ' is the result of replacing zero or more occurrences of τ in φ^{τ}_{α} by occurrences of β

which we may abbreviate as:

(h) $\Gamma_1, \Gamma_2 \vdash \forall\beta\chi$

Note separately that since we know by (29.1) that $\vdash \forall\beta\chi \rightarrow (\exists\gamma(\gamma = \tau') \rightarrow \chi^{\tau'}_{\beta})$, we have by now familiar reasoning:

(i) $\Gamma_1, \Gamma_2 \vdash \forall\beta\chi \rightarrow (\exists\gamma(\gamma = \tau') \rightarrow \chi^{\tau'}_{\beta})$

From (i) and (h) it follows by familiar reasoning:

(j) $\Gamma_1, \Gamma_2 \vdash \exists\gamma(\gamma = \tau') \rightarrow \chi^{\tau'}_{\beta}$

From (j) and (d) it follows that $\Gamma_1, \Gamma_2 \vdash \chi^{\tau'}_{\alpha}$, i.e.,

(k) $\Gamma_1, \Gamma_2 \vdash \tau = \tau' \rightarrow (\varphi^{\tau}_{\alpha} \rightarrow \varphi')$, where φ' is the result of replacing zero or more occurrences of τ in φ^{τ}_{α} by occurrences of τ'

Hence, from (ϑ) and (k), it follows by familiar reasoning that $\Gamma_1, \Gamma_2 \vdash \varphi^{\tau}_{\alpha} \rightarrow \varphi'$. From this last result and (ξ), it follows that $\Gamma_1, \Gamma_2 \vdash \varphi'$, which is what we had to show. $\bowtie$

(74.3) (Exercise)

(75) No matter whether α, β are both individual variables or both relation variables of the same arity, we may reason as follows. By $\equiv$I (64.5), it suffices to show both directions of the biconditional. ($\rightarrow$) Assume $\alpha = \beta$, for conditional proof. Note that we have established $\alpha = \alpha$ (71.1) as a $\Box$-theorem. So we may apply RN to this theorem to conclude $\Box\alpha = \alpha$. Then by Rule SubId (74.2), it follows that $\Box\alpha = \beta$. ($\leftarrow$) This is an instance of the T schema (32.2). ⋈

(76) ($\rightarrow$) By taking the individual variables z, w as instances of the previous theorem and applying GEN twice, we know:

(ϑ) $\forall z \forall w(z = w \equiv \Box z = w)$

Now assume $\iota x\varphi = \iota y\psi$, for conditional proof. Then by (73.1) and (73.2), it follows both that:

(a) $\exists z(z = \iota x\varphi)$

(b) $\exists z(z = \iota y\psi)$

provided z doesn't occur free in φ or ψ. Note however that the following is an instance of (29.1):

$$(\vartheta) \rightarrow (\exists z(z = \iota x\varphi) \rightarrow \forall w(\iota x\varphi = w \rightarrow \Box \iota x\varphi = w))$$

From this result, (ϑ) and (a), it follows by two applications of MP that:

(ξ) $\forall w(\iota x\varphi = w \rightarrow \Box \iota x\varphi = w)$

But now we have the following instance of (29.1):

$$(\xi) \rightarrow (\exists z(z = \iota y\psi) \rightarrow (\iota x\varphi = \iota y\psi \rightarrow \Box \iota x\varphi = \iota y\psi))$$

From this result, (ξ) and (b), it follows by two applications of MP that:

$$\iota x\varphi = \iota y\psi \rightarrow \Box \iota x\varphi = \iota y\psi$$

($\leftarrow$) Assume $\Box(\iota x\varphi = \iota y\psi)$. Then by the T schema, $\iota x\varphi = \iota y\psi$. ⋈

(77.1) Assume $\Gamma_1 \vdash \forall\alpha\varphi$ and $\Gamma_2 \vdash \exists\beta(\beta = \tau)$. Assume further that τ is substitutable for α in φ. By (46.7), it follows both that:

(a) $\Gamma_1, \Gamma_2 \vdash \forall\alpha\varphi$

(b) $\Gamma_1, \Gamma_2 \vdash \exists\beta(\beta = \tau)$

Since we've taken instances of (21.1) as axioms, it follows by (46.1) that:

$$\vdash \forall\alpha\varphi \rightarrow (\exists\beta(\beta = \tau) \rightarrow \varphi^{\tau}_{\alpha})$$

Hence by (46.3), we know:

$$\Gamma_1, \Gamma_2 \vdash \forall\alpha\varphi \to (\exists\beta(\beta = \tau) \to \varphi^{\tau}_{\alpha})$$

From this, (a) and (46.5), it follows that:

$$\Gamma_1, \Gamma_2 \vdash \exists\beta(\beta = \tau) \to \varphi^{\tau}_{\alpha}$$

But from this, (b) and (46.5), it follows that $\Gamma_1, \Gamma_2 \vdash \varphi^{\tau}_{\alpha}$. ⋈

(77.1) [Proof of the Variant form of the rule.] Assume $\forall\alpha\varphi$ and $\exists\beta(\beta = \tau)$, where τ is substitutable for α in φ. Then by (64.1), it follows that:

$$\forall\alpha\varphi \,\&\, \exists\beta(\beta = \tau)$$

However, by applying an appropriate instance of Exportation (63.8.b) to our first quantifier axiom (29.1), it follows that:

$$(\forall\alpha\varphi \,\&\, \exists\beta(\beta = \tau)) \to \varphi^{\tau}_{\alpha}$$

So by MP, φ^{τ}_{α}. Thus, we've established $\forall\alpha\varphi, \exists\beta(\beta = \tau) \vdash \varphi^{\tau}_{\alpha}$. ⋈

(77.2) Assume that $\Gamma \vdash \forall\alpha\varphi$, that τ is substitutable for α in φ, and that τ is not a description. Since instances of (21.2) are axioms, we know by (46.1) that $\vdash \exists\beta(\beta = \tau)$, and hence, $\Gamma \vdash \exists\beta(\beta = \tau)$. So by (77.1) (with $\Gamma = \Gamma_1 = \Gamma_2$), it follows that $\Gamma \vdash \varphi^{\tau}_{\alpha}$. ⋈

(77.2) [Proof of the Variant form of the rule.] Assume $\forall\alpha\varphi$, where τ is substitutable for α in φ and τ is not a description. Then we have, as an instance of the second axiom of quantification theory (29.2), that $\exists\beta(\beta = \tau)$, for some variable β not free in τ. Once we conjoin $\forall\alpha\varphi$ and $\exists\beta(\beta = \tau)$ by &I, it follows by reasoning used in the proof of the preceding theorem that φ^{τ}_{α}. Hence we've established that $\forall\alpha\varphi \vdash \varphi^{\tau}_{\alpha}$ ⋈

(79.1) Assume $\forall\alpha\varphi$ and that τ is substitutable for α in φ and is not a description. Then by Rule ∀E (77.2), it follows that φ^{τ}_{α}. So by conditional proof (CP), $\forall\alpha\varphi \to \varphi^{\tau}_{\alpha}$. ⋈

(79.2) Assume $\forall\alpha(\varphi \to \psi)$, where α is not free in φ. Then by the special case of the Variant of Rule ∀E (77.2), it follows that $\varphi \to \psi$. Now assume φ for a nested conditional proof. Then, ψ, by MP. Since we've derived ψ from two premises in which α isn't free, we may apply GEN to ψ to conclude $\forall\alpha\psi$. Hence we may discharge the premise of our nested conditional proof to conclude $\varphi \to \forall\alpha\psi$, and then discharge the premise of our initial conditional proof to conclude that: $\forall\alpha(\varphi \to \psi) \to (\varphi \to \forall\alpha\psi)$. ⋈

(79.3) (Exercise)

(80) Follow the proof in Enderton 1972, Theorem 24F (p. 116). (In the second edition, Enderton 2001, the proof of Theorem 24F occurs on pp. 123–124.) All of the results needed for the proof are in place. [Warning: Note that in the proof of this theorem, Enderton uses φ^c_y to be the result of replacing every occurrence of the constant c in φ with an occurrence of the variable y. By contrast, we use φ^α_τ to be the result of replacing every occurrence of the constant τ in φ with an occurrence of the variable α.] ⋈

(81.1) We need not prove this lemma by induction; the recursive definitions of φ^τ_α and *substitutable for* carry the following reasoning through all the inductive cases. Assume β is substitutable for α in φ and β doesn't occur free in φ. There are two cases. In the case where φ has no free occurrences of α, then by the definition of *substitutable for*, β is trivially substitutable for α in φ. In that case, however, φ^β_α just is φ and so $(\varphi^\beta_\alpha)^\alpha_\beta = \varphi^\alpha_\beta$. Moreover by hypothesis, β doesn't occur free in φ. So by an analogous fact, $\varphi^\alpha_\beta = \varphi$. Hence $(\varphi^\beta_\alpha)^\alpha_\beta = \varphi$.

In the case where α has at least one free occurrence in φ, then without loss of generality, consider any free occurrence of α in φ. Since β is substitutable for α in φ (by hypothesis), we know that β will not be bound when substituted for α at this occurrence. Thus, β will be free at this occurrence in φ^β_α. And since φ has no free occurrences of β (by hypothesis), we know that every free occurrence of β in φ^β_α replaced a free occurrence of α in φ. Thus no free occurrence of β in φ^β_α falls under the scope of a variable binding operator that binds α. Hence, α is substitutable for β in φ^β_α.

Now we must show that $(\varphi^\beta_\alpha)^\alpha_\beta = \varphi$. Suppose, for reductio, that $(\varphi^\beta_\alpha)^\alpha_\beta \neq \varphi$. Since substitution only changes the substituted variables, then there must be some occurrence of α in $(\varphi^\beta_\alpha)^\alpha_\beta$ that is not in φ or some occurrence of α in φ that is not in $(\varphi^\beta_\alpha)^\alpha_\beta$. But both cases lead to contradiction:

> Case 1. Suppose there is an occurrence of α in $(\varphi^\beta_\alpha)^\alpha_\beta$ that is not in φ. Then there was an occurrence of β in φ that remained an occurrence of β in φ^β_α but was replaced by an occurrence of α in $(\varphi^\beta_\alpha)^\alpha_\beta$. But if an occurrence of β in φ^β_α was replaced by an occurrence of α in $(\varphi^\beta_\alpha)^\alpha_\beta$, then that occurrence of β in φ^β_α had to be a free occurrence. But that occurrence of β must have been a free occurrence in φ (had it been a bound occurrence, it would have remained a bound occurrence in φ^β_α). And this contradicts the hypothesis that β doesn't occur free in φ.
>
> Case 2. Suppose there is an occurrence of α in φ that is not in $(\varphi^\beta_\alpha)^\alpha_\beta$. Then there was an occurrence of α in φ that was replaced by an occurrence of β in φ^β_α which, in turn, remained an occurrence of β in $(\varphi^\beta_\alpha)^\alpha_\beta$. But if that occurrence of β remained an occurrence of β in the re-replacement, then it must be bound by a variable-binding operator binding β in φ^β_α. But this contradicts the hypothesis that β is substitutable for α in φ, which requires that β must remain free at every occurrence of α in φ that it replaces in φ^β_α. ⋈

(81.2) Assume τ is a constant symbol that doesn't occur in φ. If α has no free occurrences in φ, then τ is trivially substitutable for α in φ and $\varphi^\tau_\alpha = \varphi$. In that case, both $(\varphi^\tau_\alpha)^\beta_\tau = \varphi$, and $\varphi^\beta_\alpha = \varphi$. Hence $(\varphi^\tau_\alpha)^\beta_\tau = \varphi^\beta_\alpha$. So we consider only the case where α has at least one free occurrence in φ.

Suppose, for reductio, that $(\varphi^\tau_\alpha)^\beta_\tau \neq \varphi^\beta_\alpha$. Then there must be some occurrence of β in $(\varphi^\tau_\alpha)^\beta_\tau$ that is not in φ^β_α or some occurrence of β in φ^β_α that is not in $(\varphi^\tau_\alpha)^\beta_\tau$. But both cases lead to contradiction:

> Case 1. Suppose there is an occurrence of β in $(\varphi^\tau_\alpha)^\beta_\tau$ that is not in φ^β_α. Since by hypothesis, there are no occurrences of τ in φ, then there must be an occurrence of α in φ that remained an occurrence of α in φ^τ_α but became an occurrence of β in $(\varphi^\tau_\alpha)^\beta_\tau$. But if an occurrence of α in φ^τ_α became an occurrence of β in $(\varphi^\tau_\alpha)^\beta_\tau$, then that occurrence of α in φ^τ_α had to be a free occurrence. But that contradicts the fact that φ^τ_α is, by definition, the result of replacing every free occurrence of α by τ in φ.
>
> Case 2. Suppose there is an occurrence of β in φ^β_α that is not in $(\varphi^\tau_\alpha)^\beta_\tau$. Then there must be an occurrence of α in φ that became an occurrence of τ in φ^τ_α which remained an occurrence of τ in $(\varphi^\tau_\alpha)^\beta_\tau$. But that contradicts the definition of ψ^β_τ, which signifies the result of replacing every occurrence of τ in ψ by an occurrence of β. ⋈

(81.3) (Exercise)

(83.1) ($\rightarrow$) Assume $\forall\alpha\forall\beta\varphi$, for conditional proof, to show $\forall\beta\forall\alpha\varphi$. Then by Rule $\forall$E (77.2), it follows that $\forall\beta\varphi$. By a second application of this same rule it follows that φ. Since α is not free in our hypothesis $\forall\alpha\forall\beta\varphi$, we may infer $\forall\alpha\varphi$ from φ, by GEN. Since β is not free in our hypothesis, we may infer $\forall\beta\forall\alpha\varphi$ from $\forall\alpha\varphi$, by GEN. Hence, by conditional proof, $\forall\alpha\forall\beta\varphi \rightarrow \forall\beta\forall\alpha\varphi$. ($\leftarrow$) By symmetrical reasoning. ⋈

(83.2) (Exercise)

(83.3) Assume $\forall\alpha(\varphi \equiv \psi)$ and apply Rule $\forall$E (77.2) to obtain $\varphi \equiv \psi$. By $\equiv$I (64.5), it suffices to establish both directions of $\forall\alpha\varphi \equiv \forall\alpha\psi$. ($\rightarrow$) Assume $\forall\alpha\varphi$. By applying Rule $\forall$E (77.2) to our second assumption, we have φ. So by a biconditional syllogism (64.6.a), it follows that ψ. Since α is not free in either of our premises, it follows that $\forall\alpha\psi$, by GEN. Discharging our second assumption, we've established $\forall\alpha\varphi \rightarrow \forall\alpha\psi$. ($\leftarrow$) Assume $\forall\alpha\psi$. The conclusion is then reached by analogous reasoning, but by biconditional syllogism (64.6.b). ⋈

(83.4) ($\rightarrow$) Assume, for conditional proof, that $\forall\alpha(\varphi \mathbin{\&} \psi)$. Then by Rule $\forall$E (77.2), we have $\varphi \mathbin{\&} \psi$. From this we have both φ and ψ, by (64.2.a) and (64.2.b), respectively. Since α isn't free in our assumption, we may apply GEN to both conclusions to obtain $\forall\alpha\varphi$ and $\forall\alpha\psi$. Hence by (64.1), it follows that

$\forall\alpha\varphi \,\&\, \forall\alpha\psi$. [We here omit the last step of assembling the conditional to be proved, since it is now obvious.] ($\leftarrow$) Assume $\forall\alpha\varphi \,\&\, \forall\alpha\psi$, for conditional proof. It follows by (64.2.a) and (64.2.b) that $\forall\alpha\varphi$ and $\forall\alpha\psi$. Hence, by applying Rule $\forall$E (77.2) to both, we obtain both φ and ψ. So by &I, we have $\varphi \,\&\psi$. Since α isn't free in our assumption, we may apply GEN to obtain $\forall\alpha(\varphi \,\&\, \psi)$. ⋈

(83.5) Assume, for conditional proof, that $\forall\alpha_1 \ldots \forall\alpha_n\varphi$. By the special case of Rule $\forall$E (77.2), it follows that $\forall\alpha_2 \ldots \forall\alpha_n\varphi$. By analogous reasoning, we can strip off the quantifier $\forall\alpha_2$. Once we have legitimately stripped off the outermost quantifier in this way a total of n times, it follows that φ. ⋈

(83.6) ($\rightarrow$) This direction is an instance of theorem (79.3). ($\leftarrow$) Assume $\forall\alpha\varphi$. Then since α isn't free in our assumption, we may apply GEN to obtain $\forall\alpha\forall\alpha\varphi$. So by conditional proof, $\forall\alpha\varphi \rightarrow \forall\alpha\forall\alpha\varphi$. ⋈

(83.7) By hypothesis, α isn't free in φ. By $\equiv$I (64.5), it suffices to prove both directions of the biconditional. ($\rightarrow$) Assume $\varphi \rightarrow \forall\alpha\psi$. Now for a secondary conditional proof, assume φ. Then by MP, it follows that $\forall\alpha\psi$ and by Rule $\forall$E (77.2), it follows that ψ. Discharging the premise of our secondary conditional proof, it follows that $\varphi \rightarrow \psi$. Since α isn't free in φ, it isn't free in our remaining (original) assumption. So the conditions of GEN are met and we may conclude that $\forall\alpha(\varphi \rightarrow \psi)$. ($\leftarrow$) By (79.2). ⋈

(83.8) (Exercise)

(83.9) (Exercise)

(83.10) Assume $\forall\alpha(\varphi \equiv \psi) \,\&\, \forall\alpha(\psi \equiv \chi)$. By &E, this yields $\forall\alpha(\varphi \equiv \psi)$ and $\forall\alpha(\psi \equiv \chi)$. By Rule $\forall$E (79.2), it follows, respectively, that $\varphi \equiv \psi$ and $\psi \equiv \chi$. By a biconditional syllogism (64.6.e), it follows that $\varphi \equiv \chi$. Since α isn't free in our assumption, we may apply GEN to conclude $\forall\alpha(\varphi \equiv \chi)$. ⋈

(83.11) (Exercise)

(83.12) Suppose β is substitutable for α in φ and doesn't occur free in φ. Then there are two cases. *Case 1*. β just is the variable α. Then our theorem becomes: $\forall\alpha\varphi \equiv \forall\alpha\varphi^{\alpha}_{\alpha}$. But $\varphi^{\alpha}_{\alpha}$ just is φ by definition. So our theorem becomes $\forall\alpha\varphi \equiv \forall\alpha\varphi$, which is an instance of a tautology. *Case 2*. β is distinct from α. ($\rightarrow$) Assume $\forall\alpha\varphi$. Since β is substitutable for α in φ, it follows by Rule $\forall$E (77.2) that φ^{β}_{α}. Furthermore, since β doesn't occur free in φ, β doesn't occur free in our assumption $\forall\alpha\varphi$. So we may apply GEN to obtain $\forall\beta\varphi^{\beta}_{\alpha}$. ($\leftarrow$) Assume $\forall\beta\varphi^{\beta}_{\alpha}$. Since β is substitutable for α in φ and doesn't occur free in φ, it follows, by the re-replacement lemma (81.1), both (a) that α is substitutable for β in φ^{β}_{α} and (b) that $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta} = \varphi$. From (a) and the fact that α is not a description, it follows from our assumption that $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta}$, by Rule $\forall$E (77.2). But by (b), this is just φ, and since α isn't free in our assumption, it follows by GEN that $\forall\alpha\varphi$. ⋈

(84.1) (Exercise)

(84.2) (Exercise)

(85) Follow the proof in Enderton 1972, Corollary 24H (p. 117). (In the second edition, Enderton 2001, the proof of Corollary 24H occurs on p. 124.) The warning in the proof of (80), about how our notation differs from Enderton's, still apply. ⋈

(86.1) Assume $\forall\alpha\varphi$. Then by Rule $\forall$E (77.2), it follows that φ. Hence, by the rule $\exists$I (84.2), it follows that $\exists\alpha\varphi$. ⋈

(86.2) By $\equiv$I (64.5), it suffices to prove both directions of the biconditional. ($\rightarrow$) Assume $\neg\forall\alpha\varphi$. We want to show $\exists\alpha\neg\varphi$. So by the definition of $\exists$, we have to show: $\neg\forall\alpha\neg\neg\varphi$. For reductio, assume $\forall\alpha\neg\neg\varphi$. Then it follows that $\neg\neg\varphi$, by Rule $\forall$E (77.2). Hence, by double negation elimination (64.8), we may infer φ. Since α isn't free in any of our assumptions, it follows by GEN that $\forall\alpha\varphi$. Since we've reached a contradiction, we may discharge our reductio assumption and conclude, by a version of RAA (62.2), that $\neg\forall\alpha\neg\neg\varphi$, i.e., by definition, $\exists\alpha\neg\varphi$. ($\leftarrow$) Assume $\exists\alpha\neg\varphi$, for conditional proof. Assume that τ is an arbitrary such object, i.e., that $\neg\varphi^{\tau}_{\alpha}$. Assume, for reductio, that $\forall\alpha\varphi$. Then by Rule $\forall$E (77.2), we have φ^{τ}_{α}. Since we've reached a contradiction, we can discharge our reductio assumption by a version of RAA (62.2) and conclude $\neg\forall\alpha\varphi$. Since τ doesn't appear in φ or $\neg\forall\alpha\varphi$, we may apply Rule $\exists$E (85) to discharge the assumption that $\neg\varphi^{\tau}_{\alpha}$ and conclude $\neg\forall\alpha\varphi$. ⋈

(86.3) By $\equiv$I (64.5), it suffices to prove both directions of the biconditional. [Henceforth, we omit mention of this proof strategy for biconditionals.] ($\rightarrow$) Assume $\forall\alpha\varphi$. We want to show $\neg\exists\alpha\neg\varphi$. For reductio, assume $\exists\alpha\neg\varphi$. From this and (86.2), it follows that $\neg\forall\alpha\varphi$, by a biconditional syllogism (64.6.b). Since we've reached a contradiction, we may discharge our reductio assumption and conclude by a version of RAA (62.1) that $\neg\exists\alpha\neg\varphi$. ($\leftarrow$) Assume $\neg\exists\alpha\neg\varphi$, for conditional proof. Assume, for reductio, that $\neg\forall\alpha\varphi$. From this and by (86.2), it follows that $\exists\alpha\neg\varphi$, by a biconditional syllogism (64.6.a). Since we've reached a contradiction, we can discharge our reductio assumption using a version of RAA (62.2) and conclude $\forall\alpha\varphi$. ⋈

(86.4) (Exercise)

(86.5) (Exercise)

(86.6) (Exercise)

(86.7) (Exercise)

(86.8) Suppose β is substitutable for α in φ and doesn't occur free in φ. Then there are two cases: *Case 1*. β just is α. Then our theorem states $\varphi \equiv \exists\alpha(\alpha =$

α & $\varphi^{\alpha}_{\alpha}$). By hypothesis, β, i.e., α, doesn't occur free in φ and so $\varphi^{\alpha}_{\alpha} = \varphi$. So our theorem states $\varphi \equiv \exists\alpha(\alpha{=}\alpha \ \& \ \varphi)$. We leave the remainder of the proof as an exercise. *Case 2.* β and α are distinct variables of the same type. ($\rightarrow$) Assume φ, for conditional proof. Then, by definition of substitutions, we know $\varphi^{\alpha}_{\alpha}$. Since it is a theorem (71.1) that $\alpha = \alpha$, we have, by &I, that $\alpha = \alpha \ \& \ \varphi^{\alpha}_{\alpha}$. Hence, by $\exists$I, it follows that $\exists\beta(\beta = \alpha \ \& \ \varphi^{\beta}_{\alpha})$. ($\leftarrow$) Assume, for conditional proof:

(ϑ) $\exists\beta(\beta = \alpha \ \& \ \varphi^{\beta}_{\alpha})$,

Assume τ is an arbitrary such entity; formally, the metavariable τ denotes a new constant that doesn't appear in (ϑ), that has the same type as the variable β, and that is therefore substitutable for β in the matrix of (ζ). So we know:

(ζ) $\tau = \alpha \ \& \ (\varphi^{\beta}_{\alpha})^{\tau}_{\beta}$

We may detach the two conjuncts of (ζ) by &E. Since β is substitutable for α in φ and doesn't occur free in φ, and τ is substitutable for α in φ, the Re-replacement theorem (81.3) tells us that $(\varphi^{\beta}_{\alpha})^{\tau}_{\beta} = \varphi^{\tau}_{\alpha}$. So it follows from the right conjunct of (ζ) that φ^{τ}_{α}. But from this latter conclusion and the left conjunct of (ζ), it follows by Rule SubId Special Case (74.2) that we may substitute α for every occurrence of τ in φ^{τ}_{α}, to obtain $(\varphi^{\tau}_{\alpha})^{\alpha}_{\tau}$. But, by Re-replacement lemma (81.2), since τ is a constant that doesn't appear in φ, this is just $\varphi^{\alpha}_{\alpha}$, i.e., φ. By $\exists$E (85), we can discharge (ζ) and conclude φ. ⋈

(86.9) Let τ be any term other than a description and is substitutable for α in φ. Now choose any variable of the same type as α, say β, that is substitutable for α in φ and that doesn't occur free in φ. Then by applying GEN to (86.8), we know the following applies to φ:

(ϑ) $\forall\alpha(\varphi \equiv \exists\beta(\beta{=}\alpha \ \& \ \varphi^{\beta}_{\alpha}))$

Since τ is substitutable for α in φ, it is substitutable for α in the matrix of (ϑ). So we may instantiate τ into $\forall\alpha$ in (ϑ) by $\forall$E, to obtain:

$$\varphi^{\tau}_{\alpha} \equiv \exists\beta(\beta{=}\tau \ \& \ \varphi^{\beta}_{\alpha}))$$

But given our choice of β, we know that by commuting an appropriate instance of (86.7), the following applies to φ:

$$\exists\beta(\beta{=}\tau \ \& \ \varphi^{\beta}_{\alpha}) \equiv \exists\alpha(\alpha = \tau \ \& \ \varphi)$$

So, by a biconditional syllogism (64.6.e), it follows that:

$$\varphi^{\tau}_{\alpha} \equiv \exists\alpha(\alpha = \tau \ \& \ \varphi)$$ ⋈

(86.10) Suppose α, β are distinct variables and consider any formula φ in which β is substitutable for α and doesn't occur free. ($\rightarrow$) Assume:

(ζ) $\varphi \,\&\, \forall\beta(\varphi^{\beta}_{\alpha} \rightarrow \beta = \alpha)$

to show $\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$. By (83.2), it suffices to show $\forall\beta(\varphi^{\beta}_{\alpha} \rightarrow \beta = \alpha) \,\&\, \forall\beta(\beta = \alpha \rightarrow \varphi^{\beta}_{\alpha})$. By applying &E (64.2.b) to (ζ), we already have $\forall\beta(\varphi^{\beta}_{\alpha} \rightarrow \beta = \alpha)$. So by &I, it remains to show $\forall\beta(\beta = \alpha \rightarrow \varphi^{\beta}_{\alpha})$. By hypothesis, β doesn't occur free in φ and, hence, doesn't occur free in our assumption (ζ). So by GEN, it remains to show $\beta = \alpha \rightarrow \varphi^{\beta}_{\alpha}$. So assume $\beta = \alpha$, which by the symmetry of identity (71.2), yields $\alpha = \beta$. Now by applying &E to (ζ), we know φ. Since β is, by hypothesis, substitutable for α in φ, it follows by Rule SubId Special Case (74.2) that φ^{β}_{α}. ($\leftarrow$) Assume:

(ϑ) $\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$

for conditional proof. If we can show:

(a) φ

(b) $\forall\beta(\varphi^{\beta}_{\alpha} \rightarrow \beta = \alpha)$

then by &I we are done. (a) Since, by hypothesis, β is substitutable for α in φ and isn't free in φ, it follows by the Re-replacement lemma (81.1) that α is substitutable for β in φ^{β}_{α}. Hence α is substitutable for β in $\varphi^{\beta}_{\alpha} \equiv \beta = \alpha$. So we may, by Rule $\forall$E (77.2), instantiate $\forall\beta$ in (ϑ) to α, and thereby obtain:

$$(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)^{\alpha}_{\beta}$$

By the definition of $\equiv$ and the definition of ψ^{α}_{β} (23), this becomes:

(ξ) $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta} \equiv (\beta = \alpha)^{\alpha}_{\beta}$

Since the Re-replacement lemma is operative, the left condition of (ξ) is just φ. By definition, the right condition of (ξ) is $\alpha = \alpha$ (this is obvious, but we leave the strict proof, by way of the cases in the definition of =, as an exercise). Hence, (ξ) resolves to:

$$\varphi \equiv \alpha = \alpha$$

But since we know $\alpha = \alpha$ by (71.1), it follows that φ, by biconditional syllogism. (b) By (83.2), it follows from (ϑ) that:

$$\forall\beta(\varphi^{\beta}_{\alpha} \rightarrow \beta = \alpha) \,\&\, \forall\beta(\beta = \alpha \rightarrow \varphi^{\beta}_{\alpha})$$

So $\forall\beta(\varphi^{\beta}_{\alpha} \rightarrow \beta = \alpha)$ follows by &E. ⋈

(86.11) (Exercise)

(86.12) Assume $\neg\exists\alpha\varphi \,\&\, \neg\exists\alpha\psi$. From the first conjunct and (86.4) it follows that $\forall\alpha\neg\varphi$, and from the second conjunct and the same theorem it follows that $\forall\alpha\neg\psi$. Hence, by Rule $\forall$E, it follows, respectively, that $\neg\varphi$ and $\neg\psi$, which by

&I gives us $\neg\varphi \,\&\, \neg\psi$. By $\vee$I (64.3.b), we may infer $(\varphi \,\&\, \psi) \vee (\neg\varphi \,\&\, \neg\psi)$. So by (63.5.i), we know $\varphi \equiv \psi$. Since α isn't free in our assumption, it follows by GEN that $\forall\alpha(\varphi \equiv \psi)$. ⋈

(86.13) (Exercise)

(86.14) (Exercise)

(88) Assume:

(a) $\forall x(\varphi \to \Box\varphi)$

(b) $\exists!x\varphi$

Before we show $\exists!x\Box\varphi$, note that from (b), it follows by definition (87.1) of the unique existence quantifier, where z is a variable that is substitutable for x in φ and doesn't occur free in φ:

(ϑ) $\exists x(\varphi \,\&\, \forall z(\varphi^z_x \to z = x))$

So suppose a is an arbitrary such object, i.e., suppose:

(ξ) $\varphi^a_x \,\&\, \forall z(\varphi^z_x \to z = a)$

Now to show $\exists!x\Box\varphi$, we have to show $\exists x(\Box\varphi \,\&\, \forall z(\Box\varphi^z_x \to z = x))$. By $\exists$I, it suffices to show that a is a witness to this claim, i.e., show $\Box\varphi^a_x \,\&\, \forall z(\Box\varphi^z_x \to z = a)$. But the first conjunct, $\Box\varphi^a_x$, is jointly implied by (a) and the first conjunct of (ξ). To show the second conjunct, it suffices by GEN to show $\Box\varphi^z_x \to z = a$. So assume $\Box\varphi^z_x$. Then, by the T schema (32.2), we have φ^z_x. But from this and the second conjunct of (ξ), it follows that $z = a$. This conclusion remains once we discharge (ξ) by $\exists$E. ⋈

(89) The T schema $\Box\varphi \to \varphi$ (32.2) is an axiom, and so are its closures. Hence, $\mathcal{A}(\Box\varphi \to \varphi)$ is an axiom. Note independently that actuality distributes over a conditional: by the definition of $\equiv$ (7.4.c) and &E (64.2.a) we may infer $\mathcal{A}(\chi \to \theta) \to (\mathcal{A}\chi \to \mathcal{A}\theta)$ from axiom (31.2). So if we let χ be $\Box\varphi$ and let θ be φ, we know: $\mathcal{A}(\Box\varphi \to \varphi) \to (\mathcal{A}\Box\varphi \to \mathcal{A}\varphi)$. Hence, $\mathcal{A}\Box\varphi \to \mathcal{A}\varphi$. However, axiom (33.2) is $\Box\varphi \equiv \mathcal{A}\Box\varphi$, from which it follows, by the definition of $\equiv$ (7.4.c) and &E (64.2.a), that $\Box\varphi \to \mathcal{A}\Box\varphi$. So by hypothetical syllogism, $\Box\varphi \to \mathcal{A}\varphi$. ⋈

(90.1) By applying the definition of $\equiv$ (7.4.c) to axiom (31.4) and then detaching the second conjunct by &E (64.2.b), we know:

(ζ) $\mathcal{A}\mathcal{A}\varphi \to \mathcal{A}\varphi$

Independently, by analogous reasoning, we may infer from axiom (31.2) that its right-to-left direction holds, but let us express this with different Greek metavariables so as to avoid clash of variables later:

$$(\mathcal{A}\chi \to \mathcal{A}\theta) \to \mathcal{A}(\chi \to \theta)$$

Now consider the following instance of the above schema, in which we let χ be the formula $\mathcal{A}\varphi$ and let θ be the formula φ:

(ξ) $(\mathcal{A}\mathcal{A}\varphi \to \mathcal{A}\varphi) \to \mathcal{A}(\mathcal{A}\varphi \to \varphi)$

It then follows from (ζ) and (ξ) that $\mathcal{A}(\mathcal{A}\varphi \to \varphi)$. ⋈

(90.2) By reasoning analogous to that used in the previous theorem we can derive the left-to-right direction of (31.4):

(ζ) $\mathcal{A}\varphi \to \mathcal{A}\mathcal{A}\varphi$

As in the previous theorem, we independently know the right-to-left direction of axiom (31.2) holds and we again express this with different Greek metavariables, to avoid clash of variables later:

$$(\mathcal{A}\chi \to \mathcal{A}\theta) \to \mathcal{A}(\chi \to \theta)$$

Now consider the following instance of the above schema, in which we let χ be the formula φ and let θ be the formula $\mathcal{A}\varphi$:

(ξ) $(\mathcal{A}\varphi \to \mathcal{A}\mathcal{A}\varphi) \to \mathcal{A}(\varphi \to \mathcal{A}\varphi)$

It then follows from (ζ) and (ξ) that $\mathcal{A}(\varphi \to \mathcal{A}\varphi)$. ⋈

(90.3) The principle of Adjunction (63.10.a) is $\varphi \to (\psi \to (\varphi \,\&\, \psi))$. Since this is a $\Box$-theorem, we may apply RN to obtain:

(c) $\Box(\varphi \to (\psi \to (\varphi \,\&\, \psi)))$

Now theorem (89) is that $\Box\chi \to \mathcal{A}\chi$, so it follows from (c) that:

(d) $\mathcal{A}(\varphi \to (\psi \to (\varphi \,\&\, \psi)))$

Distributing actuality over a conditional by the left-to-right direction of (31.2) again, (d) implies:

(e) $\mathcal{A}\varphi \to \mathcal{A}(\psi \to (\varphi \,\&\, \psi))$

By applying the same distribution law independently to the consequent of (e) we obtain:

(f) $\mathcal{A}(\psi \to (\varphi \,\&\, \psi)) \to (\mathcal{A}\psi \to \mathcal{A}(\varphi \,\&\, \psi))$

It now follows from (e) and (f) by hypothetical syllogism that:

(g) $\mathcal{A}\varphi \to (\mathcal{A}\psi \to \mathcal{A}(\varphi \,\&\, \psi))$

From (g), it follows by Importation (63.8.b) that $(\mathcal{A}\varphi \,\&\, \mathcal{A}\psi) \to \mathcal{A}(\varphi \,\&\, \psi)$. ⋈

(90.4) To avoid clash of variables, we may rewrite (90.3) as:

$$(\mathcal{A}\chi \,\&\, \mathcal{A}\theta) \to \mathcal{A}(\chi \,\&\, \theta)$$

Now consider the following instance of the above schema, in which we've set χ equal to $\mathcal{A}\varphi \to \varphi$ and set θ equal to $\varphi \to \mathcal{A}\varphi$:

(ϑ) $(\mathcal{A}(\mathcal{A}\varphi \to \varphi) \,\&\, \mathcal{A}(\varphi \to \mathcal{A}\varphi)) \to \mathcal{A}((\mathcal{A}\varphi \to \varphi) \,\&\, (\varphi \to \mathcal{A}\varphi))$

But by &I, we may conjoin theorems (90.1) and (90.2) to produce:

(ξ) $\mathcal{A}(\mathcal{A}\varphi \to \varphi) \,\&\, \mathcal{A}(\varphi \to \mathcal{A}\varphi)$

Hence it follows from (ϑ) and (ξ) that:

$$\mathcal{A}((\mathcal{A}\varphi \to \varphi) \,\&\, (\varphi \to \mathcal{A}\varphi))$$

But by definition of $\equiv$ (7.4.c), this last result becomes:

$$\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$$ ⋈

(91.1) Theorem (90.4) is:

(ϑ) $\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$

But as an instance of the left-to-right direction of axiom (31.4), we know:

(ξ) $\mathcal{A}(\mathcal{A}\varphi \equiv \varphi) \to \mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$

So from (ϑ) and (ξ), it follows that:

(ζ) $\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$

If $\mathcal{A}\ldots\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$ is simply $\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$, then we're done. Otherwise, repeat the above of reasoning starting with (ζ) to conclude:

$$\mathcal{A}\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$$

And so on, as many more times as needed, to obtain the finite initial string of actuality operators in $\mathcal{A}\ldots\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$. Of course, more formally, this should be cast as an argument by induction; though we left this as an exercise. ⋈

(91.2) – **(91.4)** (Exercises) [Note: The proofs are outlined in the text.]

(92) Suppose $\Gamma \vdash \varphi$, i.e., that there is a derivation of φ from Γ. We show by induction on the length of the derivation that $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$, i.e., that there is a derivation of $\mathcal{A}\varphi$ from $\mathcal{A}\Gamma$.

Base Case. If $n = 1$, then the derivation of φ from Γ consists of a single formula, namely, φ itself. So, by the definition of $\Gamma \vdash \varphi$, φ must be in $\mathbf{\Lambda} \cup \Gamma$. So either (a) φ is in $\mathbf{\Lambda}$ or (b) φ is in Γ.

If (a), then φ is an axiom. On the one hand, if φ is any axiom other that (30)★, its actualization closure $\mathcal{A}\varphi$ is an axiom, by the fact that we took all the closures of these axioms as axioms. So $\vdash \mathcal{A}\varphi$ by (46.1) and $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$ by (46.3). On the other hand, if φ is an axiom by being an instance of (30)★, then by theorem (90.4), its actualization is a theorem. And if φ is an axiom by being a universal closure of an instance of (30)★, then by theorem (91.4), its actualization is a theorem. In either case, we again have $\vdash \mathcal{A}\varphi$ by (46.1) and $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$ by (46.3).

If (b), then $\mathcal{A}\varphi$ is in $\mathcal{A}\Gamma$, by the definition of $\mathcal{A}\Gamma$. Hence by (46.2), it follows that $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$.

Inductive Case. Suppose that the derivation of φ from Γ is a sequence S of length n, where $n > 1$. Then either $\varphi \in \mathbf{\Lambda} \cup \Gamma$ or φ follows by MP from two previous members of the sequence, namely, $\psi \to \varphi$ and ψ. If $\varphi \in \mathbf{\Lambda} \cup \Gamma$, then using the reasoning in the base case, it follows that $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$. If φ follows from previous members $\psi \to \varphi$ and ψ by MP, then by the definition of a derivation, we know both that $\Gamma \vdash \psi \to \varphi$ and $\Gamma \vdash \psi$, where these are sequences of less than n. Since our IH is that the theorem holds for all such derivations of length less than n, it follows that:

(a) $\mathcal{A}\Gamma \vdash \mathcal{A}(\psi \to \varphi)$

(b) $\mathcal{A}\Gamma \vdash \mathcal{A}\psi$

Now since it is axiomatic that actuality distributes over conditionals and vice versa (31.2), we know:

$$\vdash \mathcal{A}(\psi \to \varphi) \equiv (\mathcal{A}\psi \to \mathcal{A}\varphi)$$

By definition (7.4.c), this is just:

$$\vdash (\mathcal{A}(\psi \to \varphi) \to (\mathcal{A}\psi \to \mathcal{A}\varphi))\ \&\ ((\mathcal{A}\psi \to \mathcal{A}\varphi) \to \mathcal{A}(\psi \to \varphi))$$

So by &E (64.2.a), it follows that:

$$\vdash \mathcal{A}(\psi \to \varphi) \to (\mathcal{A}\psi \to \mathcal{A}\varphi)$$

It follows from this by (46.3) that:

$$\mathcal{A}\Gamma \vdash \mathcal{A}(\psi \to \varphi) \to (\mathcal{A}\psi \to \mathcal{A}\varphi)$$

So by (46.5), it follows from this and (a) that:

$$\mathcal{A}\Gamma \vdash \mathcal{A}\psi \to \mathcal{A}\varphi$$

And again by (46.5), it follows from this and (b) that:

$$\mathcal{A}\Gamma \vdash \mathcal{A}\varphi \qquad \bowtie$$

(94.1)$\star$ Since $\mathcal{A}\varphi \equiv \varphi$ is an instance of (30)$\star$, it follows by a tautology for biconditionals (63.5.d) that $\neg\mathcal{A}\varphi \equiv \neg\varphi$. ⋈

(94.2)$\star$ If we substitute $\neg\varphi$ for φ on both sides of (94.1)$\star$, we obtain the following instance: $\neg\mathcal{A}\neg\varphi \equiv \neg\neg\varphi$. But it is a tautology that $\neg\neg\varphi \equiv \varphi$, by (63.4.b) and (63.3.g). So by a biconditional syllogism (64.6.e), it follows that $\neg\mathcal{A}\neg\varphi \equiv \varphi$. ⋈

(95.1) Apply the Rule of Actualization to the modally strict theorem $\varphi \rightarrow \varphi$ (53) and we obtain $\mathcal{A}(\varphi \rightarrow \varphi)$. So by (31.2), it follows that $\mathcal{A}\varphi \rightarrow \mathcal{A}\varphi$. By definition of $\vee$, it follows that $\neg\mathcal{A}\varphi \vee \mathcal{A}\varphi$. Now we reason by cases (64.4.a) from the two disjuncts. If $\neg\mathcal{A}\varphi$, then by axiom (31.1), it follows that $\mathcal{A}\neg\varphi$. So by $\vee$I (64.3.b), $\mathcal{A}\varphi \vee \mathcal{A}\neg\varphi$. If $\mathcal{A}\varphi$, then by $\vee$I (64.3.a), $\mathcal{A}\varphi \vee \mathcal{A}\neg\varphi$. So, $\mathcal{A}\varphi \vee \mathcal{A}\neg\varphi$. ⋈

(95.2) By theorem (90.3), &I and the definition of $\equiv$, it suffices to show just the left-to-right direction. ($\rightarrow$) A tautology of conjunction simplification (63.9.a) is $(\varphi \,\&\, \psi) \rightarrow \varphi$. By the Rule of Actualization (92), it follows that $\mathcal{A}((\varphi \,\&\, \psi) \rightarrow \varphi)$ and by the left-to-right direction of (31.2), it follows that:

(a) $\mathcal{A}(\varphi \,\&\, \psi) \rightarrow \mathcal{A}\varphi$

By analogous reasoning from the other tautology of conjunction simplification (63.9.b), i.e., $(\varphi \,\&\, \psi) \rightarrow \psi$, we may similarly infer:

(b) $\mathcal{A}(\varphi \,\&\, \psi) \rightarrow \mathcal{A}\psi$

Now for conditional proof, assume $\mathcal{A}(\varphi \,\&\, \psi)$. Then by (a) and (b), respectively, we may conclude both $\mathcal{A}\varphi$ and $\mathcal{A}\psi$. So by &I, $\mathcal{A}\varphi \,\&\, \mathcal{A}\psi$. ⋈

(95.3) As an instance of (95.2), we have:

$$\mathcal{A}((\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)) \equiv (\mathcal{A}(\varphi \rightarrow \psi) \,\&\, \mathcal{A}(\psi \rightarrow \varphi))$$

But since $\varphi \equiv \psi$ is, by definition (7.4.c), $(\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)$, we may rewrite the left-side of the biconditional and conclude:

$$\mathcal{A}(\varphi \equiv \psi) \equiv (\mathcal{A}(\varphi \rightarrow \psi) \,\&\, \mathcal{A}(\psi \rightarrow \varphi))$$ ⋈

(95.4) ($\rightarrow$) The left-to-right direction of (31.2) is a theorem of which the following are both instances:

(a) $\mathcal{A}(\varphi \rightarrow \psi) \rightarrow (\mathcal{A}\varphi \rightarrow \mathcal{A}\psi)$

(b) $\mathcal{A}(\psi \rightarrow \varphi) \rightarrow (\mathcal{A}\psi \rightarrow \mathcal{A}\varphi)$

So by Double Composition (63.10.e), we may conjoin the antecedents of (a) and (b) into a single conjunctive antecedent and conjoin the consequents of (a) and (b) into a single conjunctive consequent, to obtain:

$$(\mathcal{A}(\varphi \to \psi) \,\&\, \mathcal{A}(\psi \to \varphi)) \to ((\mathcal{A}\varphi \to \mathcal{A}\psi) \,\&\, (\mathcal{A}\psi \to \mathcal{A}\varphi))$$

By applying the definition of $\equiv$ to the consequent, the previous claim just is:

$$(\mathcal{A}(\varphi \to \psi) \,\&\, \mathcal{A}(\psi \to \varphi)) \to (\mathcal{A}\varphi \equiv \mathcal{A}\psi)$$

($\leftarrow$) Assume $\mathcal{A}\varphi \equiv \mathcal{A}\psi$, for conditional proof. Then, by definition of $\equiv$ and &E, it follows that $\mathcal{A}\varphi \to \mathcal{A}\psi$ and $\mathcal{A}\psi \to \mathcal{A}\varphi$. But by the right-to-left direction of axiom (31.2), the first implies $\mathcal{A}(\varphi \to \psi)$ and the second implies $\mathcal{A}(\psi \to \varphi)$. So by &I, we are done. $\bowtie$

(95.5) ($\to$) By biconditional syllogism (64.6.e) from theorems (95.3) and (95.4). $\bowtie$

(95.6) As an instance of axiom (33.2), we know $\Box\neg\varphi \equiv \mathcal{A}\Box\neg\varphi$. By a classical tautology (63.5.d), it follows that $\neg\Box\neg\varphi \equiv \neg\mathcal{A}\Box\neg\varphi$. Now as an instance of axiom (31.1), we know independently that $\neg\mathcal{A}\Box\neg\varphi \equiv \mathcal{A}\neg\Box\neg\varphi$. So by a biconditional syllogism (64.6.e) it follows that $\neg\Box\neg\varphi \equiv \mathcal{A}\neg\Box\neg\varphi$. Applying the definition of $\Diamond$ to both sides, we obtain $\Diamond\varphi \equiv \mathcal{A}\Diamond\varphi$. $\bowtie$

(95.7) ($\to$) This direction is just axiom (33.1). ($\leftarrow$) This direction is an instance of the T schema. $\bowtie$

(95.8) Assume $\mathcal{A}\Box\varphi$. Then by the right-to-left direction of axiom (33.2), it follows that $\Box\varphi$. So by (89), it follows that $\mathcal{A}\varphi$. But then by (33.1), it follows that $\Box\mathcal{A}\varphi$. $\bowtie$

(95.9) Assume $\Box\varphi$. From this and axiom (33.2), it follows by biconditional syllogism that $\mathcal{A}\Box\varphi$. From this latter and theorem (95.8), it follows that $\Box\mathcal{A}\varphi$. $\bowtie$

(95.10) ($\to$) Assume $\mathcal{A}(\varphi \vee \psi)$, for conditional proof. But now assume, for reductio, $\neg(\mathcal{A}\varphi \vee \mathcal{A}\psi)$. Then by a De Morgan's Law (63.6.d), it follows that $\neg\mathcal{A}\varphi \& \neg\mathcal{A}\psi$. By &E, we have both $\neg\mathcal{A}\varphi$ and $\neg\mathcal{A}\psi$. These imply, respectively, by axiom (31.1) and biconditional syllogism (64.6.b), that $\mathcal{A}\neg\varphi$ and $\mathcal{A}\neg\psi$. We may conjoin these by &I to produce $\mathcal{A}\neg\varphi \,\&\, \mathcal{A}\neg\psi$, and by an appropriate instance of theorem (95.2), namely, $\mathcal{A}(\neg\varphi \,\&\, \neg\psi) \equiv \mathcal{A}\neg\varphi \,\&\, \mathcal{A}\neg\psi$, it follows by biconditional syllogism that:

(a) $\mathcal{A}(\neg\varphi \,\&\, \neg\psi)$

Now, independently, by the commutativity of $\equiv$ (63.3.g), we may transform an instance of De Morgan's law (63.6.d) to obtain $(\neg\varphi \,\&\, \neg\psi) \equiv \neg(\varphi \vee \psi)$ as a theorem. So we may apply the Rule of Actualization to this instance to obtain:

(b) $\mathcal{A}((\neg\varphi \,\&\, \neg\psi) \equiv \neg(\varphi \vee \psi))$

Hence, from (b) it follows by an appropriate instance of (95.5) that:

(c) $\mathcal{A}(\neg\varphi \,\&\, \neg\psi) \equiv \mathcal{A}\neg(\varphi \vee \psi))$

From (a) and (c), it follows by biconditional syllogism that $\mathcal{A}\neg(\varphi \vee \psi)$. But by axiom (31.1), it follows that $\neg\mathcal{A}(\varphi \vee \psi)$, which contradicts our initial assumption. Hence, we may conclude by reductio (RAA) version (62.1) that $\mathcal{A}\varphi \vee \mathcal{A}\psi$. ($\leftarrow$) Exercise. ⋈

(95.11) As an instance of axiom (31.3), we have:

(a) $\mathcal{A}\forall\alpha\neg\varphi \equiv \forall\alpha\mathcal{A}\neg\varphi$

By the tautology $(\psi \equiv \chi) \equiv (\neg\psi \equiv \neg\chi)$, it follows from (a) that:

(b) $\neg\mathcal{A}\forall\alpha\neg\varphi \equiv \neg\forall\alpha\mathcal{A}\neg\varphi$

Now independently, as an instance of axiom (31.1), we know:

(c) $\mathcal{A}\neg\forall\alpha\neg\varphi \equiv \neg\mathcal{A}\forall\alpha\neg\varphi$

So from (c) and (b), it follows a biconditional syllogism that:

(d) $\mathcal{A}\neg\forall\alpha\neg\varphi \equiv \neg\forall\alpha\mathcal{A}\neg\varphi$

Now, independently, since $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$ is a $\Box$-theorem (31.1), we may apply GEN to obtain: $\forall\alpha(\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi)$. By a theorem of quantification theory (83.3), it follows that $\forall\alpha\mathcal{A}\neg\varphi \equiv \forall\alpha\neg\mathcal{A}\varphi$. So by our tautology $(\psi \equiv \chi) \equiv (\neg\psi \equiv \neg\chi)$, we have:

(e) $\neg\forall\alpha\mathcal{A}\neg\varphi \equiv \neg\forall\alpha\neg\mathcal{A}\varphi$

So from (d) and (e) it follows by a biconditional syllogism that:

(f) $\mathcal{A}\neg\forall\alpha\neg\varphi \equiv \neg\forall\alpha\neg\mathcal{A}\varphi$

But applying the definition of $\exists$ to both sides of (f) yields:

(g) $\mathcal{A}\exists\alpha\varphi \equiv \exists\alpha\mathcal{A}\varphi$ ⋈

(96)★ Let φ be any formula in which z is substitutable for x and doesn't occur free. Before we prove our theorem, we first establish some simple facts. By axiom (30)★, we know $\mathcal{A}\varphi \equiv \varphi$, and by the commutativity of the biconditional (63.3.g), that $\varphi \equiv \mathcal{A}\varphi$. By GEN, it follows, respectively, that $\forall x(\mathcal{A}\varphi \equiv \varphi)$ and $\forall x(\varphi \equiv \mathcal{A}\varphi)$. Given our hypothesis about z, it follows, respectively, by (83.12), that $[\forall z(\mathcal{A}\varphi \equiv \varphi)]^z_x$ and $[\forall z(\varphi \equiv \mathcal{A}\varphi)]^z_x$. From these two claims, it follows, respectively, by the definition of substitution (23) that:

(ξ) $\forall z(\mathcal{A}\varphi^z_x \equiv \varphi^z_x)$

(ζ) $\forall z(\varphi^z_x \equiv \mathcal{A}\varphi^z_x)$

With these last two facts our theorem follows simply: ($\rightarrow$) Assume $\forall z(\mathcal{A}\varphi^z_x \equiv z = x)$. From ($\zeta$) and our assumption, it follows by (83.10) that $\forall z(\varphi^z_x \equiv z = x)$. ($\leftarrow$) Assume $\forall z(\varphi^z_x \equiv z = x)$. From ($\xi$) and our assumption, it follows, again appealing to (83.10), that $\forall z(\mathcal{A}\varphi^z_x \equiv z = x)$. ⋈

(97)★ Consider any φ in which z is substitutable for x and doesn't occur free. By axiom (34), we know $x = \imath x\varphi \equiv \forall z(\mathcal{A}\varphi^z_x \equiv z = x)$. But by our previous theorem (96)★, we know that $\forall z(\mathcal{A}\varphi^z_x \equiv z = x) \equiv \forall z(\varphi^z_x \equiv z = x)$. Hence, by the transitivity of $\equiv$ (64.6.e), it follows that $x = \imath x\varphi \ \equiv\ \forall z(\varphi^z_x \equiv z = x)$. ⋈

(98)★ Consider any φ in which z is substitutable for x and doesn't occur free. Then by the fundamental theorem (97)★ for descriptions, we know:

(a) $x = \imath x\varphi \ \equiv\ \forall z(\varphi^z_x \equiv z = x)$

But, given our hypothesis that z is substitutable for x in φ and doesn't occur free in φ, we have as an instance of (86.10) that:

$$(\varphi \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = x)) \equiv \forall z(\varphi^z_x \equiv z = x)$$

From this last claim, it follows by the commutativity of $\equiv$ (63.3.g) that:

(b) $\forall z(\varphi^z_x \equiv z = x) \equiv (\varphi \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = x))$

So by biconditional syllogism from (a) and (b) it follows that:

$$x = \imath x\varphi \equiv (\varphi \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = x)) \qquad ⋈$$

(99)★ Suppose (a) ψ is either an exemplification formula $\Pi^n\kappa_1 \dots \kappa_n$ ($n \geq 1$) or an encoding formula $\kappa_1\Pi^1$, (b) x occurs in ψ and only as one or more of the κ_i ($1 \leq i \leq n$), and (c) z is substitutable for x in φ and doesn't appear free in φ. We want to show:

(a) $\psi^{\imath x\varphi}_x \equiv \exists x(\varphi \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = x) \mathbin{\&} \psi)$

Our strategy will be to use Hintikka's schema (98)★. Since z is substitutable for x in φ and doesn't appear free in φ, we know that Hintikka's schema applies to φ, so that we have:

$$x = \imath x\varphi \ \equiv\ (\varphi \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = y))$$

By GEN, it follows that:

(b) $\forall x(x = \imath x\varphi \equiv (\varphi \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = y)))$

(b) will be used in proving both directions of (a). ($\rightarrow$) Assume:

(c) $\psi^{\imath x\varphi}_x$

for conditional proof. By hypothesis, ψ is an exemplification or encoding formula and so it follows immediately by axiom (29.5), where y is some variable that doesn't occur free in φ, that:

(d) $\exists y(y = \imath x\varphi)$

Assume that a is an arbitrary such object, so that we know $a = \imath x\varphi$. If we instantiate (b) to a by Rule ∀E (77.2), we obtain:

$$a = \imath x\varphi \equiv \varphi^a_x \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = a)$$

It follows that:

(e) $\varphi^a_x \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = a)$

by biconditional syllogism. Note independently that we've established the symmetry of identity for objects (70.2), so that by GEN, we know $\forall x \forall y(x = y \rightarrow y = x)$. In this universal claim, we may instantiate $\forall x$ to a and, given (d) and (29.1), instantiate $\forall y$ to $\imath x\varphi$, thereby inferring from the assumption that $a = \imath x\varphi$ that $\imath x\varphi = a$. From this and (c) it follows by Rule SubId Special Case (74.2) that ψ^a_x (i.e., the result of substituting a for *all* the occurrences of $\imath x\varphi$ in $\psi^{\imath x\varphi}_x$). Conjoining this last result with (e) by &I we obtain:

$$\varphi^a_x \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = a) \mathbin{\&} \psi^a_x$$

Hence, by ∃I, our desired conclusion follows:

$$\exists x(\varphi \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = x) \mathbin{\&} \psi)$$

Since we've inferred this conclusion from the assumption that $a = \imath x\varphi$, where a is arbitrary, we may discharge the supposition to reach our conclusion from (d), by ∃E (85). (←) Assume, for conditional proof:

(f) $\exists x(\varphi \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = x) \mathbin{\&} \psi)$

Assume b is an arbitrary such object, so that we know:

(g) $\varphi^b_x \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = b) \mathbin{\&} \psi^b_x$

By instantiating b into (b), we have:

(h) $b = \imath x\varphi \equiv \varphi^b_x \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = b)$

If we now detach the first two conjuncts of (g) from the third conjunct by one application of &E, we have:

(i) $\varphi^b_x \mathbin{\&} \forall z(\varphi^z_x \rightarrow z = b)$

(j) ψ^b_x

From (i) and (h) it follows by biconditional syllogism that:

(k) $b = \imath x\varphi$

From (k) and (j), it follows by Rule SubId Special Case (74.2) that $\psi_x^{\imath x\varphi}$. Thus, we may discharge (g) to reach this same conclusion from (f) by ∃E (85). ⋈

(100)⋆ Let φ be any formula in which y doesn't occur free. (→) Assume, for conditional proof:

(a) $\exists y(y = \imath x\varphi)$

Assume further that a is an arbitrarily chosen such object, so that we know:

(b) $a = \imath x\varphi$

Now we want to show $\exists!x\varphi$. By definition (87.1), we have to show, where ν is some individual variable substitutable for x in φ and doesn't occur free in φ, $\exists x(\varphi \,\&\, \forall\nu(\varphi_x^\nu \rightarrow \nu = x))$. Without loss of generality, suppose ν is the variable z. Then by applying GEN to the Hintikka schema (98)⋆ and instantiating to a, we know:

$$a = \imath x\varphi \equiv (\varphi_x^a \,\&\, \forall z(\varphi_x^z \rightarrow z = a))$$

From this and (b), it follows by biconditional syllogism that:

$$\varphi_x^a \,\&\, \forall z(\varphi_x^z \rightarrow z = a)$$

By ∃I, it follows that $\exists x(\varphi \,\&\, \forall z(\varphi_x^z \rightarrow z = x))$. By Rule ∃E (85), we may discharge the assumption that $a = \imath x\varphi$ and conclude that $\exists x(\varphi \,\&\, \forall z(\varphi_x^z \rightarrow z = x))$ follows from (a) alone. (←) By analogous reasoning. ⋈

(101.1)⋆ Assume $x = \imath x\varphi$. Then by Hintikka's schema (98)⋆, it follows that $\varphi \,\&\, \forall z(\varphi_x^z \rightarrow z = x)$, where z is substitutable for x in φ. *A fortiori*, φ. ⋈

(101.2)⋆ By applying GEN to the previous theorem (101.1)⋆, we know:

(ϑ) $\forall x(x = \imath x\varphi \rightarrow \varphi)$

By hypothesis, z is substitutable for x in φ. So it is substitutable for x in the formula $x = \imath x\varphi \rightarrow \varphi$. Let this last formula be ψ, so that (ϑ) becomes $\forall x\psi$. Hence, we may apply Rule UE to (ϑ) to infer ψ_x^z. Since the free occurrences of x in φ are bound by variable-binding operator $\imath x$ in the description $\imath x\varphi$, the formula ψ_x^z resolves to: $z = \imath x\varphi \rightarrow \varphi_x^z$. ⋈

(101.3)⋆ Let φ be any formula in which y doesn't occur free. Then we may assume, for conditional proof:

(a) $\exists y(y = \imath x\varphi)$

Suppose c is an arbitrary such object, so that we know:

(b) $c = \imath x\varphi$

Since the previous theorem (101.2)★ is generalizable to any z, instantiate the generalization to c, so that we obtain φ^c_x. But by (b) and Rule SubId, it follows that $\varphi^{\imath x\varphi}_x$. So by Rule ∃E, we can discharge our hypothesis (b) and conclude that $\varphi^{\imath x\varphi}_x$ follows from (a) alone. ⋈

(102) Let φ be any formula in which z is substitutable for x and doesn't occur free. (→) Axiom (31.3) is $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$ and by the commutativity of ≡, $\mathcal{A}\mathcal{A}\varphi \equiv \mathcal{A}\varphi$ is a theorem. By GEN, it follows, respectively, that $\forall x(\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi)$ and $\forall x(\mathcal{A}\mathcal{A}\varphi \equiv \mathcal{A}\varphi)$. Given our hypothesis about z, it follows, respectively, by (83.12), that $[\forall z(\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi)]^z_x$ and $[\forall z(\mathcal{A}\mathcal{A}\varphi \equiv \mathcal{A}\varphi)]^z_x$. By the definition of substitution (23), it follows, respectively, that:

(ξ) $\forall z(\mathcal{A}\varphi^z_x \equiv \mathcal{A}\mathcal{A}\varphi^z_x)$

(ζ) $\forall z(\mathcal{A}\mathcal{A}\varphi^z_x \equiv \mathcal{A}\varphi^z_x)$

(→) Assume $\forall z(\mathcal{A}\varphi^z_x \equiv z = x)$. From ($\zeta$) and this assumption, it follows that $\forall z(\mathcal{A}\mathcal{A}\varphi^z_x \equiv z = x)$, by (83.10). (←) Assume $\forall z(\mathcal{A}\mathcal{A}\varphi^z_x \equiv z = x)$. From ($\xi$) and this assumption, it follows that $\forall z(\mathcal{A}\varphi^z_x \equiv z = x)$, also by (83.10). ⋈

(103.1) We may reason as follows:

$$\begin{array}{lcll} x = \imath x\varphi & \equiv & \forall z(\mathcal{A}\varphi^z_x \equiv z = x) & \text{by axiom (34)} \\ & \equiv & \forall z(\mathcal{A}\mathcal{A}\varphi^z_x \equiv z = x) & \text{by theorem (102)} \\ & \equiv & x = \imath x\mathcal{A}\varphi & \text{by axiom (34)} \end{array}$$

⋈

(103.2) Assume $\exists y(y = \imath x\varphi)$, where y isn't free in φ. Independently, from the previous theorem (103.1), it follows by GEN that $\forall x(x = \imath x\varphi \ \equiv \ x = \imath x\mathcal{A}\varphi)$. Since $\imath x\varphi$ is substitutable for x in the matrix of this last universal claim, our assumption and axiom (29.1) allows us to instantiate $\forall x$ to $\imath x\varphi$, to obtain $\imath x\varphi = \imath x\varphi \ \equiv \ \imath x\varphi = \imath x\mathcal{A}\varphi$. But for reasons already mentioned, we can also instantiate $\imath x\varphi$ into $\forall x(x = x)$ — the latter being the universal generalization of theorem (70.1) — to obtain $\imath x\varphi = \imath x\varphi$. Hence $\imath x\varphi = \imath x\mathcal{A}\varphi$. ⋈

(103.3). For conditional proof, assume $\psi^{\imath x\varphi}_x$, where ψ is either an exemplification formula $\Pi^n\kappa_1 \ldots \kappa_n$ ($n \geq 1$) or an encoding formula $\kappa_1\Pi^1$, and (b) x occurs in ψ and only as one or more of the κ_i ($1 \leq i \leq n$). Then where ν is any individual variable that doesn't occur free in φ, it follows by (29.5) that $\exists\nu(\nu = \imath x\varphi)$. Assume a is an arbitrary such object, so that we have $a = \imath x\varphi$. Independently, apply GEN to theorem (103.1) to obtain $\forall x(x = \imath x\varphi \equiv x = \imath x\mathcal{A}\varphi)$. If we instantiate this to a, we may conclude $a = \imath x\varphi \equiv a = \imath x\mathcal{A}\varphi$. Hence $a = \imath x\mathcal{A}\varphi$, by biconditional syllogism, and so we may conclude $\exists\nu(\nu = \imath x\mathcal{A}\varphi)$. So we've reached this conclusion once we discharge our assumption about a, by Rule ∃E. ⋈

(103.4). By (29.5) and (103.2). ⋈

(104) For both the left-to-right and the right-to-left directions, follow the proof of (98)★, but instead of appealing to the ★-theorem (97)★, appeal to the necessary axiom (34) and reason with respect to $\mathcal{A}\varphi$ instead of φ. ⋈

(105) Assume $\mathcal{A}(\varphi \equiv \psi)$ and that y doesn't occur free in φ or ψ. By GEN, it suffices to show $x = \imath x\varphi \equiv x = \imath x\psi$. ($\rightarrow$) Assume $x = \imath x\varphi$. Then by the strict version of Hintikka's Schema (104) it follows that:

(ϑ) $\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = x)$

Note that by the right-to-left direction of the strict Hintikka's Schema, to show $x = \imath x\psi$ we have to show:

(ξ) $\mathcal{A}\psi \,\&\, \forall z(\mathcal{A}\psi^z_x \rightarrow z = x)$

By &I, it suffices to show the conjuncts separately. Before we begin, note the following series of consequences that we may infer from our first assumption $\vdash (\varphi \equiv \psi)$:

(a)	$\vdash \mathcal{A}\varphi \equiv \mathcal{A}\psi$	from our initial assumption, by (95.5)
(b)	$\vdash \forall x(\mathcal{A}\varphi \equiv \mathcal{A}\psi)$	from (a) by GEN
(c)	$\vdash \mathcal{A}\varphi^z_x \equiv \mathcal{A}\psi^z_x$	also from (b), by $\forall$E

Now, from (a) and the first conjunct of (ϑ) we obtain the first conjunct of (ξ), i.e., $\mathcal{A}\psi$, by biconditional elimination ($\equiv$E). To show the second conjunct of (ξ), it suffices, by GEN, to show $\mathcal{A}\psi^z_x \rightarrow z = x$. So assume $\mathcal{A}\psi^z_x$, to show $z = x$ by conditional proof. Then by (c) and $\equiv$E, we may infer $\mathcal{A}\varphi^z_x$. But by the second conjunct of (ϑ), it follows that $z = x$. ($\leftarrow$) Exercise. ⋈

(106) For both directions, follow the proof of (99)★, but instead of appealing to (98)★, appeal to the modally-strict version of Hintikka's schema (104) and reason with respect to $\mathcal{A}\varphi$ instead of φ. ⋈

(107.1) Suppose y doesn't occur free in φ. ($\rightarrow$) Assume $\exists y(y = \imath x\varphi)$. Assume that a is an arbitrary such object, so that we have $a = \imath x\varphi$. Now we have to show $\exists! x\mathcal{A}\varphi$. By definition (87.1), we have to show $\exists x(\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = x))$, where z is an individual variable that is substitutable for x in φ and that doesn't occur free in φ. So suppose, without loss of generality, that z is such a variable. Then by applying GEN to theorem (104) and instantiating to a, we know:

$$a = \imath x\varphi \equiv \mathcal{A}\varphi^a_x \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = a)$$

So it follows by biconditional syllogism that $\mathcal{A}\varphi^a_x \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = a)$. Hence, by $\exists$I, it follows that $\exists x(\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = x))$. This last conclusion remains once we discharge our assumption about a by $\exists$E. ($\leftarrow$) Use analogous reasoning in the reverse direction. ⋈

(107.2) Assume $x = \imath x\varphi$. Then by modally strict version of Hintikka's schema (104), it follows that $\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \to z = x)$, provided z is substitutable for x in φ. *A fortiori*, $\mathcal{A}\varphi$. ⋈

(107.3) (The following is analogous to the proof of (101.2)★, though in a simplified form.) Instantiate z into (107.2) to obtain: $z = \imath x\varphi \equiv \mathcal{A}\varphi^z_x$. ⋈

(107.4) Suppose y doesn't occur free in φ. Assume $\exists y(y = \imath x\varphi)$. Assume that b is an arbitrary such object, so that we have $b = \imath x\varphi$. Without loss of generality, assume that z is substitutable for x in φ and doesn't occur free in φ. (We can say this becase if z fails these conditions with respect to φ, we know we can pick some other variable and appeal in what follows to a universally-generalized alphabetic-variant of the modally-strict version of Hintikka's schema, which we know now how to prove.) Then by the modally-strict version of Hintikka's schema (104) and &E, it follows that $\mathcal{A}\varphi^b_x$. From this and our second assumption it follows by Rule SubId Special Case that $\mathcal{A}\varphi^{\imath x\varphi}_x$. But then this follows from our first assumption by $\exists$E. ⋈

(108.1) Assume $\exists!x\Box\varphi$. By definition (87), it follows that:

(ζ) $\exists x(\Box\varphi \,\&\, \forall z(\Box\varphi^z_x \to z = x))$

Suppose b is an arbitrary such object, i.e., that:

(ϑ) $\Box\varphi^b_x \,\&\, \forall z(\Box\varphi^z_x \to z = b)$

To show $\forall y(y = \imath x\varphi \to \varphi^y_x)$, it suffices by GEN to show $y = \imath x\varphi \to \varphi^y_x$. So assume $y = \imath x\varphi$. If we apply GEN to the modally strict version of Hintikka's Schema (104) and instantiate to y, then $y = \imath x\varphi$ implies:

(ξ) $\mathcal{A}\varphi^y_x \,\&\, \forall z(\mathcal{A}\varphi^z_x \to z = y)$

Now if we instantiate b into the second conjunct of (ξ), we obtain $\mathcal{A}\varphi^b_x \to b = y$. But the first conjunct of (ϑ) implies, by theorem (89), that $\mathcal{A}\varphi^b_x$. So $b = y$. But the first conjunct of (ϑ) also implies φ^b_x, by axiom (32.2) (i.e., by the T schema). Hence, by Rule SubId, φ^y_x, and so we may discharge (ϑ) by $\exists$E to conclude that φ^y_x follows from (ζ) and thus from our initial assumption. ⋈

(108.2) Assume $\forall x(\varphi \to \Box\varphi)$ and $\exists!x\varphi$. Then by (88), it follows that $\exists!x\Box\varphi$. But then by (108.1), we may conclude $\forall y(y = \imath x\varphi \to \varphi^y_x)$. ⋈

(110.1) Assume $\Gamma \vdash_{\Box} \varphi \to \psi$, i.e., that there is a modally-strict derivation of $\varphi \to \psi$ from Γ. Then the conditions of RN are met and we may apply RN to conclude $\Box\Gamma \vdash \Box(\varphi \to \psi)$. Since instances of the K schema are axioms, we know $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, by (46.1). So $\Box\Gamma \vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, by (46.3), and by (46.6), it follows that $\Box\Gamma \vdash \Box\varphi \to \Box\psi$. ⋈

(110.2) Assume $\Gamma \vdash_{\Box} \varphi \rightarrow \psi$, i.e., that there is a modally-strict derivation of $\varphi \rightarrow \psi$ from Γ. Since the metarules of contraposition (60) apply generally to all derivations, they apply to modally-strict derivations, and so it follows by (60.1) that $\Gamma \vdash_{\Box} \neg\psi \rightarrow \neg\varphi$. Hence by RM (110.1), it follows that $\Box\Gamma \vdash \Box\neg\psi \rightarrow \Box\neg\varphi$. So, again by our metarule of contraposition (60.1), it follows that $\Box\Gamma \vdash \neg\Box\neg\varphi \rightarrow \neg\Box\neg\psi$. But by definition of $\Diamond$, this last result implies: $\Box\Gamma \vdash \Diamond\varphi \rightarrow \Diamond\psi$. ⋈

(111.1) By the first axiom (21.1) governing conditionals, we have $\varphi \rightarrow (\psi \rightarrow \varphi)$. Since this is a $\Box$-theorem, we may apply RM to conclude $\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$. ⋈

(111.2) Since we have the tautology (58.3), i.e., $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$, as a non-contingent theorem, it follows by RM that $\Box\neg\varphi \rightarrow \Box(\varphi \rightarrow \psi)$. ⋈

(111.3) ($\rightarrow$) A tautology of conjunction simplification (63.9.a) is $(\varphi \,\&\, \psi) \rightarrow \varphi$. Since this is a $\Box$-theorem, we may apply RM (110) to obtain:

(a) $\Box(\varphi \,\&\, \psi) \rightarrow \Box\varphi$

By analogous reasoning from $(\varphi \,\&\, \psi) \rightarrow \psi$ (63.9.b), we obtain:

(b) $\Box(\varphi \,\&\, \psi) \rightarrow \Box\psi$

Assume $\Box(\varphi \,\&\, \psi)$ for conditional proof. Then from (a) and (b), respectively, we may infer $\Box\varphi$ and $\Box\psi$. Hence by &I, $\Box\varphi \,\&\, \Box\psi$. ($\leftarrow$) The principle of Adjunction (63.10.a) is $\varphi \rightarrow (\psi \rightarrow (\varphi \,\&\, \psi))$. Since this is a $\Box$-theorem, we may apply RM to obtain:

(c) $\Box\varphi \rightarrow \Box(\psi \rightarrow (\varphi \,\&\, \psi))$

The consequent of (c) can be used to form an instance of K (32.1):

(d) $\Box(\psi \rightarrow (\varphi \,\&\, \psi)) \rightarrow (\Box\psi \rightarrow \Box(\varphi \,\&\, \psi))$

By hypothetical syllogism (56.1), if follows from (c) and (d) that $\Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \,\&\, \psi))$. Then by Importation (63.8.b), it follows that $(\Box\varphi \,\&\, \Box\psi) \rightarrow \Box(\varphi \,\&\, \psi)$. ⋈

(111.4) As an instance of (111.3), we have:

$$\Box((\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)) \equiv (\Box(\varphi \rightarrow \psi) \,\&\, \Box(\psi \rightarrow \varphi))$$

But, then, by applying the definition of $\equiv$ to the antecedent, we have:

$$\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \rightarrow \psi) \,\&\, \Box(\psi \rightarrow \varphi))$$ ⋈

(111.5) The following are both instances of the K axiom (32.1):

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

$$\Box(\psi \rightarrow \varphi) \rightarrow (\Box\psi \rightarrow \Box\varphi)$$

So by Double Composition (63.10.e), we may conjoin the two antecedents into a single conjunctive antecedent and conjoin the two consequents into a single conjunctive consequent, to obtain:

$$(\Box(\varphi \to \psi) \,\&\, \Box(\psi \to \varphi)) \to ((\Box\varphi \to \Box\psi) \,\&\, (\Box\psi \to \Box\varphi))$$

By applying the definition of $\equiv$ to the consequent, the previous claim just is:

$$(\Box(\varphi \to \psi) \,\&\, \Box(\psi \to \varphi)) \to (\Box\varphi \equiv \Box\psi) \qquad \bowtie$$

(111.6) Theorem (111.4) is $\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \to \psi) \,\&\, \Box(\psi \to \varphi))$. From this, it follows by definition of the main connective $\equiv$ and an application of &E (64.2.a) that $\Box(\varphi \equiv \psi) \to (\Box(\varphi \to \psi) \,\&\, \Box(\psi \to \varphi))$. But from this and (111.5), it follows by hypothetical syllogism (56.1) that $\Box(\varphi \equiv \psi) \to (\Box\varphi \equiv \Box\psi)$. $\bowtie$

(111.7) Assume, for conditional proof, $\Box\varphi \,\&\, \Box\psi$, which by &E, yields both $\Box\varphi$ and $\Box\psi$. From the latter, it follows by (111.1) that $\Box(\varphi \to \psi)$. From the former, it follows by (111.1) that $\Box(\psi \to \varphi)$. Hence, by &I, it follows that $\Box(\varphi \to \psi)\&\Box(\psi \to \varphi)$. So by biconditional syllogism, this last conclusion together with (111.4) imply $\Box(\varphi \equiv \psi)$. $\bowtie$

(111.8) Assume $\Box(\varphi \,\&\, \psi)$. From this and (111.3), it follows by $\equiv$E (64.6.a) that $\Box\varphi \,\&\, \Box\psi$. So by (111.7), $\Box(\varphi \equiv \psi)$. $\bowtie$

(111.9) If we instantiate the previous theorem to $\neg\varphi$ and $\neg\psi$, we have:

(ϑ) $\Box(\neg\varphi \,\&\, \neg\psi) \to \Box(\neg\varphi \equiv \neg\psi)$

But by the commutativity of the biconditional, it follows from (63.5.d) that $(\neg\varphi \equiv \neg\psi) \equiv (\varphi \equiv \psi)$ is a modally strict theorem. But then we may use this with the Rule of Substitution to infer from (ϑ):

(ϑ) $\Box(\neg\varphi \,\&\, \neg\psi) \to \Box(\varphi \equiv \psi)$ $\bowtie$

(112.1) Our global assumption is:

(ξ) $\vdash \Box(\psi \equiv \chi)$

Since instances of the T schema (32.2) are axioms, we know $\vdash \Box(\psi \equiv \chi) \to (\psi \equiv \chi)$, by (46.1). From this and our global assumption, it follows by (46.6) that:

(ϑ) $\vdash \psi \equiv \chi$

We frequently appeal to either (ξ) or (ϑ) in establishing the following cases of the consequent of the rule:

(.a) Show $\vdash \neg\psi \equiv \neg\chi$. Since instances of (63.5.d) are theorems, we know:

$$\vdash \psi \equiv \chi \equiv \neg\psi \equiv \neg\chi$$

From this and (ϑ), it follows by biconditional syllogism (64.6.a) that:

$$\vdash \neg\psi \equiv \neg\chi$$

(.b) Show $\vdash (\psi \to \theta) \equiv (\chi \to \theta)$. Note that since instances of (63.5.e) are theorems, we know:

$$\vdash (\psi \equiv \chi) \to ((\psi \to \theta) \equiv (\chi \to \theta))$$

From this and (ϑ), it follows by (46.6) that $\vdash (\psi \to \theta) \equiv (\chi \to \theta)$.

(.c) Show $\vdash (\theta \to \psi) \equiv (\theta \to \chi)$. By reasoning analogous to the previous case, but starting with an instance of (63.5.f) instead of (63.5.e).

(.d) Show $\vdash \forall\alpha\psi \equiv \forall\alpha\chi$. From ($\vartheta$), it follows by GEN that $\vdash \forall\alpha(\psi \equiv \chi)$. But by our proof of (83.3), we know:

$$\vdash \forall\alpha(\psi \equiv \chi) \to (\forall\alpha\psi \equiv \forall\alpha\chi)$$

Hence it follows by (46.6) that $\vdash \forall\alpha\psi \equiv \forall\alpha\chi$.

(.e) Show $\vdash \mathcal{A}\psi \equiv \mathcal{A}\chi$. Our proof of (89) establishes that:

$$\vdash \Box(\psi \equiv \chi) \to \mathcal{A}(\psi \equiv \chi)$$

From this and our global assumption (ξ), it follows that:

(ζ) $\vdash \mathcal{A}(\psi \equiv \chi)$,

by (46.6).[270] But by (95.5), we know that $\mathcal{A}(\psi \equiv \chi) \equiv (\mathcal{A}\psi \equiv \mathcal{A}\chi)$, so that we know:[271]

$$\vdash \mathcal{A}(\psi \equiv \chi) \equiv (\mathcal{A}\psi \equiv \mathcal{A}\chi)$$

Hence it follows that $\vdash \mathcal{A}\psi \equiv \mathcal{A}\chi$, by (64.6.a).

(.f) Show $\vdash \Box\psi \equiv \Box\chi$. By our proof of (111.6), we know:

$$\vdash \Box(\psi \equiv \chi) \to (\Box\psi \equiv \Box\chi)$$

But it follows from this an our global assumption (ξ) that $\vdash \Box\psi \equiv \Box\chi$, by biconditional syllogism (64.6.a). ⋈

[270] Although we could have established $\vdash \mathcal{A}(\psi \equiv \chi)$ by citing (ϑ) and appealing to the necessitation-averse axiom (30)★, we have refrained from doing so. If we had done so, our Rule of Substitution would have become a non-strict rule, since its proof would depend on an axiom that fails to be necessarily true. Any conclusion drawn using the rule derived in this manner would have been a ★-theorem. By appealing to (89), we prove this case without an appeal to any ★-theorems.

[271] Again, (95.5) could have been proved using an appeal to (30)★, but for the reasons given in footnote 270, we are relying on the proof that makes no appeal to ★-theorems.

(112.2) [Informal proof; see below for a strict proof] The informal proof appeals to the fact that (112.1) includes all the cases that *ground* the ways ψ can appear as a *proper* subformula of φ (in the case where ψ just is φ, ψ is trivially a subformula of φ). From (112.1), therefore, it follows that if both (a) φ is any formula that can be generated from ψ using $\neg$, $\rightarrow$, $\forall\alpha$, $\mathcal{A}$, and $\Box$, and (b) φ' is generated from χ in exactly the same way that φ is generated from ψ, then if $\Box(\psi \equiv \chi)$ is a theorem, so is $\varphi \equiv \varphi'$. But, by the definition of subformula (8), this applies to every formula φ of our language, since every formula of our language can be generated from exemplification and encoding formulas using $\neg$, $\rightarrow$, $\forall\alpha$, $\mathcal{A}$, and $\Box$. ⋈

(112.2) [Strict proof] Assume:

(ϑ) $\vdash \Box(\psi \equiv \chi)$

and let φ' be the result of substituting the formula χ for zero or more occurrences of ψ where the latter is a subformula of φ. We then show, by induction on the complexity of φ, that $\vdash \varphi \equiv \varphi'$. Note that if no occurrences of ψ in φ are replaced by χ, then $\varphi' = \varphi$, and we simply have to show $\vdash \varphi \equiv \varphi$. But $\varphi \equiv \varphi$ is theorem (63.4.a). Note also that if ψ is a subformula of φ because $\psi = \varphi$, then $\varphi' = \chi$ and (ϑ) becomes $\vdash \Box(\varphi \equiv \varphi')$. Since instances of the T schema (32.2) are axioms, we know $\vdash \Box(\varphi \equiv \varphi') \rightarrow (\varphi \equiv \varphi')$, by (46.1). It follows that $\vdash \varphi \equiv \varphi'$, by (46.6). So, we may assume in what follows that ψ is a proper subformula of φ (i.e., a subformula of φ not identical with φ), and that in φ', χ has been substituted for at least one occurrence of ψ in φ.

Base Case. φ is an exemplification formula of the form $\Pi^n\kappa_1 \ldots \kappa_n$ ($n \geq 0$) or an encoding formula of the form $\kappa\Pi^1$. Then, φ has no proper subformulas, and the theorem is true trivially by failure of the antecedent.

Inductive Case 1. $\varphi = \neg\theta$. Then $\varphi' = \neg\theta'$. Since our IH implies:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$,

it follows from this and (ϑ) that $\vdash \theta \equiv \theta'$. Since we've proved instances of the tautology (63.5.d), we know:

$\vdash (\theta \equiv \theta') \equiv (\neg\theta \equiv \neg\theta')$

From this and $\vdash \theta \equiv \theta'$, it follows by biconditional syllogism (64.6.a) that $\vdash \neg\theta \equiv \neg\theta'$, i.e., $\vdash \varphi \equiv \varphi'$.

Inductive Case 2. $\varphi = \theta \rightarrow \omega$. Then φ' must be $\theta' \rightarrow \omega'$. Since our IHs are:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \omega \equiv \omega'$

it follows from these and (ϑ) that $\vdash \theta \equiv \theta'$ and $\vdash \omega \equiv \omega'$. Hence by &I (64.1), it follows that:

(a) $\vdash (\theta \equiv \theta') \,\&\, (\omega \equiv \omega)'$

But one can prove (as an exercise) the tautology:

$$((\theta \equiv \theta') \,\&\, (\omega \equiv \omega')) \to ((\theta \to \omega) \equiv (\theta' \to \omega'))$$

It therefore follows from the theoremhood of this tautology and (a), by (46.6), that:

$$\vdash (\theta \to \omega) \equiv (\theta' \to \omega')$$

i.e., $\vdash \varphi \equiv \varphi'$.

Inductive Case 3. $\varphi = \forall\alpha\theta$. Then $\varphi' = \forall\alpha\theta'$. Since our IH implies:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$,

it follows from this and (ϑ) that $\vdash \theta \equiv \theta'$. So by GEN it follows that $\vdash \forall\alpha(\theta \equiv \theta')$. But by our proof of (83.3), we know:

$$\vdash \forall\alpha(\theta \equiv \theta') \to (\forall\alpha\theta \equiv \forall\alpha\theta')$$

Hence it follows by biconditional syllogism (64.6.a) that $\vdash \forall\alpha\theta \equiv \forall\alpha\theta'$, i.e., $\vdash \varphi \equiv \varphi'$.

Inductive Case 4. $\varphi = \mathcal{A}\theta$. Then $\varphi' = \mathcal{A}\theta'$. Since our IH implies:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$,

it follows from this and (ϑ) that $\vdash \theta \equiv \theta'$. So by the Rule of Actualization (92), we have $\vdash \mathcal{A}(\theta \equiv \theta')$. But since (95.5) is a theorem, we know $\vdash \mathcal{A}(\theta \equiv \theta') \equiv (\mathcal{A}\theta \equiv \mathcal{A}\theta')$. So by biconditional syllogism (64.6.a), we have $\vdash \mathcal{A}\theta \equiv \mathcal{A}\theta'$, i.e., $\vdash \varphi \equiv \varphi'$.

Inductive Case 5. $\varphi = \Box\theta$. Then $\varphi' = \Box\theta'$. Since our IH implies:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$,

it follows from this and (ϑ) that $\vdash \theta \equiv \theta'$. Since no $\star$-theorems were cited to draw this conclusion, it follows by the Rule of Necessitation that $\vdash \Box(\theta \equiv \theta')$. But by our proof of (111.6), we know:

$$\vdash \Box(\theta \equiv \theta') \to (\Box\theta \equiv \Box\theta')$$

Hence it follows by (46.6) that $\vdash \Box\psi \equiv \Box\chi$, i.e., $\varphi \equiv \varphi'$. $\bowtie$

(113) Assume (a) $\vdash_\Box \psi \equiv \chi$, (b) φ' is the result of substituting the formula χ for zero or more occurrences of ψ where the latter is a subformula of φ, and (c) $\Gamma \vdash \varphi$. Then from (a) it follows by RN that $\vdash \Box(\psi \equiv \chi)$. So by (112.2), $\vdash \varphi \equiv \varphi'$. By the definition of $\equiv$ and &E, it follows that $\vdash \varphi \to \varphi'$. From this last result, it follows by (46.10) that $\varphi \vdash \varphi'$. From this and (c) it follows by (46.8) that $\Gamma \vdash \varphi'$. $\bowtie$

(115) Let φ' be any alphabetic variant of φ. It suffices to prove the (somewhat more easily established) Variant rule $\varphi \dashv\vdash \varphi'$, for by the following argument, we may obtain the stated rule from the Variant:

> ($\rightarrow$) Assume $\Gamma \vdash \varphi$. From this, and the left-to-right direction of the Variant rule, $\varphi \vdash \varphi'$, it follows by (46.8) that $\Gamma \vdash \varphi'$. ($\leftarrow$) By analogous reasoning.

So we turn to a proof of the Variant rule by induction on the complexity of φ, with a secondary induction on the complexity of terms τ occurring in φ. Note that it suffices to prove $\varphi \vdash \varphi'$, since alphabetic variance is a symmetric relation. However, sometimes it is useful to deploy the full, bidirectional inductive hypothesis, $\psi \dashv\vdash \psi'$, where ψ is any formula of lesser complexity than φ.

Formula Induction: Base Case. φ is $\Pi^n\kappa_1 \ldots \kappa_n$ $(n \geq 0)$ or $\kappa\Pi^1$. These cases are proved as part of the Term Induction Base Case.

Term Induction: Base Case. The κ_i and Π^n in φ are all simple. Then the only alphabetic variants of φ are φ itself. So $\varphi' = \varphi$, and since $\varphi \vdash \varphi$ by a special case of (46.2), it follows that $\varphi \vdash \varphi'$. This applies not only when φ is $\Pi^n\kappa_1 \ldots \kappa_n$ $(n \geq 0)$ but also when φ is $\kappa\Pi^1$.

Term Induction: Inductive Case 1. *Case A*. φ is $\Pi^n\kappa_1 \ldots \kappa_n$, where Π^n is $[\lambda\nu_1 \ldots \nu_n\, \psi^*]$ and the κ_i are all simple. Then φ' is $[\lambda\nu_1 \ldots \nu_n\, \psi^*]'\kappa_1 \ldots \kappa_n$, where $[\lambda\nu_1 \ldots \nu_n\, \psi^*]'$ is some alphabetic variant of Π^n. By α-Conversion (36.1), we know that:

$$[\lambda\nu_1 \ldots \nu_n\, \psi^*] = [\lambda\nu_1 \ldots \nu_n\, \psi^*]'$$

Now assume $[\lambda\nu_1 \ldots \nu_n \psi^*]\kappa_1 \ldots \kappa_n$. By the Rule of Substitution of Alphabetically-Variant Relation Terms (68) it follows that $[\lambda\nu_1 \ldots \nu_n\, \psi^*]'\kappa_1 \ldots \kappa_n$. So by conditional proof we have:

$$[\lambda\nu_1 \ldots \nu_n\, \psi^*]\kappa_1 \ldots \kappa_n \rightarrow [\lambda\nu_1 \ldots \nu_n\, \psi^*]'\kappa_1 \ldots \kappa_n$$

which by (46.10) yields that:

$$[\lambda\nu_1 \ldots \nu_n\, \psi^*]\kappa_1 \ldots \kappa_n \vdash [\lambda\nu_1 \ldots \nu_n\, \psi^*]'\kappa_1 \ldots \kappa_n$$

i.e., $\varphi \vdash \varphi'$.[272]

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[272]Strictly speaking, to apply the Rule of Substitution of Alphabetically-Variant Relation terms, we have to let α be an n-place relation variable, χ be $\alpha\kappa_1 \ldots \kappa_n$, and $\tau = [\lambda\nu_1 \ldots \nu_n\, \psi^*]$. Then we know:

$$\varphi = \chi^\tau_\alpha$$

The Rule of Substitution of Alphabetically-Variant Relation Terms (68) then allows us to conclude:

$$\chi^\tau_\alpha \vdash \chi'$$

where $\tau' = [\lambda\nu_1 \ldots \nu_n\, \psi^*]'$ and χ' is the result of substituting τ' for zero or more occurrences of τ in χ^τ_α. Since φ' results from φ by substituting τ' for a single occurrence of τ in φ, φ' is such a χ' and it follows that $\varphi \vdash \varphi'$.

Case B. φ is $\kappa\Pi^1$, where Π^1 is $[\lambda\nu\ \psi^*]$ and κ is simple. Then φ' is $\kappa[\lambda\nu\ \psi^*]'$, where $[\lambda\nu\ \psi^*]'$ is an alphabetic variant of Π^1. The conclusion follows by reasoning similar similar to *Case A*, but starting with the α-Conversion instance $[\lambda\nu\ \psi^*] = [\lambda\nu\ \psi^*]'$.

Term Induction: Inductive Case 2. *Case A*. φ is $\Pi^n\kappa_1\ldots\kappa_n$, where one or more of the κ_i is a description and Π^n is simple. Without loss of generality, suppose κ_1 is the description $\imath x\theta$, so that φ is $\Pi^n\imath x\theta\kappa_2\ldots\kappa_n$. Then φ' is $\Pi^n(\imath x\theta)'\kappa_2\ldots\kappa_n$, where $(\imath x\theta)'$ is some alphabetic variant of $\imath x\theta$. So by Metatheorem $\langle 8.3\rangle$, $(\imath x\theta)'$ must have the form $\imath y\theta'^{y}_{x}$, for some variable y that is substitutable for x in θ' and not free in θ'. Then φ' is $\Pi^n\imath y\theta'^{y}_{x}\kappa_2\ldots\kappa_n$. Note that it is *not* axiomatic that $\imath x\theta = \imath y\theta'^{y}_{x}$, since this identity claim is not valid.[273] To show $\varphi \vdash \varphi'$, assume φ, i.e., $\Pi^n\imath x\theta\kappa_2\ldots\kappa_n$. Now let ψ be the formula $\Pi^n x\kappa_2\ldots\kappa_n$, then φ is $\psi^{\imath x\theta}_{x}$ and the following is an instance of the theorem asserting the *necessary* version of Russell's analysis for descriptions (106), where z a variable meeting the requirements of the theorem:[274]

$$\Pi^n\imath x\theta\kappa_2\ldots\kappa_n \equiv \exists x(\mathcal{A}\theta \,\&\, \forall z(\mathcal{A}\theta^z_x \rightarrow z=x) \,\&\, \Pi^n x\kappa_2\ldots\kappa_n)$$

Since we've assumed the left side, we may conclude by biconditional syllogism that:

$$(\xi)\ \ \exists x(\mathcal{A}\theta \,\&\, \forall z(\mathcal{A}\theta^z_x \rightarrow z=x) \,\&\, \Pi^n x\kappa_2\ldots\kappa_n)$$

Now since y is substitutable for x in θ and not free in θ, the following alphabetic-variant of (ξ) follows by (83.12):

$$(\zeta)\ \ \exists y(\mathcal{A}\theta^y_x \,\&\, \forall z(\mathcal{A}\theta^z_x \rightarrow z=y) \,\&\, \Pi^n y\kappa_2\ldots\kappa_n)$$

Note independently that our IH implies both that $\theta \vdash \theta'$and $\theta' \vdash \theta$. By the Deduction Theorem, it follows from the former that $\vdash \theta \rightarrow \theta'$ and from the latter that $\vdash \theta' \rightarrow \theta$. Hence, by &I (64.1), it follows that $\vdash (\theta' \rightarrow \theta) \,\&\, (\theta \rightarrow \theta')$, and so, $\vdash \theta \equiv \theta'$. Since this is a $\Box$-theorem, it follows from this and (ζ) by the Rule of Substitution that:

$$(\zeta')\ \ \exists y(\mathcal{A}\theta'^{y}_{x} \,\&\, \forall z(\mathcal{A}\theta'^{z}_{x} \rightarrow z=y) \,\&\, \Pi^n y\kappa_2\ldots\kappa_n)$$

Now if we let ψ be the formula $\Pi^n y\kappa_2\ldots\kappa_n$, then $\psi^{\imath y\theta'^{y}_{x}}_{y}$ is $\Pi^n\imath y\theta'^{y}_{x}\kappa_2\ldots\kappa_n$ and the following is also an instance of Russell's analysis (106):

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[273] As we've remarked upon before, once we expand the definiendum into primitive notation, we can see that the resulting definiens is false in those interpretations where $\imath x\theta$ doesn't have a denotation. For in such interpretations, the exemplification formulas $\imath x\theta =_E \imath y\theta'^{y}_{x}$, $A!\imath x\theta$, and $A!\imath y\theta^{y}_{x}$, all of which appear as conjuncts in the expanded identity claim, are false, making the entire, disjunctive identity claim false.

[274] We cite the necessary version of Russell's analysis to avoid using the $\star$-theorem (99)$\star$ in the proof of this derived rule. Recall the discussion in Remarks (50) and (93) that explains why we've studiously avoided the construction of non-strict rules.

$$\Pi^n \imath y \theta'^{y}_{x} \kappa_2 \ldots \kappa_n \equiv \exists y(\mathcal{A}\theta'^{y}_{x} \,\&\, \forall z(\mathcal{A}\theta'^{z}_{x} \to z = y) \,\&\, \Pi^n y \kappa_2 \ldots \kappa_n)$$

From this last fact and (ζ'), it follows by biconditional syllogism that:

$$\Pi^n \imath y \theta'^{y}_{x} \kappa_2 \ldots \kappa_n$$

i.e., φ'. So we've established that $\varphi \vdash \varphi'$.

Case B. φ is $\kappa\Pi^1$, where κ is a description, say $\imath x\theta$. Then φ' is $\imath x\theta'\Pi^1$, where $\imath x\theta'$ is some alphabetic variant of $\imath x\theta$. Then $\varphi \vdash \varphi'$ follows by reasoning analogous to *Case A*.

Formula Induction: Inductive Case 1. φ is $\neg\psi$, $\Box\psi$ or $\mathcal{A}\psi$. Then, by Metatheorem ⟨8.3⟩, φ' is either $\neg(\psi')$, $\Box(\psi')$ or $\mathcal{A}(\psi')$, where ψ' is some alphabetic variant of ψ. Our IH implies both that $\psi \vdash \psi'$ and $\psi' \vdash \psi$. By the Deduction Theorem, it follows from the former that $\vdash \psi \to \psi'$ and from the latter that $\vdash \psi' \to \psi$. Hence, by &I (64.1), it follows that $\vdash (\psi' \to \psi) \,\&\, (\psi \to \psi')$, and so by definition, $\vdash \psi \equiv \psi'$. Since this is a $\Box$-theorem, it follows by RN that $\vdash \Box(\psi \equiv \psi')$, and hence by derived Rules of Necessary Equivalence (112.1.a), (112.1.e), and (112.1.f), that:

$$\vdash \neg\psi \equiv \neg\psi'$$

$$\vdash \mathcal{A}\psi \equiv \mathcal{A}\psi'$$

$$\vdash \Box\psi \equiv \Box\psi'$$

as the case may be. Hence, it follows in each case that:

$$\vdash \varphi \equiv \varphi'$$

By definition, this last result is $\vdash (\varphi \to \varphi') \,\&\, (\varphi' \to \varphi)$. So by &E (64.2.a) we have $\vdash \varphi \to \varphi'$ and by (46.10), that $\varphi \vdash \varphi'$.

Formula Induction: Inductive Case 2. φ is $\psi \to \chi$. Then in light of Metatheorem ⟨8.3⟩(d), φ' is $\psi' \to \chi'$, where ψ' and χ' are alphabetic variants of ψ and χ, respectively. Our IH implies both that $\psi \dashv\vdash \psi'$ and that $\chi \dashv\vdash \chi'$. By the reasoning used in previous inductive cases, we know that these latter imply:

(a) $\vdash \Box(\psi \equiv \psi')$

(b) $\vdash \Box(\chi \equiv \chi')$

But from (a), it follows by a Rule of Necessary Equivalence (112.2) that:[275]

(c) $\vdash \psi \to \chi \equiv \psi' \to \chi$

And from (b), it follows by this same Rule of Necessary Equivalence (112.2) that:

[275] The meaning of φ' in the Rule of Necessary Equivalence should not be confused with the meaning of φ' in the present theorem.

(d) $\vdash \psi' \to \chi \equiv \psi' \to \chi'$

Hence from (c) and (d), it follows by biconditional syllogism (64.6.e) that:

$$\vdash (\psi \to \chi) \equiv (\psi' \to \chi')$$

i.e., $\vdash \varphi \equiv \varphi'$. Thus, by the reasoning used in at the end of Inductive Case 1 for Formulas, we have established $\varphi \vdash \varphi'$.

Formula Induction: Inductive Case 3. φ is $\forall\alpha\psi$. Then by Metatheorem ⟨8.3⟩, φ' has the form $\forall\beta(\psi'^{\beta}_{\alpha})$, for some ψ' that is an alphabetic variant of ψ and some variable β substitutable for α in ψ' and not free in ψ'. Although, as noted at the outset, it suffices to prove $\varphi \vdash \varphi'$, we also prove $\varphi' \vdash \varphi$, since this particular direction involves an interesting application of the Re-replacement lemma. ($\to$) By our IH, it follows that $\psi \dashv\vdash \psi'$, and so by reasoning developed in Inductive Case 1 for Formulas, it follows that $\vdash \Box(\psi \equiv \psi')$. Hence by a derived Rule of Necessary Equivalence (112.1.d), it follows that $\vdash \forall\alpha\psi \equiv \forall\alpha\psi'$. This in turn implies $\vdash \forall\alpha\psi \to \forall\alpha\psi'$, which by (46.10), yields:

(a) $\forall\alpha\psi \vdash \forall\alpha\psi'$

Now since β is, by hypothesis, a variable substitutable for α, we have as an instance of Rule $\forall$E (77.2) that $\forall\alpha\psi' \vdash \psi'^{\beta}_{\alpha}$. By hypothesis, β isn't free in ψ' and so isn't free in the premise of this last conclusion. So we may apply GEN to infer:

(b) $\forall\alpha\psi' \vdash \forall\beta(\psi'^{\beta}_{\alpha})$

Hence from (a) and (b), we obtain by (46.8) that:

$$\forall\alpha\psi \vdash \forall\beta(\psi'^{\beta}_{\alpha})$$

i.e., $\varphi \vdash \varphi'$. ($\leftarrow$) Assume $\forall\beta(\psi'^{\beta}_{\alpha})$. Since β is, by hypothesis, substitutable for α and doesn't occur free in ψ', it follows by the Re-replacement lemma (81.1) that α is substitutable for β in ψ'^{β}_{α}. By Rule $\forall$E (77.2), we can instantiate our assumption to α, to obtain $(\psi'^{\beta}_{\alpha})^{\alpha}_{\beta}$, which by Re-replacement lemma (81.1) is just ψ'. Our IH is $\psi \dashv\vdash \psi'$, and so it follows that ψ. So we have therefore established $\forall\beta(\psi'^{\beta}_{\alpha}) \vdash \psi$. Since α isn't free in the premise, it follows by GEN that $\forall\beta(\psi'^{\beta}_{\alpha}) \vdash \forall\alpha\psi$, i.e., $(\forall\alpha\psi)' \vdash \forall\alpha\psi$, i.e., $\varphi' \vdash \varphi$. ⋈

(116.1) Assume φ. Then by the Rule of Alphabetic Variants (115), it follows that φ'. So $\varphi \to \varphi'$, by conditional proof. By analogous reasoning, it follows that $\varphi' \to \varphi$. Hence by &I (64.1), it follows that $(\varphi \to \varphi') \,\&\, (\varphi' \to \varphi)$. So, by definition of $\equiv$, this is just $\varphi \equiv \varphi$. ⋈

(116.2) Assume $\exists y(y = \imath\nu\varphi)$. Now by applying GEN to theorem (70.1), we know $\forall x(x = x)$. So it follows by Rule $\forall$E that $\imath\nu\varphi = \imath\nu\varphi$. Now by definition (35.4), (35.5) and the ensuing discussion in (35), we know that since

$\iota\nu\varphi$ and $(\iota\nu\varphi)'$ are alphabetically-variant terms, the formulas $\iota\nu\varphi = \iota\nu\varphi$ and $\iota\nu\varphi = (\iota\nu\varphi)'$ are alphabetic variants. So by the Rule of Alphabetic Variants (115), it follows that $\iota\nu\varphi = (\iota\nu\varphi)'$. ⋈

(117.1) The tautology $\varphi \equiv \neg\neg\varphi$ (63.4.b) is a $\Box$-theorem. So by RN, we have $\Box(\varphi \equiv \neg\neg\varphi)$. Since φ is a subformula of $\Box\varphi$, we may, by a Rule of Necessary Equivalence (112.2), conclude: $\Box\varphi \equiv \Box\neg\neg\varphi$. ⋈

(117.2) ($\rightarrow$) Assume $\neg\Box\varphi$, for conditional proof. We want to show $\Diamond\neg\varphi$. By definition of $\Diamond$, it remains to show $\neg\Box\neg\neg\varphi$. For reductio, assume $\Box\neg\neg\varphi$. From this and (117.1), it follows by biconditional syllogism that $\Box\varphi$, which contradicts our initial assumption. Hence, we may discharge our reductio assumption and conclude by a version of RAA (62.2) that $\neg\Box\neg\neg\varphi$. ($\leftarrow$) Assume $\Diamond\neg\varphi$, i.e., $\neg\Box\neg\neg\varphi$, for conditional proof. We want to show $\neg\Box\varphi$. So, for reductio, assume $\Box\varphi$. From this and (117.1), it follows by biconditional syllogism that $\Box\neg\neg\varphi$, which contradicts our initial assumption. Hence, we may discharge our reductio assumption by a version of RAA (62.1) and conclude that $\neg\Box\varphi$. ⋈

(117.3) [With Rule of Substitution] From tautologies (58.1) and (58.2) it follows by $\equiv$I (64.5) that $\varphi \equiv \neg\neg\varphi$ is a $\Box$-theorem. As an instance of this last fact, we therefore know:

(ϑ) $\Box\varphi \equiv \neg\neg\Box\varphi$

But since $\varphi \equiv \neg\neg\varphi$ is a $\Box$-theorem, we may use the Rule of Substitution to substitute $\neg\neg\varphi$ for the very last occurrence of φ in (ϑ) to obtain:

(ξ) $\Box\varphi \equiv \neg\neg\Box\neg\neg\varphi$

But, by definition of $\Diamond$, this implies $\Box\varphi \equiv \neg\Diamond\neg\varphi$. ⋈

(117.3) [Without Rule of Substitution] ($\rightarrow$) Assume $\Box\varphi$, for conditional proof. We want to show $\neg\Diamond\neg\varphi$. For reductio, assume $\Diamond\neg\varphi$. From this and (117.2), it follows by biconditional syllogism (64.6.b) that $\neg\Box\varphi$, which contradicts our initial assumption. Discharging our reductio assumption by RAA, it follows that $\neg\Diamond\neg\varphi$. ($\leftarrow$) Assume $\neg\Diamond\neg\varphi$, for conditional proof. We want to show $\Box\varphi$. Assume $\neg\Box\varphi$, for reductio. From this and (117.2), it follows by biconditional syllogism (64.6.a) that $\Diamond\neg\varphi$, which contradicts our initial assumption. Discharging our reductio assumption, it follows by RAA that $\Box\varphi$. ⋈

(117.4) (Exercise)

(117.5) By the special case of (46.2), we know $(\varphi \rightarrow \psi) \vdash_{\Box} (\varphi \rightarrow \psi)$. So by RM$\Diamond$, it follows that $\Box(\varphi \rightarrow \psi) \vdash \Diamond\varphi \rightarrow \Diamond\psi$. Hence, by conditional proof, it follows that $\Box(\varphi \rightarrow \psi) \rightarrow \Diamond\varphi \rightarrow \Diamond\psi$. ⋈

(117.6) As an instance of (111.3), we have:

$$\Box(\neg\varphi \,\&\, \neg\psi) \equiv (\Box\neg\varphi \,\&\, \Box\neg\psi)$$

If follows from this and an appropriate instance of theorem (117.3), by $\equiv$E (64.6.f) that:

$$(\vartheta)\quad \neg\Diamond\neg(\neg\varphi \,\&\, \neg\psi) \equiv (\Box\neg\varphi \,\&\, \Box\neg\psi)$$

Similarly, since $\Box\neg\varphi \equiv \neg\Diamond\varphi$ and $\Box\neg\psi \equiv \neg\Diamond\psi$ are instances of our non-contingent theorem (117.4), we may use two simultaneous applications of the Rule of Substitution to infer from (ϑ) that:

$$\neg\Diamond\neg(\neg\varphi \,\&\, \neg\psi) \equiv (\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$$

Since De Morgan's Law (63.6.b) is a non-continent theorem, we may transform our last displayed result using the Rule of Substitution into:

$$\neg\Diamond(\varphi \vee \psi) \equiv (\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$$

By an appropriate instance of (63.5.d), we can negate both sides of the biconditional to obtain:

$$\neg\neg\Diamond(\varphi \vee \psi) \equiv \neg(\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$$

But given the equivalence of a formula and its double negation (63.4.b), it is a theorem that $\neg\neg\Diamond(\varphi \vee \psi) \equiv \Diamond(\varphi \vee \psi)$. So we may use the Rule of Substitution to conclude:

$$\Diamond(\varphi \vee \psi) \equiv \neg(\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$$

But De Morgan's Law (63.6.b) is a $\Box$-theorem, and so we may, with the Rule of Substitution, use the instance asserting the equivalence of $\neg(\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$ and $\Diamond\varphi \vee \Diamond\psi$ to obtain:

$$\Diamond(\varphi \vee \psi) \equiv (\Diamond\varphi \vee \Diamond\psi) \qquad \bowtie$$

(117.7) By simple conditional proofs and the rules for $\vee$I (64.3.a) and (64.3.b), we can establish the following $\Box$-theorems:

$$\varphi \to (\varphi \vee \psi)$$

$$\psi \to (\varphi \vee \psi)$$

Hence it follows by RM that:

$$\Box\varphi \to \Box(\varphi \vee \psi)$$

$$\Box\psi \to \Box(\varphi \vee \psi)$$

So by an appropriate instance of the tautology (63.10.d), it follows that $(\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi)$. ⋈

(117.8) The tautologies of conjunction simplification (63.9) are $(\varphi \,\&\, \psi) \rightarrow \varphi$ and $(\varphi \,\&\, \psi) \rightarrow \psi$. Since these are non-contintent theorems, it follows by RM$\Diamond$ that $\Diamond(\varphi \,\&\, \psi) \rightarrow \Diamond\varphi$ and $\Diamond(\varphi \,\&\, \psi) \rightarrow \Diamond\psi$. So assume $\Diamond(\varphi \,\&\, \psi)$, for conditional proof. It follows that both $\Diamond\varphi$ and $\Diamond\psi$. So by &I, we have $\Diamond\varphi \,\&\, \Diamond\psi$. Hence, by conditional proof, $\Diamond(\varphi \,\&\, \psi) \rightarrow (\Diamond\varphi \,\&\, \Diamond\psi)$. ⋈

(117.9) As an instance of (117.6), we have $\Diamond(\neg\varphi \vee \psi) \equiv (\Diamond\neg\varphi \vee \Diamond\psi)$. But $\Box$-theorem (117.2) is $\Diamond\neg\varphi \equiv \neg\Box\varphi$. So by the Rule of Substitution, it follows that $\Diamond(\neg\varphi \vee \psi) \equiv (\neg\Box\varphi \vee \Diamond\psi)$. Since it can be established (exercise) as a $\Box$-theorem that $(\neg\varphi \vee \psi) \equiv (\varphi \rightarrow \psi)$, we may use the Rule of Substitution to conclude $\Diamond(\varphi \rightarrow \psi) \equiv (\neg\Box\varphi \vee \Diamond\psi)$. And since it is an instance of the $\Box$-theorem just assigned as an exercise that $(\neg\Box\varphi \vee \Diamond\psi) \equiv (\Box\varphi \rightarrow \Diamond\psi)$, a final application of the Rule of Substitution yields: $\Diamond(\varphi \rightarrow \psi) \equiv (\Box\varphi \rightarrow \Diamond\psi)$. ⋈

(117.10) By (117.2), it is a $\Box$-theorem that $\neg\Box\varphi \equiv \Diamond\neg\varphi$. Hence, by RN, it follows that $\Box(\neg\Box\varphi \equiv \Diamond\neg\varphi)$. So by (111.6), it follows that $\Box\neg\Box\varphi \equiv \Box\Diamond\neg\varphi$. By the relevant instance of the biconditional tautology (63.5.d), we can negate both sides to obtain: $\neg\Box\neg\Box\varphi \equiv \neg\Box\Diamond\neg\varphi$. And by the definition of $\Diamond$, it follows that $\Diamond\Box\varphi \equiv \neg\Box\Diamond\neg\varphi$. ⋈

(117.11) ($\rightarrow$) Assume $\Diamond\Diamond\varphi$. Then, by the definition of the second occurrence of $\Diamond$, this becomes $\Diamond\neg\Box\neg\varphi$. But as an instance of (117.2), we know $\neg\Box\Box\neg\varphi \equiv \Diamond\neg\Box\neg\varphi$. So, by biconditional syllogism (64.6.b), it follows that $\neg\Box\Box\neg\varphi$. So by CP, $\Diamond\Diamond\varphi \rightarrow \neg\Box\Box\neg\varphi$. ($\leftarrow$) Assume $\neg\Box\Box\neg\varphi$. Complete the proof by reversing the reasoning in the left-to-right direction. ⋈

(117.12) As an instance of the K axiom (32.1), we know:

(ϑ) $\Box(\neg\psi \rightarrow \varphi) \rightarrow (\Box\neg\psi \rightarrow \Box\varphi)$

Independently, as instances of (63.5.k), we know there is a modally strict proof of the following:

(a) $(\neg\psi \rightarrow \varphi) \equiv (\neg\neg\psi \vee \varphi)$

(b) $(\Box\neg\psi \rightarrow \Box\varphi) \equiv (\neg\Box\neg\psi \vee \Box\varphi)$

By the modally strict equivalence $\neg\neg\psi \equiv \psi$ and Rule of Substitution, (a) implies the following by a modally strict proof:

(c) $(\neg\psi \rightarrow \varphi) \equiv (\psi \vee \varphi)$

So from the (c) and the Rule of Substitution, we can infer the following from (ϑ):

(ξ) $\Box(\psi \vee \varphi) \rightarrow (\Box\neg\psi \rightarrow \Box\varphi)$

Now by definition of $\Diamond$, (b) implies:

(d) $(\Box\neg\psi \rightarrow \Box\varphi) \equiv (\Diamond\psi \vee \Box\varphi)$

So from (ξ) and the left-to-right direction of (d), it follows by hypothetical syllogism that:

(ζ) $\Box(\psi \vee \varphi) \rightarrow (\Diamond\psi \vee \Box\varphi)$

But we know that there is a modally strict proof of the fact that disjuncts of a disjunction commute (63.3.e). So, by applying appropriate equivalences and the Rule of Substitution to the antecedent and consequence of (ζ), we have:

$$\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Diamond\psi) \qquad \bowtie$$

(118) As an instance of the T schema (32.2), we know $\Box\neg\varphi \rightarrow \neg\varphi$. So, by the derived rule of contraposition (60), it follows that $\neg\neg\varphi \rightarrow \neg\Box\neg\varphi$. But as an instance of (58.2), we know $\varphi \rightarrow \neg\neg\varphi$. So by hypothetical syllogism (56.1), it follows that $\varphi \rightarrow \neg\Box\neg\varphi$. Hence, by the definition of $\Diamond$ in (7.4.e), it follows that $\varphi \rightarrow \Diamond\varphi$. $\bowtie$

(119.1) Assume $\Diamond\Box\varphi$. By (117.10), it follows by biconditional syllogism that $\neg\Box\Diamond\neg\varphi$. But note that the following is an instance of the 5 schema: $\Diamond\neg\varphi \rightarrow \Box\Diamond\neg\varphi$. So by MT (59.1), it follows that $\neg\Diamond\neg\varphi$, i.e., $\Box\varphi$. Hence, by CP, $\Diamond\Box\varphi \rightarrow \Box\varphi$. $\bowtie$

(119.2) ($\rightarrow$) Assume $\Box\varphi$. Then by the T$\Diamond$ schema (118), it follows that $\Diamond\Box\varphi$. So by CP, $\Box\varphi \rightarrow \Diamond\Box\varphi$. ($\leftarrow$) By (119.1) $\bowtie$

(119.3) ($\rightarrow$) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ is just the 5 schema. ($\leftarrow$) Assume $\Box\Diamond\varphi$. Then by the T schema (32.2), it follows that $\Diamond\varphi$. So by CP, $\Box\Diamond\varphi \rightarrow \Diamond\varphi$. $\bowtie$

(119.4) By (118), we know $\varphi \rightarrow \Diamond\varphi$. And as an instance of the 5 schema (32.3), we know: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$. So by hypothetical syllogism (56.1), it follows that $\varphi \rightarrow \Box\Diamond\varphi$. $\bowtie$

(119.5) As an instance of the B schema (119.4), we have $\neg\varphi \rightarrow \Box\Diamond\neg\varphi$. It follows from this by contraposition that $\neg\Box\Diamond\neg\varphi \rightarrow \neg\neg\varphi$. Since the equivalence of $\neg\neg\varphi$ and φ is a $\Box$-theorem, it follows by the Rule of Substitution that:

(ϑ) $\neg\Box\Diamond\neg\varphi \rightarrow \varphi$

Now, independently, by (117.10), it is a $\Box$-theorem that $\Diamond\Box\varphi \equiv \neg\Box\Diamond\neg\varphi$, and by the commutativity of $\equiv$, that $\neg\Box\Diamond\neg\varphi \equiv \Diamond\Box\varphi$. So by the Rule of Substitution, we may transform (ϑ) into $\Diamond\Box\varphi \rightarrow \varphi$. $\bowtie$

(119.6) $\Box\varphi \to \Box\Diamond\Box\varphi$ is an instance of the B axiom (119.4). Independently, since (119.1), i.e., $\Diamond\Box\varphi \to \Box\varphi$, is a $\Box$-theorem, it follows by RM that $\Box\Diamond\Box\varphi \to \Box\Box\varphi$. So by hypothetical syllogism, $\Box\varphi \to \Box\Box\varphi$. ⋈

(119.7) (Exercise)

(119.8) As an instance of (119.6) we have $\Box\neg\varphi \to \Box\Box\neg\varphi$. By a rule of contraposition, this implies $\neg\Box\Box\neg\varphi \to \neg\Box\neg\varphi$. By the definition of $\Diamond$, this becomes $\neg\Box\Box\neg\varphi \to \Diamond\varphi$. But it is a $\Box$-theorem (117.11) that $\Diamond\Diamond\varphi \equiv \neg\Box\Box\neg\varphi$, which by commutativity of $\equiv$ is $\neg\Box\Box\neg\varphi \equiv \Diamond\Diamond\varphi$. So by the Rule of Substitution, it follows that $\Diamond\Diamond\varphi \to \Diamond\varphi$. ⋈

(119.9) (Exercise)

(119.10) As an instance of (117.12), we know:

$$\Box(\varphi \vee \Box\psi) \to (\Box\varphi \vee \Diamond\Box\psi)$$

Since (119.2) establishes a modally strict equivalence between $\Diamond\Box\psi$ and $\Box\psi$, the Rule of Substitution allows us to infer the following from the above:

$$(\vartheta)\quad \Box(\varphi \vee \Box\psi) \to (\Box\varphi \vee \Box\psi)$$

Now, independently, as an instance of (117.7), we know:

$$(\Box\varphi \vee \Box\Box\psi) \to \Box(\varphi \vee \Box\psi)$$

Since (119.7) establishes a modally strict equivalence between $\Box\Box\psi$ and $\Box\psi$, the Rule of Substitution allows us to infer the following from the above:

$$(\xi)\quad (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \Box\psi)$$

Hence, from (ϑ) and (ξ), it follows by definition of $\equiv$ that:

$$\Box(\varphi \vee \Box\psi) \equiv \Box(\varphi \vee \Box\psi)$$ ⋈

(119.11) (Exercise)

(119.12) As an instance of (119.10), we know:

$$\Box(\neg\varphi \vee \Box\neg\psi) \equiv (\Box\neg\varphi \vee \Box\neg\psi)$$

By (63.5.d), we can negate both sides, to conclude:

$$(\vartheta)\quad \neg\Box(\neg\varphi \vee \Box\neg\psi) \equiv \neg(\Box\neg\varphi \vee \Box\neg\psi)$$

Since $\neg\Box\chi \equiv \Diamond\neg\chi$ (117.2) and $\Box\neg\chi \equiv \neg\Diamond\chi$ (117.4) are modally strict equivalences, we can use the Rule of Substitution multiple times to infer the following from (ϑ):

$$\Diamond\neg(\neg\varphi \vee \neg\Diamond\psi) \equiv \neg(\neg\Diamond\varphi \vee \neg\Diamond\psi)$$

By using the De Morgan law (63.6.a) and applying the the Rule of Substitution to both sides, it follows that:

$$\Diamond(\varphi \,\&\, \Diamond\psi) \equiv (\Diamond\varphi \,\&\, \Diamond\psi) \qquad \bowtie$$

(119.13) (Exercise)

(119.14) ($\rightarrow$) For the left-to-right direction, our proof strategy is as follows:

(a) Show $\Box(\varphi \rightarrow \Box\psi) \rightarrow (\Diamond\varphi \rightarrow \psi)$, by a modally strict proof.

(b) Conclude $\Box(\Box(\varphi \rightarrow \Box\psi) \rightarrow (\Diamond\varphi \rightarrow \psi))$ from (a) by RN.

(c) Conclude $\Box\Box(\varphi \rightarrow \Box\psi) \rightarrow \Box(\Diamond\varphi \rightarrow \psi)$ from (b) by the K axiom (32.1).

(d) Show that the left-to-right direction of our theorem, i.e., $\Box(\varphi \rightarrow \Box\psi) \rightarrow \Box(\Diamond\varphi \rightarrow \psi)$ follows from (c).

It remains to show (a) and (d).

For (a), assume $\Box(\varphi \rightarrow \Box\psi)$. Then by K$\Diamond$ (117.5), it follows that $\Diamond\varphi \rightarrow \Diamond\Box\psi$. But the B$\Diamond$ schema (119.5) is $\Diamond\Box\psi \rightarrow \psi$. So $\Diamond\varphi \rightarrow \psi$ follows by hypothetical syllogism from our last two results.

For (d), assume $\Box(\varphi \rightarrow \Box\psi)$. Then by the 4 schema (119.6), it follows that $\Box\Box(\varphi \rightarrow \Box\psi)$. So by (c), it follows that $\Box(\Diamond\varphi \rightarrow \psi)$.

($\leftarrow$) We leave the right-to-left direction as an exercise. $\bowtie$

(120.1) Assume $\Box(\varphi \rightarrow \Box\varphi)$. Then by (119.14), it follows that $\Box(\Diamond\varphi \rightarrow \varphi)$. So by the T schema (32.2):

(ϑ) $\Diamond\varphi \rightarrow \varphi$

But it also follows from our initial assumption by the T schema that:

(ξ) $\varphi \rightarrow \Box\varphi$

Now we want to show $\Diamond\varphi \equiv \Box\varphi$. ($\rightarrow$) This direction follows by hypothetical syllogism (56.1) from (ϑ) and (ξ). ($\leftarrow$) Assume $\Box\varphi$. Then by the T schema, φ, and by the T$\Diamond$ schema (118), $\Diamond\varphi$. $\bowtie$

(120.2) Assume $\Box(\varphi \rightarrow \Box\varphi)$. Then by (120.1):

(ϑ) $\Diamond\varphi \equiv \Box\varphi$

We want to show $\neg\Box\varphi \equiv \Box\neg\varphi$. ($\rightarrow$) Assume $\neg\Box\varphi$. Then from (ϑ), it follows by biconditional syllogism (64.6.d) that $\neg\Diamond\varphi$, i.e., $\Box\neg\varphi$, by (117.4). ($\leftarrow$) Assume $\Box\neg\varphi$. Then $\neg\varphi$ by the T schema, and so $\Diamond\neg\varphi$, by the T$\Diamond$ schema. Hence, $\neg\Box\varphi$, by (117.2). $\bowtie$

(120.3) Assume both the antecedent and the antecedent of the consequent:

(ϑ) $\Box(\varphi \to \Box\varphi) \,\&\, \Box(\psi \to \Box\psi)$

(ξ) $\Box\varphi \equiv \Box\psi$

From (ξ) it follows that $(\Box\varphi \& \Box\psi) \vee (\neg\Box\varphi \& \neg\Box\psi)$, by an appropriate instance of (63.5.i). We now reason by cases from the two disjuncts to show, in each case, that $\Box(\varphi \equiv \psi)$:

- Assume $\Box\varphi \,\&\, \Box\psi$. Then it follows from (111.7) that $\Box(\varphi \equiv \psi)$, by a biconditional syllogism (64.6.a).
- Assume $\neg\Box\varphi \,\&\, \neg\Box\psi$. Note independently that by (120.2), the conjuncts of (ϑ) imply, respectively:

  $\neg\Box\varphi \equiv \Box\neg\varphi$

  $\neg\Box\psi \equiv \Box\neg\psi$

  So we may easily derive $\Box\neg\varphi \,\&\, \Box\neg\psi$ from our local assumption. But then by (111.3), it follows that $\Box(\neg\varphi \,\&\, \neg\psi)$, and by (111.9) that $\Box(\varphi \equiv \psi)$.

⋈

(121.1) Assume $\Gamma \vdash_{\Box} \Diamond\varphi \to \psi$, i.e., that there is a modally-strict derivation of $\Diamond\varphi \to \psi$ from Γ. So by Rule RM, it follows that $\Box\Gamma \vdash \Box\Diamond\varphi \to \Box\psi$. By (119.4), the instances of the B schema are theorems, so by (46.3) we have $\Box\Gamma \vdash \varphi \to \Box\Diamond\varphi$. Hence, by (55.1), it follows that $\Box\Gamma \vdash \varphi \to \Box\psi$. ⋈

(121.2) Assume $\Gamma \vdash_{\Box} \varphi \to \Box\psi$. Then by RM$\Diamond$ (110.2), it follows that $\Box\Gamma \vdash \Diamond\varphi \to \Diamond\Box\psi$. But the schema B$\Diamond$ (119.5) is a theorem, and so by (46.3) we know $\Box\Gamma \vdash \Diamond\Box\psi \to \psi$. Hence, by (55.1) it follows that $\Box\Gamma \vdash \Diamond\varphi \to \psi$. ⋈

(122.1)

1.	$\forall\alpha\Box\varphi \to \Box\varphi$	instance of (79.3)
2.	$\Diamond\forall\alpha\Box\varphi \to \Diamond\Box\varphi$	from (1) by RM$\Diamond$ (110.2)
3.	$\Diamond\Box\varphi \to \varphi$	instance of B$\Diamond$ (119.5)
4.	$\Diamond\forall\alpha\Box\varphi \to \varphi$	from (2),(3) by (56.1)
5.	$\forall\alpha(\Diamond\forall\alpha\Box\varphi \to \varphi)$	from (5) by GEN
6.	$\forall\alpha(\Diamond\forall\alpha\Box\varphi \to \varphi) \to (\Diamond\forall\alpha\Box\varphi \to \forall\alpha\varphi)$	instance of (79.2)
7.	$\Diamond\forall\alpha\Box\varphi \to \forall\alpha\varphi$	from (5),(6) by MP
8.	$\forall\alpha\Box\varphi \to \Box\forall\alpha\varphi$	from (7), by Rule (121.1) ⋈

(122.2) As an instance of the $\Box$-theorem (79.3), we have $\forall\alpha\varphi \to \varphi$. So by Rule RM (110.1), it follows that $\Box\forall\alpha\varphi \to \Box\varphi$. By GEN, it follows that $\forall\alpha(\Box\forall\alpha\varphi \to \Box\varphi)$. But since α isn't free in $\Box\forall\alpha\varphi$, it follows by an appropriate instance of (79.2) that $\Box\forall\alpha\varphi \to \forall\alpha\Box\varphi$. ⋈

(122.3) As an instance of BF (122.1) we have $\forall\alpha\Box\neg\varphi \to \Box\forall\alpha\neg\varphi$. By a rule of contraposition, it follows that $\neg\Box\forall\alpha\neg\varphi \to \neg\forall\alpha\Box\neg\varphi$. Given the instance

$\neg\Box\forall\alpha\neg\varphi \equiv \Diamond\neg\forall\alpha\neg\varphi$ of the $\Box$-theorem (117.2), the Rule of Substitution yields $\Diamond\neg\forall\alpha\neg\varphi \to \neg\forall\alpha\Box\neg\varphi$. And given the instance $\neg\forall\alpha\Box\neg\varphi \equiv \exists\alpha\neg\Box\neg\varphi$ of the $\Box$-theorem (86.2), we apply the Rule of Substitution again to obtain: $\Diamond\neg\forall\alpha\neg\varphi \to \exists\alpha\neg\Box\neg\varphi$. We can transform the antecedent using the definition of $\exists$ and the consequent using the definition of $\Diamond$ to obtain: $\Diamond\exists\alpha\varphi \to \exists\alpha\Diamond\varphi$. ⋈

(122.4) As an instance of (122.2), we have $\Box\forall\alpha\neg\varphi \to \forall\alpha\Box\neg\varphi$. By a rule of contraposition, it follows that $\neg\forall\alpha\Box\neg\varphi \to \neg\Box\forall\alpha\neg\varphi$. From the instance $\neg\forall\alpha\Box\neg\varphi \equiv \exists\alpha\neg\Box\neg\varphi$ of the $\Box$-theorem (86.2), we may apply the Rule of Substitution to obtain $\exists\alpha\neg\Box\neg\varphi \to \neg\Box\forall\alpha\neg\varphi$. And given the instance $\neg\Box\forall\alpha\neg\varphi \equiv \Diamond\neg\forall\alpha\neg\varphi$ of the $\Box$-theorem (117.2), we may again apply the Rule of Substitution to obtain: $\exists\alpha\neg\Box\neg\varphi \to \Diamond\neg\forall\alpha\neg\varphi$. We can transform the antecedent using the definition $\Diamond$ and the consequent using the definition of $\exists$ to obtain: $\exists\alpha\Diamond\varphi \to \Diamond\exists\alpha\varphi$. ⋈

(123.1) Assume $\exists\alpha\Box\varphi$, for conditional proof. Now let τ be an arbitrary such α, so that we have $\Box\varphi^{\tau}_{\alpha}$ (i.e., τ is an arbitrary constant that is substitutable for, and has the same type as the variable α in φ). Independently, note that from the premise φ^{τ}_{α} it follows by $\exists$I (84.2) that $\exists\alpha\varphi$. Hence, by RN, the premise $\Box\varphi^{\tau}_{\alpha}$ implies the conclusion $\Box\exists\alpha\varphi$. Hence by Rule $\exists$E, (85), the premise $\exists\alpha\Box\varphi$ implies the conclusion $\Box\exists\alpha\varphi$. So by conditional proof, $\exists\alpha\Box\varphi \to \Box\exists\alpha\varphi$. ⋈

(123.2) As an instance of the $\Box$-theorem (79.3), we know $\forall\alpha\varphi \to \varphi$. By RM$\Diamond$ (110.2), then, it follows that $\Diamond\forall\alpha\varphi \to \Diamond\varphi$. So by GEN, it follows that $\forall\alpha(\Diamond\forall\alpha\varphi \to \Diamond\varphi)$. But since α isn't free in $\Diamond\forall\alpha\varphi$, it follows by an appropriate instance of (79.2) that $\Diamond\forall\alpha\varphi \to \forall\alpha\Diamond\varphi$. ⋈

(123.3) From (86.5), by RM$\Diamond$. ⋈

(123.4) Assume:

$$\Box\forall\alpha(\varphi \to \psi) \,\&\, \Box\forall\alpha(\psi \to \chi)$$

to show $\Box\forall\alpha(\varphi \to \chi)$ by conditional proof. Then since a conjunction of necessities implies a necessary conjunction (111.3), it follows that:

$$(\vartheta)\quad \Box(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi))$$

Note, independently, that the following is (83.9):

$$(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \forall\alpha(\varphi \to \chi)$$

Since this is a modally strict theorem, its necessitation follows by RN:

$$(\zeta)\quad \Box[(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \forall\alpha(\varphi \to \chi)]$$

But as an instance of the K axiom (32.1), we know:

$$(\xi)\quad \Box[(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \forall\alpha(\varphi \to \chi)] \to \\ \Box(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \Box\forall\alpha(\varphi \to \chi)$$

From (ξ) and (ζ) it follows by MP that:

$$\Box(\forall\alpha(\varphi \rightarrow \psi)\,\&\,\forall\alpha(\psi \rightarrow \chi)) \rightarrow \Box\forall\alpha(\varphi \rightarrow \chi)$$

And from this last result and (ϑ), it follows by MP that $\Box\forall\alpha(\varphi \rightarrow \chi)$. ⋈

(123.5) By reasoning analogous to (123.4) but starting with (83.10) and using (111.6) instead of the K axiom. ⋈

(124.1) ($\rightarrow$) From the $\Box$-theorem (75), it follows *a fortiori* that $\alpha = \beta \rightarrow \Box\alpha = \beta$. Since this is a $\Box$-theorem, it follows by (121.2) that $\Diamond\alpha = \beta \rightarrow \alpha = \beta$. ($\leftarrow$) This is an instance of the T$\Diamond$ schema (118). ⋈

(124.2) ($\rightarrow$) From (124.1), it follows that $\Diamond\alpha = \beta \rightarrow \alpha = \beta$, by the definition of $\equiv$ and &E. The contraposition of this result is $\neg\alpha = \beta \rightarrow \neg\Diamond\alpha = \beta$. But as an instance of (117.4), we know $\Box\neg\alpha = \beta \equiv \neg\Diamond\alpha = \beta$. So by commuting this equivalence and using it with the Rule of Substitution, it follows that $\neg\alpha = \beta \rightarrow \Box\neg\alpha = \beta$. By applying infix notation, this is equivalent to $\alpha \neq \beta \rightarrow \Box\alpha \neq \beta$. ($\leftarrow$) This is an instance of the T schema. ⋈

(124.3) ($\rightarrow$) From the previous $\Box$-theorem (124.2), it follows *a fortiori* that $\alpha \neq \beta \rightarrow \Box\alpha \neq \beta$, by definition of $\equiv$ and &E. Since this is a modally strict theorem, we may apply (121.2) to conclude $\Diamond\alpha \neq \beta \rightarrow \alpha \neq \beta$. ($\leftarrow$) This is an instance of the T$\Diamond$ schema (118). ⋈

(125.1) Before we begin the proof proper, we note the following facts. As an instance of theorem (72.2), we have:

$$\forall x\Box(x = x)$$

Furthermore, the following is an instance of (29.1):

$$\forall x\Box(x = x) \rightarrow (\exists y(y = \imath x\varphi) \rightarrow \Box\imath x\varphi = \imath x\varphi)$$

Hence by MP we know the following fact:

(ϑ) $\exists y(y = \imath x\varphi) \rightarrow \Box\imath x\varphi = \imath x\varphi$

Now to prove our theorem, assume $\exists y(y = \imath x\varphi)$, for conditional proof. Then by MP and (ϑ), we have $\Box\imath x\varphi = \imath x\varphi$. From this last fact and our assumption that $\exists y(y = \imath x\varphi)$, it follows by Rule $\exists$I (84.1) that $\exists y\Box(y = \imath x\varphi)$. ⋈

(125.2) Assume $\exists y(y = \imath x\varphi)$. then by (125.1), it follows that $\exists\Box y(y = \imath x\varphi)$. But then by the Buridan schema (123.1), it follows that $\Box\exists y(y = \imath x\varphi)$. ⋈

(126.1) Since axiom (37), $xF \rightarrow \Box xF$, is a necessary axiom, it has a modally strict proof. So by RN, $\Box(xF \rightarrow \Box xF)$. So by theorem (120.1), $\Diamond xF \equiv \Box xF$. ⋈

(126.2) ($\rightarrow$) This direction is simply our rigidity of encoding axiom (37). ($\leftarrow$) Assume $\Box xF$. Then by the T schema (32.2), xF. ⋈

(126.3) (→) Since our axiom (37), i.e., $xF \to \Box xF$, is a $\Box$-theorem, it follows by the rule (121.2) that $\Diamond xF \to xF$. (←) Assume xF. Then, by T$\Diamond$ (118), it follows that $\Diamond xF$. ⋈

(126.4) (→) Assume:

(ϑ) $xF \equiv yG$

To show $\Box xF \equiv \Box yG$, we show both directions:

> (→) Assume $\Box xF$. Then by the T schema (32.2), xF. So by (ϑ), yG. Then by axiom (37), $\Box yG$.
>
> (←) By analogous reasoning.

(←) Assume $\Box xF \equiv \Box yG$. Again, we show both directions:

> (→) Assume xF. Then by axiom (37), $\Box xF$. From this and our assumption, $\Box yG$ follows by biconditional elimination (64.6.a). Hence, by the T schema (32.2), yG.
>
> (←) By analogous reasoning. ⋈

(126.5) (→) This direction is immediate from an appropriate instance of theorem (111.6), which asserts that $\Box(\varphi \equiv \psi) \to (\Box\varphi \equiv \Box\psi)$. (←) As an instance of (120.3), we know:

(ϑ) $(\Box(xF \to \Box xF) \,\&\, \Box(yG \to \Box yG)) \to ((\Box xF \equiv \Box yG) \to \Box(xF \equiv yG))$

But $xF \to \Box xF$ and $yG \to \Box yG$ are both just instances of axiom (37). Hence, by RN, we have both $\Box(xF \to \Box xF)$ and $\Box(yG \to \Box yG)$. So it follows from (ϑ) that $(\Box xF \equiv \Box yG) \to \Box(xF \equiv yG)$. ⋈

(126.6) By the commutativity of the biconditional (63.3.g), (126.5) converts to $(\Box xF \equiv \Box yG) \equiv \Box(xF \equiv yG)$. Then (126.4) and this last result imply $(xF \equiv yG) \equiv \Box(xF \equiv yG)$, by $\equiv$E (64.6.e). ⋈

(126.7) Theorem (126.3) is that $\Diamond xF \equiv xF$. So by a classical tautology (63.5.d), it follows that $\neg\Diamond xF \equiv \neg xF$, which by the commutativity of $\equiv$ implies $\neg xF \equiv \neg\Diamond xF$. Independently, as an instance of (117.4), we know that $\Box\neg xF \equiv \neg\Diamond xF$, and by commutativity, $\neg\Diamond xF \equiv \Box\neg xF$. So by the transitivity of $\equiv$, it follows that $\neg xF \equiv \Box\neg xF$. ⋈

(126.8) Theorem (126.2) is that $xF \equiv \Box xF$. So by a classical tautology (63.5.d), it follows that $\neg xF \equiv \neg\Box xF$. Independently, as an instance of (117.2), we know that $\neg\Box xF \equiv \Diamond\neg xF$. So by the transitivity of $\equiv$, it follows that $\neg xF \equiv \Diamond\neg xF$, which commutes to $\Diamond\neg xF \equiv \neg xF$. ⋈

(126.9) (Exercise)

(126.10) (→) Since axiom (37), i.e., $xF \to \Box xF$, is a theorem, we may apply the Rule of Actualization to obtain $\mathcal{A}(xF \to \Box xF)$. But then by axiom (31.2), it follows that $\mathcal{A}xF \to \mathcal{A}\Box xF$. Now as an instance of the right-to-left direction of axiom (33.2), we know $\mathcal{A}\Box xF \to \Box xF$. So, by hypothetical syllogism, $\mathcal{A}xF \to \Box xF$. But as an instance of the T schema, we know $\Box xF \to xF$. Hence, again by hypothetical syllogism, $\mathcal{A}xF \to xF$. (←) Axiom (37) is $xF \to \Box xF$ and as an instance of theorem (89) we know $\Box xF \to \mathcal{A}xF$. So by hypothetical syllogism, it follows that $xF \to \mathcal{A}xF$. ⋈

(127.1) By hypothesis, $y_1, \ldots, y_n$ are substitutable, respectively, for $x_1, \ldots, x_n$ in φ^* and ψ^*, and don't occur free in φ and ψ^*. Now assume:

(ϑ) $\varphi^* \equiv \psi^*$

Since the conditions of the Re-replacement Lemma (81.1) are met by hypothesis, we may reason using β-Conversion (36.2) as follows:

$$\begin{array}{lll}
[\lambda y_1 \ldots y_n\, \varphi^{*y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n & \equiv [\varphi^{*y_1,\ldots,y_n}_{x_1,\ldots,x_n}]^{x_1,\ldots,x_n}_{y_1,\ldots,y_n} & \text{by } \beta\text{-Conversion} \\
& \equiv \varphi^* & \text{by Re-replacement} \\
& \equiv \psi^* & \text{by } (\vartheta) \\
& \equiv [\psi^{*y_1,\ldots,y_n}_{x_1,\ldots,x_n}]^{x_1,\ldots,x_n}_{y_1,\ldots,y_n} & \text{by Re-replacement} \\
& \equiv [\lambda y_1 \ldots y_n\, \psi^{*y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n & \text{by } \beta\text{-Conversion}
\end{array}$$

We've therefore proved from our assumption that:

$$[\lambda y_1 \ldots y_n\, \varphi^{*y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n \equiv [\lambda y_1 \ldots y_n\, \psi^{*y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n$$

⋈

(127.2)

(128) This follows from the facts that (1) every instance of the Strengthened β-Conversion is an alphabetic variant of the stated version of β-Conversion (36.2) and (2) alphabetic variants are interderivable (115). ⋈

(129.1) The axiom schema of β-Conversion is:

$$[\lambda y_1 \ldots y_n\, \varphi^*]x_1 \ldots x_n \equiv \varphi^{*x_1,\ldots,x_n}_{y_1,\ldots,y_n}$$

But we have taken the closures of this schema as axioms. So the following is an axiom:

$$\Box\forall x_1 \ldots \forall x_n([\lambda y_1 \ldots y_n\, \varphi^*]x_1 \ldots x_n \equiv \varphi^{*x_1,\ldots,x_n}_{y_1,\ldots,y_n})$$

Hence, by $\exists$I, it follows that:

$$\exists F^n \Box\forall x_1 \ldots \forall x_n(F^n x_1 \ldots x_n \equiv \varphi^*) \qquad ⋈$$

(129.2) (Exercise)

(130) Assume:

(ϑ) $\forall x(xF \equiv xG)$

as a premise for conditional proof, to show $F = G$. By definition (16.1), we have to show $\Box\forall x(xF \equiv xG)$. By the Barcan formula, it suffices to show $\forall x\Box(xF \equiv xG)$. Now from our initial assumption, it follows by Rule $\forall$E that:

(ζ) $xF \equiv xG$

Now by (126.2), we know both:

(ξ) $xF \equiv \Box xF$

(ω) $xG \equiv \Box xG$

So starting with $\Box xF$, we can appeal to (ξ) (right-to-left), then (ζ) (left-to-right), and then (ω) (left-to-right), to show $\Box xF \equiv \Box xG$. From this, we may infer $\Box(xF \equiv xG)$, by (126.5) (set y in (126.5) to x). Since x isn't free in our initial premise (ϑ), we may apply GEN to conclude $\forall x\Box(xF \equiv xG)$. ⋈

(131.1) As a 0-place instance of η-Conversion (36.3), we have $[\lambda\, p] = p$, where p is a 0-place relation variable. By GEN, it follows that:

(ϑ) $\forall p([\lambda\, p] = p)$

Since λ binds no variables in $[\lambda\, p]$, every propositional formula φ^* in the language is a 0-place relation term that is substitutable for p in the matrix formula of (ϑ), i.e., every φ^* is substitutable for p in $[\lambda\, p] = p$. Moreover, φ^* is not a description. So we may apply our Rule $\forall$E to conclude $[\lambda\, \varphi^*] = \varphi^*$. ⋈

(131.2) By (131.1), we know $[\lambda\, \varphi^*] = \varphi^*$. Moreover, as an instance of the tautology $\psi \equiv \psi$, we know $[\lambda\, \varphi^*] \equiv [\lambda\, \varphi^*]$. (Recall that $[\lambda\, \varphi^*]$ is a 0-place relation term, and hence a formula.) So by Rule SubId (74.2) it follows that $[\lambda\, \varphi^*] \equiv \varphi^*$. ⋈

(131.3) We want to show that $\varphi^* = \varphi^{*\prime}$, where $\varphi^{*\prime}$ is any alphabetic variant of φ^*. By the 0-place case of α-Conversion (36.1), we know:

(ϑ) $[\lambda\, \varphi^*] = [\lambda\, \varphi^*]'$

where $[\lambda\, \varphi^*]'$ is any alphabetic variant of $[\lambda\, \varphi^*]$. By η-Conversion for 0-place λ-expressions and propositional formulas that we just proved (131.1), we also know:

(ξ) $[\lambda\, \varphi^*] = \varphi^*$

Hence, by from (ϑ) and (ξ) it follows by Rule SubId that:

$$\varphi^* = [\lambda\, \varphi^*]'$$

Moreover, by another instance of η-Conversion for 0-place λ-expressions and propositional formulas (131.1), we also know:

(ζ) $[\lambda\, \varphi^*]' = \varphi^{*\prime}$

Hence, it follows from (ζ) and the previous result that:

$$\varphi^* = \varphi^{*\prime} \qquad \bowtie$$

(131.4) We want to show that $\mathcal{A}(\varphi \equiv \psi) \to \chi^* = \chi^{*\prime}$, where $\chi^{*\prime}$ is the result of substituting $\iota x\psi$ for zero or more occurrences of $\iota x\varphi$ anywhere the latter occurs in χ^*. So assume $\mathcal{A}(\varphi \equiv \psi)$. Then it follows from the 0-place case of ι-Conversion (36.4) that:

(ϑ) $[\lambda\, \chi^*] = [\lambda\, \chi^{*\prime}]$

By η-Conversion for 0-place λ-expressions and propositional formulas that we just proved (131.1), we also know:

(ξ) $[\lambda\, \chi^*] = \chi^*$

Hence, by from (ϑ) and (ξ) it follows by Rule SubId that:

$$\chi^* = [\lambda\, \chi^{*\prime}]$$

Moreover, by another instance of η-Conversion for 0-place λ-expressions and propositional formulas (131.1), we also know:

(ζ) $[\lambda\, \chi^{*\prime}] = \chi^{*\prime}$

Hence, it follows from (ζ) and the previous result that:

$$\chi^* = \chi^{*\prime} \qquad \bowtie$$

(135) Assume $\Diamond\neg\forall x(Fx \equiv Gx)$, which by (117.2), implies $\neg\Box\forall x(Fx \equiv Gx)$. For reductio, assume $F = G$. Then by an appropriate instance of our axiom for the substitution of identicals (25), it follows that $\Box\forall x(Fx \equiv Fx) \to \Box\forall x(Fx \equiv Gx)$. But $Fx \equiv Fx$ is a tautology, which by GEN yields $\forall x(Fx \equiv Fx)$ and by RN yields $\Box\forall F(Fx \equiv Fx)$. Hence, $\Box\forall x(Fx \equiv Gx)$. Contradiction. $\bowtie$

(137.1) By β-conversion (36.2), we know:

(ϑ) $[\lambda y_1 \ldots y_n\, \neg Fy_1 \ldots y_n]x_1 \ldots x_n \equiv \neg Fx_1 \ldots x_n$

Now definition (136.1) is that $\overline{F^n} = [\lambda y_1 \ldots y_n \neg F y_1 \ldots y_n]$, which by symmetry of relation identity (67.5) yields $[\lambda y_1 \ldots y_n \neg F y_1 \ldots y_n] = \overline{F^n}$. Hence, by substituting the latter for the former in (ϑ), it follows by substitution of identicals that $\overline{F^n} x_1 \ldots x_n \equiv \neg F^n x_1 \ldots x_n$. ⋈

(137.2) (Exercise)

(137.3) By β-Conversion for 0-place terms (131.2), we know $[\lambda\, \neg p] \equiv \neg p$. But definition (136.2) is $\overline{p} = [\lambda\, \neg p]$, which by symmetry of proposition identity (67.8) yields $[\lambda\, \neg p] = \overline{p}$. So by substitution of identicals it follows that $\overline{p} \equiv \neg p$. ⋈

(137.4) (Exercise)

(137.5) Suppose, for reductio, that $F^n = \overline{F^n}$, which by symmetry of relation identity (67.5) implies $\overline{F^n} = F^n$. Since theorem (137.1) is that $\overline{F^n} x_1 \ldots x_n \equiv \neg F^n x_1 \ldots x_n$, it follows by substitution of identicals that $F^n x_1 \ldots x_n \equiv \neg F^n x_1 \ldots x_n$, which is a contradiction (63.1.b). ⋈

(137.6) Assume for reductio that $p = \overline{p}$, which by symmetry of proposition identity (67.8) implies $\overline{p} = p$. Then, by substitution of identicals into theorem (137.3), i.e., $\overline{p} \equiv \neg p$, it follows that $p \equiv \neg p$, which is a contradiction (63.1.b). ⋈

(137.7) By definition of $\overline{p}$ (136.2), we have $\overline{p} = [\lambda \neg p]$. Moreover, by η-conversion for 0-place terms (131.1), we have $[\lambda\, \neg p] = \neg p$. So it follows by the transitivity of identity for propositions (67.9) that $\overline{p} = \neg p$. ⋈

(137.8) (Exercise)

(139.1) ($\rightarrow$) Assume *NonContingent*(F). Then by the 1-place case of definition (138.3), we know *Necessary*$(F) \vee$ *Impossible*(F), and so by definitions (138.1) and (138.2):

(ϑ) $\Box \forall x F x \vee \Box \forall x \neg F x$

Now we know by special cases of (137.2) and (137.1), respectively, that the following are $\Box$-theorems: $\neg \overline{F} x \equiv F x$ and $\overline{F} x \equiv \neg F x$. Applying the commutativity of $\equiv$ to each, we therefore have the following $\Box$-theorems:

(a) $Fx \equiv \neg \overline{F} x$

(b) $\neg F x \equiv \overline{F} x$

Hence, by the Rule of Substitution, (ϑ) and (a) yield $\Box \forall x \neg \overline{F} x \vee \Box \forall x \neg F x$, and from this latter and (b), by the Rule of Substitution, we have:

$$\Box \forall x \neg \overline{F} x \vee \Box \forall x \overline{F} x$$

So by the commutativity of $\vee$, it follows that:

$$\Box\forall x\overline{F}x \vee \Box\forall x\neg\overline{F}x$$

i.e., $Necessary(\overline{F}) \vee Impossible(\overline{F})$, by (138.1) and (138.2). Hence $NonContingent(\overline{F})$, by (138.3). (←) Reverse the reasoning. ⋈

(137.9) Assume $p = q$. By Rule ReflId (74.1) and the fact that $\neg p$ is a 0-place term, we know $\neg p = \neg p$. Hence $\neg p = \neg q$, by Rule SubId (74.2). ⋈

(137.10) (Exercise)

(139.2) (→) Assume $Contingent(F)$. By definition (138.4), this is $\neg(Necessary(F) \vee Impossible(F))$, and so by definitions (138.1) and (138.2):

$$\neg(\Box\forall xFx \vee \Box\forall x\neg Fx)$$

By De Morgan's Law (63.6.d), it follows that:

$$\neg\Box\forall xFx \ \&\ \neg\Box\forall x\neg Fx$$

Using (117.2) on both conjuncts, it follows that:

$$\Diamond\neg\forall xFx \ \&\ \Diamond\neg\forall x\neg Fx$$

Using a quantifier-negation □-theorem (86.2) with the Rule of Substitution on the left conjunct, and the definition of ∃ (7.4.d) on the right conjunct, it follows that:

$$\Diamond\exists x\neg Fx \ \&\ \Diamond\exists xFx$$

Finally, by the commutativity of & (63.3.b), it follows that:

$$\Diamond\exists xFx \ \&\ \Diamond\exists x\neg Fx$$

(←) Reverse the reasoning. ⋈

(139.3) (→) Assume $Contingent(F)$. Then by definition (138.4), we know $\neg(Necessary(F) \vee Impossible(F))$, and so by definitions (138.1) and (138.2):

(ϑ) $\neg(\Box\forall xFx \vee \Box\forall x\neg Fx)$

Now we know by special cases of (137.2) and (137.1), respectively, that the following are □-theorems: $\neg\overline{F}x \equiv Fx$ and $\overline{F}x \equiv \neg Fx$. Applying the commutativity of ≡ to both, we therefore have, respectively:

(a) $Fx \equiv \neg\overline{F}x$

(b) $\neg Fx \equiv \overline{F}x$

Hence, by the Rule of Substitution, (ϑ) and (a) imply $\neg(\Box\forall x\neg\overline{F}x \vee \Box\forall x\neg Fx)$, and the latter and (b) imply, by the Rule of Substitution:

(ξ) $\neg(\Box\forall x\neg\overline{F}x \vee \Box\forall x\overline{F}x)$

Now the following equivalence, based on the commutativity of $\vee$, is a $\Box$-theorem:

$$(\Box\forall x\neg\overline{F}x \vee \Box\forall x\overline{F}x) \vee (\Box\forall x\overline{F}x \vee \Box\forall x\neg\overline{F}x)$$

From this $\Box$-theorem, we can use the Rule of Substitution to transform (ξ) into:

$$\neg(\Box\forall x\overline{F}x \vee \Box\forall x\neg\overline{F}x)$$

i.e., $\neg(Necessary(\overline{F}) \vee Impossible(\overline{F}))$, by (138.1) and (138.2). Hence, $Contingent(\overline{F})$, by (138.3). ($\leftarrow$) Reverse the reasoning. ⋈

(140.1) By definition (138.1), we have to show $\Box\forall x_1 Lx_1$. Note that as an instance of the $\Box$-theorem (53), we have $E!y \to E!y$. But the following is an instance of β-conversion (128):

$$[\lambda x\, E!x \to E!x]y \equiv E!y \to E!y$$

So by biconditional syllogism, we have $[\lambda x\, E!x \to E!x]y$, i.e., Ly. So, by GEN, $\forall y Ly$ is a $\Box$-theorem. So by RN, $\Box\forall y Ly$. This conclusion suffices, since it and $\Box\forall x_1 Lx_1$ are alphabetic variants, which by (115) are interderivable. ⋈

(140.2) By the commutativity of $\equiv$, it follows from the 1-place case of theorem (137.2) that $Fx_1 \equiv \neg\overline{F}x_1$. By two applications of GEN, it then follows that $\forall F\forall x_1(Fx_1 \equiv \neg\overline{F}x_1)$. So by $\forall$E, we may instantiate this to L to obtain:

(a) $\forall x_1(Lx_1 \equiv \neg\overline{L}x_1)$

Since theorem (140.1), by definition (138.1), implies $\Box\forall x_1 Lx_1$, it follows that, by an appropriate instance of the T schema (32.2), that:

(b) $\forall x_1 Lx_1$

Hence using an appropriate instance of (83.3), it follows from (a) and (b) that:

$$\forall x_1\neg\overline{L}x_1$$

Since this is a $\Box$-theorem, it follows by RN that $\Box\forall x_1\neg\overline{L}x_1$, i.e., by definition (138.2), that $Impossible(\overline{L})$. ⋈

(140.3) (Exercise)

(140.4) This follows either by theorems (140.3) and (139.1), or by theorem (140.2) and definition (138.3). ⋈

(140.5) (Exercise) [Hint: The simplest way to prove this is to use (140.3), (140.4), and facts about the distinctness of properties that are negations of one another. Another way to prove this is to use (135).]

(141) ($\rightarrow$) We may reason as follows:[276]

$$\begin{array}{rll} \Diamond\exists x(Fx \,\&\, \Diamond\neg Fx) & \rightarrow\ \exists x\Diamond(Fx \,\&\, \Diamond\neg Fx) & \text{by BF}\Diamond\ (122.3) \\ & \rightarrow\ \exists x(\Diamond Fx \,\&\, \Diamond\neg Fx) & \text{by (119.12) and (113)} \\ & \rightarrow\ \exists x(\Diamond\neg Fx \,\&\, \Diamond Fx) & \text{by (63.3.b) and (113)} \\ & \rightarrow\ \exists x\Diamond(\neg Fx \,\&\, \Diamond Fx) & \text{by (119.12) and (113)} \\ & \rightarrow\ \Diamond\exists x(\neg Fx \,\&\, \Diamond Fx) & \text{by CBF}\Diamond\ (122.4) \end{array}$$

($\leftarrow$) By analogous reasoning. ⋈

(142.1) Axiom (32.4) is:

$$\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x) \,\&\, \Diamond\neg\exists x(E!x \,\&\, \Diamond\neg E!x)$$

So, by &E, we know:

$$\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x)$$

By applying GEN to (141), we also know:

$$\forall F[\Diamond\exists x(Fx \,\&\, \Diamond\neg Fx) \equiv \Diamond\exists x(\neg Fx \,\&\, \Diamond Fx)]$$

and by instantiating $E!$ for $\forall F$, it follows that:

$$\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x) \equiv \Diamond\exists x(\neg E!x \,\&\, \Diamond E!x)$$

Hence, by biconditional syllogism, it follows that $\Diamond\exists x(\neg E!x \,\&\, \Diamond E!x)$. ⋈

(142.2) In light of (139.2), it suffices to establish:

(ϑ) $\Diamond\exists x E!x \,\&\, \Diamond\exists x\neg E!x$.

Note independently that from (123.3) and $\Diamond(\exists\alpha\varphi \,\&\, \exists\alpha\psi) \rightarrow (\Diamond\exists\alpha\varphi \,\&\, \Diamond\exists\alpha\psi)$, which is an instance of (117.8), it follows by hypothetical syllogism that:

(ξ) $\Diamond\exists\alpha(\varphi \,\&\, \psi) \rightarrow (\Diamond\exists\alpha\varphi \,\&\, \Diamond\exists\alpha\psi)$

But clearly the consequent of (ξ) implies $\Diamond\exists\alpha\varphi$. So it follows by hypothetical syllogism that:

(ζ) $\Diamond\exists\alpha(\varphi \,\&\, \psi) \rightarrow \Diamond\exists\alpha\varphi$

As instances of (ζ), we have both of the following:

(a) $\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x) \rightarrow \Diamond\exists x E!x$

(b) $\Diamond\exists x(\neg E!x \,\&\, \Diamond E!x) \rightarrow \Diamond\exists x\neg E!x$

[276] The following proof by Uri Nodelman is much simpler than my original.

But now the left conjunct of (ϑ) follows from (a) and the left conjunct of axiom (32.4) by MP, and the right conjunct of (ϑ) follows from (b) and (142.1) by MP. $\bowtie$

(142.3) From (142.2) and an appropriate instance of (139.3) by biconditonal syllogism. $\bowtie$

(142.4) From (142.2), (142.3), and an appropriate instance of the 1-place case of (137.5). $\bowtie$

(143.1) Assume $NonContingent(F)$. By definition (138.3), we have $Necessary(F) \vee Impossible(F)$. By Double Negation Introduction, it follows that $\neg\neg(Necessary(F) \vee Impossible(F))$. Hence, by the definition of contingent properties (138.4) it follows that $\neg Contingent(F)$. Now assume, for reductio, that $\exists G(Contingent(G) \,\&\, G = F)$. Let P be an arbitrary such property, so that we have $Contingent(P) \,\&\, P = F$. Applying the second conjunct and the substitution of identicals, the first conjunct implies $Contingent(F)$, which is a contradiction. Since we've proved a contradiction from $Contingent(P) \,\&\, P = F$, it follows by $\exists$E that the contradiction follows our reductio assumption. Hence, $\neg\exists G(Contingent(G) \,\&\, G = F)$. $\bowtie$

(143.2) (Exercise) [Hint: Use reasoning similar to that of (143.1).]

(143.3) (Exercise)

(143.4) By four applications of $\exists$I to (143.3). $\bowtie$

(144.1) ($\rightarrow$) Assume $NonContingent(p)$. Then by the 0-place case of definition (138.3), we know $Necessary(p) \vee Impossible(p)$, and so by definitions (138.1) and (138.2):

(ϑ) $\Box p \vee \Box\neg p$

We also know (137.4) and (137.3), i.e., that the following are $\Box$-theorems: $\neg\overline{p} \equiv p$ and $\overline{p} \equiv \neg p$. Applying the commutativity of $\equiv$ to both, we therefore have, as $\Box$-theorems:

(a) $p \equiv \neg\overline{p}$

(b) $\neg p \equiv \overline{p}$

From (ϑ) and (a), it follows by the Rule of Substitution that $\Box\neg\overline{p} \vee \Box\neg p$, and from this and (b), the Rule of Substitution implies: $\Box\neg\overline{p} \vee \Box\overline{p}$. By the commutativity of $\vee$, this implies: $\Box\overline{p} \vee \Box\neg\overline{p}$, i.e., $Necessary(\overline{p}) \vee Impossible(\overline{p})$, by (138.1) and (138.2). Hence, $NonContingent(\overline{p})$, by (138.3). ($\leftarrow$) Reverse the reasoning. $\bowtie$

(144.2) Assume $Contingent(p)$. Then by the 0-place case of definition (138.4), we know $\neg(Necessary(p) \vee Impossible(p))$, and so by definitions (138.1) and (138.2):

$$\neg(\Box p \vee \Box\neg p)$$

By De Morgan's Law (63.6.d), it follows that:

$$\neg\Box p \;\&\; \neg\Box\neg p$$

Using (117.2) on the left conjunct and applying the definition of $\Diamond$ to the right conjunct, we obtain:

$$\Diamond\neg p \;\&\; \Diamond p$$

Finally, by the commutativity of & (63.3.b), it follows that:

$$\Diamond p \;\&\; \Diamond\neg p$$

($\leftarrow$) Reverse the reasoning. ⋈

(144.3) ($\rightarrow$) Assume *Contingent*(p). Then by the 0-place case of definition (138.4), we know $\neg(\mathit{Necessary}(p) \vee \mathit{Impossible}(p))$, and so by definitions (138.1) and (138.2):

(ϑ) $\neg(\Box p \vee \Box\neg p)$

We also know (137.4) and (137.3), i.e., that the following are $\Box$-theorems: $\neg\overline{p} \equiv p$ and $\overline{p} \equiv \neg p$. Applying the commutativity of $\equiv$ to each, we therefore have as $\Box$-theorems:

(a) $p \equiv \neg\overline{p}$

(b) $\neg p \equiv \overline{p}$

From (ϑ) and (a), it follows by the Rule of Substitution that $\neg(\Box\neg\overline{p} \vee \Box\neg p)$, and from this result and (b), it follows by the same rule that:

$$\neg(\Box\neg\overline{p} \vee \Box\overline{p})$$

So we may use an appropriate instance of the commutativity of $\vee$ (which is a $\Box$-theorem) and the Rule of Substitution to transform the last formula into:

$$\neg(\Box\overline{p} \vee \Box\neg\overline{p})$$

i.e., $\neg(\mathit{Necessary}(\overline{p}) \vee \mathit{Impossible}(\overline{p}))$, by (138.1) and (138.2). Hence, *Contingent*($\overline{p}$), by (138.3). ($\leftarrow$) Reverse the reasoning. ⋈

(145.1) Since $E!x \rightarrow E!x$ is an instance of tautology (53), it follows by GEN that $\forall x(E!x \rightarrow E!x)$. Since this is a $\Box$-theorem, it follows by RN that $\Box\forall x(E!x \rightarrow E!x)$. By definition of p_0, then, it follows that $\Box p_0$. Hence by the 0-place case of definition (138.1), it follows that *Necessary*(p_0). ⋈

(145.2) By the reasoning in (145.1), we can establish the following as a theorem:

(a) $\Box p_0$

Note that by the commutativity of $\equiv$, it follows from theorem (137.4) that $p \equiv \neg\overline{p}$, and hence by GEN that $\forall p(p \equiv \neg\overline{p})$. Instantiating this last claim to p_0, we obtain $p_0 \equiv \neg\overline{p_0}$. So from this $\Box$-theorem, we may apply the Rule of Substitution to (a) to obtain $\Box\neg\overline{p_0}$. Hence, by the 0-place case of definition (138.2), it follows that *Impossible*$(\overline{p_0})$. ⋈

(145.3) (Exercise)

(145.4) (Exercise)

(145.5) (Exercise)

(146.1) (Exercise)

(146.2) (Exercise)

(146.3) (Exercise)

(146.4) (Exercise)

(147.1) Assume *NonContingent*(p). By the 0-place case of definition (138.3), we have *Necessary*$(p) \vee$ *Impossible*(p). By Double Negation Introduction, it follows that $\neg\neg$(*Necessary*$(p) \vee$ *Impossible*(p)). Hence, by the definition of contingent properties (138.4) it follows that $\neg$*Contingent*(p). Now assume, for reductio, that $\exists q$(*Contingent*(q) & $q = p$). Assume further that q_1 is an arbitrary such property, so that *Contingent*(q_1) & $q_1 = p$. Applying the second conjunct and the substitution of identicals, the first conjunct implies *Contingent*(p), which is a contradiction. Since we've proved a contradiction from *Contingent*(q_1) & $q_1 = p$, we may discharge our second assumption, by $\exists$E, to conclude that a contradiction follows our reductio assumption. Hence, $\neg\exists q$(*Contingent*(q) & $q = p$). ⋈

(147.2) (Exercise)

(147.3) (Exercise)

(147.4) By four applications of $\exists$I to (147.3). ⋈

(149.1) – **(149.2)** (Exercises)

(149.3) Assume *ContingentlyTrue*(p). Then by definition (148.1), we know:

(ϑ) $p \,\&\, \Diamond\neg p$

By (148.2), we have to show: $\neg\overline{p} \,\&\, \Diamond\overline{p}$. By (137.4), the first conjunct of (ϑ) implies $\neg\overline{p}$. So it remains to show $\Diamond\overline{p}$. But by the commutativity of the biconditional, (137.3) yields a modally strict proof of $\neg p \equiv \overline{p}$. Hence it follows by the Rule of Substitution from the second conjunct of (ϑ) that $\Diamond\overline{p}$. ⋈

(149.4) Assume *ContingentlyFalse*(p). Then by definition (148.2), we know:

(ϑ) $\neg p \,\&\, \Diamond p$

By (148.1), we have to show: $\overline{p} \,\&\, \Diamond\neg\overline{p}$. By (137.3), the first conjunct of (ϑ) implies $\overline{p}$. So it remains to show $\Diamond\overline{p}$. But by the commutativity of the biconditional, (137.4) yields a modally strict proof of $p \equiv \neg\overline{p}$. Hence it follows by the Rule of Substitution from the second conjunct of (ϑ) that $\Diamond\neg\overline{p}$. ⋈

(150.1) Where q_0 is $\exists x(E!x \,\&\, \Diamond\neg E!x)$, axiom (32.4) becomes $\Diamond q_0 \,\&\, \Diamond\neg q_0$. With this, we can now prove our theorem by disjunctive syllogism from the tautology $q_0 \vee \neg q_0$. From q_0 and the second conjunct of (32.4), we have $q_0 \,\&\, \Diamond\neg q_0$ and, hence, $ContingentlyTrue(q_0)$. From $\neg q_0$ and the first conjunct of (32.4), we have $\neg q_0 \,\&\, \Diamond q_0$ and, hence, $ContingentlyFalse(q_0)$. ⋈

(150.2) By (150.1), we know:

(ϑ) $ContingentlyTrue(q_0) \vee ContingentlyFalse(q_0)$

But, independently, it is a modally strict consequence of (149.1) that $ContingentlyTrue(q_0) \equiv ContingentlyFalse(\overline{q}_0)$. So by the Rule of Substitution, it follows from (ϑ) that:

$$ContingentlyFalse(\overline{q}_0) \vee ContingentlyFalse(q_0)$$

So by the commutativity of disjunction, $ContingentlyFalse(q_0) \vee ContingentlyFalse(\overline{q}_0)$. ⋈

(150.3) By (150.1), we know:

$$ContingentlyTrue(q_0) \vee ContingentlyFalse(q_0)$$

So we may reason by cases from the two disjuncts. If $ContingentlyTrue(q_0)$, then $\exists p ContingentlyTrue(p)$. If $ContingentlyFalse(q_0)$, then $ContingentlyTrue(\overline{q_0})$, by (149.4). So $\exists p ContingentlyTrue(p)$. ⋈

(150.4) By (150.1), we know:

$$ContingentlyTrue(q_0) \vee ContingentlyFalse(q_0)$$

So we may reason by cases from the two disjuncts. If $ContingentlyTrue(q_0)$, then by (149.3), $ContingentlyFalse(\overline{q_0})$. So $\exists p ContingentlyFalse(p)$. If, on the other hand, $ContingentlyFalse(q_0)$, then $\exists p ContingentlyFalse(p)$. ⋈

(150.5) Assume $ContingentlyTrue(p)$ and $Necessary(q)$. Assume for reductio that $p = q$. Then $Necessary(p)$, i.e., $\Box p$, i.e., $\neg\Diamond\neg p$. But since p is contingently false, it follows by definition (148) that $\Diamond\neg p$. Contradiction. ⋈

(150.6) Assume $ContingentlyFalse(p)$ and $Impossible(q)$. Assume for reductio that $p = q$. Then $Impossible(p)$, i.e., $\Box\neg p$, i.e., $\neg\Diamond p$. But since p is contingently false, it follows by definition (148) that $\Diamond p$. Contradiction. ⋈

(152.1) By definition (18), we have to show $\neg(O! = A!)$. For reductio, assume $O! = A!$. By definitions (11.1) and (11.2) it follows that $[\lambda x\, \Diamond E!x] = [\lambda x\, \neg\Diamond E!x]$. Now the following is an instance of β-Conversion (128):

$$[\lambda x \diamond E!x]x \equiv \diamond E!x$$

So by Rule SubId (74.2), it follows that:

(ϑ) $[\lambda x \neg\diamond E!x]x \equiv \diamond E!x$

But the following is also an instance of Strengthened β-Conversion:

(ξ) $[\lambda x \neg\diamond E!x]x \equiv \neg\diamond E!x$

So from (ϑ) and (ξ) it follows by biconditional syllogism (64.6.f) that $\diamond E!x \equiv \neg\diamond E!x$, which by (63.1.b) is a contradiction. ⋈

(152.2) The following are instances of β-Conversion (128):

(a) $[\lambda x \diamond E!x]x \equiv \diamond E!x$

(b) $[\lambda x \neg\diamond E!x]x \equiv \neg\diamond E!x$

From (b), it follows by (63.5.d) that:

(c) $\neg[\lambda x \neg\diamond E!x]x \equiv \neg\neg\diamond E!x$

Note independently that $\diamond E!x \equiv \neg\neg\diamond E!x$ an instance of a tautology (63.4.b), which by the commutativity of $\equiv$ (63.8) becomes $\neg\neg\diamond E!x \equiv \diamond E!x$. It follows from this and (c) by biconditional syllogism that:

(d) $\neg[\lambda x \neg\diamond E!x]x \equiv \diamond E!x$

By the commutativity of the biconditional, (d) implies:

(e) $\diamond E!x \equiv \neg[\lambda x \neg\diamond E!x]x$

So from (a) and (e) it follows by a biconditional syllogism that:

(f) $[\lambda x \diamond E!x]x \equiv \neg[\lambda x \neg\diamond E!x]x$

But (f) is, by definitions (11.1), (11.2), just is $O!x \equiv \neg A!x$. ⋈

(152.3) (Exercise)

(152.4) By (139.2), it suffices to establish $\diamond\exists x O!x \,\&\, \diamond\exists x\neg O!x$. By &I, it suffices to prove both conjuncts.

To prove the first conjunct, note that by theorem (142.2), *Contingent*($E!$), which by (139.2), is equivalent to: $\diamond\exists x E!x \,\&\, \diamond\exists x\neg E!x$. By detaching the first conjunct and applying BF$\diamond$ (122.3), we obtain $\exists x\diamond E!x$. This latter, given the Rule of Substitution and the instance of β-Conversion (128) that $[\lambda x \diamond E!x]x \equiv \diamond E!x$ (a $\square$-theorem), yields $\exists x([\lambda x \diamond E!x]x)$. But by definition of $O!$ (11.1), this is just $\exists x O!x$. So by T$\diamond$ (118), we have $\diamond\exists x O!x$.

To prove the second conjunct, pick any instance of the comprehension axiom for abstract objects. It has the form: $\exists x(A!x \,\&\, \ldots)$. By (86.5), it follows that

$\exists x A!x$. Now we know by the previous theorem (152.2) that $O!x \equiv \neg A!x$. It is easy to prove from this that $A!x \equiv \neg O!x$ (exercise). This is a $\Box$-theorem and so by the Rule of Substitution, we may infer $\exists x \neg O!x$ from $\exists x A!x$. But by T$\Diamond$, it then follows that $\Diamond \exists x \neg O!x$. ⋈

(152.5) (Exercise)

(152.6) By (18), we have to show $\neg(\overline{O!} = \overline{A!})$. Suppose, for reductio, that $\overline{O!} = \overline{A!}$. By applying the definition of relation negation (136.1) to both terms, it follows that $[\lambda y\, \neg O!y] = [\lambda y\, \neg A!y]$. But as an instance of β-Conversion, we have: $[\lambda y\, \neg O!y]x \equiv \neg O!x$. So by Rule SubId, it follows that $[\lambda y\, \neg A!y]x \equiv \neg O!x$. Since the β-Conversion instance $[\lambda y \neg A!y]x \equiv \neg A!x$ is a $\Box$-theorem, it follows by the Rule of Substitution that $\neg A!x \equiv \neg O!x$ (Note that the Rule of Substitution saves a step here; without it, we have to commute the sides of the last instance of β-Conversion and then apply transivitity of $\equiv$.) But by (152.2), we know $O!x \equiv \neg A!x$, and so by a biconditional syllogism, it follows that $O!x \equiv \neg O!x$, which by (63.1.b) is a contradiction. Hence $\neg(\overline{O!} = \overline{A!})$, i.e., $\overline{O!} \neq \overline{A!}$. ⋈

(152.7) From (152.2) by (63.5.d), (58) and the Rule of Substitution. ⋈

(152.8) From (152.4) and (139.3). ⋈

(152.9) From (152.5) and (139.3). ⋈

(153.1) Assume $O!x$, for conditional proof. Then by definition of $O!$ (11.1), we know $[\lambda x\, \Diamond E!x]x$. So by β-Conversion (128), it follows that $\Diamond E!x$. This implies $\Box\Diamond E!x$, by the 5 schema (32.3). Since the β-Conversion (128) instance $[\lambda x\, \Diamond E!x]x \equiv \Diamond E!x$ is a $\Box$-theorem, it follows by the Rule of Substitution that $\Box[\lambda x\, \Diamond E!x]x$, which by the definition of $O!$ yields $\Box O!x$. ⋈

(153.2) Assume $A!x$, for conditional proof. Then by definition of $A!$ (11.2), we know $[\lambda x\, \neg\Diamond E!x]x$. So by β-Conversion (128), it follows that $\neg\Diamond E!x$. This implies $\Box\neg E!x$, by (117.4). By the 4 schema (119.6) it follows that $\Box\Box\neg E!x$. Now using our modally-strict equivalence (117.4), we apply the Rule of Substitution to obtain $\Box\neg\Diamond E!x$. By an appropriate instance of β-Conversion (128), we can again appeal to the Rule of Substitution to infer $\Box[\lambda x\, \neg\Diamond E!x]x$, which by the definition of $A!$ yields $\Box A!x$. ⋈

(153.3) We may apply Derived Rule (121.2) to the $\Box$-theorem (153.1) to conclude $\Diamond O!x \to O!x$. ⋈

(153.4) We may apply Derived Rule (121.2) to the $\Box$-theorem (153.2) to conclude $\Diamond A!x \to A!x$. ⋈

(153.5) ($\to$) By hypothetical syllogism from (153.3) and (153.1). ($\leftarrow$) By the T and T$\Diamond$ schemata. ⋈

(153.6) (→) By hypothetical syllogism from (153.4) and (153.2). (←) By the T and T$\diamond$ schemata. ⋈

(153.7) (→) Assume $O!x$. By (153.1), it follows that $\Box O!x$. By (89), it follows that $\mathcal{A}O!x$. (←) Assume $\mathcal{A}O!x$. By definition of $O!$ (11.1), this is just $\mathcal{A}[\lambda x\, \diamond E!x]x$. Now β-Conversion (128) yields the $\Box$-theorem:

(ϑ) $[\lambda x\, \diamond E!x]x \equiv \diamond E!x$

So by the Rule of Substitution, it follows that $\mathcal{A}\diamond E!x$. But then by the right-to-left direction of (95.6), it follows that $\diamond E!x$. Hence by (ϑ), it follows that $[\lambda x\, \diamond E!x]x$, i.e., $O!x$, by definition of $O!$. ⋈

(153.8) (→) Assume $A!x$. By (153.2), it follows that $\Box A!x$. By (89), it follows that $\mathcal{A}A!x$. (←) Assume $\mathcal{A}A!x$. By definition of $A!$ (11.2), this is just $\mathcal{A}[\lambda x\, \neg\diamond E!x]x$. Now β-Conversion (128) yields the $\Box$-theorem:

(ϑ) $[\lambda x\, \neg\diamond E!x]x \equiv \neg\diamond E!x$

So by the Rule of Substitution, it follows that $\mathcal{A}\neg\diamond E!x$. Note independently that it is an instance of a $\Box$-theorem (117.4) that $\Box\neg E!x \equiv \neg\diamond E!x$, which commutes to $\neg\diamond E!x \equiv \Box\neg E!x$. So by the Rule of Substitution, it follows that $\mathcal{A}\Box\neg E!x$. By the right-to-left direction of axiom (33.2), it follows that $\Box\neg E!x$, which by our theorem (117.4) implies $\neg\diamond E!x$. Hence by (ϑ), it follows that $[\lambda x\, \neg\diamond E!x]x$, i.e., $A!x$, by definition of $A!$. ⋈

(155.1) (→) Assume *WeaklyContingent*(F). Then by definition (154) and &E, we know both *Contingent*(F) and $\forall x(\diamond Fx \to \Box Fx)$. From the former, it follows that *Contingent*($\overline{F}$), by (139.3). So, by the definition of *WeaklyContingent*($\overline{F}$), it remains to establish $\forall x(\diamond \overline{F}x \to \Box \overline{F}x)$. Assume $\diamond \overline{F}x$, to establish $\Box \overline{F}x$ by conditional proof. Since it is a $\Box$-theorem (137.1) that $\overline{F}x \equiv \neg Fx$, it follows by the Rule of Substitution that $\diamond\neg Fx$, i.e., $\neg\Box Fx$. Now for reductio, assume $\neg\Box\overline{F}x$, i.e., by definition of relation negation, $\neg\Box[\lambda y\, \neg Fy]x$. Since $[\lambda y\, \neg Fy]x \equiv \neg Fx$ is a $\Box$-theorem (it is an instance of β-Conversion), it follows by the Rule of Substitution that $\neg\Box\neg Fx$, i.e., $\diamond Fx$, by definition of $\diamond$. Since we already know $\forall x(\diamond Fx \to \Box Fx)$, it follows that $\Box Fx$. Contradiction. So, by reductio, $\Box\overline{F}x$, and by conditional proof, $\diamond\overline{F}x \to \Box\overline{F}x$. Since we've now discharged the assumption in which the variable x is free, we may invoke GEN to conclude $\forall x(\diamond\overline{F}x \to \Box\overline{F}x)$. (←) Exercise. ⋈

(155.2) (Exercise)

(156.1) By (152.4), we know *Contingent*($O!$). By the left-to-right direction of (153.5), we know $\forall x(\diamond O!x \to \Box O!x)$. Hence, by &I and definition (154), *WeaklyContingent*($O!$). ⋈

(156.2) By (152.5), we know *Contingent*($A!$). By the left-to-right direction of (153.6), we know: $\forall x(\Diamond A!x \rightarrow \Box A!x)$. Hence by &I and definition (154), *Weakly-Contingent*($A!$). ⋈

(156.3) (Exercise) [Hint: Show that $E!$ fails the second conjunct in the definition of contingently necessary, i.e., fails to be such that $\forall x(\Diamond E!x \rightarrow \Box E!x)$. Use either (32.4) or (142).]

(156.4) (Exercise) [Hint: Show that L fails the first conjunct in the definition of contingently necessary. Appeal to previous theorems.] ⋈

(156.5) (Exercise)

(156.6) (Exercise)

(156.7) (Exercise)

(157.1) ($\rightarrow$) Assume $x =_E y$, to show $\Box x =_E y$. Then, by theorem (69.1) and &E, we know the following three comma-separated claims:

(ϑ) $O!x, O!y, \Box\forall F(Fx \equiv Fy)$

By (153.1), the first two claims of (ϑ) imply $\Box O!x$ and $\Box O!y$, respectively. From the third claim in (ϑ), it follows that $\Box\Box\forall F(Fx \equiv Fy)$, by the 4 schema (119.6). Assembling what we have established using &I, we have:

$$\Box O!x \,\&\, \Box O!y \,\&\, \Box\Box\forall F(Fx \equiv Fy)$$

By a basic theorem of K (111.3), a conjunction of necessary truths is equivalent to a necessary conjunction. Hence, it follows that:[277]

(ξ) $\Box(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$

But if we commute $\Box$-theorem (69.1), we have the $\Box$-theorem:

$$(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \equiv x =_E y$$

So by the Rule of Substitution, we can infer from (ξ) that $\Box x =_E y$. Hence, by conditional proof, we've established: $x =_E y \rightarrow \Box x =_E y$. ($\leftarrow$) This direction is an instance of the T schema. ⋈

(157.2) ($\rightarrow$) From (157.1), it follows *a fortiori* that $x =_E y \rightarrow \Box x =_E y$. Since this is a $\Box$-theorem, it follows by (121.2) that $\Diamond x =_E y \rightarrow x =_E y$. ($\leftarrow$) This direction is an instance of the T$\Diamond$ schema. ⋈

(159) We reason as follows:

[277] Strictly speaking, we have to first assemble $\Box O!x \,\&\, \Box O!y$ from what we know, apply (111.3) to obtain $\Box(O!x \& O!y)$, then conjoin this with $\Box\Box\forall F(Fx \equiv Fy)$ to obtain $\Box(O!x \& O!y) \,\&\, \Box\Box\forall F(Fx \equiv Fy)$, and finally apply (111.3) a second time to obtain $\Box(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$.

$$\begin{array}{rcll} x \neq_E y & \equiv & \neq_E xy & \text{by (158.2)} \\ & \equiv & \overline{=_E}xy & \text{by (158.1)} \\ & \equiv & [\lambda y_1 y_2\, \neg(=_E y_1 y_2)]xy & \text{by (136.1)} \\ & \equiv & \neg(=_E xy) & \text{by } \beta\text{-Conversion} \\ & \equiv & \neg(x =_E y) & \text{by (13)} \qquad \bowtie \end{array}$$

(160.1) We know by (117.4) that $\Box\neg x =_E y \equiv \neg\Diamond x =_E y$. Independently, from (157.2) it follows by biconditional law (63.5.d) that $\neg\Diamond x =_E y \equiv \neg x =_E y$. So by biconditional syllogism, it follows from our first two results that:

(a) $\Box\neg x =_E y \equiv \neg x =_E y$

Given that $\neg x =_E y \equiv x \neq_E y$ is a $\Box$-theorem derivable by commuting theorem (159), a single application of the Rule of Substitution to (a) yields $\Box x \neq_E y \equiv x \neq_E y$. By the commutativity of $\equiv$ we are done. $\bowtie$

(160.2) ($\rightarrow$) It follows *a fortiori* from (160.1) that $x \neq_E y \rightarrow \Box x \neq_E y$. Since this is a $\Box$-theorem, it follows by (121.2), $\Diamond x \neq_E y \rightarrow x \neq_E y$. ($\leftarrow$) This is an instance of the T$\Diamond$ schema. $\bowtie$

(161.1) ($\rightarrow$) Assume $x =_E y$. Then by (157.1), it follows that $\Box x =_E y$. So by theorem (89), it follows that $\mathcal{A}x =_E y$. ($\leftarrow$) Assume $\mathcal{A}x =_E y$. By (69.1), it is a $\Box$-theorem that:

$$x =_E y \equiv (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$$

So it follows by the Rule of Substitution that:

$$\mathcal{A}(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$$

But by theorem (95.2), an actualized conjunction of truths is equivalent to a conjunction of actualized truths. So it follows that:[278]

$$\mathcal{A}O!x \,\&\, \mathcal{A}O!y \,\&\, \mathcal{A}\Box\forall F(Fx \equiv Fy)$$

By the equivalence (153.7), the first two conjuncts imply, respectively, $O!x$ and $O!y$. By biconditional syllogism, the third conjunct and axiom (33.2) imply $\Box\forall F(Fx \equiv Fy)$. So, by using by &I to assemble what we've established, it follows that:

$$O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)$$

[278] Again, strictly speaking, we have to treat the previously displayed formula as:

$$\mathcal{A}((O!x \,\&\, O!y) \,\&\, \Box\forall F(Fx \equiv Fy))$$

Then we have to distribute the $\mathcal{A}$ to the two conjuncts, and then repeat the process to turn $\mathcal{A}(O!x \,\&\, O!y)$ into $\mathcal{A}O!x \,\&\, \mathcal{A}O!y$.

So by (69.1), it follows that $x =_E y$. ⋈

(161.2) By theorem (159), we know $x \neq_E y \equiv \neg(x =_E y)$. And by the tautology $(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi)$, we may infer from (161.1) that $\neg(x =_E y) \equiv \neg\mathcal{A}x =_E y$. So it follows by biconditional syllogism that $x \neq_E y \equiv \neg\mathcal{A}x =_E y$. But by commuting an instance of axiom (31.1), we have $\neg\mathcal{A}x =_E y \equiv \mathcal{A}\neg x =_E y$. So by another biconditional syllogism, it follows that:

(ϑ) $x \neq_E y \equiv \mathcal{A}\neg x =_E y$

Now commute (159) and we obtain the $\Box$-theorem that $\neg(x =_E y) \equiv x \neq_E y$. So it follows from ($\vartheta$) by the Rule of Substitution that $x \neq_E y \equiv \mathcal{A}x \neq_E y$. ⋈

(161.3) Let α, β be variables of the same type. ($\rightarrow$) Assume $\alpha = \beta$. Then, by (75), $\Box\alpha = \beta$. So by theorem (89), it follows that $\mathcal{A}\alpha = \beta$. ($\leftarrow$) We reason by cases, where α, β are either both (A) objects, (B) properties, (C) propositions, or (D) n-place relations ($n \geq 2$).

Case A. ($\rightarrow$) Assume $\mathcal{A}x = y$. Then by definition of = (15), this is just:

$$\mathcal{A}(x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)))$$

But by theorem (95.10), i.e., that $\mathcal{A}(\varphi \vee \psi) \equiv (\mathcal{A}\varphi \vee \mathcal{A}\psi)$, it follows that:

(ϑ) $\mathcal{A}x =_E y \vee \mathcal{A}(A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$

We now reason by cases, using the disjuncts of (ϑ) as our two cases:

- Assume $\mathcal{A}x =_E y$. Then by (161.1), it follows that $x =_E y$. So by theorem (69.2), it follows that $x = y$.

- Assume $\mathcal{A}(A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$. Since a conjunction of actualized truths is equivalent to an actualized conjunction (95.2), it follows that:

  $$\mathcal{A}A!x \,\&\, \mathcal{A}A!y \,\&\, \mathcal{A}\Box\forall F(xF \equiv yF)$$

  From the first two conjuncts, it follows, respectively, that $A!x$ and $A!y$, by the equivalence (153.8). From the third conjunct, it follows, by the axiom (33.2), that $\Box\forall F(xF \equiv yF)$. So, using &I to assemble what we know, we have $A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)$. By $\vee$I, this implies $x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$. By definition of = (15), this is just $x = y$.

So reasoning by cases from (ϑ), we have established $x = y$.

Case B. Assume $\mathcal{A}F = G$. Then, by definition, this becomes $\mathcal{A}\Box\forall x(xF \equiv xG)$. But by axiom (33.2), this latter is equivalent to $\Box\forall x(xF \equiv xG)$ which, by definition, is $F = G$.

Case C. Assume $\mathcal{A}(p = q)$. By definition, this is just $\mathcal{A}([\lambda y\, p] = [\lambda y\, q])$. By *Case B*, this implies $[\lambda y\, p] = [\lambda y\, q]$ which, by definition, is $p = q$.

Case D. Exercise. ⋈

(161.4) By the tautology $(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi)$, (161.3) implies that $\neg\alpha = \beta \equiv \neg\mathcal{A}\alpha = \beta$. Now if commute an appropriate instance of axiom (31.1), we know $\neg\mathcal{A}\alpha = \beta \equiv \mathcal{A}\neg\alpha = \beta$. So by biconditional syllogism, $\neg\alpha = \beta \equiv \mathcal{A}\neg\alpha = \beta$. By definition of $\neq$ (18), it follows that $\alpha \neq \beta \equiv \mathcal{A}\alpha \neq \beta$. ⋈

(162) By (161.3), we know $x = y \equiv \mathcal{A}x = y$, which commutes to $\mathcal{A}x = y \equiv x = y$. By GEN, it follows that $\forall x(\mathcal{A}x = y \equiv x = y)$. By the Rule of Alphabetic Variants, it follows that

(ϑ) $\forall z(\mathcal{A}z = y \equiv z = y)$

Now, independently, we may apply GEN to axiom (34), to obtain:

> $\forall x(x = \imath x\varphi \equiv \forall z(\mathcal{A}\varphi^z_x \equiv z = x))$, provided z is substitutable for x in φ and doesn't occur free in φ

Since y is substitutable for x in the matrix of this universal claim, we may instantiate to the variable y and obtain:

> $y = \imath x\varphi \equiv \forall z(\mathcal{A}\varphi^z_x \equiv z = y)$, provided z is substitutable for x in φ and doesn't occur free in φ

Now let φ be the formula $x = y$. Then z is substitutable for x in φ and doesn't occur free in φ. So as an instance of our last result, we know:

$$y = \imath x(x = y) \equiv \forall z(\mathcal{A}z = y \equiv z = y)$$

But from this last fact and (ϑ), it follows that $y = \imath x(x = y)$, by biconditional syllogism. ⋈

(164.1) The axiom of η-Conversion is a particular formula of our language for every $n \geq 0$:

$$[\lambda x_1 \ldots x_n\, F^n x_1 \ldots x_n] = F^n \tag{36.3}$$

The occurrences of F^n are free, even when we eliminate the identity symbol and expand the above into primitive notation. Since we've taken the universal generalizations of our axiom schemata as axioms, the following is therefore an axiom for η-Conversion, for every $n \geq 0$:

(ϑ) $\forall F^n([\lambda x_1 \ldots x_n\, F^n x_1 \ldots x_n] = F^n)$

Without loss of generality, pick a variable, say G, that is not free in n-place relation term Π^n, so that following is an instance of axiom (29.2):

(ξ) $\exists G^n(G^n = \Pi^n)$

By hypothesis, $x_1,\ldots,x_n$ aren't free in Π^n, and so there are no occurrences of $x_1,\ldots,x_n$ that could be captured by $\lambda x_1\ldots x_n$ if we substitute Π^n for F^n in (36.3). So Π^n is substitutable for F^n. Hence, it follows from (ϑ) and (ξ) by Rule ∀E that $[\lambda x_1\ldots x_n\,\Pi^n x_1\ldots x_n] = \Pi^n$. ⋈

(164.2) By α-Conversion (36.1), we may equate alphabetically-variant λ-expressions. So the following is an instance of α-Conversion, where the ν_i are any qualifying object variables (i.e., any distinct variables not free in Π^n):

(ϑ) $[\lambda x_1\ldots x_n\,\Pi^n x_1\ldots x_n] = [\lambda \nu_1\ldots \nu_n\,\Pi^n \nu_1\ldots \nu_n]$

By the symmetry of identity (71.2), it follows from (ϑ) that:

(ζ) $[\lambda \nu_1\ldots \nu_n\,\Pi^n \nu_1\ldots \nu_n] = [\lambda x_1\ldots x_n\,\Pi^n x_1\ldots x_n]$

Hence, from (ζ) and (164.1), it follows by the transitivity of identity (71.3) that $[\lambda \nu_1\ldots \nu_n\,\Pi^n \nu_1\ldots \nu_n] = \Pi^n$.[279] ⋈

(164.3) Suppose ρ' is an immediate η-variant of ρ. Then, by definition (163.4), ρ' results from ρ either (i) by replacing one n-place relation term Π^n in ρ by an η-expansion $[\lambda \nu_1\ldots \nu_n\,\Pi^n \nu_1\ldots \nu_n]$ or (ii) by replacing one elementary λ-expression $[\lambda \nu_1\ldots \nu_n\,\Pi^n \nu_1\ldots \nu_n]$ in ρ by its η-contraction Π^n. Note that the following is an instance of (164.2):

(a) $[\lambda \nu_1\ldots \nu_n\,\Pi^n \nu_1\ldots \nu_n] = \Pi^n$

By symmetry of identity, we also know:

(b) $\Pi^n = [\lambda \nu_1\ldots \nu_n\,\Pi^n \nu_1\ldots \nu_n]$

We also know that since ρ is not a description, Rule ReflId (74.1) has the following instance:

(c) $\rho = \rho$

Then in case (i), we use may (b) to infer $\rho = \rho'$ from (c) by Rule SubId (74.2), and in case (ii), we may use (a) to infer $\rho = \rho'$ from (c) by Rule SubId. ⋈

(165) Whenever ρ' is an η-variant of ρ, then by our definition of η-variants (163.5), there is a finite sequence of λ-expressions such that each member of the sequence is an immediate η-variant of the preceding member of the sequence. So our theorem follows by a finite number of applications of both (164.3) and the transitivity of identity (71.3). ⋈

[279] A simpler proof might be to regard the present theorem as an alphabetic variant of (164.1) and cite the equivalence and interderivability of alphabetically-variant formulas (115). One can argue that the present theorem is an an alphabetic variant of (164.1) in virtue of the fact that they differ only by terms that are alphabetically-variant.

(166) ($\rightarrow$) Assume $[\lambda\, p] = [\lambda\, q]$. By 0-place η-conversion (36.3), we know both $[\lambda\, p] = p$ and $[\lambda\, q] = q$. So by two applications of Rule SubId, substituting p for $[\lambda\, p]$ and q for $[\lambda\, q]$ into our assumption, we obtain $p = q$. ($\leftarrow$) (Exercise) ⋈

(167.1) By excluded middle (63.2), either $\diamond E!x \vee \neg\diamond E!x$. But by the commutativity of $\equiv$, the next two claims follow from instances of β-Conversion (36.2):

$$\diamond E!x \equiv [\lambda y\, \diamond E!y]x$$

$$\neg\diamond E!x \equiv [\lambda y\, \neg\diamond E!y]x$$

So by a disjunctive syllogism (64.4.e), it follows that $[\lambda y\, \diamond E!y]x \vee [\lambda y\, \neg\diamond E!y]x$. By our rule of Substitution of Alphabetically-Variant Relation Terms (68), we may infer $[\lambda x\, \diamond E!x]x \vee [\lambda x\, \neg\diamond E!x]x$, which by definitions (11.1) and (11.2), becomes $O!x \vee A!x$. ⋈

(167.2) Assume, for reductio, that $\exists x(O!x \,\&\, A!x)$. Assume further that b is an arbitrary such x, so that we have $O!b \,\&\, A!b$ and, by &E, both $O!b$ and $A!b$. From the former, it follows that $\neg A!b$, by (152.2). Contradiction. ⋈

(168.1) Assume $O!x$ (to show $x =_E x$). So by (63.3.a), it follows that $O!x \,\&\, O!x$. Independently, by (63.4.a), we know $Fx \equiv Fx$, and so by GEN, $\forall F(Fx \equiv Fx)$, and by RN, $\Box\forall F(Fx \equiv Fx)$. So we have established: $O!x \,\&\, O!x \,\&\, \Box\forall F(Fx \equiv Fx)$. So by theorem (69.1), it follows that $x =_E x$. ⋈

(168.2) Assume $x =_E y$, for conditional proof. Then, by (69.1), it follows that $O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)$. By &E, we know: $O!x$, $O!y$, and $\Box\forall F(Fx \equiv Fy)$. By the commutativity of the biconditional (63.3.g), we know it is a $\Box$-theorem that $(Fx \equiv Fy) \equiv (Fy \equiv Fx)$. So by the Rule of Substitution, it follows that $\Box\forall F(Fy \equiv Fx)$. Using &I to conjoin what we have established, we obtain: $O!y \,\&\, O!x \,\&\, \Box\forall F(Fy \equiv Fx)$, which by (69.1), yields $y =_E x$. ⋈

(168.3) Assume $x =_E y$, and $y =_E z$. Then, by (69.1) and &E, we know all of the following, comma-separated claims:

$$O!x,\ O!y,\ \Box\forall F(Fx \equiv Fy)$$

$$O!y,\ O!z,\ \Box\forall F(Fy \equiv Fz)$$

We leave it as an exercise to show that from $\Box\forall F(Fx \equiv Fy)$ and $\Box\forall F(Fy \equiv Fz)$, it follows that $\Box\forall F(Fx \equiv Fz)$. So using &I to conjoin some of what we know, we have $O!x \,\&\, O!z \,\&\, \Box\forall F(Fx \equiv Fz)$, which by (69.1), yields $x =_E z$. ⋈

(169) Assume $O!x \vee O!y$. We may reason by cases from the disjuncts. Case 1. Assume $O!x$. ($\rightarrow$) Assume $x = y$. From $O!x$ and (168.1), it follows that $x =_E x$. Hence, by substitution of identicals, $x =_E y$. ($\leftarrow$) Assume $x =_E y$. Then by (69.2), $x = y$. Case 2. Assume $O!y$. The reasoning is analogous. ⋈

(170) Suppose $O!x$, $O!y$, and $\forall F(Fx \equiv Fy)$ (to show $x =_E y$). Instantiate *being identical*$_E$ *to* x, i.e., $[\lambda z\, z =_E x]$, into the hypothesis $\forall F(Fx \equiv Fy)$ to obtain $[\lambda z\, z =_E x]x \equiv [\lambda z\, z =_E x]y$. But, by (168.1), we know $x =_E x$, and so by β-Conversion (128), we know $[\lambda z\, z =_E x]x$. So by biconditional syllogism, it follows that $[\lambda z\, z =_E x]y$. By β-Conversion (128), this implies $y =_E x$, and by (168.2), $x =_E y$.[280] ⋈

(171) Assume $O!x$, $O!z$. ($\rightarrow$) Assume $x \neq z$. Then $x \neq_E z$, by (69.2). For reductio, assume $[\lambda y\, y =_E x] = [\lambda y\, y =_E z]$. Since $O!x$, we know by the reflexivity of $=_E$ (168.1) that $x =_E x$. So by β-Conversion (36.2), it follows that $[\lambda y\, y =_E x]x$. But then by substitution of identicals, $[\lambda y\, y =_E z]x$. By β-Conversion, $x =_E z$. But by (159), this contradicts our assumption. ($\leftarrow$) It suffices to prove the contrapositive. So assume $[\lambda y\, y =_E x] = [\lambda y\, y =_E z]$. Then by the reasoning just given (inside the reductio), $x =_E z$. ⋈

(172.1) Suppose $A!x$, $A!y$, and $\forall F(xF \equiv yF)$, for conditional proof. Let P be an arbitrary property. So $xP \equiv yP$. From (126.2), we know $xP \equiv \Box xP$, which commutes to $\Box xP \equiv xP$. So by transitivity of $\equiv$, $\Box xP \equiv yP$. But we also know from (126.2) that $yP \equiv \Box yP$. So, by transitivity of $\equiv$, it follows that $\Box xP \equiv \Box yP$. So by (126.5) (set both F and G in (126.5) to P), it follows that $\Box(xP \equiv yP)$. Since P was arbitrary, we have $\forall F\Box(xF \equiv yF)$. Thus, by the Barcan formula, $\Box\forall F(xF \equiv yF)$. So we may conjoin what we have supposed and what we know to obtain: $A!x\,\&\,A!y\,\&\,\Box\forall F(xF \equiv yF)$, which by (15), is just $x = y$. ⋈

(172.2) (Exercise)

(173) Assume $O!x$. Then by (153.1), it follows that $\Box O!x$. Since the closures of the instances of (38) are axioms, we also know: $\Box(O!x \rightarrow \neg\exists F xF)$. So by the K axiom (32.1), it follows that $\Box\neg\exists F xF$. ⋈

(174.1) By applying &E to axiom (32.4), we have $\Diamond\exists x(E!x\,\&\,\Diamond\neg E!x)$. This implies, by (123.3), that $\Diamond(\exists x E!x\,\&\,\exists x\Diamond\neg E!x)$. Hence by (117.8), $\Diamond\exists x E!x\,\&\,\Diamond\exists x\Diamond\neg E!x$, which by &E gives us $\Diamond\exists x E!x$. ⋈

(174.2) By (174.1), we know $\Diamond\exists x E!x$. Hence, by BF$\Diamond$ (122.3), it follows that:

(ϑ) $\exists x\Diamond E!x$

But independently, by β-Conversion, we know $[\lambda y \Diamond E!y]x \equiv \Diamond E!x$, which by the commutativity of the biconditional (63.3.g) yields $\Diamond E!x \equiv [\lambda y\, \Diamond E!y]x$. Since this latter fact is modally strict, we can use it and the Rule of Substitution to infer from (ϑ) that $\exists x([\lambda y\, \Diamond E!y]x)$. Hence, by definition of $O!$ (11.1), this becomes $\exists x O!x$. Since we established this last fact by modally strict means, it follow by RN that $\Box\exists x O!x$. ⋈

[280] Thanks go to Johannes Korbmacher for suggesting a simplification of an earlier version of this proof.

(174.3) From modally strict theorem (152.2), i.e., that $O!x \equiv \neg A!x$, and the previous theorem (174.2), it follows, by the Rule of Substitution, that $\Box\exists x \neg A!x$. From this and modally strict theorem (86.2), it follows by the Rule of Substitution that $\Box\neg\forall x A!x$. ⋈

(175.1) Let φ be the formula $F = F$. Then by Comprehension for Abstract Objects (39), $\exists x(A!x \,\&\, \forall F(xF \equiv F = F))$. By (86.5), it follows that $\exists x A!x \,\&\, \exists x \forall F(xF \equiv F = F))$. So by &E, $\exists x A!x$. Since this last result is a theorem proved by modally strict means, it follows by RN that $\Box\exists x A!x$. ⋈

(175.2) From modally strict theorem (152.3), i.e., $A!x \equiv \neg O!x$, and the previous theorem (175.1), it follows, by the Rule of Substitution, that $\Box\exists x \neg O!x$. Hence from this and modally strict theorem (86.2), i.e., that $\exists x \neg\varphi \equiv \neg\forall x \varphi$, it follows that $\Box\neg\forall x O!x$, also by the Rule of Substitution. ⋈

(175.3) As part of the proof in (175.1), we established $\exists x A!x$ as a modally strict theorem. By definition, this implies $\exists x[\lambda y\, \neg\Diamond E!y]x$. Suppose a is such an object, so that we know $[\lambda y\, \neg\Diamond E!y]a$. Then by β-Conversion, $\neg\Diamond E!a$. By (117.4), $\Box\neg E!a$. Hence $\neg E!a$, by the T schema. Hence, by $\exists$I and $\exists$E, $\exists x \neg E!x$. By (86.2), the implies $\neg\forall x E!x$. Since the derivation of this last result is modally strict, it follows that $\Box\neg\forall x E!x$. ⋈

(176) By contraposing (38) and eliminating the double negation, we have: $\exists F xF \rightarrow \neg O!x$. But by appealing to (152.2), it is easy to establish that $\neg O!x \rightarrow A!x$ (exercise). So by hypothetical syllogism, $\exists F xF \rightarrow A!x$ ⋈

(177) Let φ be any formula with no free xs. Then by (39), $\exists x(A!x \& \forall F(xF \equiv \varphi))$. Assume a is an arbitrary such object, so that we have:

(ϑ) $A!a \,\&\, \forall F(aF \equiv \varphi)$

If we can then show:

(ξ) $\forall y[(A!y \,\&\, \forall F(yF \equiv \varphi)) \rightarrow y = a]$

Then our theorem follows by conjoining (ϑ) and (ξ), existentially generalizing on a from the resulting conjunction, to obtain, by definition (87.1):

$$\exists! x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

Finally, this conclusion would then be established once we discharge our assumption about a using $\exists$E. We therefore need to show (ξ) on the assumption that (ϑ). Pick an arbitrary object, say b, and assume $A!b \,\&\, \forall F(bF \equiv \varphi)$, for conditional proof, to show $b = a$. By applying &E to our assumptions about a and b, we know the following:

$$\forall F(aF \equiv \varphi)$$

$$\forall F(bF \equiv \varphi)$$

It is now straightforward to establish $\forall F(aF \equiv bF)$ by the laws of quantified biconditionals (83.11) and (83.10). Since we also know $A!a$ and $A!b$ by &E, we may now appeal to (172.1) to infer $b = a$. So by conditional proof, $(A!b \& \forall F(bF \equiv \varphi)) \rightarrow b = a$. Since b was arbitrary, it follows that (ξ), completing the derivation of (ξ) from (ϑ). ⋈

(178.1) – **(178.6)** These are all instances of theorem (177). ⋈

(179.1) We want to establish:

$$\mathcal{A}\exists!\alpha\varphi \equiv \exists!\alpha\mathcal{A}\varphi$$

By definition of unique existence (87.2), for some variable β that is substitutable for α in φ and that doesn't occur free in φ, we have to show:

$$(\vartheta)\ \ \mathcal{A}\exists\alpha\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)\ \equiv\ \exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$$

Our proof strategy is to start with the left condition of (ϑ) and establish a string of equivalences that ends with the right condition of (ϑ). Our first equivalence is an instance of theorem (95.11), which allows us to commute the actuality operator with the existential quantifier in the left condition of (ϑ):

$$\text{(a)}\ \ \mathcal{A}\exists\alpha\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)\ \equiv\ \exists\alpha\mathcal{A}\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$$

Note independently that the actuality operator also commutes with the universal quantifer, by axiom (31.3). The necessitations of the instances of this axiom are axioms, so we know $\Box(\mathcal{A}\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)\ \equiv\ \forall\beta\mathcal{A}(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha))$. Hence, by a Rule of Necessary Equivalence (112.2), we know the following about the right condition of (a):

$$\text{(b)}\ \ \exists\alpha\mathcal{A}\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)\ \equiv\ \exists\alpha\forall\beta\mathcal{A}(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$$

Again, note independently that by applying RN to an instance of the $\Box$-theorem (95.5), we have as a theorem: $\Box(\mathcal{A}(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)\ \equiv\ (\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \mathcal{A}\beta = \alpha))$. Hence, by the Rule of Necessary Equivalence (112.2) again, we know the following about the right condition of (b):

$$\text{(c)}\ \ \exists\alpha\forall\beta\mathcal{A}(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)\ \equiv\ \exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \mathcal{A}\beta = \alpha)$$

Again, note independently that by commuting $\Box$-theorem (161.3) and applying RN to the result, we have as a theorem: $\Box(\mathcal{A}\alpha = \beta \equiv \alpha = \beta)$. Hence, again by the same Rule of Necessary Equivalence (112.2), we know the following about the right condition of (c):

$$\text{(d)}\ \ \exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \mathcal{A}\beta = \alpha)\ \equiv\ \exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$$

Consequently, by the chain of biconditionals (a) – (d) and biconditional syllogism, it follows that the left condition of (a) is equivalent to the right condition of (d), i.e.,

$$\mathcal{A}\exists\alpha\forall\beta(\varphi_{\alpha}^{\beta} \equiv \beta = \alpha) \equiv \exists\alpha\forall\beta(\mathcal{A}\varphi_{\alpha}^{\beta} \equiv \beta = \alpha)$$

But this is just (ϑ). ⋈

(179.2) By biconditional syllogism from (107.1) and (179.1). ⋈

(180) Let φ be any formula in which x, y don't occur free. Then by Strengthened Comprehension (177), it is a theorem that:

$$\exists!x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

So by the rule of actualization RA (92), it follows that:

$$\mathcal{A}\exists!x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

Since, by hypothesis, y isn't free in φ, it follows from this and an instance of theorem (179.2), by biconditional syllogism, that:

$$\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))) \qquad ⋈$$

(182)★ Our theorem is an instance of (101.2)★ if we set φ in (101.2)★ to the formula $A!x \,\&\, \forall F(xF \equiv \varphi)$ and choose variables appropriately, without loss of generality. ⋈

(183) Let ψ be the formula $A!x \,\&\, \forall F(xF \equiv \varphi)$, where x doesn't occur free in φ and y is substitutable for x in φ. Then if we assume the antecedent of our theorem, our assumption can be expressed as:

(ϑ) $y = \imath x\psi$

Without loss of generality, choose z be to some variable substitutable for x in φ. Then, independently, as an instance of the modal version of Hintikka's schema (104), we know:

$$y = \imath x\psi \rightarrow (\mathcal{A}\psi_{x}^{y} \,\&\, \forall z(\mathcal{A}\varphi_{x}^{z} \rightarrow z = y))$$

From the last fact and (ϑ), it follows that:

$$\mathcal{A}\psi_{x}^{y} \,\&\, \forall z(\mathcal{A}\varphi_{x}^{z} \rightarrow z = y)$$

i.e.,

$$\mathcal{A}(A!y \,\&\, \forall F(yF \equiv \varphi)) \,\&\, \forall z(\mathcal{A}(A!z \,\&\, \forall F(zF \equiv \varphi)) \rightarrow z = y)$$

By (95.2), $\mathcal{A}$ distibutes over a conjunction and so the first conjunct of our last result implies:

$$\mathcal{A}A!y \,\&\, \mathcal{A}\forall F(yF \equiv \varphi)$$

The first conjunct of this result, namely $\mathcal{A}A!y$, implies $A!y$, by the modally-strict theorem (153.8). ⋈

(184)★ Suppose x doesn't occur free in φ and G is substitutable for F in φ. Without loss of generality, let y be a variable that doesn't occur free in φ. Then we know by (180) that $\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))$. Assume a is an arbitrary such object, so that we have: $a = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$. Again, without loss of generality, let z be a variable that is substitutable for x and that doesn't occur free in the matrix of our canonical description. Then by the Hintikka schema (98)★, it follows that:

$$A!a \,\&\, \forall F(aF \equiv \varphi) \,\&\, \forall z((A!z \,\&\, \forall F(zF \equiv \varphi)) \rightarrow z = a)$$

So by &E, $\forall F(aF \equiv \varphi)$. Then since G is, by hypothesis, substitutable for F in φ, it is substitutable for F in $\forall F(aF \equiv \varphi)$. So by Rule $\forall$E, we may instantiate $\forall F$ to the variable G and conclude:

$$(\vartheta)\quad aG \equiv \varphi^{G}_{F}$$

But by assumption, $a = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$. It follows from this and (ϑ) that $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi^{G}_{F}$, by Rule SubId. This last conclusion can be considered established once we discharge our assumption about a by citing $\exists$E. ⋈

(186) Consider any formula φ where x doesn't occur free and G is substitutable for F. We want to show (without appealing to any ★-theorems):

$$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \mathcal{A}\varphi^{G}_{F}$$

In what follows, we take ψ to be: $A!x \,\&\, \forall F(xF \equiv \varphi)$. Hence, we want to show:

$$\imath x\psi G \equiv \mathcal{A}\varphi^{G}_{F}$$

Since $\imath x\psi$ is a canonical description, it follows by (180) that $\exists y(y = \imath x\psi)$, where y is some variable not free in ψ. So by (107.4), it follows that $\mathcal{A}\psi^{\imath x\psi}_{x}$, i.e.,

$$\text{(a)}\quad \mathcal{A}(A!\imath x\psi \,\&\, \forall F(\imath x\psi F \equiv \varphi))$$

Now it is a $\Box$-theorem (95.2) that actuality distributes over a conjunction, so it follows from (a) that:

$$\text{(b)}\quad \mathcal{A}A!\imath x\psi \,\&\, \mathcal{A}\forall F(\imath x\psi F \equiv \varphi)$$

By &E, we detach the second conjunct of (b), $\mathcal{A}\forall F(\imath x\psi F \equiv \varphi)$, and since the necessary axiom (31.3) guarantees that the actuality operator commutes with the universal quantifier, it follows that $\forall F\mathcal{A}(\imath x\psi F \equiv \varphi)$. Since G is substitutable for F in this last result (given that, by hypothesis, it is substitutable for F in φ), we may instantiate this last result to G, to obtain $\mathcal{A}(\imath x\psi G \equiv \varphi^G_F)$. Now we've also established as a $\Box$-theorem (95.5) that the actuality operator distributes over the biconditional. So it follows that:

(c) $\mathcal{A}\imath x\psi G \equiv \mathcal{A}\varphi^G_F$

But note also that by theorem (126.10), it follows that $\mathcal{A}\imath x\psi G \equiv \imath x\psi G$, which commutes to:

(d) $\imath x\psi G \equiv \mathcal{A}\imath x\psi G$

Hence from (d) and (c) it follows by biconditional syllogism that $\imath x\psi G \equiv \mathcal{A}\varphi^G_F$, which is what we had to show. ⋈

(187.1) Assume $\Box\varphi^G_F$. It follows that $\mathcal{A}\varphi^G_F$, by theorem (89). But then by the right-to-left direction of the theorem (186), we may conclude $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$. ⋈

(187.2) Assume, for conditional proof:

(a) $\Box\varphi^G_F$

Then it follows from (a) by a 'paradox' of strict implication (111.1) that:

(b) $\Box(\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \to \varphi^G_F)$

Put this result aside for the moment. By our previous theorem (187.1), it also follows from (a) that:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$$

From this it follows by the rigidity of encoding that:

$$\Box\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$$

This implies, by the same 'paradox' of strict implication, that:

(c) $\Box(\varphi^G_F \to \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G)$

By &I we may conjoin (b) and (c), so that by (111.4), we've derived:

$$\Box(\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi^G_F) \qquad ⋈$$

(189.1) Suppose φ is a rigid condition on properties (188.1), so that we know $\vdash \Box\forall F(\varphi \to \Box\varphi)$. Now assume the antecedent of our theorem, so that we know both:

(a) $A!x$

(b) $\forall F(xF \equiv \varphi)$

To show $\Box(A!x \& \forall F(xF \equiv \varphi))$, it suffices, by the right-to-left direction of (111.3), to show both $\Box A!x$ and $\Box\forall F(xF \equiv \varphi)$. But $\Box A!x$ follows from (a) by (153.2). So it remains to show $\Box\forall F(xF \equiv \varphi)$. By BF (122.1), it remains to show $\forall F\Box(xF \equiv \varphi)$. By GEN, it remains to show $\Box(xF \equiv \varphi)$. But if we can establish:

(ϑ) $\Box(\varphi \to \Box\varphi)$

(ξ) $\Box(xF \to \Box xF)$

(ζ) $\Box xF \equiv \Box\varphi$

then $\Box(xF \equiv \varphi)$ follows by (120.3) — just consider the instance of (120.3) in which φ is set to φ and ψ is set to xF. So it remains to show (ϑ), (ξ), and (ζ).

(ϑ) follows from our initial hypothesis that φ is a rigid condition on properties; from the theorem that $\Box\forall F(\varphi \to \Box\varphi)$, we can apply CBF (122.2) and $\forall$E to obtain $\Box(\varphi \to \Box\varphi)$.

(ξ) follows from axiom (37) by RN.

(ζ) requires a proof of both directions. ($\to$) Assume $\Box xF$. Then xF by the T schema (32.2). But this implies φ, once we apply $\forall$E to (b) and reason by biconditional syllogism. From φ and the result of applying the T schema to (ϑ), it follows that $\Box\varphi$. ($\leftarrow$) Assume $\Box\varphi$. Then φ, by the T schema. So by (b), we have xF, which implies $\Box xF$ by axiom (37). ⋈

(189.2) Suppose φ is a rigid condition on properties. Now let ψ be the formula $A!x \,\&\, \forall F(xF \equiv \varphi)$, in which x doesn't occur free in φ. Then it follows from (189.1) by GEN that:

$$\forall x(\psi \to \Box\psi)$$

So by theorem (88), it follows that:

$$\exists!x\psi \to \forall y(y = \imath x\psi \to \psi^y_x)$$

But we know $\exists!x\psi$ since that just is Strenghtened Comprehension for Abstract Objects (177), i.e.,

$$\exists!x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

Hence, $\forall y(y = \imath x\psi \to \psi^y_x)$, i.e.,

$$\forall y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)) \to (A!y \,\&\, \forall F(yF \equiv \varphi)))$$

So by $\forall$E, we're done. ⋈

(189.3) Let φ be a rigid condition on properties in which both x doesn't occur free and G is substitutable for F. Now without loss of generality, suppose y is a variable that doesn't occur free in φ. Then by (180):

$$\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))$$

Suppose a is an arbitrary such individual, so that we know:

(ϑ) $a = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$

Since φ is a rigid condition on properties, it follows by (189.2) that $A!a\,\&$ $\forall F(aF \equiv \varphi)$. Instantiating the second conjunct, we obtain: $aG \equiv \varphi^G_F$. But by (ϑ), it follows that:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi^G_F \qquad \bowtie$$

(192.1) By the Comprehension Principle for Abstract Objects (39), we know:

$$\exists x(A!x \,\&\, \forall F(xF \equiv F \neq F))$$

Suppoe a is an arbitrary such object, so that we know:

(ξ) $A!a \,\&\, \forall F(aF \equiv F \neq F)$

Now given the definition of $Null(x)$ (191.1) as $A!x \,\&\, \neg\exists FxF$, it suffices, by &I and $\exists$I, to show both:

(a) $A!a \,\&\, \neg\exists FaF$

(b) $\forall y((A!y \,\&\, \neg\exists FyF) \rightarrow y = a)$

(a) Since the first conjunct of (ξ) is $A!a$, it remains to show $\neg\exists FaF$. Suppose for reductio that $\exists FaF$, and suppose further that P is an arbitrary such property, so that we know aP. Then by the right conjunct of (ξ), it follows that $P \neq P$, contradicting the fact that $P = P$, which we know by theorem (67.1).

(b) By GEN, it suffices to show $(A!y \,\&\, \neg\exists FyF) \rightarrow y = a$. So assume $A!y \,\&\, \neg\exists FyF$. But the second conjunct, $\neg\exists FyF$, and the fact we just established, namely $\neg\exists FaF$, jointly imply $\forall F(aF \equiv yF)$, by (86.12). Since we also know that both y and a are abstract, it follows by (172.1) that $y = a$. $\bowtie$

(192.2) (Exercise)

(192.3) Since (192.1) is a modally strict theorem, it follows that $\mathcal{A}\exists!xNull(x)$, by the Rule of Actualization. Since y doesn't occur free in this last result, it follows by (179.2) that $\exists y(y = \imath xNull(x))$. $\bowtie$

(192.4) (Exercise)

(194.1) Assume $Null(x)$. Then, by definition (191.1) and &E, we know both (a) $A!x$ and (b) $\neg\exists FxF$. Now to show $\Box Null(x)$, we have to show $\Box(A!x \& \neg\exists FxF)$. By &I and (111.3), it suffices to show $\Box A!x$ and $\Box\neg\exists FxF$. By (153.2), (a) implies $\Box A!x$. Now to show $\Box\neg\exists FxF$, suppose $\neg\Box\neg\exists FxF$, for reductio. Then by definition (7.4.e), $\Diamond\exists FxF$. So by BF$\Diamond$ (122.3), $\exists F\Diamond xF$. Suppose P is an arbitrary such property, so that we know $\Diamond xP$. Then by the left-to-right direction of (126.3), it follows that xP, and hence, $\exists FxF$, by $\exists$I, which contradicts (b). $\bowtie$

(194.2) Assume $Universal(x)$. Then, by definition (191.2) and &E, we know both (a) $A!x$ and (b) $\forall FxF$. Now to show $\Box Universal(x)$, we have to show $\Box(A!x \,\&\, \forall FxF)$. By &I and (111.3), it suffices to show $\Box A!x$ and $\Box\forall FxF$. By (153.2), (a) implies $\Box A!x$. Now to show $\Box\forall FxF$, suppose $\neg\Box\forall FxF$, for reductio. Then by (117.2), $\Diamond\neg\forall FxF$. Since $\neg\forall FxF \equiv \exists F\neg xF$ is an instance of the modally strict theorem (86.2), it follows by the Rule of Substitution that $\Diamond\exists F\neg xF$. So by BF$\Diamond$ (122.3), $\exists F\Diamond\neg xF$. Suppose P is an arbitrary such property, so that we know $\Diamond\neg xP$. Then by the left-to-right direction of (126.3), it follows that $\neg xP$, and hence, $\exists F\neg xF$, i.e., $\neg\forall FxF$, which contradicts (b). $\bowtie$

(194.3) Let ψ be the formula $Null(x)$. Then as an instance of (108.2), we know:

$$\forall x(\psi \to \Box\psi) \to (\exists!x\psi \to \forall y(y = \imath x\psi \to \psi^y_x))$$

By applying GEN to (194.1), we know $\forall x(\psi \to \Box\psi)$. Hence:

$$\exists!x\psi \to \forall y(y = \imath x\psi \to \psi^y_x)$$

But (192.1) is $\exists!x\psi$. Hence:

$$\forall y(y = \imath x\psi \to \psi^y_x)$$

Since $\boldsymbol{a}_\varnothing$ is logically proper, by (192.3) and (193.1), it follows that:

$$\boldsymbol{a}_\varnothing = \imath x\psi \to \psi^{\boldsymbol{a}_\varnothing}_x$$

Hence by definition (193.1), $\psi^{\boldsymbol{a}_\varnothing}_x$, i.e., $Null(\boldsymbol{a}_\varnothing)$. $\bowtie$

(194.4) (Exercise)

(196.1) [Note: Readers who have gotten this far should now be well-equipped to follow the more obvious shortcuts in reasoning we take in this proof and the ones that follow.] Consider the following instance of comprehension, in which there is a free occurrence of the 2-place relation variable R:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(A!y \,\&\, F = [\lambda z\, Rzy] \,\&\, \neg yF)))$$

Assume a is an arbitrary such object, so that we have:

$$(\vartheta)\quad A!a \,\&\, \forall F(aF \equiv \exists y(A!y \,\&\, F = [\lambda z Rzy] \,\&\, \neg yF))$$

Now consider the property $[\lambda z\, Rza]$ and ask the question whether a encodes this property. Assume $\neg a[\lambda z\, Rza]$. Then, from the second conjunct of (ϑ) it follows that:

$$\neg\exists y(A!y \,\&\, [\lambda z\, Rza] = [\lambda z\, Rzy] \,\&\, \neg y[\lambda z\, Rza]))),$$

i.e., by quantifier negation:

(ξ) $\forall y(A!y \,\&\, [\lambda z\, Rza] = [\lambda z\, Rzy] \rightarrow y[\lambda z\, Rza]))),$

i.e., for any abstract object y, if $[\lambda z\, Rza] = [\lambda z\, Rzy]$, then $y[\lambda z\, Rza]$. Instantiate this universal claim to a. We know $A!a$ by the left conjunct of (ϑ) and and we know $[\lambda z\, Rza] = [\lambda z\, Rza]$ by Rule ReflId. So if we instantiate $\forall y$ in (ξ) to a, it follows that $a[\lambda z Rza]$, contrary to assumption. So we've established by reductio that $a[\lambda z\, Rza]$. Then by the second conjunct of (ϑ), there is an abstract object, say b, such that both $[\lambda z\, Rza] = [\lambda z\, Rzb]$ and $\neg b[\lambda z\, Rza]$. But since $a[\lambda z\, Rza]$ and $\neg b[\lambda z\, Rza]$, it follows by the contrapositive of Rule SubId that $a \neq b$.[281] So, by $\exists$I and two applications of $\exists$E, there are abstract objects x and y such that $x \neq y$, yet such that $[\lambda z\, Rzx] = [\lambda z\, Rzy]$. By GEN, this theorem holds for every R. ⋈

(196.2) By reasoning analogous to that used in the proof of (196.1). ⋈

(196.3) Consider the following instance of comprehension in which there is a free occurrence of the 1-place property variable P:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(A!y \,\&\, F = [\lambda z\, Py] \,\&\, \neg yF)))$$

By reasoning analogous to that used in the proof of (196.1), it is straightforward to establish that there are distinct abstract objects, say k, l, such that $[\lambda z Pk]$ is identical to $[\lambda z Pl]$. But, then by the definition of proposition identity (16.3), it follows that $[\lambda\, Pk]$ is identical to $[\lambda\, Pl]$. So, there are distinct abstract objects, x, y such that $[\lambda\, Px] = [\lambda\, Py]$, and this holds for any property P. ⋈

(197) Let R_1 be the relation $[\lambda xy \forall F(Fx \equiv Fy)]$. So by (196.1), there are abstract objects, say a, b, such that $a \neq b$ and $[\lambda z\, R_1 za] = [\lambda z\, R_1 zb]$. But, from the definition of R_1, β-Conversion (128), and the easily-established fact that $\forall F(Fa \equiv Fa)]$, it is easily provable that $R_1 aa$. Hence, by β-Conversion (128), it follows that $[\lambda z\, R_1 za]a$. But, by Rule SubId, it then follows that $[\lambda z\, R_1 zb]a$. Hence, by β-Conversion (128), $R_1 ab$, which by the definition of R_1 and β-Conversion (128), yields $\forall F(Fa \equiv Fb)$. Hence, $\exists x, y(A!x \,\&\, A!y \,\&\, a \neq b \,\&\, \forall F(Fx \equiv Fy))$. ⋈

(199.1) Since $[\lambda y\, p]$ is not a description, we know by Rule ReflId that $[\lambda y\, p] = [\lambda y\, p]$. But then, we may also apply $\exists$I to conclude: $\exists F(F = [\lambda y\, p])$. And by GEN, we have $\forall p \exists F(F = [\lambda y\, p])$. ⋈

(199.2) As an instance of β-conversion (36.2), we know $[\lambda y p]x \equiv p$. So by GEN, it follows that $\forall x([\lambda y\, p]x \equiv p)$. Since no $\star$-theorems have been used, it follows by RN that $\Box \forall x([\lambda y\, p]x \equiv p)$. Now assume $F = [\lambda y\, p]$. Then by substitution of

[281] Since the *derived* Rule SubId is: φ^{τ}_{α}, $\tau = \tau' \,/\, \varphi'$, its contrapositive rule is: φ^{τ}_{α}, $\neg\varphi' \,/\, \tau \neq \tau'$. We leave its justification as an exercise.

identicals into what we've already established, we know $\Box\forall x(Fx \equiv p)$. So by conditional proof, $F = [\lambda y\, p] \to \Box\forall x(Fx \equiv p)$. ⋈

(199.3) Assume *Propositional*(F). Then by definition, $\exists p(F = [\lambda y\, p])$. We want to show that this is a necessary truth. Assume q_1 is such a proposition, so that we know $F = [\lambda y\, q_1]$. Hence by (75), $\Box(F = [\lambda y\, q_1])$. So, by $\exists$I and $\exists$E, $\exists p\Box(F = [\lambda y\, p])$. Then by the Buridan Formula (123.1), $\Box\exists p(F = [\lambda y\, p])$. Hence, $\Box$*Propositional*(F) ⋈

(201) By the definitions of propositional (198) and indiscriminate (200) properties, we have to show:

$$\exists p(F = [\lambda y\, p]) \to \Box(\exists x Fx \to \forall x Fx)$$

Note that this follows, by hypothetical syllogism, from:

(A) $\exists p(F = [\lambda y\, p]) \to \Box\exists p(F = [\lambda y\, p])$

(B) $\Box\exists p(F = [\lambda y\, p]) \to \Box(\exists x Fx \to \forall x Fx)$

(A) follows from (199.3) by the definition of propositional property (198). So it remains to show (B). But if we can show $\exists p(F = [\lambda y\, p]) \to (\exists x Fx \to \forall x Fx)$ by a modally strict proof, then (B) follows by RM. So assume $\exists p(F = [\lambda y\, p])$ and $\exists x Fx$. Suppose q_1 and a are such a proposition and object, respectively, so that we know $F = [\lambda y\, q]$ and Fa. Then $[\lambda y\, q]a$, and so by β-Conversion q. But it is an instance of β-Conversion that $[\lambda y\, q]x \equiv q$. Hence $[\lambda y\, q]x$, i.e., Fx. Since x doesn't occur free in any of our assumptions, we may infer $\forall x Fx$ by GEN. This conclusion remains once we discharge our assumptions by $\exists$E. Hence by conditional proof:

$$(\exists p(F = [\lambda y\, p])\ \&\ \exists x Fx) \to \forall x Fx$$

And by exportation (63.8.a), we've developed a modally strict proof of:

$$\exists p(F = [\lambda y\, p]) \to (\exists x Fx \to \forall x Fx) \qquad ⋈$$

(202.1) As an instance of (21.1), we know:

$$\forall x Fx \to (\exists x Fx \to \forall x Fx)$$

Since this theorem is modally strict, it follows by RM that:

$$\Box\forall x Fx \to \Box(\exists x Fx \to \forall x Fx)$$

But applying definition (138.1) to the antecedent and definition (200) to the consequent, and we have:

$$\mathit{Necessary}(F) \to \mathit{Indiscriminate}(F) \qquad ⋈$$

(202.2) As an instance of (58.3), we know:

$$\neg\exists xFx \to (\exists xFx \to \forall xFx)$$

So by (86.4) and the Rule of Substitution:

$$\forall x\neg Fx \to (\exists xFx \to \forall xFx)$$

Since this theorem is modally strict, it follows by RM that:

$$\Box\forall x\neg Fx \to \Box(\exists xFx \to \forall xFx)$$

But applying definition (138.2) to the antecedent and definition (200) to the consequent, and we have:

$$\mathit{Impossible}(F) \to \mathit{Indiscriminate}(F) \qquad \bowtie$$

(202.3.a) Suppose, for reductio, that $E!$ is indiscriminate. Then, by definition:

$$(\vartheta)\quad \Box(\exists xE!x \to \forall xE!x)$$

Now independently we know $\Diamond\exists xE!x$ (174.1). But this and (ϑ) imply $\Diamond\forall xE!x$, by (117.5). But we also know independently that $\Box\neg\forall xE!x$ (175.3). This implies $\neg\Diamond\forall xE!x$. Contradiction. $\bowtie$

(202.3.b) Suppose, for reductio, that $\overline{E!}$ is indiscriminate. Then, by definition:

$$(\vartheta)\quad \Box(\exists x\overline{E!}x \to \forall x\overline{E!}x)$$

Now independently, we know $\Box\neg\forall xE!x$ (175.3). This is equivalent to $\Box\exists x\neg E!x$ (exercise). In turn, this is equivalent to $\Box\exists x\overline{E!}x$ (exercise). But from this last result and (ϑ) it follows that $\Box\forall x\overline{E!}x$, by the K schema. Independently, however, we know $\Diamond\exists xE!x$ (174.1), i.e., by definition, $\neg\Box\neg\exists xE!x$. But this is equivalent to $\neg\Box\forall x\neg E!x$ (exercise), which in turn is equivalent to $\neg\Box\forall x\overline{E!}x$ (exercise). Contradiction. $\bowtie$

(202.3.c) Suppose, for reductio, that $O!$ is indiscriminate. Then, by definition:

$$(\vartheta)\quad \Box(\exists xO!x \to \forall xO!x)$$

Now independently we know $\Box\exists xO!x$ (174.2). Hence, by the K schema, it follows that $\Box\forall xO!x$. By the T schema, $\forall xO!x$. But an application of the T schema to (175.2) yields $\neg\forall xO!x$. Contradiction. $\bowtie$

(202.3.d) Suppose, for reductio, that $A!$ is indiscriminate. Then, by definition:

$$(\vartheta)\quad \Box(\exists xA!x \to \forall xA!x)$$

Now independently we know $\Box\exists x A!x$ (175.1). Hence, by the K schema, it follows that $\Box\forall x A!x$. By the T schema, $\forall x A!x$. But an application of the T schema to (174.3) yields $\neg\forall x A!x$. Contradiction. ⋈

(202.4.a) – (202.4.d) (Exercises)

(203.1) Assume $\Diamond\exists p(F=[\lambda y\, p])$, to show $\exists p(F=[\lambda y\, p])$. By the Barcan Formula, it follows that $\exists p\Diamond(F=[\lambda y\, p])$. Assume q_1 is an arbitrary such p, so that we know $\Diamond(F=[\lambda y\, q_1])$. By the definition of property identity (16.1), it follows that $\Diamond\Box\forall x(xF \equiv x[\lambda y\, q_1])$. By 5$\Diamond$ (119.1), it follows that $\Box\forall x(xF \equiv x[\lambda y\, q_1])$. Again, by the definition of property identity, we have $F=[\lambda y\, q_1]$. So by $\exists$I (84.2), $\exists p(F=[\lambda y\, p])$. Hence, $\exists p(F=[\lambda y\, p])$, by $\exists$E (85). ⋈

(203.2) By contraposition on the previous theorem (203.1), we know $\neg\exists p(F=[\lambda y\, p]) \to \neg\Diamond\exists p(F=[\lambda y\, p])$. By applying the quantifier negation equivalence (86.4) and the Rule of Substitution to the antecedent, we have $\forall p\neg(F=[\lambda y p]) \to \neg\Diamond\exists p(F=[\lambda y p])$. By applying the Rule of Substitution to the consequent twice, first using (117.4) and then using the quantifier negation equivalence (86.4), we obtain: $\forall p\neg(F=[\lambda y\, p]) \to \Box\forall p\neg(F=[\lambda y\, p])$. But by the definition of $\neq$ (18), this becomes: $\forall p(F\neq[\lambda y\, p]) \to \Box\forall p(F\neq[\lambda y\, p])$. ⋈

(203.3) From (203.1) by (121.1), given that the proof of (203.1) is modally-strict. ⋈

(203.4) From (203.2) by (121.2), given that the proof of (203.2) is modally-strict. ⋈

(204.1) Assume $\Diamond\forall F(xF \to \exists p(F=[\lambda y\, p]))$, for conditional proof. By the Buridan$\Diamond$ formula (123.2), this implies:

(ϑ) $\forall F\Diamond(xF \to \exists p(F=[\lambda y\, p]))$

Now we want to show $\forall F(xF \to \exists p(F=[\lambda y p]))$, so let Q be an arbitrarily chosen property and assume xQ, for conditional proof, to show $\exists p(Q=[\lambda y\, p])$. It follows that $\Box xQ$, by axiom (37). Now if we instantiate (ϑ) to Q, it follows that $\Diamond(xQ \to \exists p(Q=[\lambda y\, p]))$. This together with $\Box xQ$ yields $\Diamond\exists p(Q=[\lambda y\, p])$, by (117.9). But by our previous theorem (203.1), it follows that $\exists p(Q=[\lambda y\, p])$. By discharging our second assumption, we've established $xQ \to \exists p(Q=[\lambda y\, p])$. Since Q was arbitrary, it follows that $\forall F(xF \to \exists p(F=[\lambda y\, p])$. ⋈

(204.2) From (204.1), by (121.1), given that (204.1) is a $\Box$-theorem. ⋈

(212.1) By definition (211) and the Comprehension Principle for Abstract Objects (39). ⋈

(212.2) By definition (211) and the Strengthened Comprehension for Abstract Objects (177). ⋈

(213) It follows from (212.2) that $\mathcal{A}\exists!x TruthValueOf(x,p)$, by the Rule of Actualization. Hence by (179.2), $\exists y(y = \iota x TruthValueOf(x,p))$.[282] ⋈

(217.1)★ Since z is a variable substitutable for x in $TruthValueOf(x,p)$ and doesn't occur free in $TruthValueOf(x,p)$, we have the following instance of theorem (97)★, where φ in that theorem is set to $TruthValueOf(x,p)$:

$$x = \iota x TruthValueOf(x,p) \equiv \forall z(TruthValueOf(z,p) \equiv z = x)$$

So by GEN:

$$(\vartheta)\quad \forall x(x = \iota x TruthValueOf(x,p) \equiv \forall z(TruthValueOf(z,p) \equiv z = x))$$

Independently, $\exists y(y = \iota x TruthValueOf(x,p))$ is an instance of theorem (213). So by definition (214), we know $\exists y(y = p^\circ)$. From this last result and (ϑ), it follows by Rule ∀E (77.1) Variant that:

$$p^\circ = \iota x TruthValueOf(x,p) \equiv \forall z(TruthValueOf(z,p) \equiv z = p^\circ)$$

So, by definition (214) and biconditional syllogism, it follows that:

$$\forall z(TruthValueOf(z,p) \equiv z = p^\circ)$$

So by instantiating this last result to x, and we have:

$$TruthValueOf(x,p) \equiv x = p^\circ$$ ⋈

(217.2)★ We can obtain this result in one of two basic ways. In both cases, we use the fact that $\exists y(y = p^\circ)$, which was established as part of the reasoning in (217.1)★.

(1) Since z is substitutable for x in $TruthValueOf(x,p)$ and doesn't occur free in $TruthValueOf(x,p)$, we have the following instance of (101.2)★:

$$z = \iota x TruthValueOf(x,p) \to TruthValueOf(z,p)$$

So by GEN:

$$\forall z(z = \iota x TruthValueOf(x,p) \to TruthValueOf(z,p))$$

Since $\exists y(y = p^\circ)$, we may instantiate the above to p°, to obtain:

$$p^\circ = \iota x TruthValueOf(x,p) \to TruthValueOf(p^\circ,p)$$

But the antecedent holds by definition (214). So $TruthValueOf(p^\circ,p)$.

(2) Alternatively, if we apply GEN to (217.1)★ and instantiate the result to p°, we obtain:

[282] Alternatively, we know by (180) that $\exists y(y = \iota x(A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))))$. So, by definition of $TruthValueOf$ (211) and the fact that a definiendum and definiens can be exchanged in any context (208.5), it follows that $\exists y(y = \iota x TruthValueOf(x,p))$.

(ϑ) $\mathit{TruthValueOf}(p^\circ, p) \equiv p^\circ = p^\circ$

But if we apply GEN to (70.1), we have $\forall x(x = x)$. Since $\exists y(y = p^\circ)$, it follows by Rule $\forall$E (77.1) Variant that $p^\circ = p^\circ$. Hence, it follows by biconditional syllogism from (ϑ) that $\mathit{TruthValueOf}(p^\circ, p)$. ⋈

(217.3)★ From (217.2)★, our theorem follows by definition (211) and &E. ⋈

(217.4)★ By instantiating (217.3)★ to the property $[\lambda y\, r]$, it follows that:

(ϑ) $p^\circ[\lambda y\, r] \equiv \exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q])$

Now we have to show $p^\circ\Sigma r \equiv (r \equiv p)$. ($\rightarrow$) Assume that $p^\circ\Sigma r$, for conditional proof. By (216), it follows that $p^\circ[\lambda y\, r]$. From this and (ϑ), we may conclude that:

(ξ) $\exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q])$

Let q_1 be an arbitrary such proposition, so that we know:

(ζ) $(q_1 \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q_1]$

From the right conjunct of (ζ), it follows by the definition of proposition identity (16.3) that $r = q_1$, which by symmetry yields $q_1 = r$. But from this and the left conjunct of (ζ), it follows by Rule SubId that $r \equiv p$. ($\leftarrow$) Suppose that $r \equiv p$. By Rule ReflId (74.1), $[\lambda y\, r] = [\lambda y\, r]$. Conjoining our assumption and this last fact, we obtain: $(r \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, r]$. By $\exists$I, it follows that $\exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q])$. So, by ($\vartheta$), it follows that $p^\circ[\lambda y\, r]$. From this and the fact established in the proof of (217.3)★ that $A!p^\circ$, it follows that $p^\circ\Sigma r$, by (216). ⋈

(217.5)★ (Exercise)

(218)★ Suppose p_1 and q_1 are arbitrarily chosen propositions. ($\rightarrow$) Assume $p_1{}^\circ = q_1{}^\circ$. Since we know that $p_1{}^\circ\Sigma p_1$ by (217.5)★, it follows by Rule SubId that $q_1{}^\circ\Sigma p_1$. Hence by (217.4)★, it follows that $p_1 \equiv q_1$. ($\leftarrow$) Assume $p_1 \equiv q_1$. We want to show that $p_1{}^\circ = q_1{}^\circ$. Now it follows from (217.2)★ and the definition of *ExtensionOf* that both $A!p_1{}^\circ$ and $A!q_1{}^\circ$. So by theorem (172.1), it remains only to show $\forall F(p_1{}^\circ F \equiv q_1{}^\circ F)$. Since F isn't free in our assumptions, it suffices by GEN to show $p_1{}^\circ F \equiv q_1{}^\circ F$. ($\rightarrow$) Assume $p_1{}^\circ F$. From this and (217.3)★ it follows that:

$$\exists q((q \equiv p_1) \,\&\, F = [\lambda y\, q])$$

Let r_1 be an arbitrary such proposition, so that we know:

(ϑ) $(r_1 \equiv p_1) \,\&\, F = [\lambda y\, r_1]$

From the first conjunct of (ϑ) and our global assumption that $p_1 \equiv q_1$, it follows that $r_1 \equiv q_1$. Conjoining this result with the second conjunct of (ϑ) and we have: $(r_1 \equiv q_1) \,\&\, F = [\lambda y\, r_1]$. So by ∃I, we have:

$$\exists q((q \equiv q_1) \,\&\, F = [\lambda y\, q])$$

From this it follows by (217.3)★ that $q_1 {}^{\circ} F$. (←) By reversing the reasoning. ⋈

(220.1) (→) Assume p. By GEN, it suffices to show the equivalence of (ϑ) and (ξ):

(ϑ) $\exists q(q \,\&\, F = [\lambda y\, q])$

(ξ) $\exists q((q \equiv p) \,\&\, F = [\lambda y\, q])$

(→) Assume (ϑ). Suppose q_1 is such a proposition, so that we know q_1 is true and $F = [\lambda y\, q_1]$. But since p is true, it is materially equivalent to q_1. So it follows that $(q_1 \equiv p) \,\&\, F = [\lambda y\, q_1]$, and by ∃I, that ($\xi$). (←) Assume ($\xi$). Suppose q_2 is an arbitrary such proposition, so that we know $q_2 \equiv p$ and $F = [\lambda y\, q_2]$. But then since p is true, q_2 is true. So $q_2 \,\&\, F = [\lambda y\, q_2]$, and by ∃I, ($\vartheta$).

(←) Our assumption is:

$$\forall F[\exists q(q \,\&\, F = [\lambda y\, q]) \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q])]$$

We want to show p. For reductio, assume $\neg p$. Instantiate the property $[\lambda y\, \neg p]$ into our assumption, to obtain:

(ζ) $\exists q(q \,\&\, [\lambda y\, \neg p] = [\lambda y\, q]) \equiv \exists q((q \equiv p) \,\&\, [\lambda y\, \neg p] = [\lambda y\, q])$

Now note independently that by the reflexivity of identity we know $[\lambda y\, \neg p] = [\lambda y\, \neg p]$. Hence we know:

$$\neg p \,\&\, [\lambda y\, \neg p] = [\lambda y\, \neg p]$$

So, by ∃I,

$$\exists q(q \,\&\, [\lambda y\, \neg p] = [\lambda y\, q])$$

From this last conclusion and (ζ), it follows that:

$$\exists q((q \equiv p) \,\&\, [\lambda y\, \neg p] = [\lambda y\, q])$$

Assume q_1 is an arbitrary such proposition, so that we know $(q_1 \equiv p) \& [\lambda y \neg p] = [\lambda y\, q_1]$. By the definition of identity for propositions, the second conjunct implies $\neg p = q_1$. But $\neg p$ is true by reductio assumption. So it follows that q_1 is true. But since q_1 is equivalent to p, p is true. Contradiction. ⋈

(220.2) (→) Assume $\neg p$. By GEN, it suffices to show the equivalence of (ϑ) and (ξ):

(ϑ) $\exists q(\neg q \,\&\, F = [\lambda y\, q])$

(ξ) $\exists q((q \equiv p) \,\&\, F = [\lambda y\, q])$

We show each direction separately:

> ($\rightarrow$) Assume (ϑ). Suppose q_1 is an arbitrary such proposition, so that we know $\neg q_1$ and $F = [\lambda y\, q_1]$. But since $\neg p$, it follows that $q_1 \equiv p$. Hence $(q_1 \equiv p) \,\&\, F = [\lambda y\, q_1]$, and by $\exists$I, that (ξ).
>
> ($\leftarrow$) Assume (ξ). Suppose q_2 is an arbitrary such proposition, so that we know $q_2 \equiv p$ and $F = [\lambda y\, q_2]$. But then since $\neg p$, it follows that $\neg q_2$. So $\neg q_2 \,\&\, F = [\lambda y\, q_2]$, and by $\exists$I, (ϑ).

($\leftarrow$) (Exercise) $\bowtie$

(222)$\star$ By (217.2)$\star$, we know *TruthValueOf*(q°, q). Hence, $\exists p$*TruthValueOf*(q°, p). So by definition (221), it follows that *TruthValue*(q°). $\bowtie$

(223.1) Assume:

(ϑ) $A!x \,\&\, \forall F(xF \equiv \exists q(q \,\&\, F = [\lambda y\, q]))$

To show *TruthValue*(x), we have to show, by definition (221):

$$\exists p(\mathit{TruthValueOf}(x, p))$$

We can do this if we take p_0, defined in item (145) as $\forall x(E!x \rightarrow E!x)$, as our witness. So we have to show:

$$\mathit{TruthValueOf}(x, p_0)$$

By (211), we have to show:

(ξ) $A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p_0) \,\&\, F = [\lambda y\, q]))$

Note independently that in the proof of *Necessary*(p_0) in (145.1), we established that p_0 is true by a modally-strict proof. So it follows from (220.1) that:

$$\exists q(q \,\&\, F = [\lambda y\, q]) \equiv \exists q((q \equiv p_0) \,\&\, F = [\lambda y\, q])$$

Since we've established this by a modally-strict proof, it follows from (ϑ) by the Rule of Substitution that (ξ). $\bowtie$

(223.2) (Exercise)

(224.1)$\star$ By definition of $\top$ (219.1), we know:

(ϑ) $\top = \imath x(A!x \,\&\, \forall F(xF \equiv \exists p(p \,\&\, F = [\lambda y\, p])))$

Hence by (101.2)$\star$, it follows that:

$$A!\top \,\&\, \forall F(\top F \equiv \exists p(p \,\&\, F = [\lambda y\, p]))$$

Hence by lemma (223.1), it follows that $TruthValue(\top)$. ⋈

(224.2)★ (Exercise)

(224.3)★ (Exercise)

(225) Consider the following two instances of Object Comprehension (39):

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists p(p \,\&\, F = [\lambda y\, p])))$$

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists p(\neg p \,\&\, F = [\lambda y\, p]))$$

Let a, b be arbitrary such objects, respectively, so that we know:

(ϑ) $A!a \,\&\, \forall F(aF \equiv \exists p(p \,\&\, F = [\lambda y\, p]))$

(ξ) $A!b \,\&\, \forall F(bF \equiv \exists p(\neg p \,\&\, F = [\lambda y\, p]))$

We now develop modally-strict arguments that show:

(1) $TruthValue(a) \,\&\, TruthValue(b)$

(2) $a \neq b$

(3) $\forall z(TruthValue(z) \rightarrow z = a \vee z = b)$

Note that (1) follows from (223.1) and (223.2), respectively, given (ϑ) and (ξ).

To show (2), then since we know by the left conjuncts of (ϑ) and (ξ), respectively, that $A!a$ and $A!b$, it suffices, by theorem (172.1), to show a encodes a property that b fails to encode, or vice versa. So consider the property $[\lambda y\, p_0]$ (where p_0 has been previously defined as $\forall x(E!x \rightarrow E!x)$), which we know exists by (29.2). By the right conjunct of (ϑ), we know:

$$a[\lambda y\, p_0] \equiv \exists p(p \,\&\, [\lambda y\, p_0] = [\lambda y\, p])$$

But since p_0 is true and $[\lambda y\, p_0] = [\lambda y\, p_0]$ is provable by Rule ReflId (74.1), we can infer the right side of the above biconditional by $\exists$I, so that we may conclude $a[\lambda y\, p_0]$. Now, for reductio, assume $b[\lambda y\, p_0]$. Then by the right conjunct of (ξ), it follows that $\exists p(\neg p \,\&\, [\lambda y\, p_0] = [\lambda y\, p])$. Suppose p_1 is an arbitrary such proposition, so that we know $\neg p_1 \,\&\, [\lambda y\, p_0] = [\lambda y\, p_1]$. By the definition of proposition identity (16.3), it follows that $p_0 = p_1$. Hence by Rule SubId (74.2), it follows that $\neg p_0$, which contradicts the fact that p_0 is true.

To show (3), it suffices by GEN to show $TruthValue(z) \rightarrow z = a \vee z = b$, since z isn't free in any assumption. Assume $TruthValue(z)$. Then $\exists p(TruthValueOf(z, p))$. Suppose p_3 is an arbitrary such proposition, so that we know $TruthValueOf(z, p_3)$. Then by (211), we know:

(ζ) $A!z \,\&\, \forall F(zF \equiv \exists q((q \equiv p_3) \,\&\, F = [\lambda y\, q]))$

Now we reason by disjunctive syllogism, from the tautology $p_3 \vee \neg p_3$, to show $z = a \vee z = b$.

- Assume p_3, to show $z = a$. Since both z and a are abstract, it suffices to show $\forall G(zG \equiv aG)$ by theorem (172.1). Since G isn't free in our assumption, it suffices, by GEN, to show $zG \equiv aG$. But since p_3 is true, we may reason as follows:

$$\begin{array}{llll} zG & \equiv & \exists q((q \equiv p_3) \,\&\, F = [\lambda y\, q]) & \text{by right conjunct of } (\zeta) \\ & \equiv & \exists q(q \,\&\, F = [\lambda y\, q]) & \text{by (220.1) and the truth of } p_3 \\ & \equiv & aG & \text{by right conjunct of } (\vartheta) \end{array}$$

- Assume $\neg p_3$, to show $z = b$. By analogous reasoning, using (223.2).

Hence, by disjunctive syllogism, $z = a \vee z = b$.

Since the proofs of (1), (2), and (3) make no appeal to any $\star$-theorems, they constitute a modally-strict proof that there are exactly two truth-values. $\bowtie$

(226.1)$\star$ Assume *TruthValueOf*(x, p). By definition (211), it follows that:

(ϑ) $A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$

Independently, by the reasoning in (224.1)$\star$, we know:

(a) $A!\top$

(b) $\forall F(\top F \equiv \exists q(q \,\&\, F = [\lambda y\, q]))$

Now we want to show $p \equiv x = \top$. ($\rightarrow$) Assume p. Since we know both x and $\top$ are abstract, it suffices by (172.1) to show that $\forall F(xF \equiv \top F)$. Note that since p is true, we know by theorem (220.1) that:

$$\forall F[\exists q(q \,\&\, F = [\lambda y\, q]) \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q])]$$

By (83.10), it follows from (b) and this last result that:

$$\forall F(\top F \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$$

By (83.11), we can commute the conditions of a quantified biconditional, to obtain:

(c) $\forall F(\exists q((q \equiv p) \,\&\, F = [\lambda y\, q]) \equiv \top F)$

Again by (83.10), the right conjunct of (ϑ) and (c) jointly imply $\forall F(xF \equiv \top F)$.

($\leftarrow$) Assume $x = \top$. Then by Rule SubId, it follows from the right conjunct of (ϑ) that:

(d) $\forall F(\top F \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$

By by commuting the biconditional in (b) (83.11) and applying (83.10) to the result and (d), it follows that:

$$\forall F[\exists q((q \equiv p)\,\&\,F = [\lambda y\, q]) \equiv \exists q((q \equiv p)\,\&\,F = [\lambda y\, q])]$$

Hence by (220.1), it follows that p. ⋈

(226.2)★ (Exercise)

(227.1)★ By definition of p° (214) and (101.2)★, we know *TruthValueOf*(p°, p). So by the previous theorem (226.1)★, it follows that $p \equiv p^{\circ} = \top$. ⋈

(227.2)★ (Exercise)

(227.3)★ ($\rightarrow$) Assume p. Then by (227.1)★, $p^{\circ} = \top$. But we know independently by (217.5)★, that $p^{\circ}\Sigma p$. Hence by Rule SubId, $\top\Sigma p$. ($\leftarrow$) Assume $\top\Sigma p$. Then, by definition (216), $\top[\lambda y\, p]$. By the reasoning in (224.1)★, we already know that:

$$\forall F(\top F \equiv \exists q(q\,\&\,F = [\lambda y\, q])$$

Hence, $\exists q(q\,\&\,[\lambda y\, p] = [\lambda y\, q])$. Suppose q_1 is an arbitrary such proposition, so that we know both that q_1 is true and that $[\lambda y\, p] = [\lambda y\, q_1]$. Then by definition of proposition identity, $p = q_1$. Hence p is true since q_1 is. ⋈

(227.4)★ (Exercise)

(227.5)★ (Exercise)

(227.6)★ (Exercise)

(229) ($\rightarrow$) Assume *ExtensionOf*(x, p). So by definition (228):

(ϑ) $A!x\,\&\,\forall F(xF \rightarrow \exists q(F = [\lambda y\, q]))\,\&\,\forall q((x\Sigma q) \equiv (q \equiv p))$

Since the first conjunct is $A!x$, it remains by definition (211) to show:

$$\forall F(xF \equiv \exists q((q \equiv p)\,\&\,F = [\lambda y\, q]))$$

By GEN, it suffices to show:

$$xF \equiv \exists q((q \equiv p)\,\&\,F = [\lambda y\, q])$$

We prove both directions:

($\rightarrow$) Assume xF. Then by the second conjunct of (ϑ), it follows that $\exists q(F = [\lambda y\, q])$. Let q_1 be an arbitrary such proposition, so that we know $F = [\lambda y\, q_1]$. Hence, $x[\lambda y\, q_1]$, and since we know $A!x$ by the first conjunct of (ϑ), it follows by definition (216) that $x\Sigma q_1$. But then from the third conjunct of (ϑ), it follows that $q_1 \equiv p$. So we've established $(q_1 \equiv p)$ & $F = [\lambda y\, q_1]$. Hence, $\exists q((q \equiv p)\,\&\,F = [\lambda y\, q])$.

($\leftarrow$) Assume $\exists q((q \equiv p) \,\&\, F = [\lambda y\, q])$. Let q_2 be an arbitrary such proposition, so that we know:

(ξ) $(q_2 \equiv p) \,\&\, F = [\lambda y\, q_2]$

Now by the third conjunct of (ϑ), it follows that $(x\Sigma q_2) \equiv (q_2 \equiv p)$. From this and the first conjunct of (ξ), it follows that $x\Sigma q_2$. By definition (216), it follows that $x[\lambda y\, q_2]$. But then by the second conjunct of (ξ), it follows that xF.

($\leftarrow$) Assume $\mathit{TruthValueOf}(x,p)$. So by definition (211), we know:

(ζ) $A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$

Since $A!x$, it remains by definition (228) to show:

(a) $\forall F(xF \rightarrow \exists q(F = [\lambda y\, q]))$

(b) $\forall q((x\Sigma q) \equiv (q \equiv p))$

(a) follows *a fortiori* from the second conjunct of (ζ). For (b), it suffices to show $(x\Sigma q_3) \equiv (q_3 \equiv p)$, for an arbitrary proposition q_3:

($\rightarrow$) Assume $x\Sigma q_3$. Then by definition (216), $x[\lambda y q_3]$. So by the second conjuct of (ζ), it follows that:

$$\exists q((q \equiv p) \,\&\, [\lambda y\, q_3] = [\lambda y\, q])$$

Suppose q_4 is an arbitrary such proposition, so that we know: $(q_4 \equiv p) \,\&\, [\lambda y\, q_3] = [\lambda y\, q_4]$. But the second conjunct of this last result and the definition of identity for propositions, we know $q_3 = q_4$. So from the first conjunct, it follows that $q_3 \equiv p$.

($\leftarrow$) Assume $q_3 \equiv p$. Then by Rule ReflId (74.1) and &I, $(q_3 \equiv p) \,\&\, [\lambda y\, q_3] = [\lambda y\, q_3]$. Hence, $\exists q((q \equiv p) \,\&\, [\lambda y\, q_3] = [\lambda y\, q])$. So by the second conjunct of (ζ), it follows that $x[\lambda y\, q_3]$. But since $A!x$, it follows by (216) that $x\Sigma q_3$.

⋈

(230.1) By (212.2), $\exists! x \mathit{TruthValueOf}(x,p)$. Since the proof of (229) is modally strict, we may use it and the Rule of Substitution to infer $\exists! x \mathit{ExtensionOf}(x,p)$. ⋈

(230.2) (Exercise)

(230.3) As an instance of theorem (105), we know:

$$\mathcal{A}(\mathit{ExtensionOf}(x,p) \equiv \mathit{TruthValueOf}(x,p)) \rightarrow \\ \forall x(x = \iota x \mathit{ExtensionOf}(x,p) \rightarrow x = \iota x \mathit{TruthValueOf}(x,p))$$

But the antecedent follows by applying the Rule of Actualization to theorem (229). Hence:

$$\forall x(x = \iota x TruthValueOf(x,p) \to x = \iota x ExtensionOf(x,p))$$

Instantiating to p° (which we know to be well-defined), we obtain:

$$p^\circ = \iota x TruthValueOf(x,p) \to p^\circ = \iota x ExtensionOf(x,p)$$

But the antecedent holds by definition (214). Hence $p^\circ = \iota x ExtensionOf(x,p)$, which by the symmetry of identity gives us $\iota x ExtensionOf(x,p) = p^\circ$. ⋈

(230.4) (Exercise)

(236) Assume $ExtensionOf(x,G)$ and $ExtensionOf(y,H)$. By definition (234), these assumptions yield, respectively:

(a) $A!x \,\&\, \forall F(xF \equiv \forall z(Fz \equiv Gz))$

(b) $A!y \,\&\, \forall F(yF \equiv \forall z(Fz \equiv Hz))$

($\to$) Assume $x = y$. Then by Rule SubId, it follows from (a) that:

(c) $A!y \,\&\, \forall F(yF \equiv \forall z(Fz \equiv Gz))$

Hence, by (83.11) and (83.10), the right conjuncts of (b) and (c) imply:

(d) $\forall F[\forall z(Fz \equiv Hz) \equiv \forall z(Fz \equiv Gz)]$

Now if we instantiate (d) to G, we know:

(e) $\forall z(Gz \equiv Hz) \equiv \forall z(Gz \equiv Gz)$

But the right side is easily derived from the tautology $Gz \equiv Gz$ and an application of GEN. So the left side of (e), $\forall z(Gz \equiv Hz)$, follows by biconditional syllogism.

($\leftarrow$) Assume:

(f) $\forall z(Gz \equiv Hz)$

Since we know $A!x$ and $A!y$ by the left conjuncts of (a) and (b), it suffices by theorem (172.1) to show $\forall F(xF \equiv yF)$, and by GEN, that $xF \equiv yF$:

> ($\to$) Assume xF. Then by the right conjunct of (a), it follows that $\forall z(Fz \equiv Gz)$. But from this and (f), it follows that $\forall z(Fz \equiv Hz)$. Hence, by the right conjunct of (b), it follows that yF.
>
> ($\leftarrow$) Assume yF. Then by the right conjunct of (b), it follows that $\forall z(Fz \equiv Hz)$. Now since our assumption (f), which asserts the material equivalence of G and H, is a symmetrical condition on G and H (exercise), it follows that $\forall z(Hz \equiv Gz)$. But material equivalence is also a transitive condition on properties (exercise). Hence it follows from $\forall z(Fz \equiv Hz)$ and $\forall z(Hz \equiv Gz)$ that $\forall z(Fz \equiv Gz)$. So by the right conjunct of (a), xF. ⋈

(237.1) As an instance of the Comprehension Principle for Abstract Objects (39), we know:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \forall z(Fz \equiv Gz)))$$

So by definition (234), $\exists x ExtensionOf(x, G)$. ⋈

(237.2) (Exercise)

(238) It follows from (237.2) that $\mathcal{A}\exists!x ExtensionOf(x, G)$, by the Rule of Actualization. Hence by (179.2), $\exists y(y = \imath x ExtensionOf(x, G))$. ⋈[283]

(240) $\overline{L}$ is the negation of L, where the latter is defined as $[\lambda x\, E!x \to E!x]$. Now abbreviate $[\lambda x\, E!x \,\&\, \Diamond\neg E!x]$ as P. Then by axiom (32.4), β-Conversion, and the Rule of Substitution, we know:

(ϑ) $\Diamond\exists x Px \,\&\, \Diamond\neg\exists x Px$

Since $\overline{L}$ is an impossible property (i.e., necessarily, nothing exemplifies $\overline{L}$), we can establish both of the following:

(ζ) $\Diamond\neg\forall z(\overline{L}z \equiv Pz)$.
Proof. We first establish $\exists zPz \to \neg\forall z(\overline{L}z \equiv Pz)$. So assume $\exists zPz$. By definition of $\overline{L}$, we know $\neg\exists z\overline{L}z$. Hence by (86.13), it follows that $\neg\forall z(Pz \equiv \overline{L}z)$, i.e., $\neg\forall z(\overline{L}z \equiv Pz)$. Hence, by conditional proof $\exists zPz \to \neg\forall z(\overline{L}z \equiv Pz)$. Since this proof is modally strict, it follows by Rule RM (110) that $\Diamond\exists zPz \to \Diamond\neg\forall z(\overline{L}z \equiv Pz)$. But the first conjunct of (ϑ) is equivalent to $\Diamond\exists zPz$. So it follows that $\Diamond\neg\forall z(\overline{L}z \equiv Pz)$.

(ξ) $\Diamond\forall z(\overline{L}z \equiv Pz)$
Proof. We first establish $\neg\exists zPz \to \forall z(\overline{L}z \equiv Pz)$. So assume $\neg\exists zPz$. Again, by definition of $\overline{L}$, we know $\neg\exists z\overline{L}z$. Hence by (86.12), it follows that $\forall z(Pz \equiv \overline{L}z)$, i.e., $\forall z(\overline{L}z \equiv Pz)$. Hence, by conditional proof $\neg\exists zPz \to \forall z(\overline{L}z \equiv Pz)$. Since this proof is modally strict, it follows by Rule RM that $\Diamond\neg\exists zPz \to \Diamond\forall z(\overline{L}z \equiv Pz)$. But the second conjunct of (ϑ) is equivalent to $\Diamond\neg\exists zPz$. So it follows that $\Diamond\forall z(\overline{L}z \equiv Pz)$.

Conjoining (ξ) and (ζ), we have:

$$\Diamond\forall z(\overline{L}z \equiv Pz) \,\&\, \Diamond\neg\forall z(\overline{L}z \equiv Pz)$$

Hence by the right-to-left direction of (119.12):

$$\Diamond(\forall z(\overline{L}x \equiv Pz) \,\&\, \Diamond\neg\forall z(\overline{L}z \equiv Pz)) \qquad ⋈$$

[283] Alternatively, we know by (180) that $\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \forall z(Fz \equiv Gz))))$. So, by definition of *ExtensionOf* (234), it follows that $\exists y(y = \imath x ExtensionOf(x, G))$.

(242.1)★ [The reasoning that follows is analogous to the reasoning in (217.1)★ but is in a much more compressed form.] By theorem (97)★, we know:

(ϑ) $x = \iota x ExtensionOf(x,G) \equiv \forall z(ExtensionOf(z,G) \equiv z = x)$

Independently, we know $\exists y(y = \epsilon G)$, by (238) and (239). So if we apply GEN to (ϑ) and instantiate the result to ϵG, we obtain:

$$\epsilon G = \iota x ExtensionOf(x,G) \equiv \forall z(ExtensionOf(z,G) \equiv z = \epsilon G)$$

So, by definition (239) and biconditional syllogism, it follows that:

$$\forall z(ExtensionOf(z,G) \equiv z = \epsilon G)$$

So by instantiating this last result to x, and we have:

$$Extension\,(x,G) \equiv x = \epsilon G \qquad \bowtie$$

(242.2)★ By definition (239), we know $\epsilon G = \iota x ExtensionOf(x,G)$. Hence by (101.2)★, it follows that $ExtensionOf(\epsilon G, G)$. ⋈

(242.3)★ By (242.2)★ and the definition of *ExtensionOf* (234), it follows that $\forall F(\epsilon GF \equiv \forall z(Fz \equiv Gz))$. ⋈

(242.4)★ By instantiating (242.3)★ to G, it follows that:

$$\epsilon GG \equiv \forall z(Gz \equiv Gz)$$

But the right side is easily established by GEN from the tautology $Gz \equiv Gz$. ⋈

(243)★ Since ϵF and ϵG are logically proper, we can instantiate them into pre-Basic Law V (236). Substituting ϵF for x, F for G, ϵG for y, and G for H, we obtain:

$$(ExtensionOf(\epsilon F, F) \,\&\, ExtensionOf(\epsilon G, G)) \rightarrow (\epsilon F = \epsilon G \equiv \forall z(Fz \equiv Gz))$$

But we also know by (242.2)★ that $ExtensionOf(\epsilon F, F)$ and $ExtensionOf(\epsilon G, G)$. Hence, $\epsilon F = \epsilon G \equiv \forall z(Fz \equiv Gz)$. ⋈[284]

(245) Assume $ExtensionOf(x,H)$. So by definition (234) and &E, it follows that:

[284] Alternatively: ($\rightarrow$) Assume $\epsilon F = \epsilon G$. By (242.3)★, we know that $\forall H(\epsilon FH \equiv \forall x(Hx \equiv Fx))$. Since $\epsilon F = \epsilon G$, it follows by Rule SubId that $\forall H(\epsilon GH \equiv \forall x(Hx \equiv Fx))$. Instantiating to G, it follows that $\epsilon GG \equiv \forall x(Gx \equiv Fx)$. Since ϵGG by (242.4)★, it follows that $\forall x(Gx \equiv Fx)$, which in turn implies $\forall x(Fx \equiv Gx)$. ($\leftarrow$) Take $\forall x(Fx \equiv Gx)$ as a global assumption. As noted at at the beginning of Remark (241), ϵF and ϵG are (identical to) canonical individuals, and so by (183), they are both abstract. So to show they are identical it suffices, by (172.1), to show they encode the same properties. By GEN, it suffices to show $\epsilon FH \equiv \epsilon FG$:

($\rightarrow$) Assume ϵFH. Then by the left-to-right direction of (242.3)★, $\forall x(Hx \equiv Fx)$. From this and our global assumption it follows that $\forall x(Hx \equiv Gx)$. But by the right-to-left direction of (242.3)★, it follows that ϵGH.

($\leftarrow$) By analogous reasoning.

$$\forall F(xF \equiv \forall z(Fz \equiv Hz))$$

We want to show $\forall y(y \in x \equiv Hy)$. Since y isn't free in our assumption, it suffices by GEN to show $y \in x \equiv Hy$.

($\rightarrow$) Assume $y \in x$. Then by definition of membership (244), it follows that $\exists G(\mathit{ExtensionOf}(x,G) \,\&\, Gy)$. Suppose P is an arbitrary such property, so that we know both $\mathit{ExtensionOf}(x,P) \,\&\, Py$. Note independently that it follows from pre-Basic Law V (236) that:

(ϑ) $(\mathit{ExtensionOf}(x,P) \,\&\, \mathit{ExtensionOf}(x,H)) \rightarrow (x=x \equiv \forall z(Pz \equiv Hz))$

But we know both conjuncts of the antecedent. So we may infer from (ϑ) that $x = x \equiv \forall z(Pz \equiv Hz)$. Since $x = x$ is a known theorem (70.1), it follows that $\forall z(Pz \equiv Hz)$. But since we also know Py, it follows that Hy.

($\leftarrow$) Assume Hy. But given our initial assumption, we have $\mathit{ExtensionOf}(x,H) \,\&\, Hy$. So $\exists G(\mathit{ExtensionOf}(x,G) \,\&\, Gy)$. Hence, by definition of membership (244), $y \in x$. ⋈

(247.1) (Exercise)

(247.2) (Exercise)

(248.1) (Exercise)

(248.2) (Exercise)

(250) (Exercise)

(251)★ By (242.2)★, we know that $\mathit{ExtensionOf}(\epsilon G, G))$. By definition (246), it follows that $\mathit{ClassOf}(\epsilon G, G))$. Hence, $\exists F(\mathit{ClassOf}(\epsilon G, F))$, which by definition (249) yields $\mathit{Class}(\epsilon G)$. ⋈

(252) If we instantiate the universal generalization of (248) to the variable F, we have $\exists x \mathit{ClassOf}(x,F)$. Let a be an arbitrary such object, so that we know $\mathit{ClassOf}(a,F)$. From this, it follows by (247.2) that $\forall y(y \in a \equiv Fy)$. Hence, we may conjoin our results to conclude $\mathit{ClassOf}(a,F) \,\&\, \forall y(y \in a \equiv Fy)$. By $\exists$I, it follows that $\exists x(\mathit{ClassOf}(x,F) \,\&\, \forall y(y \in x \equiv Fy))$. By GEN, we're done. ⋈

(257) ($\rightarrow$) Exercise. ($\leftarrow$) Assume:

(a) $\forall z(z \in c \equiv z \in c')$

where c and c' are both classes. Then it follows by definition (249) that there are properties, say P and Q, such that $\mathit{ClassOf}(c,P)$ and $\mathit{ClassOf}(c',Q)$, respectively. But by theorem (247.2), it follows from the first that $\forall z(z \in c \equiv Pz)$, i.e.,

(b) $\forall z(Pz \equiv z \in c)$

and it follows from the second that:

(c) $\forall z(z \in c' \equiv Qz)$

From (b) and (a), it follows by (83.10) that $\forall z(Pz \equiv z \in c')$, and from this latter conclusion and (c) it follows also by (83.10) that $\forall z(Pz \equiv Qz)$. But, independently, by the definition of *ClassOf* (246), we may transform pre-Basic Law V (236) into:

$$(ClassOf(c,G) \,\&\, ClassOf(c',H)) \rightarrow (c = c' \equiv \forall z(Gz \equiv Hz))$$

Since $ClassOf(c,P)$ and $ClassOf(c',Q)$ by assumption, it follows that:

$$c = c' \equiv \forall z(Pz \equiv Qz)$$

Since we've established the right side of the above biconditional, it follows that $c = c'$. ⋈

(259.1) By our conventions for bound occurrences of restricted variables (254) and the definition of *Empty* (258), we have to show:

$$\exists x[Class(x) \,\&\, \neg\exists y(y \in x)]$$

Consider the impossible property $\overline{L}$, where L was defined as $[\lambda x\, E!x \rightarrow E!x]$ (140) and $\overline{F}$ was defined as $[\lambda y\, \neg Fy]$ (136). Then by the Fundamental Theorem for Natural Classes and Logical Sets (252), we know:

$$\exists x(Class(x) \,\&\, \forall y(y \in x \equiv \overline{L}y))$$

Suppose a is an arbitrary such object, so that we know:

(ϑ) $Class(a) \,\&\, \forall y(y \in a \equiv \overline{L}y)$

If we can show $\neg\exists y(y \in a)$, then given the first conjunct of (ϑ), we're done. Suppose, for reductio, that $\exists y(y \in a)$. Let b is an arbitrary such object, so that we know $b \in a$. Then by the second conjunct of (ϑ), it follows that $\overline{L}b$. But recall that in the proof of (140.2), we developed a (modally-strict) proof of $\forall x \neg\overline{L}x$. So $\neg\overline{L}b$. Contradiction. Hence, $\neg\exists y(y \in a)$. ⋈

(259.2) By definition (258) and our conventions for restricted variables, we have to show:

$$\exists x[Class(x) \,\&\, \neg\exists y(y \in x) \,\&\, \forall z((Class(z) \,\&\, \neg\exists y(y \in z)) \rightarrow z = x)]$$

By (259.1), we already know $\exists x(Class(x) \,\&\, \neg\exists y(y \in x))$. So suppose a is an arbitrary such object, so that $Class(a) \,\&\, \neg\exists y(y \in a)$. Then, since z isn't free in our assumption, it suffices by GEN to show $(Class(z) \,\&\, \neg\exists y(y \in z)) \rightarrow z = a$. So assume $Class(z) \,\&\, \neg\exists y(y \in z)$. From the second conjunct and the fact that $\neg\exists y(y \in a)$, it follows by (86.12) that $\forall x(x \in z \equiv x \in a)$, i.e., that z and a have the same members. Hence, by the principle of extensionality (257), it follows that $z = a$. ⋈

(259.3) (Exercise)

(261.1) By definition (260) and our conventions for bound occurrences of restricted variables (254), we have to show:

$$\exists x[Class(x) \,\&\, \forall y(y \in x)]$$

Consider the necessary property L (= $[\lambda x\, E!x \rightarrow E!x]$). Then by the Fundamental Theorem for Natural Classes and Logical Sets (252), we know:

$$\exists x(Class(x) \,\&\, \forall y(y \in x \equiv Ly))$$

Suppose b is an arbitrary such object, so that we know:

$$(\vartheta)\quad Class(b) \,\&\, \forall y(y \in b \equiv Ly)$$

If we can show $\forall y(y \in b)$, then in light of the first conjunct of (ϑ), we're done. By GEN, it suffices to show $y \in b$. Since by the second conjunct of (ϑ) we know $y \in b \equiv Ly$, it suffices to show Ly. But by definition of L and Strengthened β-Conversion, we know $Ly \equiv (E!y \rightarrow E!y)$. But the right side of this biconditional is a tautology. ⋈

(261.2) (Exercise)

(263.1) By definition (262) and our conventions for restricted variables, we have to show:

$$(Class(z) \,\&\, Class(w)) \rightarrow \exists x(Class(x) \,\&\, UnionOf(x, z, w))$$

By definition of *UnionOf* (262), we have to show:

$$(Class(z) \,\&\, Class(w)) \rightarrow \exists x[Class(x) \,\&\, \forall y(y \in x \equiv (y \in z \,\vee\, y \in w))]$$

So assume $Class(z)$ and $Class(w)$. Then, we know by (249) that there are properties P and Q such that $ClassOf(z, P)$ and $ClassOf(w, Q)$. Now consider the property $[\lambda y\, Py \vee Qy]$, which we know exists by (29.2). Call this property R. Then by the Fundamental Theorem for Natural Classes and Logical Sets (252), we know:

$$\exists x(Class(x) \,\&\, \forall y(y \in x \equiv Ry))$$

Suppose a is an arbitrary such object, so that we know:

$$(\vartheta)\quad Class(a) \,\&\, \forall y(y \in a \equiv Ry)$$

Since $Class(a)$, it remains, by ∃I, to show $\forall y(y \in a \equiv (y \in z \vee y \in w))$. By GEN, it suffices to show $y \in a \equiv (y \in z \vee y \in w)$. This is established by the following series of biconditionals, all of which are consequences of what we have established so far:

$$\begin{array}{rcll} y \in a & \equiv & Ry & \text{2nd conjunct } (\vartheta) \\ & \equiv & [\lambda x\, Px \vee Qx]y & \text{by definition of } R \\ & \equiv & Py \vee Qy & \text{by } \beta\text{-Conversion (128)} \\ & \equiv & y \in z \vee Qy & \text{by } \mathit{ClassOf}(z,P)\text{, (247.2), RN, and (112.2)} \\ & \equiv & y \in z \vee y \in w & \text{by } \mathit{ClassOf}(w,Q)\text{, (247.2), RN, and (112.2)} \bowtie \end{array}$$

(263.2) By (263.1), $\exists c\mathit{UnionOf}(c,c',c'')$. Assume c_1 is an arbitrary such class, so that we know (i) $\mathit{Class}(c_1)$ and (ii) $\mathit{UnionOf}(c_1,c',c'')$. It thus remains to show:

(ϑ) $\forall z[(\mathit{Class}(z) \,\&\, \mathit{UnionOf}(z,c',c'')) \rightarrow z = c_1]$

Note that it follows from (ii) by definition of *UnionOf* (262) that:

$$\forall y(y \in c_1 \equiv (y \in c' \vee y \in c''))$$

By (83.11), this is equivalent to:

(ξ) $\forall y((y \in c' \vee y \in c'') \equiv y \in c_1)$

It also follows by definition of *UnionOf* that to show (ϑ), we have to show:

$$\forall z[(\mathit{Class}(z) \,\&\, \forall y(y \in z \equiv (y \in c' \vee y \in c''))) \rightarrow z = c_1]$$

Given GEN, assume $\mathit{Class}(z) \,\&\, \forall y(y \in z \equiv (y \in c' \vee y \in c''))$, to prove $z = c_1$. The right conjunct of this last assumption is:

(ζ) $\forall y(y \in z \equiv (y \in c' \vee y \in c''))$

But (ζ) and (ξ) together imply $\forall y(y \in z \equiv y \in c_1)$, by (83.10). Since both $\mathit{Class}(z)$ and $\mathit{Class}(c_1)$, by separate hypotheses, it follows by the principle of extensionality (257) that $z = c_1$. $\bowtie$

(264) By applying the Rule of Actualization to theorem (263.2), it follows that:

$$\mathcal{A}\exists! c\forall y(y \in c \equiv (y \in c' \vee y \in c''))$$

So by theorem (179.2), it follows that:

$$\exists z(z = \iota c\forall y(y \in c \equiv (y \in c' \vee y \in c'')))$$

Hence, by definition (262):

$$\exists z(z = \iota c\mathit{UnionOf}(c,c',c'')) \qquad \bowtie$$

(268.1) Since c is a class, we know by (249) that there is a property, say P, such that $\mathit{ClassOf}(c,P)$. So consider the property $\overline{P}$, i.e., $[\lambda y\, \neg Py]$, which we know exists by (29.2). If we formulate the Fundamental Theorem for Natural Classes and Logical Sets (252) so that it uses restricted variables and instantiate to $\overline{P}$, we know:

$$\exists c'\forall y(y \in c' \equiv \overline{P}y)$$

Suppose c_1 is an arbitrary such class, so that we know:

(ϑ) $\forall y(y \in c_1 \equiv \overline{P}y)$

By definition (267) and now familiar reasoning, we simply show $y \in c_1 \equiv y \notin c$. We do this as follows:

$$\begin{array}{lcll} y \in c_1 & \equiv & \overline{P}y & \text{by } (\vartheta) \\ & \equiv & [\lambda x\, \neg Px]y & \text{by definition of } \overline{P} \\ & \equiv & \neg Py & \text{by } \beta\text{-Conversion (128)} \\ & \equiv & \neg y \in c & \text{by (247.2), } \mathit{ClassOf}(c,P) \\ & \equiv & y \notin c & \text{by definition of } \notin \qquad \bowtie \end{array}$$

(268.2) Suppose c_1 is an arbitrarily chosen class witnessing (268.1), so that by definition (267) we know $\forall y(y \in c_1 \equiv y \notin c)$. To show uniqueness, assume, for an arbitrarily chosen class c_2 that $\forall y(y \in c_2 \equiv y \notin c)$. Then it follows by properties of the biconditional that $\forall y(y \in c_2 \equiv y \in c_1)$, and so $c_2 = c_1$ by the principle of extensionality (257). ⋈

(268.3) (Exercise)

(270.1) Since c' and c'' are classes, there are, by (249), properties P and Q such that $\mathit{ClassOf}(c',P)$ and $\mathit{ClassOf}(c'',Q)$. Now consider the property $[\lambda y\, Py \,\&\, Qy]$ and call this property R. By the Fundamental Theorem for Natural Classes and Logical Sets (252) (formulated with restricted variables), we know:

$$\exists c\forall y(y \in c \equiv Ry)$$

Suppose c_1 is an arbitrary such class, so that we know $\forall y(y \in c_1 \equiv Ry)$. By definition (269), it remains to show:

$$\forall y(y \in c_1 \equiv (y \in c' \,\&\, y \in c''))$$

We leave this as an exercise. ⋈

(270.2) (Exercise)

(271) (Exercise)

(273)★ By (271), we know

$$\exists z(z = \imath c\mathit{IntersectionOf}(c,c',c''))$$

Assume c_1 is an arbitrary such object, so that we know:

(ϑ) $c_1 = \imath c\forall y(y \in c \equiv y \in c' \,\&\, y \in c'')$

So by definition (269), we know:

$$c_1 = \imath c \forall y(y \in c \equiv y \in c' \,\&\, y \in c'')$$

Then by (101.2)★, it follows that:

$$\forall y(y \in c_1 \equiv y \in c' \,\&\, y \in c'')$$

But by definition of $c' \cap c''$ (272), it also follows from (ϑ) that $c_1 = c' \cap c''$. Hence, by Rule SubId,

$$\forall y(y \in c' \cap c'' \equiv y \in c' \,\&\, y \in c'')$$

Instantiating to z:

$$z \in c' \cap c'' \equiv z \in c' \,\&\, z \in c'' \qquad \bowtie$$

(274.1) We want to show:

$\exists c \forall y(y \in c \equiv \varphi^*)$, provided c doesn't occur free in φ^*

For any such φ^*, consider the property $[\lambda y\, \varphi^*]$. By the Fundamental Theorem for Natural Classes and Logical Sets (252) (formulated with restricted variables), we know:

$$\exists c \forall y(y \in x \equiv [\lambda y\, \varphi^*]y)$$

Suppose c_1 is an arbitrary such class, so that we know:

(ϑ) $\forall y(y \in c_1 \equiv [\lambda y\, \varphi^*]y)$

It then remains to show:

$$\forall y(y \in c_1 \equiv \varphi^*)$$

But this is immediate from (ϑ) by the Rule of Substitution and the modally strict instance of Strengthened β-Conversion: $[\lambda y\, \varphi^*]y \equiv \varphi$. $\bowtie$

(274.1) (Alternative Proof From First Principles) The following proves our theorem by an appeal to the Comprehension Principle for Abstract Objects (39) instead of theorem (248.1). We do this only to make it a bit clearer how the result comes about. We want to show:

$\exists c \forall y(y \in c \equiv \varphi^*)$, provided c doesn't occur free in φ^*

i.e., by eliminating the restricted variable, we have to show:

$\exists x(Class(x) \,\&\, \forall y(y \in x \equiv \varphi^*))$, provided x doesn't occur free in φ^*

By expanding the definition of $Class(x)$, we have to show:

(ϑ) $\exists x[\exists G(ClassOf(x,G)) \,\&\, \forall y(y \in x \equiv \varphi^*)]$, provided x doesn't occur free in φ^*

To prove this, let φ^* be any formula in which x doesn't occur free. By (29.2), $[\lambda y\, \varphi^*]$ exists. Hence, the following is an instance of the Comprehension Principle for Abstract Objects:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \forall z(Fz \equiv [\lambda y\, \varphi^*]z)))$$

Let b be an arbitrary such object, so that we know:

$$A!b \,\&\, \forall F(bF \equiv \forall z(Fz \equiv [\lambda y\, \varphi^*]z))$$

So by definitions (234) and (246), $\mathit{ClassOf}(b, [\lambda y\, \varphi^*])$. So $\exists G(\mathit{ClassOf}(b, G))$. So if we can show $\forall y(y \in b \equiv \varphi^*)$, then by &I and $\exists$I we will have established (ϑ) and be done.

To show $\forall y(y \in b \equiv \varphi^*)$, it suffices by GEN to show $y \in b \equiv \varphi^*$. But since $\mathit{ClassOf}(b, [\lambda y\, \varphi^*])$, it follows by (247.2) that :

$$y \in b \equiv [\lambda y\, \varphi^*]y$$

Hence, by a biconditional syllogism with the appropriate instance of Strengthened β-Conversion (i.e., $[\lambda y\, \varphi^*]y \equiv \varphi^*$), it follows that:

$$y \in b \equiv \varphi^* \qquad \bowtie$$

(274.2) (Exercise)

(275) Suppose φ^* is a propositional formula in which c doesn't occur free. Then by (274.2), we know:

$$\exists! c \forall y(y \in c \equiv \varphi^*))$$

Hence, by the Rule of Actualization, it follows that:

$$\mathcal{A}\exists! c \forall y(y \in c \equiv \varphi^*))$$

So by (179.2), it follows that $\exists z(z = \imath c \forall y(y \in c \equiv \varphi^*))$. $\bowtie$

(277.1)$\star$ Let φ^* be any propositional formula in which c doesn't occur free. Then by (275), it follows that $\exists z(z = \imath c \forall y(y \in c \equiv \varphi^*))$. Assume a is an arbitrary such object, so that we know:

(ϑ) $a = \imath c \forall y(y \in c \equiv \varphi^*)$

By (101.2)$\star$, it follows from (ϑ) that $\forall y(y \in a \equiv \varphi^*)$, and so, by $\forall$E, $y \in a \equiv \varphi^*$. But by definition of $\{y|\varphi^*\}$ (276), it also follows from (ϑ) that $a = \{y|\varphi^*\}$. Hence, by Rule SubId, it follows that $y \in \{y \,|\, \varphi^*\} \equiv \varphi^*$. $\bowtie$

(277.2)$\star$ Apply GEN to (277.1)$\star$ and instantiate the result to z. $\bowtie$

(277.3)$\star$ Let φ^* be any formula in which the variable x doesn't occur free, or in which the restricted class variable c doesn't occur free. Then our proof strategy is to show:

(a) $Class(\{y \mid \varphi^*\})$

(b) $Class(\epsilon[\lambda y\, \varphi^*])$

(c) $\forall z(z \in \{y \mid \varphi^*\} \equiv z \in \epsilon[\lambda y\, \varphi^*])$

If we can prove these three claims, we are done, since it follows by the Principle of Extensionality (257) that $\{y \mid \varphi^*\}$ and $\epsilon[\lambda y\, \varphi^*]$ are identical.

By our discussion of bound restricted variables in (254.3), (a) is an immediate a consequence of the definition (276) of $\{y|\varphi^*\}$. We have also established (b), since it is an instance of (251)★. So it remains to show (c), namely, that $\{y \mid \varphi^*\}$ and $\epsilon[\lambda y\, \varphi^*]$ have the same members. Note first that by theorem (247.2), it follows from (b) that:

(ϑ) $\forall x(x \in \epsilon[\lambda y\, \varphi^*] \equiv [\lambda y\, \varphi^*]x)$

So we may establish $z \in \{y \mid \varphi^*\} \equiv z \in \epsilon[\lambda y\, \varphi^*]$ by the following reasoning:

$$\begin{array}{lcll} z \in \{y \mid \varphi^*\} & \equiv & \varphi^{*z}_{\ y} & \text{by (277.2)}\bigstar \\ & \equiv & [\lambda y\, \varphi^*]z & \text{by } \beta\text{-Conversion} \\ & \equiv & z \in \epsilon[\lambda y\, \varphi^*] & \text{by } (\vartheta) \end{array}$$

Hence, by GEN, $\{y \mid \varphi^*\}$ and $\epsilon[\lambda y\, \varphi^*]$ have the same members. ⋈

(277.4)★ Since Gy is a propositional formula in which s isn't free, we have, as an instance of theorem (277.3)★, that $\{y \mid Gy\} = \epsilon[\lambda y\, Gy]$. Hence by η-Conversion, $\{y \mid Gy\} = \epsilon G$. ⋈

(278.1) Let c' be an arbitrarily chosen class and φ^* be any propositional formula in which c doesn't occur free. Since c' is a class, we know there exists a property, say P, such that $ClassOf(c', P)$. Since φ^* has no encoding subformulas, the property $[\lambda y\, \varphi^*]$ exists, by (29.2). Call this property Q. Then $[\lambda y\, Py \,\&\, Qy]$ also exists, by (29.2). Call this property R.

By the Fundamental Theorem for Natural Classes and Logical Sets (252) (formulated with restricted variables), we know:

$$\exists c \forall y(y \in x \equiv Ry)$$

Suppose c_1 is an arbitrary such class, so that we know:

(ϑ) $\forall y(y \in c_1 \equiv Ry)$

It then remains to show:

$$y \in c_1 \equiv y \in c' \,\&\, \varphi^*$$

We do so as follows:

$$\begin{array}{rcll}
y \in c_1 & \equiv & Ry & (247.2) \text{ and } \mathit{ClassOf}(c_1, R) \\
& \equiv & [\lambda y\, Py \,\&\, Qy]y & \text{definition of } R \\
& \equiv & Py \,\&\, Qy & \text{by } \beta\text{-Conversion } (128) \\
& \equiv & Py \,\&\, [\lambda y\, \varphi^*]y & \text{definition of } Q \\
& \equiv & Py \,\&\, \varphi^* & (128), \text{RN, and } (112.2) \\
& \equiv & y \in c' \,\&\, \varphi^* & (247.2), \mathit{ClassOf}(c', P), (113) \bowtie
\end{array}$$

(278.2) (Exercise)

(279) (Exercise)

(281)★ (Exercise)

(282)★ By definition and our conventions in (254.3), we know that $\{y|y \in c' \& \varphi^*\}$ and $c' \cap \{y|\varphi^*\}$ are both classes. Consequently, by the Principle of Extensionality, it remains only to show:

$$z \in \{y \,|\, y \in c' \,\&\, \varphi^*\} \equiv z \in c' \cap \{y \,|\, \varphi^*\}$$

We first note that by instantiating c'' in (273)★ to $\{y|\varphi^*\}$, we know the following fact:

$$(\vartheta) \quad z \in c' \cap \{y \,|\, \varphi^*\} \equiv (z \in c' \,\&\, z \in \{y \,|\, \varphi^*\})$$

We now establish what we have to show by the following biconditional chain:

$$\begin{array}{rcll}
z \in \{y \,|\, y \in c' \,\&\, \varphi^*\} & \equiv & z \in c' \,\&\, \varphi^{*z}_{\ \ y} & \text{by } (281)\bigstar \\
& \equiv & z \in c' \,\&\, z \in \{y \,|\, \varphi^*\} & \text{by } (277.2)\bigstar, \text{RN, and } (112.2) \\
& \equiv & z \in c' \cap \{y \,|\, \varphi^*\} & \text{by } (\vartheta) \qquad \bowtie
\end{array}$$

(283) Let R be any 2-place relation and consider any class c'. By definition, we know $\mathit{ClassOf}(c', P)$, for some P. So $[\lambda x\, \exists z(Pz \,\&\, Rzx)]$ is also a property. By the Fundamental Theorem for Natural Classes, $\exists c \forall y(y \in c \equiv [\lambda x\, \exists z(Pz \,\&\, Rzx)]y)$. Then by an appropriate instance of β-Conversion and the Rule of Substitution:

$$(\vartheta) \quad \exists c \forall y(y \in c \equiv \exists z(Pz \,\&\, Rzy)).$$

But from the fact that $\mathit{ClassOf}(c', P)$, we know by (247.2), that $\forall y(y \in c' \equiv Py)$. Hence, it follows by modally strict reasoning that $z \in c' \equiv Pz$. So from this and (ϑ) it follows by the Rule of Substitution that $\exists c \forall y(y \in c \equiv \exists z(z \in c' \,\&\, Rzy))$. ⋈

(284.1) Assume $O!z$. Then the consequent, $\exists c \forall y(y \in c \equiv y =_E z)$, is an instance of set comprehension (274.1). [Indeed, $\exists c \forall y(y \in c \equiv y =_E z)$, is an instance of (274.1) whether we assume $O!z$ or not. But we postpone discussion of why we don't prove the stronger result until Remark (285).] ⋈

(284.2) Assume $O!z$. Then the consequent, $\exists! c \forall y(y \in c \equiv y =_E z)$, is an instance of set comprehension (274.2). [Again, since $\exists c \forall y(y \in c \equiv y =_E z)$, is an instance of (274.2) whether we assume $O!z$ or not, we postpone discussion of why we don't prove the stronger result until Remark (285).] ⋈

(284.3) (Exercise)

(286.1) Assume $O!x$ and $O!z$. Consider the property $[\lambda y\, y =_E x \vee y =_E z]$. This property exists by (29).2). Call this property P. Then by the Fundamental Theorem, $\exists c \forall y(y \in c \equiv Py)$. Suppose c_1 is an arbitrary such class. By now familiar reasoning, it remains to show $y \in c_1 \equiv y =_E x \vee y =_E z$. (Exercise) ⋈

(286.2) (Exercise)

(287.1) Let c' be any class. Then there is some property, say P, such that $ClassOf(c, P)$. For conditional proof, assume $O!x$. Now consider the property $[\lambda z\, Pz \vee z =_E x]$, which clearly exists. Call this property Q. Then by the Fundamental Theorem, $\exists c \forall y(y \in c \equiv Qy)$. Suppose c_1 is an arbitrary such class. By now familiar reasoning, it remains to show $y \in c_1 \equiv (y \in c' \vee y =_E x)$. (Exercise) ⋈

(287.2) (Exercise)

(291.1) – **(291.4)** (Exercises)

(293.1)★ (Exercise)

(293.2)★ (Exercise)

(295) ($\rightarrow$) Assume $\forall z([\lambda w\, w \| u]z \equiv [\lambda w\, w \| v]z)$. Instantiating to u, it follows that $[\lambda w\, w \| u]u \equiv [\lambda w\, w \| v]u$. Reducing both the left and right conditions by strengthened β-Conversion and the Rule of Substitution, we may conclude $u \| u \equiv u \| v$. But *being parallel to* is, by hypothesis, an equivalence relation with respect to lines, and so $u \| u$. Hence, $u \| v$.

($\leftarrow$) Let $u \| v$ be our global assumption. We want to show $\forall z([\lambda w\, w \| u]z \equiv [\lambda w\, w \| v]z)$. By GEN, it suffices to show $[\lambda w\, w \| u]z \equiv [\lambda w\, w \| v]z$.

> ($\rightarrow$) Assume $[\lambda w\, w \| u]z$. Then by strengthened β-Conversion, z is a line and $z \| u$. But from this and our global assumption that $u \| v$, it follows that $z \| v$, by the fact that $\|$ is a transitive relation on lines. Hence, by the right-to-left direction of strengthened β-Conversion and the fact that z is a line, $[\lambda w\, w \| v]z$.
>
> ($\leftarrow$) Assume $[\lambda w\, w \| v]z$. Then by strengthened β-Conversion, z is a line and $z \| v$. Independently, by the symmetry of $\|$, our global assumption $u \| v$ implies that $v \| u$. Hence $z \| u$, by the fact that $\|$ is a transitive relation on lines. So by strengthened β-Conversion and the fact that z is a line, $[\lambda w\, w \| u]z$. ⋈

(297.1) By (237.1), $\exists x ExtensionOf(x, [\lambda v\, v\|u])$. Hence by definition (296), $\exists x DirectionOf(x, [\lambda v\, v\|u])$. ⋈

(297.2) (Exercise)

(297.3) (Exercise)

(299)★ By definition (298), we know both:

$$\vec{u} = \imath x DirectionOf(x, u)$$

$$\vec{v} = \imath x DirectionOf(x, v)$$

But by definition (296), we also know that $DirectionOf(x, u)$ is a definiendum with $ExtensionOf(x, [\lambda w\, w\|u])$ as its definiens, and that $DirectionOf(x, v)$ is a definiendum with $ExtensionOf(x, [\lambda w\, w\|v])$ as its definiens. Hence, we can appeal to Rule SubDefForm (208.5) to derive the following identities, respectively, from our initial identities:[285]

$$\vec{u} = \imath x ExtensionOf(x, [\lambda w\, w\|u])$$

$$\vec{v} = \imath x ExtensionOf(x, [\lambda w\, w\|v])$$

But by definition of ϵG (239), we know:

$$\imath x ExtensionOf(x, [\lambda w\, w\|u]) = \epsilon[\lambda w\, w\|u]$$

$$\imath x ExtensionOf(x, [\lambda w\, w\|v]) = \epsilon[\lambda w\, w\|v]$$

Hence, by the transitivity of identity, it follows that:

$$(\vartheta)\quad \vec{u} = \epsilon[\lambda w\, w\|u]$$

$$(\xi)\quad \vec{v} = \epsilon[\lambda w\, w\|v]$$

So we may now reason as follows:

$$\begin{array}{lcll}
\vec{u}=\vec{v} & \equiv & \vec{u}=\vec{v} & \text{by idempotence of} \equiv \text{(63.4.a)} \\
\vec{u}=\vec{v} & \equiv & \epsilon[\lambda w\, w\|u] = \epsilon[\lambda w\, w\|v] & \text{by } (\vartheta), (\xi), \text{Rule SubId (74.2)} \\
 & \equiv & \forall z([\lambda w\, w\|u]z \equiv [\lambda w\, w\|v]z) & \text{by Basic Law V (243)}\star \\
 & \equiv & u\|v & \text{by (295)} \qquad \bowtie
\end{array}$$

---

[285] Alternatively, we can obtain the identities that follow in the text by appealing to (296) to conclude both:

$$DirectionOf(x, u) \equiv ExtensionOf(x, [\lambda w\, w\|u])$$

$$DirectionOf(x, v) \equiv ExtensionOf(x, [\lambda w\, w\|v])$$

Then we apply the Rule of Actualization to obtain the actualized form of these equivalences. Then the resulting actualized forms together with our initial identities yield the identities that follow in the text, by theorem (105).

(301) (Exercise)

(303) (Exercise)

(305.1) – **(305.3)** (Exercises)

(307)$\star$ (Exercise)

(309) (Exercise)

(311) (Exercise)

(312.1) ($\rightarrow$) Assume $\forall w([\lambda z\, \widetilde{R}zx]w \equiv [\lambda z\, \widetilde{R}zy]w)$. Instantiating to x, it follows that $[\lambda z\, \widetilde{R}zx]x \equiv [\lambda z\, \widetilde{R}zy]x$. Reducing both the left and right conditions by strengthened β-Conversion and the Rule of Substitution, we may conclude $\widetilde{R}xx \equiv \widetilde{R}xy$. But $\widetilde{R}$ is, by hypothesis, an equivalence relation, and so $\widetilde{R}xx$. Hence, $\widetilde{R}xy$.

($\leftarrow$) Let $\widetilde{R}xy$ be our global assumption. We want to show $\forall w([\lambda z\, \widetilde{R}zx]w \equiv [\lambda z\, \widetilde{R}zy]w)$. By GEN, it suffices to show $[\lambda z\, \widetilde{R}zx]w \equiv [\lambda z\, \widetilde{R}zy]w$:

> ($\rightarrow$) Assume $[\lambda z\, \widetilde{R}zx]w$. Then by strengthened β-Conversion, $\widetilde{R}wx$. But from this and our global assumption that $\widetilde{R}xy$, it follows that $\widetilde{R}wy$, by the fact that $\widetilde{R}$ is a transitive relation. Hence, by the right-to-left direction of strengthened β-Conversion, $[\lambda z\, \widetilde{R}zy]w$.
>
> ($\leftarrow$) Assume $[\lambda z\, \widetilde{R}zy]w$. Then by strengthened β-Conversion, $\widetilde{R}wy$. Independently, by the symmetry of $\widetilde{R}$, our global assumption $\widetilde{R}xy$ implies that $\widetilde{R}yx$. Hence $\widetilde{R}wx$, by the fact that $\widetilde{R}$ is a transitive relation. So by strengthened β-Conversion, $[\lambda z\, \widetilde{R}zx]w$. $\bowtie$

(312.2) (Exercise)

(312.3) (Exercise)

(314.1) (Exercise)

(314.2) (Exercise)

(314.3) (Exercise)

(316)$\star$ From definitions of $\widehat{x_{\widetilde{R}}}$ (315), $\widetilde{R}$-*AbstractionOf*(w,x) (313), and ϵG (239), it follows by reasoning analogous to that used at the outset of the proof of (299)$\star$ that:

(ϑ) $\widehat{x_{\widetilde{R}}} = \epsilon[\lambda z\, \widetilde{R}zx]$

(ξ) $\widehat{y_{\widetilde{R}}} = \epsilon[\lambda z\, \widetilde{R}zy]$

So we may now reason as follows:

$\widehat{x_{\widetilde{R}}} = \widehat{y_{\widetilde{R}}}$	$\equiv$	$\widehat{x_{\widetilde{R}}} = \widehat{y_{\widetilde{R}}}$	by idempotence of $\equiv$ (63.4.a)
$\widehat{x_{\widetilde{R}}} = \widehat{y_{\widetilde{R}}}$	$\equiv$	$\epsilon[\lambda z\, \widetilde{R}zx] = \epsilon[\lambda z\, \widetilde{R}zy]$	by (ϑ), (ξ), Rule SubId (74.2)
	$\equiv$	$\forall w([\lambda z\, \widetilde{R}zx]w \equiv [\lambda z\, \widetilde{R}zy]w)$	by Basic Law V (243)★
	$\equiv$	$\widetilde{R}xy$	by (312) ⋈

(319.1) (Exercise)

(319.2) (Exercise)

(320) By reasoning analogous to that used in (213) and (238). ⋈

(322) By RN, it suffices to show $\forall F(F = G \rightarrow \Box F = G)$, and by GEN, it suffices to show: $F = G \rightarrow \Box F = G$. But this is just the left-to-right direction of the modally strict theorem (124). (This argument reprises the one in Remark (190), in the discussion of Example (b).) ⋈

(323.1) By (322) and (188.1), we know that $F = G$ is a rigid condition on properties. So the following is an instance of (189.2):

$$y = \imath x(A!x \,\&\, \forall F(xF \equiv F = G)) \rightarrow (A!y \,\&\, \forall F(yF \equiv F = G))$$

By GEN, the logical propriety of $\boldsymbol{a}_G$, and $\forall$E, it follows that:

$$\boldsymbol{a}_G = \imath x(A!x \,\&\, \forall F(xF \equiv F = G)) \rightarrow (A!\boldsymbol{a}_G \,\&\, \forall F(\boldsymbol{a}_G F \equiv F = G))$$

So by definition (321), $A!\boldsymbol{a}_G \,\&\, \forall F(\boldsymbol{a}_G F \equiv F = G)$. ⋈

(323.2) From (323.1) by definition (318). ⋈

(324) Assume *ThinFormOf*(x, G). Then by definition of *ThinFormOf* (318), $A!x \,\&\, \forall F(xF \equiv F = G)$, from which it follows that $\forall F(xF \equiv F = G)$. Instantiating to G, we have $xG \equiv G = G$. But the right side is an instance of the reflexivity of property identity (67.1). Hence xG. ⋈

(326) We take *ThinFormOf*(x, G) as our global assumption. By GEN, it suffices to show $Gy \equiv$ *ParticipatesIn*(y, x). ($\rightarrow$) Assume Gy. Then by conjoining our two assumptions and applying $\exists$I, it follows that $\exists F($*ThinFormOf*$(x, F) \,\&\, Fy)$. So by the definition of *ParticipatesIn* (325), it follows that *ParticipatesIn*(y, x). ($\leftarrow$) Assume *ParticipatesIn*(y, x). By definition of *ParticipatesIn* (325), it follows that $\exists F($*ThinFormOf*$(x, F) \,\&\, Fy)$. Assume P is an arbitrary such property, so that we know both (a) *ThinFormOf*(x, P) and (b) Py. From (a), it follows that xP, by (324). But our global assumption is *ThinFormOf*(x, G), from which it follows by definition of *ThinFormOf* (318) that $\forall F(xF \equiv F = G)$. Hence $xP \equiv P = G$. Since we've established xP, it follows that $P = G$. So from (b) it follows that Gy, by Rule SubId. ⋈

(327) Since $\boldsymbol{a}_G$ is logically proper, we can instantiate (326) to obtain:

$$ThinFormOf(\boldsymbol{a}_G, G) \to \forall y(Gy \equiv ParticipatesIn(y, \boldsymbol{a}_G))$$

But we know the antecedent by (323.2). Hence:

$$\forall y(Gy \equiv ParticipatesIn(y, \boldsymbol{a}_G))$$

By ∀E, $Gx \equiv ParticipatesIn(x, \boldsymbol{a}_G)$. ⋈

(328) Assume $Gx \& Gy \& x \neq y$. Note that by definition (321) and theorem (320), $\boldsymbol{a}_G$ is logically proper and so we may instantiate it into the reflexivity of object identity (70.1) to obtain $\boldsymbol{a}_G = \boldsymbol{a}_G$. Independently, it follows from the first conjunct of our assumption and (327) that $Participates(x, \boldsymbol{a}_G)$. By similar reasoning from the second conjunct of our assumption, we can derive $Participates(y, \boldsymbol{a}_G)$. Hence, we have established that:

$$\boldsymbol{a}_G = \boldsymbol{a}_G \; \& \; Participates(x, \boldsymbol{a}_G) \; \& \; Participates(y, \boldsymbol{a}_G)$$

The desired consequent of our theorem then follows by ∃I. ⋈

(329.1) (Exercise)

(329.2) (Exercise)

(330.1) By (323.2), we know $ThinFormOf(\boldsymbol{a}_G, G)$. So instantiating $\boldsymbol{a}_G$ into (324), we may detach the consequent of the result by MP to conclude $\boldsymbol{a}_G G$. ⋈

(330.2) By (330.1), we know $\boldsymbol{a}_G G$. By the second conjunct of (323.1), we know $\forall F(\boldsymbol{a}_G F \equiv F = G)$. Conjoining the two results and existentially quantifying on G, we have: $\exists H(\boldsymbol{a}_G H \; \& \; \forall F(\boldsymbol{a}_G F \equiv F = H))$, i.e., $\exists! H \boldsymbol{a}_G H$, by (87.1). ⋈

(333) (Exercise)

(335.1) Assume $ThinForm(x)$. Then by definition (332.1), $\exists F(ThinFormOf(x, F))$. Assume P is an arbitrary such property, so that we know $ThinFormOf(x, P)$. Then by applying definition (318), it follows that $A!x$. ⋈

(335.2) Assume $ThinForm(x)$. By GEN, it suffices to show $\Box\forall y(A!y \to Fy) \to Fx$. So assume $\Box\forall y(A!y \to Fy)$. Note that from our first assumption and (335.1), it follows that $A!x$. Independently, from our second assumption and the T schema, it follows that $\forall y(A!y \to Fy)$, and so $A!x \to Fx$. Hence Fx. ⋈

(335.3) By instantiating $A!$ for G in the left conjunct of (323.1). ⋈

(335.4) Assume $ThinForm(x)$. Then by (335.1), it follows that $A!x$. But then by (327), it follows that $Participates(x, \boldsymbol{a}_{A!})$. ⋈

(335.5) Assume $ThinForm(x)$. By GEN, it suffices to show $\Box\forall y(A!y \to Fy) \to ParticipatesIn(x, \boldsymbol{a}_F)$. So assume $\Box\forall y(A!y \to Fy)$. Then by (335.2), it follows from our two assumptions that Fx. By (327), it follows that $ParticipatesIn(x, \boldsymbol{a}_F)$. ⋈

(336.1) By (333), $\mathit{ThinForm}(\boldsymbol{a}_{A!})$. So by (335.4), $\mathit{ParticipatesIn}(\boldsymbol{a}_{A!}, \boldsymbol{a}_{A!})$. By &I and $\exists$I, we're done. ⋈

(336.2) By (333), $\mathit{ThinForm}(\boldsymbol{a}_{O!})$. So by (327), $\forall y(O!y \equiv \mathit{ParticipatesIn}(y, \boldsymbol{a}_{O!}))$. Instantiating to $\boldsymbol{a}_{O!}$, we have: $O!\boldsymbol{a}_{O!} \equiv \mathit{ParticipatesIn}(\boldsymbol{a}_{O!}, \boldsymbol{a}_{O!})$. But by (329.2), we know $\neg O!\boldsymbol{a}_{O!}$. Hence, $\neg\mathit{ParticipatesIn}(\boldsymbol{a}_{O!}, \boldsymbol{a}_{O!})$. By &I and $\exists$I, we're done. ⋈

(340.1) (Exercise)

(340.2) (Exercise)

(340.3) ($\rightarrow$) Assume $G \Leftrightarrow F$, i.e., by (339.1) and (339.2), we know both:

(a) $\Box\forall x(Gx \rightarrow Fx)$

(b) $\Box\forall x(Fx \rightarrow Gx)$

Since H doesn't have a free occurrence in our assumption, it suffices by GEN to show $G \Rightarrow H \equiv F \Rightarrow H$:

> ($\rightarrow$) Assume $G \Rightarrow H$. Then $\Box\forall x(Gx \rightarrow Hx)$, by (339.1). So from (b) and this latter conclusion, it follows that $\Box\forall x(Fx \rightarrow Hx)$, by (123.4). Hence, $F \Rightarrow H$ by definition (339.1).
>
> ($\leftarrow$) By analogous reasoning using (a).

($\leftarrow$) Assume $\forall H(G \Rightarrow H \ \equiv F \Rightarrow H)$. By definition of $\Leftrightarrow$ (339), we have to show both (a) $G \Rightarrow F$ and (b) $F \Rightarrow G$. To show (a), instantiate G into our assumption, to obtain $G \Rightarrow G \ \equiv F \Rightarrow G$. The left condition follows from the fact that $\Leftrightarrow$ is an equivalence condition (340.2) and so reflexive. Hence, $F \Rightarrow G$. To show (b), reason analogously by instantiating our assumption to F. ⋈

(340.4) (Exercise)

(340.5) Before we begin, we first prove the following lemma:

(ϑ) $\Box\forall x\neg Fx \ \rightarrow \ F \Rightarrow H$

Proof:

1.	$\neg Fx \rightarrow (Fx \rightarrow Hx)$	instance of (58.3)
2.	$\forall x(\neg Fx \rightarrow (Fx \rightarrow Hx))$	from 1, by GEN
3.	$\forall x\neg Fx \rightarrow \forall x(Fx \rightarrow Hx)$	from 2, by axiom (29.3)
4.	$\Box(\forall x\neg Fx \rightarrow \forall x(Fx \rightarrow Hx))$	from 3, by RN
5.	$\Box\forall x\neg Fx \rightarrow \Box\forall x(Fx \rightarrow Hx)$	from 4, by K (32.1)
6.	$\Box\forall x\neg Fx \ \rightarrow \ F \Rightarrow H$	from 5, by definition (339.1)

Now to prove our theorem, assume $\mathit{Impossible}(G) \& \mathit{Impossible}(F)$. Hence, by &E and definition (138.2), we know:

(ζ) $\Box\forall x\neg Gx$

(ξ) $\Box\forall x\neg Fx$

Now by (340.3), it suffices to show $\forall H(G \Rightarrow H \equiv F \Rightarrow H)$, and by GEN, $G \Rightarrow H \equiv F \Rightarrow H$. ($\rightarrow$) From ($\xi$) and ($\vartheta$), it follows that $F \Rightarrow H$. Hence, $G \Rightarrow H \rightarrow F \Rightarrow H$, by axiom (21.1). ($\leftarrow$) Since F and H are free in (ϑ), the latter implies, $\forall F\forall H(\Box\forall x\neg Fx \rightarrow F \Rightarrow H)$, by GEN. Hence instantiating to G and H, we obtain $\Box\forall x\neg Gx \rightarrow G \Rightarrow H$. From this and ($\zeta$), it follows that $G \Rightarrow H$. Hence $F \Rightarrow H \rightarrow G \Rightarrow H$, by axiom (21.1). ⋈

(342.1) (Exercise)

(342.2) (Exercise)

(345) (Exercise)

(346.1) By (345) and (188.1), we know that $G \Rightarrow F$ is a rigid condition on properties. So the following is an instance of (189.2):

$$y = \imath x(A!x \,\&\, \forall F(xF \equiv G \Rightarrow F)) \rightarrow (A!y \,\&\, \forall F(yF \equiv G \Rightarrow F))$$

By GEN, the logical propriety of Φ_G, and $\forall$E, it follows that:

$$\Phi_G = \imath x(A!x \,\&\, \forall F(xF \equiv G \Rightarrow F)) \rightarrow (A!\Phi_G \,\&\, \forall F(\Phi_G F \equiv G \Rightarrow F))$$

So by definition (321), $A!\Phi_G \,\&\, \forall F(\Phi_G F \equiv G \Rightarrow F)$. ⋈

(346.2) From (346.1) by definition (341). ⋈

(349.1) Assume $\mathit{FormOf}(x, G)$. By GEN, it suffices to show:

$$Gy \equiv \mathit{ParticipatesIn}_{\text{PTA}}(y, x)$$

($\rightarrow$) Assume Gy. Then by conjoining our two assumptions and applying $\exists$I, it follows that $\exists F(\mathit{FormOf}(x, F) \,\&\, Fy)$. So by the definition of $\mathit{ParticipatesIn}_{\text{PTA}}$ (348.1), it follows that $\mathit{ParticipatesIn}_{\text{PTA}}(y, x)$. ($\leftarrow$) Assume $\mathit{ParticipatesIn}_{\text{PTA}}(y, x)$. By definition of $\mathit{ParticipatesIn}_{\text{PTA}}$ (348.1), it follows that $\exists F(\mathit{FormOf}(x, F) \,\&\, Fy)$. Assume P is an arbitrary such property, so that we know both (a) $\mathit{FormOf}(x, P)$ and (b) Py. From (a), it follows that $\forall F(xF \equiv P \Rightarrow F)$, by (341). But we also know $\mathit{FormOf}(x, G)$, from which it also follows by (341) that $\forall F(xF \equiv G \Rightarrow F)$. By the laws of quantified biconditional, (83.11) and (83.10), it thus follows that $\forall F(P \Rightarrow F \equiv G \Rightarrow F)$. But by facts about property implication and equivalence (340.3), it follows that $P \Leftrightarrow G$, i.e., by (340.1), that $\Box\forall x(Px \equiv Gx)$. Hence, since we know Py, it follows that Gy. ⋈

(349.2) Assume $\mathit{FormOf}(x, G)$. By GEN, it suffices to show:

$$yG \rightarrow \mathit{ParticipatesIn}_{\text{PH}}(y, x)$$

So assume yG. Then by conjoining our two assumptions and applying $\exists$I, it follows that $\exists F(FormOf(x,F)\,\&\,yF)$. So by the definition of $ParticipatesIn_{\text{PH}}$ (348.2), it follows that $ParticipatesIn_{\text{PH}}(y,x)$. ⋈

(351.1) Instantiate Φ_G into (349.1) to obtain:

$$FormOf(\Phi_G,G) \rightarrow \forall y(Gy \equiv ParticipatesIn_{\text{PTA}}(y,\Phi_G))$$

Then by (346.2), it follows that $\forall y(Gy \equiv ParticipatesIn_{\text{PTA}}(y,\Phi_G))$. Instantiate this result to x and we obtain: $Gx \equiv ParticipatesIn_{\text{PTA}}(x,\Phi_G)$. ⋈

(351.2) Instantiate Φ_G into (349.2) to obtain:

$$FormOf(\Phi_G,G) \rightarrow \forall y(yG \rightarrow ParticipatesIn_{\text{PH}}(y,\Phi_G))$$

Then by (346.2), it follows that, $\forall y(yG \rightarrow ParticipatesIn_{\text{PH}}(y,\Phi_G))$. Instantiate this result to x and we obtain: $xG \rightarrow ParticipatesIn_{\text{PH}}(x,\Phi_G)$. ⋈

(352) Assume $ParticipatesIn_{\text{PTA}}(y,x)$. Then $\exists F(FormOf(x,F)\ \&\ Fy)$, by (348.1). Suppose P is an arbitrary such property, so that we know both $FormOf(x,P)$ and Py. By GEN, it suffices to show $xF \rightarrow Fy$. So assume xF. Then $P \Rightarrow F$, by the established fact that $FormOf(x,P)$ and definition (341). So $\Box\forall y(Py \rightarrow Fy)$, by (339.1). By the T schema, $\forall y(Py \rightarrow Fy)$, and by $\forall$E, $Py \rightarrow Fy$. But Py is already known. Hence Fy. ⋈

(353.1) Assume $Gx\,\&\,Gy\,\&\,x\neq y$. Since Φ_G is logically proper (exercise), we may instantiate it into the reflexivity of object identity (70.1) to obtain $\Phi_G = \Phi_G$. Independently, it follows from the first conjunct of our assumption and (351.1) that $ParticipatesIn_{\text{PTA}}(x,\Phi_G)$. By similar reasoning from the second conjunct of our assumption, we can derive $ParticipatesIn_{\text{PTA}}(y,\Phi_G)$. Hence, we have established that:

$$\Phi_G = \Phi_G\ \&\ ParticipatesIn_{\text{PTA}}(x,\Phi_G)\ \&\ ParticipatesIn_{\text{PTA}}(y,\Phi_G)$$

By $\exists$I, we're done. ⋈

(353.2) (Exercise)

(354.1) By reasoning analogous to the proof of (329.1).

(354.2) By reasoning analogous to the proof of (329.2).

(354.3) (Exercise)

(355) By instantiating G into the second conjunct of (346.1), it follows that $\Phi_G G \equiv G \Rightarrow G$. But the right-hand side, by definition (339.1), is just $\Box\forall x(Gx \rightarrow Gx)$, which is easily derivable by applying GEN and then RN to the tautology $Gx \rightarrow Gx$. So $\Phi_G G$. ⋈

(357.1) (Exercise)

(357.2) (Exercise)

(357.3) (Exercise)

(358.1) Instantiate the first conjunct of (346.1) to $A!$. ⋈

(358.2) By (358.1), we know $A!\Phi_{A!}$. It follows that $\mathit{ParticipatesIn}_{\text{PTA}}(\Phi_{A!},\Phi_{A!})$, by (351.1). ⋈

(358.3) By (357.1), we know $\mathit{Form}(\Phi_{A!})$. Conjoin this with (358.2) and existentially generalize. ⋈

(358.4) Assume $A! \Rightarrow H$. Since G isn't free in our assumption, it suffices by GEN to show $H\Phi_G$. By definition (339), it follows from our assumption that $\Box\forall x(A!x \to Hx)$, and by the T schema, that $\forall x(A!x \to Hx)$. By the first conjunct of (346.1), we know $A!\Phi_G$. Hence $H\Phi_G$. ⋈

(358.5) (Exercise)

(358.6) Assume $A! \Rightarrow \overline{H}$. By GEN, it suffices to show $\neg H\Phi_G$. By (358.4), it follows from our assumption that $\forall G(\overline{H}\Phi_G)$, and hence $\overline{H}\Phi_G$. By theorem (137.1), it follows that $\neg H\Phi_G$. ⋈

(358.7) (Exercise)

(359.1) Assume $A! \Rightarrow H$. Then by (358.4), it follows that $\forall G(H\Phi_G)$. But since theorem (351.1) is modally strict, there is a modally strict proof of its instance $H\Phi_G \equiv \mathit{ParticipatesIn}_{\text{PTA}}(\Phi_G,\Phi_H)$. Hence, by the Rule of Substitution, $\forall G(\mathit{ParA}(\Phi_G,\Phi_H)$. ⋈

(359.2) Assume $G \Rightarrow H$. Hence by the second conjunct of (346.1), it follows that $\Phi_G H$. But, independently, if we substitute Φ_G for x and H for G in (351.1), we know $\Phi_G H \to \mathit{ParticipatesIn}_{\text{PH}}(\Phi_G,\Phi_H)$. Hence, $\mathit{ParticipatesIn}_{\text{PH}}(\Phi_G,\Phi_H)$. ⋈

(364) Assume $\mathit{TruthValue}(x)$. Then by (221), $\exists p(\mathit{TruthValueOf}(x,p))$. Let p_1 be an arbitrary such proposition, so that we know $\mathit{TruthValueOf}(x,p_1)$. Hence by (211), it follows that:

(ϑ) $A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p_1) \,\&\, F = [\lambda y\, q]))$

Note that if we let φ be the formula $\exists q((q \equiv p_1) \,\&\, F = [\lambda y\, q])$, then it is straightforward to show that every property such that φ is a propositional property, i.e., that:

(ξ) $\forall G(\varphi^G_F \to \mathit{Propositional}(G))$

Proof. Assume φ^G_F. Then $\exists q((q \equiv p_1) \,\&\, G = [\lambda y\, q])$. Let q_1 be an arbitrary such proposition, so that we know $(q_1 \equiv p_1) \,\&\, G = [\lambda y\, q_1]$. It follows from the second conjunct that $\exists p(G = [\lambda y\, p])$. So $\mathit{Propositional}(G)$, by definition (198).

Now since the first conjunct of (ϑ) is that $A!x$, it remains to show, by definition (363), that $\forall F(xF \rightarrow \mathit{Propositional}(F))$. By GEN, it suffices to show $xF \rightarrow \mathit{Propositional}(F)$. So assume xF. Then by the second conjunct of (ϑ) it follows that:

$$\exists q((q \equiv p_1) \,\&\, F = [\lambda y\, q])$$

Hence, we've established φ, i.e., φ^F_F. So by (ξ), it follows that $\mathit{Propositional}(F)$. ⋈

(366.1) ($\rightarrow$) Suppose $\mathit{Situation}(x)$. Then, by the definition of *Situation* (363), we know:

(ϑ) $A!x \,\&\, \forall F(xF \rightarrow \exists p(F = [\lambda y\, p]))$

To show $\Box(\vartheta)$, it suffices to show that both conjuncts of (ϑ) are necessary, by right-to-left direction of theorem (111.3), i.e., by the fact that $(\Box\varphi \,\&\, \Box\psi) \rightarrow \Box(\varphi \,\&\, \psi)$. But the first conjunct of (ϑ), i.e., $A!x$, implies $\Box A!x$, by (153.2). And the second conjunct of (ϑ) also implies its own necessity, by (204.2). ($\leftarrow$) Exercise. ⋈

(366.2) (Exercise)

(366.3) (Exercise)

(366.4) ($\rightarrow$) Assume $\mathcal{A}\mathit{Situation}(x)$. Independently, it is a consequence of (366.1) that $\mathit{Situation}(x) \rightarrow \Box\mathit{Situation}(x)$. So by RA:

$$\mathcal{A}(\mathit{Situation}(x) \rightarrow \Box\mathit{Situation}(x))$$

So by (31.2):

$$\mathcal{A}\mathit{Situation}(x) \rightarrow \mathcal{A}\Box\mathit{Situation}(x)$$

Since the antecedent holds by assumption, it follows that $\mathcal{A}\Box\mathit{Situation}(x)$. Hence by (33.2), $\Box\mathit{Situation}(x)$. Hence by the T schema, $\mathit{Situation}(x)$. ($\leftarrow$) Assume $\mathit{Situation}(x)$. Then by (366.1), $\Box\mathit{Situation}(x)$. So by (89), $\mathcal{A}\mathit{Situation}(x)$. ⋈

(367.1) ($\rightarrow$) Assume $s \models p$. Then by definition (365), $s\Sigma p$. But s is a situation, and so by definition (363), s is abstract. Hence by (126.2), $s[\lambda y\, p]$. So by axiom (37), $\Box s[\lambda y\, p]$. Hence, by reversing the definitions we've already applied, $\Box s \models p$. ($\leftarrow$) (Exercise) ⋈

(367.2) – **(367.4)** (Exercises)

(367.5) Theorem (367.2) is that $\Diamond s \models p \equiv s \models p$. So by a classical tautology (), $\neg\Diamond s \models p \equiv \neg s \models p$, which by commutativity, implies $\neg s \models p \equiv \neg\Diamond s \models p$. Independently, as an instance of (117.4), we know $\Box\neg s \models p \equiv \neg\Diamond s \models p$, and so by commutativity, $\neg\Diamond s \models p \equiv \Box\neg s \models p$. So by transitivity, $\neg s \models p \equiv \Box\neg s \models p$. ⋈

(368) ($\rightarrow$) Exercise. ($\leftarrow$) Assume $\forall p(s \models p \equiv s' \models p)$. Since both s and s' are situations, it follows that they are abstract objects, by definition (363). By (172.1), it suffices to show that s and s' encode the same properties:

> ($\rightarrow$) Assume sF. Then since s is a situation, it follows that $\exists p(F = [\lambda y\, p])$, by definition (363). Suppose p_1 is an arbitrary such proposition, so that we know $F = [\lambda y\, p_1]$. So $s[\lambda y\, p_1]$. Since situations are, by definition, abstract, it follows by definition (216) that $s\Sigma p_1$. Hence by definition (365), $s \models p_1$. But our initial hypothesis is that the same propositions are true in s and s'. So $s' \models p_1$, i.e., by definitions (365) and (216), $s'[\lambda y\, p_1]$. So $s'F$.
>
> ($\leftarrow$) By analogous reasoning. ⋈

(370.1) (Exercise)

(370.2) Assume $s \trianglelefteq s'$ and $s \neq s'$. For reductio, assume $s' \trianglelefteq s$. From the assumption that $s \trianglelefteq s'$, it follows that $\forall p(s \models p \rightarrow s' \models p)$, by definition (369). Similarly, from the reductio assumption, it follows that $\forall p(s' \models p \rightarrow s \models p)$. Hence, $\forall p(s \models p \equiv s' \models p)$. So by theorem (368), $s = s'$. Contradiction. ⋈

(370.3) Assume (a) $s \trianglelefteq s'$ and (b) $s' \trianglelefteq s''$. To show $s \trianglelefteq s''$, assume $s \models p$. From this and (a), it follows from definition of $\trianglelefteq$ (369) that $s' \models p$. From this and (b), it follows from (369) that $s'' \models p$. ⋈

(371.1) ($\rightarrow$) Exercise. ($\leftarrow$) By exportation (63.8.a), the anti-symmetry of $\trianglelefteq$ (370.2) becomes $s \trianglelefteq s' \rightarrow (s \neq s' \rightarrow \neg(s' \trianglelefteq s))$. By contraposition of the consequent, we get $s \trianglelefteq s' \rightarrow (s' \trianglelefteq s \rightarrow s = s')$. So by importation (63.8.b), $(s \trianglelefteq s' \,\&\, s' \trianglelefteq s) \rightarrow s = s'$. ⋈

(371.2) ($\rightarrow$) Exercise. ($\leftarrow$) Let $\forall s''(s'' \trianglelefteq s \equiv s'' \trianglelefteq s')$ be our global assumption. Given theorem (368), it suffices to show $\forall p(s \models p \equiv s' \models p)$:

> ($\rightarrow$) Assume $s \models p$. By instantiating our global assumption to s, we know $s \trianglelefteq s \equiv s \trianglelefteq s'$. But by (370.1), we know $s \trianglelefteq s$. Hence $s \trianglelefteq s'$. So by definition of $\trianglelefteq$ (369), it follows that $s' \models p$.
>
> ($\leftarrow$) Assume $s' \models p$. Then $s \models p$ follows by analogous reasoning once we instantiate our global assumption to s' and let our instance of (370.1) be $s' \trianglelefteq s'$. ⋈

(373) (Exercise)

(375.1) By the Comprehension Principle for Abstract Objects (39), we know:

$$\exists x(A!x \,\&\, \forall F(xF \equiv F \neq F))$$

Suppoe a is an arbitrary such object, so that we know:

(ξ) $A!a \,\&\, \forall F(aF \equiv F \neq F)$

Now given the definition of $NullSituation(x)$ (374.1), it suffices, by &I and ∃I, to show both:

(a) $Situation(a) \,\&\, \neg\exists p(a \models p)$

(b) $\forall y((Situation(y) \,\&\, \neg\exists p(y \models p)) \to y = a)$

(a) To show the first conjunct, we have to show both $A!a$ and $\forall F(aF \to \exists p(F = [\lambda y\, p])$, by the definition of situation. But $A!a$ is known by the first conjunct of (ξ). Note that the second conjunct of (ξ) implies that $\neg\exists G aG$. So $\neg aF$, and hence by failure of the antecedent, $aF \to \exists p(F = [\lambda y\, p])$. So, by GEN, we completed the proof of the first conjunct. For the second conjunct, suppose for reductio that some proposition, say q_1, is such that $a \models q_1$. Then since a is both a situation and abstract, it follows by definitions (365) and (216) that $a[\lambda y\, q_1]$, contradicting the second conjunct of (ξ), which we know implies that a doesn't encode any properties.

(b) Assume $Situation(y) \,\&\, \neg\exists p(y \models p)$. Since we now know both $\neg\exists p(y \models p)$, by assumption, and $\neg\exists p(a \models p)$, by (a), it follows that $\forall p(y \models p \equiv a \models p)$, by (86.12). Since both y and a are situations, it follows by (368) that $y = a$. ⋈

(375.2) (Exercise)

(375.3) By the Rule of Actualization, (375.1) implies $\mathcal{A}\exists!xNullSituation(x)$. Hence by the right-to-left direction of (179.2), it follows that $\exists y(y = \imath xNullSituation(x))$. ⋈

(375.4) (Exercise)

(377.1) Assume $NullSituation(x)$. Then, by definition (374.1) and &E, we know both (a) $Situation(x)$ and (b) $\neg\exists p(x \models p)$. Now to show $\Box NullSituation(x)$, we have to show $\Box(Situation(x) \,\&\, \neg\exists p(x \models p))$. By &I and (111.3), it suffices to show $\Box Situation(x)$ and $\Box\neg\exists p(x \models p)$. By the left to right direction of (366.1), (a) implies $\Box Situation(x)$. Now to show $\Box\neg\exists p(x \models p)$, suppose $\neg\Box\neg\exists p(x \models p)$, for reductio. Then by definition (7.4.e), $\Diamond\exists p(x \models p)$. So by BF$\Diamond$ (122.3), $\exists p\Diamond x \models p$. Suppose p_1 is an arbitrary such proposition, so that we know $\Diamond x \models p_1$. Then by the left-to-right direction of (367.2), it follows that $x \models p_1$. But then, $\exists p(x \models p)$, by ∃I, which contradicts (b). ⋈

(377.2) (Exercise)

(377.3) Let ψ be the formula $NullSituation(x)$. Then as an instance of (108.2), we know:

$$\forall x(\psi \to \Box\psi) \to (\exists!x\psi \to \forall y(y = \imath x\psi \to \psi^y_x))$$

By applying GEN to (377.1), we know $\forall x(\psi \rightarrow \Box\psi)$. Hence:

$$\exists!x\psi \rightarrow \forall y(y = \imath x\psi \rightarrow \psi^y_x)$$

But (375.1) is $\exists!x\psi$. Hence:

$$\forall y(y = \imath x\psi \rightarrow \psi^y_x)$$

Since $\boldsymbol{s}_\varnothing$ is logically proper, by (376.1) and (375.3), it follows that:

$$\boldsymbol{s}_\varnothing = \imath x\psi \rightarrow \psi^{\boldsymbol{s}_\varnothing}_x$$

Hence by definition (376.1), $\psi^{\boldsymbol{s}_\varnothing}_x$, i.e., $NullSituation(\boldsymbol{s}_\varnothing)$. ⋈

(377.4) (Exercise)

(378.1) (Exercise)

(378.2) By the Rule of Actualization, (378.1) implies:

$$\mathcal{A}(NullSituation(x) \equiv Null(x))$$

So by (105), it follows that:

$$\forall y(y = \imath xNullSituation(x) \equiv y = \imath xNull(x))$$

Since $\boldsymbol{s}_\varnothing$ is logically proper, it follows that:

$$\boldsymbol{s}_\varnothing = \imath xNullSituation(x) \equiv \boldsymbol{s}_\varnothing = \imath xNull(x)$$

By definition (376.1), it follows that:

$$\boldsymbol{s}_\varnothing = \imath xNull(x)$$

Hence, by definition (193.1), $\boldsymbol{s}_\varnothing = \boldsymbol{a}_\varnothing$. ⋈

(378.3) Before we begin, note that by theorem (194.4), we know $Universal(\boldsymbol{a_V})$, and so by definition (191.2), that:

(ϑ) $A!\boldsymbol{a_V}\ \&\ \forall F\, \boldsymbol{a_V}F$

Furthermore, by theorem (377.4), we know *Trivial Situation*$(\boldsymbol{s_V})$, and so by definition (374.2), that:

(ζ) $Situation(\boldsymbol{s_V})\ \&\ \forall p(\boldsymbol{s_V} \models p)$

Since the first conjunct of (ζ) implies $A!\boldsymbol{s_V}$, we've established that $\boldsymbol{a_V}$ and $\boldsymbol{s_V}$ are abstract. Hence, to establish our theorem, it suffices to show:

$$\exists F(\boldsymbol{a_V}F\ \&\ \neg \boldsymbol{s_V}F)$$

by (172.1). So we have to find a witness to this existential claim. Either $A!$ or $O!$ will work, and for the present proof, we have chosen the former. So we have to show:

$$\boldsymbol{a_V} A! \,\&\, \neg \boldsymbol{s_V} A!$$

Now the first conjunct follows immediately by instantiating the second conjunct of (ϑ) to $A!$. So it remains to show the second conjunct. For reductio, suppose $\boldsymbol{s_V} A!$. Now since $\boldsymbol{s_V}$ is a situation, every property it encodes is a propositional property. So $\exists p(A! = [\lambda y\, p])$. Let p_1 be an arbitary such proposition, so that we know $A! = [\lambda y\, p_1]$. From this identity it follows, by an appropriate instance of axiom (25), that:

(ξ) $(\exists x A!x \,\&\, \exists z \neg A!z) \to (\exists x[\lambda y\, p_1]x \,\&\, \exists z \neg[\lambda y\, p_1]z)$

But we can show that the antecedent of (ξ) is true:

> Clearly, there are abstract objects and so the first conjunct of the antecedent is true. To see that there are objects that aren't abstract, note that the first conjunct of axiom (32.4) is that $\Diamond \exists x(E!x \,\&\, \Diamond \neg E!x)$. So by BF$\Diamond$, $\exists x \Diamond(E!x \,\&\, \Diamond \neg E!x)$. Let a be an arbitrary such object, so that we know $\Diamond(E!a \,\&\, \Diamond \neg E!a)$. It follows *a fortiori* that $\Diamond E!a$. So it follows that $\neg A!a$, by now familiar reasoning. Hence, $\exists z \neg A!z$.

From (ξ) and its antecedent we have $\exists x[\lambda y\, p_1]x \,\&\, \exists z \neg[\lambda y\, p_1]z$. But this yields a contradiction, for suppose b and c are arbitrary such an objects, so that we know both $[\lambda y\, p_1]b$ and $\neg[\lambda y\, p_1]c$. Then by Strengthened β-Conversion, it follows from the first that p_1 and from the second that $\neg p_1$. $\bowtie$

(380) Suppose φ is a condition on propositional properties. Then by (379), we know there is a modally strict proof of:

(ϑ) $\forall H(\varphi^H_F \to \mathit{Propositional}(H))$

Now if we let φ be $\mathit{Situation}(x)$, ψ be $\forall F(xF \equiv \varphi)$, and χ be $A!x$, then the theorem we have to prove has the form:

$$(\varphi \,\&\, \psi) \equiv (\chi \,\&\, \psi)$$

So in light of the right-to-left direction of the tautology (63.10.g):

$$((\varphi \,\&\, \psi) \equiv (\chi \,\&\, \psi)) \equiv (\psi \to (\varphi \equiv \chi))$$

it suffices to show $\psi \to (\varphi \equiv \chi)$, i.e., $\forall F(xF \equiv \varphi) \to (\mathit{Situation}(x) \equiv A!x)$. So assume:

(ζ) $\forall F(xF \equiv \varphi)$

($\to$) Assume $\mathit{Situation}(x)$. But $A!x$ follows immediately by definition (363) of *Situation*.

($\leftarrow$) Assume $A!x$. In virtue of the definition (363) of *Situation*, it remains only to show $\forall F(xF \to \mathit{Propositional}(F))$. By GEN, we need only show: $xF \to \mathit{Propositional}(F)$. So assume xF. Then by (ζ), it follows that:

$$\varphi$$

Now if we instantiate (ϑ) to F, it follows that:

$$\varphi \to \mathit{Propositional}(F)$$

Hence, from our last two displayed results, follows that $\mathit{Propositional}(F)$. ⋈

(381.1) Suppose φ is a condition on propositional properties in which x doesn't occur free. Since x doesn't occur free in φ, it follows by the Comprehension Principle for Abstract Objects (39) that:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

Since φ is a condition on propositional properties, it follows by the fact that (380) is a modally strict theorem and the Rule of Substitution that:

$$\exists x(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi)) \qquad ⋈$$

(381.2) (Exercise)

(382.1) Suppose φ is a condition on propositional properties in which x doesn't occur free. Then we can derive our theorem:

$$(\vartheta)\ \ \exists y(y = \imath x(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi)))$$

by applying the Rule of Actualization to (381.2) and then using the right-to-left direction of (179.2) to derive (ϑ). ⋈

(382.2) Let φ be a condition on propositional properties in which x doesn't occur free. Now by applying the Rule of Actualization to (380), we know:

$$\mathcal{A}[(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi)) \equiv (A!x \,\&\, \forall F(xF \equiv \varphi))]$$

Hence, by theorem (105), it follows that:

$$(\vartheta)\ \ \forall x[(x = \imath x(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi))) \equiv (x = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))]$$

Since we know the description $\imath x(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi))$ is logically proper (382.1), we may instantiante it in (ϑ), and then appeal to the reflexivity of identity to complete the derivation of our theorem. ⋈

(383) Suppose φ is a rigid condition on propositional properties in which x isn't free. Given our conventions for interpreting bound restricted variables, we have to show:

$$y = \imath x(\mathit{Situation}(x) \,\&\, \forall F(xF \equiv \varphi)) \to \forall F(yF \equiv \varphi))$$

So assume:

(ϑ) $y = \imath x(Situation(x) \,\&\, \forall F(xF \equiv \varphi))$

Since φ is, by hypothesis, a condition on propositional properties in which x doesn't occur free, we may appeal to the identity (382.2) and infer from (ϑ) that:

(ξ) $y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$

But since φ is also, by hypothesis, a rigid condition on properties in which x doesn't occur free, it follows from (ξ) by (189.2) that:

$$A!y \,\&\, \forall F(yF \equiv \varphi)$$

A fortiori, $\forall F(yF \equiv \varphi)$. ⋈

(389.1) There are a number of ways to prove this theorem; for example, by considering the null situation $\boldsymbol{s}_{\varnothing}$. But we can also show that there is an actual situation that encodes at least one property. Consider the following instance of situation comprehension (381.1):

$$\exists s \forall F(sF \equiv F = [\lambda y\, q \rightarrow q])$$

where q is any proposition you please. Let s_3 be an arbitrary such situation, so that we know:

(ϑ) $\forall F(s_3F \equiv F = [\lambda y\, q \rightarrow q])$

Now to show $Actual(s_3)$, it suffices by GEN to show $(s_3 \models p) \rightarrow p$. So assume $s_3 \models p$. Then by definitions (365) and (216), $s[\lambda y\, p]$. By (ϑ), it follows that $[\lambda y\, p] = [\lambda y\, q \rightarrow q]$. Hence by the definition of proposition identity (16.3), it follows that $p = (q \rightarrow q)$. But it is a theorem (53) that $q \rightarrow q$. Hence, by Rule SubId, p. ⋈

(387) Theorem (150.3) tells us that there are contingently true propositions. So, by definition (148), we know $\exists p(p \,\&\, \Diamond\neg p)$. Suppose q_1 is such a proposition, so that we know:

(ϑ) $q_1 \,\&\, \Diamond\neg q_1$

Now since the formula $F = [\lambda y\, q_1]$ is clearly a condition on propositional properties, we know the following is an instance of Comprehension for Situations (381.1):

$$\exists s \forall F(sF \equiv F = [\lambda y\, q_1])$$

Suppose s_1 is such a situation, so that we know:

(ξ) $\forall F(s_1F \equiv F = [\lambda y\, q_1])$

We now show that s_1 is a witness that proves our theorem by establishing:

(A) $Actual(s_1)$

(B) $\diamond\neg Actual(s_1)$

(A) To show $Actual(s_1)$, we have to show $\forall p(s_1 \models p \rightarrow p)$. So by GEN, assume $s_1 \models p$. Then by (365), $s_1 \Sigma p$, and by (216), $s_1[\lambda y\, p]$. Hence by (ξ), it follows that $[\lambda y\, p] = [\lambda y\, q_1]$. So by the definition of proposition identity (16.3), $p = q_1$. But by (ϑ), we know q_1. Hence p.

(B) Assume, for reductio, that $\neg\diamond\neg Actual(s_1)$. Then, by (117.3), $\Box Actual(s_1)$, and so by definition of actual (386), $\Box\forall p(s_1 \models p \rightarrow p)$. By CBF (122.2), it follows that $\forall p\Box(s_1 \models p \rightarrow p)$. So, in particular:

(ζ) $\Box(s_1 \models q_1 \rightarrow q_1)$

But by (ξ), we know $s_1[\lambda y\, q_1] \equiv [\lambda y\, q_1] = [\lambda y\, q_1]$. Since the right side is true by the laws of identity, it follows that $s_1[\lambda y q_1]$. So by the rigidity of encoding (37), $\Box s_1[\lambda y q_1]$. Since s_1 is known to be abstract and a situation, this implies not just $\Box s_1 \Sigma q_1$ but also $\Box s_1 \models q_1$, by (216) and (365), respectively. But by the K axiom, this last conclusion and (ζ) together imply $\Box q_1$, i.e., $\neg\diamond\neg q_1$, which contradicts the second conjunct of (ϑ). ⋈

(389.2) Consider the trivial situation $\boldsymbol{s_V}$. By theorem (378.2) and definition (374.2), every proposition is true in $\boldsymbol{s_V}$. Hence, $\boldsymbol{s_V} \models (p \,\&\, \neg p)$, where p is any proposition you please. For reductio, suppose $Actual(\boldsymbol{s_V})$. Then by definition of *Actual*, $p \,\&\, \neg p$. Contradiction. Hence $\neg Actual(\boldsymbol{s_V})$. By $\exists$I, we're done. ⋈

(389.3) Let q be any proposition, and consider the (necessarily) false proposition $q \,\&\, \neg q$ which we know exists by (133). Assume $Actual(s)$. Then, by definition (386):

(ϑ) $\forall p(s \models p \rightarrow p)$

Now suppose, for reductio, that $s \models (q \,\&\, \neg q)$. Then by ($\vartheta$), it follows that $q \,\&\, \neg q$, which is a contradiction. Hence, $Actual(s) \rightarrow \neg s \models (q \,\&\, \neg q)$. So by GEN, $\forall s(Actual(s) \rightarrow \neg s \models (q \,\&\, \neg q))$, and by $\exists$I, $\exists p\forall s(Actual(s) \rightarrow \neg s \models p)$. ⋈

(390) Let φ be the formula $s'F \vee s''F$. Then since every property H such that φ is a property encoded by a situation, every property H such φ is propositional. So the following in an instance of Comprehension for Situations (381.1):

$$\exists s\forall F(sF \equiv s'F \vee s''F)$$

Let s_4 be an arbitrary such situation, so that we know:

(ϑ) $\forall F(s_4F \equiv s'F \vee s''F)$

If we can show that both (a) $s' \trianglelefteq s_4$ and (b) $s'' \trianglelefteq s_4$, then by $\exists$I, we're done. (a) Assume $s' \models p$. Then by definition (365), $s'\Sigma p$, and by definition (216), $s'[\lambda y\, p]$. Hence by $\vee$I, $s'[\lambda y\, p] \vee s''[\lambda y\, p]$. So by ($\vartheta$), $s_4[\lambda y\, p]$. Hence by definitions (216) and (365), $s_4 \models p$. So by CP, $s' \models p \to s_4 \models p$, and by GEN, $\forall p(s' \models p \to s_4 \models p)$. Hence, by definition (369), $s' \trianglelefteq s_4$. (b) By analogous reasoning. ⋈

(391.1) Assume *Actual*(s) and $s \models p$ true. By definition of *Actual* (386), it follows that p is true. But, by GEN, β-Conversion implies: $\forall x([\lambda y\ p]x \equiv p)$. In particular, $[\lambda y\ p]s \equiv p$. Hence, $[\lambda y\ p]s$. ⋈

(391.2) (Exercise) [Hint: Use (390).]

(394) Assume *Actual*(s). Then by definition (386):

(ϑ) $\forall p(s \models p \to p)$

Now suppose, for *reductio*, that $\neg$*Consistent*(s). So by Rule $\neg\neg$E (64.8) and the definition of *Consistent*, there is a proposition q, say q_1, such that both $s \models q_1$ and $s \models \neg q_1$. Then, it follows from the first and (ϑ) that q_1 and it follows from the second and (ϑ) that $\neg q_1$. Contradiction. ⋈

(396.1) (Exercise)

(396.2) Assume (a) $s \models p$ and (b) $\neg\Diamond p$. By definitions (395) and (386), we have to show $\neg\Diamond\forall q(s \models q \to q)$. By (117.4), it suffices to show $\Box\neg\forall q(s \models q \to q)$. But since there is a modally strict proof of the equivalence of $\neg\forall q(s \models q \to q)$ and $\exists q(s \models q \,\&\, \neg q)$ (exercise), it suffices by the Rule of Substitution to show $\Box\exists q(s \models q \,\&\, \neg q)$. Now from (a) we know $\Box s \models p$ by (367.1), and from (b) we know $\Box\neg p$ by (117.4). So by &I and (111.3) , we know $\Box(s \models p \,\&\, \neg p)$. By $\exists$I, it follows that $\exists q\Box(s \models q \,\&\, \neg q)$. But by the Buridan formula (123.1), it follows that $\Box\exists q(s \models q \,\&\, \neg q)$. ⋈

(397.1) ($\to$) Assume $\neg$*Consistent*(s). By definition (392) and Rule $\neg\neg$E (64.8), it follows that $\exists p(s \models p \,\&\, s \models \neg p)$. Let q_1 be an arbitrary such proposition, so that we know:

(ϑ) $s \models q_1 \,\&\, s \models \neg q_1$

Then by (367.1), it follows that both $\Box s \models q_1$ and $\Box s \models \neg q_1$. But a conjunction of necessities is equivalent to a necessary conjunction (111.3), and so it follows that $\Box(s \models q_1 \,\&\, s \models \neg q_1)$. Hence, by $\exists$I, $\exists p\Box(s \models p \,\&\, s \models \neg p)$, and this conclusion holds once we discharge (ϑ) by $\exists$E. Thus, by the Buridan formula (123.1), it follows that $\Box\exists p(s \models p \,\&\, s \models \neg p)$. But we may apply the relevant instance of the modally-strict theorem that $\varphi \equiv \neg\neg\varphi$ and the Rule of Substitution to obtain:

$$\Box\neg\neg\exists p(s \models p \,\&\, s \models \neg p)$$

Hence, by definition of *Consistent* (392), it follows that $\Box\neg Consistent(s)$. ($\leftarrow$) Exercise. ⋈

(397.2) (Exercise)

(398.1) Assume *Possible*(s). Then by definition (395),

(ϑ) $\Diamond Actual(s)$

Note independently that (394) is a theorem, and so by RN, we also know:

(ξ) $\Box(Actual(s) \to Consistent(s))$

Hence by an instance of the law K$\Diamond$ of modal logic (117.5), it follows from (ϑ) and (ξ) that $\Diamond Consistent(s)$. But then, by (397.2), *Consistent*(s). ⋈

(398.2) [The following proof rehearses some of the discussion in Remark (393).] Let s_1 be the situation:

$$\imath s\forall F(sF \equiv F = [\lambda y\, q_0 \,\&\, \neg q_0]))$$

where q_0 is the proposition $\exists x(E!x \,\&\, \Diamond\neg E!x)$. We leave it as an exercise to show s_1 is identical to a strictly canonical situation, i.e., to show, when φ is the formula $F = [\lambda y\, q_0 \,\&\, \neg q_0]$, that $\Box\forall F(\varphi \to Propositional(F))$ and $\Box\forall F(\varphi\Box\varphi)$. Hence by (383), it follows by definition of s_1 that:

$$\forall F(s_1F \equiv F = [\lambda y\, q_0 \,\&\, \neg q_0])$$

So we know both that $s_1[\lambda y\, q_0 \,\&\, \neg q_0]$ and that s_1 encodes no other properties. Hence, we know both that $s_1 \models (q_0 \,\&\, \neg q_0)$ that no other proposition is true in s_1. Consequently, there is no proposition p such that both p and $\neg p$ are true in s_1. Hence *Consistent*(s_1), by definition (392). It remains to show $\neg Possible(s_1)$. Assume, for reductio, that *Actual*(s_1). Then by definition of *Actual* (386), it follows that $q_0 \,\&\, \neg q_0$, which is a contradiction. Hence, $\neg Actual(s_1)$. Since this is a modally strict theorem, we may apply RN to obtain: $\Box\neg Actual(s_1)$, i.e., $\neg\Diamond Actual(s_1)$. So, by definition (395), $\neg Possible(s_1)$. ⋈

(403) By the conventions for our restricted variable w (255.2) and the definition of *PossibleWorld* (400), $\Diamond\forall p(w \models p \equiv p)$. *A fortiori*, $\Diamond\forall p(w \models p \to p)$. So by the definition of *Actual* (386), $\Diamond Actual(w)$. By the definition of *Possible* (395), *Possible*(w). ⋈

(404.1) By the previous theorem (403), *Possible*(w). So by (398.1), *Consistent*(w). ⋈

(404.2) (Exercise)

(405.1) We prove only the left-to-right direction since the right-to-left direction is just an instance of the T schema. ($\to$) Assume *PossibleWorld*(x). By definition (400), we know:

(ϑ) $Situation(x) \,\&\, \Diamond\forall p(s \models p \equiv p)$

By (366.1), the first conjunct of (ϑ) implies:

(ξ) $\Box Situation(x)$

Moreover, by the 5 schema (32.3), the second conjunct of (ϑ) implies:

(ζ) $\Box\Diamond\forall p(s \models p \equiv p)$

Hence, by the right-to-left direction of (111.3), the conjunction of (ξ) and (ζ) implies:

$$\Box(Situation(x) \,\&\, \Diamond\forall p(s \models p \equiv p))$$

So by the definition of possible world (400), $\Box PossibleWorld(x)$. ⋈

(405.2) ($\rightarrow$) It follows *a fortiori* from (405.1) that:

$$PossibleWorld(x) \rightarrow \Box PossibleWorld(x)$$

Since this result is a modally strict theorem, it follows by Rule (121.2) that $\Diamond PossibleWorld(x) \rightarrow PossibleWorld(x)$. ($\leftarrow$) (Exercise) ⋈

(405.3) (Exercise)

(405.4) ($\rightarrow$) Assume $\mathcal{A}PossibleWorld(x)$. Independently, it is a consequence of (405.1) that $PossibleWorld(x) \rightarrow \Box PossibleWorld(x)$. So by RA:

$$\mathcal{A}(PossibleWorld(x) \rightarrow \Box PossibleWorld(x))$$

So by (31.2):

$$\mathcal{A}PossibleWorld(x) \rightarrow \mathcal{A}\Box PossibleWorld(x)$$

Since the antecedent holds by assumption, it follows that $\mathcal{A}\Box PossibleWorld(x)$. Hence by (33.2), $\Box PossibleWorld(x)$. Hence by the T schema, $PossibleWorld(x)$. ($\leftarrow$) Assume $PossibleWorld(x)$. Then by (405.1), $\Box PossibleWorld(x)$. So by (89), $\mathcal{A}PossibleWorld(x)$. ⋈

(406.1) Assume $w \models p$. Given our discussion in Remark (402), we know that since w is a situation, we may apply definition (365) to conclude $w\Sigma p$. It also follows from the fact that w is a situation that it is abstract, by definition (363). Hence by (126.2), $w[\lambda y\, p]$. So by axiom (37), $\Box w[\lambda y\, p]$. Hence, by reversing the definitions we've already applied, $\Box w \models p$. ($\leftarrow$) (Exercise) ⋈

(406.2) – **(406.4)** (Exercises)

(408) By definition (400):

(ϑ) $\Diamond\forall p(w \models p \equiv p)$

To show that *Maximal*(w), it suffices, by GEN, to show, $w \models q \vee w \models \neg q$, where q is an arbitrarily chosen proposition. Our proof strategy will be to:

(a) show that $\Diamond(w \models q \vee w \models \neg q)$, and then

(b) appeal to various modal facts, including the rigidity of *truth at* (406.2), to derive that $w \models q \vee w \models \neg q$ from (a).

Now our proof strategy for (a) is to:

(i) show $\Box(\varphi \to \psi)$, where φ is $\forall p(w \models p \equiv p)$ and ψ is $w \models q \vee w \models \neg q$, and

(ii) use the modal law $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$ (117.5) to conclude $\Diamond\psi$ from (i) and (ϑ), i.e., from $\Box(\varphi \to \psi)$ and $\Diamond\varphi$.

So for (i), assume $\forall p(w \models p \equiv p)$. It follows both that $w \models q \equiv q$ and $w \models \neg q \equiv \neg q$. Since $q \vee \neg q$, it follows by a biconditional syllogism that $w \models q \vee w \models \neg q$. Thus, by conditional proof, that $\forall p(w \models p \equiv p) \to (w \models q \vee w \models \neg q)$. Since our proof of this is modally strict, it follows that:

(ξ) $\Box(\forall p(w \models p \equiv p) \to (w \models q \vee w \models \neg q))$

Now for (ii), it follows from (ξ) and (ϑ) by the modal law (117.5) that:

(ζ) $\Diamond(w \models q \vee w \models \neg q)$

Now for (b), if we apply to (ζ) the modal law (117.6), which asserts that possibility distributes over a disjunction, it follows that:

$$\Diamond w \models q \vee \Diamond w \models \neg q$$

But by (406.2), the left disjunct is equivalent to $w \models q$ and the right is equivalent to $w \models \neg q$. So, by biconditional syllogism, it follows that $w \models q \vee w \models \neg q$. Since q was arbitrary, we have shown: *Maximal*(w). ⋈

(410.1) Assume *Maximal*(s). Then by definition of *Maximal* (407):

(ϑ) $\forall p(s \models p \vee s \models \neg p)$

We want to show $\Box\forall p(s \models p \vee s \models \neg p)$. By the Barcan Formula (122.1), it suffices to show $\forall p\Box(s \models p \vee s \models \neg p)$. By GEN, it suffices to show $\Box(s \models p \vee s \models \neg p)$. Now from ($\vartheta$), it follows by $\forall$E that $s \models p \vee s \models \neg p$. But by (367.1), the first disjunct implies $\Box s \models p$ and the second disjunct implies $\Box s \models \neg p$. Hence, by disjunctive syllogism, $\Box s \models p \vee \Box s \models \neg p$. So by (117.7), $\Box(s \models p \vee s \models \neg p)$. ⋈

(410.2) ($\to$) By (408) and (403). ($\leftarrow$) Before we begin, we establish a few facts needed for the proof, the first of which is:

$$Maximal(s) \to (\forall p(s \models p \to p) \to \forall p(s \models p \equiv p))$$

Proof. Assume (a) $Maximal(s)$ and (b) $\forall p(s \models p \rightarrow p)$. To show $\forall p(s \models p \equiv p)$ it suffices, by $\forall$I, to show $s \models q \equiv q$, where q is arbitrary. ($\rightarrow$) This direction is immediate by instantiating (b) to q. ($\leftarrow$) Assume q. Now assume, for reductio, that $\neg s \models q$. Then by (a) and the definition of maximality (407), it follows that $s \models \neg q$. But we know, as an instance of (b) that $s \models \neg q \rightarrow \neg q$. Hence, $\neg q$. Contradiction.

Since we established this first fact by a modally strict proof, it follows by RN that:

$$\Box[Maximal(s) \rightarrow (\forall p(s \models p \rightarrow p) \rightarrow \forall p(s \models p \equiv p))]$$

So by the K axiom, we know:

$$(\vartheta) \quad \Box Maximal(s) \rightarrow \Box(\forall p(s \models p \rightarrow p) \rightarrow \forall p(s \models p \equiv p))$$

Now to establish our theorem, assume $Maximal(s)$ & $Possible(s)$. Then from $Maximal(s)$ and the previous theorem (410.1), it follows that $\Box Maximal(s)$. From this and (ϑ) it follows that:

$$\Box(\forall p(s \models p \rightarrow p) \rightarrow \forall p(s \models p \equiv p))$$

From this, it follows by theorem (117.5) that:

$$(\xi) \quad \Diamond\forall p(s \models p \rightarrow p) \rightarrow \Diamond\forall p(s \models p \equiv p)$$

But $Possible(s)$ by assumption, from which it follows by definition of *Possible* that $\Diamond Actual(s)$, and by definition of *Actual* that $\Diamond\forall p(s \models p \rightarrow p)$ But from this last fact and (ξ), it follows that $\Diamond\forall p(s \models p \equiv p)$, i.e., $PossibleWorld(s)$. ⋈

(412) ($\rightarrow$) Assume $Coherent(s)$. Then by definition (411):

$$(\vartheta) \quad \forall p(s \models \neg p \equiv \neg s \models p)$$

By GEN, it suffices to show $s \models p \equiv \neg s \models \neg p$:

($\rightarrow$) Assume $s \models p$. For reductio, assume $\neg\neg s \models \neg p$. Then $s \models \neg p$. By (ϑ), $\neg s \models p$. Contradiction. So $\neg s \models \neg p$.

($\leftarrow$) Assume $\neg s \models \neg p$. Then by (ϑ), $\neg\neg s \models p$. Hence $s \models p$.

($\leftarrow$) Assume:

$$(\xi) \quad \forall p(s \models p \equiv \neg s \models \neg p)$$

By definition of (411), we want to show (ϑ). By GEN, it suffices to show $s \models \neg p \equiv \neg s \models p$:

($\rightarrow$) Assume $s \models \neg p$. For reductio, assume $\neg\neg s \models p$. Then $s \models p$. So by (ξ), $\neg s \models \neg p$. Contradiction. So $\neg s \models p$.

(←) Assume $\neg s \models p$. Then by (ξ), it follows that $\neg\neg s \models \neg p$. Hence $s \models \neg p$.

⋈

(413) (→) Assume *Coherent*(s), i.e.,

(ϑ) $\forall p(s \models \neg p \equiv \neg s \models p)$

We first show (a) *Maximal*(s) and then (b) *Consistency*(s). (a) To show *Maximal*(s), it suffices by GEN to show $s \models q \vee s \models \neg q$, where q is an arbitrary proposition. We establish this by cases, starting from the tautology $s \models q \vee \neg s \models q$. If $s \models q$, then clearly by ∨I, it follows that $s \models q \vee s \models \neg q$. If $\neg s \models q$, then by the coherency of s (ϑ), it follows that $s \models \neg q$, and so again by ∨I, it follows that $s \models q \vee s \models \neg q$. So, reasoning by cases, we're done. (b) To show *Consistent*(s), we reason by reductio. Suppose ¬*Consistent*(s), i.e., by ¬¬E, that $\exists p(s \models p \,\&\, s \models \neg p)$. Let p_1 be an arbitrary such proposition, so that we know $s \models p_1 \,\&\, s \models \neg p_1$. But from the second conjunct of this last fact and the coherency of s (ϑ), it follows that $\neg s \models p_1$, which contradicts the first conjunct.

(←) Suppose *Maximal*(s) and *Consistent*(s). Then, by definitions (407) and (392), we know, respectively, the following:

(ϑ) $\forall p(s \models p \ \vee \ s \models \neg p)$

(ζ) $\neg\exists p(s \models p \,\&\, s \models \neg p)$

Now to show *Coherent*(s), it suffices, by definition (411) and GEN, to show $s \models \neg q \equiv \neg s \models q$, where q is arbitrary:

(→) Assume $s \models \neg q$. Now, for reductio, suppose $s \models q$. Then $s \models q \,\&\, s \models \neg q$, and so $\exists p(s \models p \,\&\, s \models \neg p)$, contradicting ($\zeta$). Hence, $\neg s \models q$.

(←) Assume $\neg s \models q$. Then by (ϑ), $s \models \neg q$. ⋈

(414) By (408) and (404.1), we know *Maximal*(w) and *Consistent*(w), respectively. Hence, by the right-to-left direction of (413), *Coherent*(w). ⋈

(416) (Exercise)

(418.1) For conditional proof, assume both:

(ϑ) $s \models p_1 \,\&\, \ldots \,\&\, s \models p_n \,\&\, ((p_1 \,\&\, \ldots \,\&\, p_n) \rightarrow q)$

(ξ) $\forall p(s \models p \equiv p)$

Then instantiate each of $p_1, \ldots, p_n$ into (ξ), to obtain:

$s \models p_1 \equiv p_1$
...
$s \models p_n \equiv p_n$

But each left-side condition in the above biconditionals is one of the first n conjuncts in (ϑ), and so it follows that: $p_1 \,\&\, \ldots \,\&\, p_n$. Hence, by the last conjunct in (ϑ), it follows that q. But now instantiate q into (ξ), and we obtain $s \models q \equiv q$. Hence $s \models q$. ⋈

(418.2) By applying RN to the previous theorem, we have:

$$\Box[(s \models p_1 \,\&\, \ldots \,\&\, s \models p_n \,\&\, ((p_1 \,\&\, \ldots \,\&\, p_n)) \rightarrow q) \rightarrow (\forall p(s \models p \equiv p) \rightarrow s \models q)]$$

This has the form:

$$\Box[(\varphi_1 \,\&\, \ldots \,\&\, \varphi_n \,\&\, \psi) \rightarrow (\chi \rightarrow \theta)]$$

By the K axiom (32.1), it follows that:

$$\Box(\varphi_1 \,\&\, \ldots \,\&\, \varphi_n \,\&\, \psi) \rightarrow \Box(\chi \rightarrow \theta)$$

But since it is a modally strict fact that a necessary conjunction is equivalent to a conjunction of necessarty truths (111.3), we may use the Rule of Substitution to infer:

$$(\Box\varphi_1 \,\&\, \ldots \,\&\, \Box\varphi_n \,\&\, \Box\psi) \rightarrow \Box(\chi \rightarrow \theta)$$

This is our theorem, once we make the obvious substitutions and apply the definition of necessary implication (415.1) to the last conjunct of the antecedent. ⋈

(419) By definition of *PossibleWorld*, we know:

(ϑ) $\Diamond\forall p(w \models p \equiv p)$

To show that *ModallyClosed*(w), assume:

(ξ) $w \models p_1 \,\&\, \ldots \,\&\, w \models p_n \,\&\, ((p_1 \,\&\, \ldots \,\&\, p_n) \Rightarrow q)$

By (406.1), we can infer the necessitation of each of the first n conjuncts of (ξ), and so by &I, we have:

$$\Box w \models p_1 \,\&\, \ldots \,\&\, \Box w \models p_n$$

Conjoining this result with the last conjunct of (ξ), we have:

$$\Box w \models p_1 \,\&\, \ldots \,\&\, \Box w \models p_n \,\&\, ((p_1 \,\&\, \ldots \,\&\, p_n) \Rightarrow q)$$

Hence, by (418.2), it follows that:

(ζ) $\Box(\forall p(w \models p \equiv p) \rightarrow w \models q)$

From (ζ) and (ϑ), it follows that: $\Diamond w \models q$, by (117.5). But by (406.2), it follows that $w \models q$. ⋈

(420.1) ($\rightarrow$) Assume $w \models (p \,\&\, q)$. Since w is by (419) modally closed, we know that the following instances of the definition (417) of modal closure obtain (where p_1 in (417) is set to $p \,\&\, q$ and $p_2, \ldots, p_n$ in (417) are vacuous):

$$[w \models (p \,\&\, q) \,\&\, ((p \,\&\, q) \Rightarrow p)] \rightarrow w \models p$$

$$[w \models (p \,\&\, q) \,\&\, ((p \,\&\, q) \Rightarrow q)] \rightarrow w \models q$$

Since the first conjunct of each antecedent is just our assumption and the second conjunct of each antecedent follows by an easily proved fact (e.g., either that $(\varphi \,\&\, \psi) \Rightarrow \varphi$ or that $(\varphi \,\&\, \psi) \Rightarrow \psi$), we can derive, respectively, $w \models p$ and $w \models q$. Hence $(w \models p) \,\&\, (w \models q)$.

($\leftarrow$) Assume $(w \models p) \,\&\, (w \models q)$. Since w is by (419) modally closed, we know that the following instance of the definition (417) of modal closure obtains (where p_1 in (417) is set to p and p_2 in (417) is set to q):

$$[w \models p \ \&\ w \models q \ \&\ ((p \,\&\, q) \Rightarrow (p \,\&\, q))] \rightarrow w \models (p \,\&\, q)$$

Now the conjunction of the first two conjuncts of the antecedent is just our assumption, and the third conjunct of the antecedent is an instance of the easily proved fact that $\varphi \Rightarrow \varphi$. It follows that $w \models (p \,\&\, q)$. ⋈

(420.2) (Exercise)

(420.3) (Exercise)

(421) Given our discussion in (401), we may interpret the variable w as doubly restricted and eliminate it so that the theorem to be proved becomes:

$$\exists! s(\mathit{PossibleWorld}(s) \,\&\, \mathit{Actual}(s))$$

By the definition of uniqueness, we have to show:

$$\exists s(\mathit{PossibleWorld}(s) \,\&\, \mathit{Actual}(s) \,\&\, \forall s'((\mathit{PossibleWorld}(s') \,\&\, \mathit{Actual}(s')) \rightarrow s' = s))$$

Now to find our witness to this existential claim, consider the formula $\exists p(p \,\&\, F = [\lambda y\ p])$, which is clearly a condition on propositional properties. So by Comprehension for Situations (381.1), we know:

$$\exists s \forall F(sF \equiv \exists p(p \,\&\, F = [\lambda y\ p]))$$

Let s_0 be an arbitrary such object, so that we know:

(ϑ) $\forall F(s_0 F \equiv \exists p(p \,\&\, F = [\lambda y\, p]))$

It suffices to show:

(A) $\mathit{PossibleWorld}(s_0)$

(B) $\mathit{Actual}(s_0)$

(C) $\forall s((\mathit{PossibleWorld}(s) \,\&\, \mathit{Actual}(s)) \rightarrow s = s_0)$

To show (A), we have show, by (400):

$$\Diamond\forall p(s_0 \models p \equiv p)$$

By the T$\Diamond$ theorem (118), it suffices to show $\forall p(s_0 \models p \equiv p)$. By GEN, it suffices to show, for an arbitrary proposition q, that $s_0 \models q \equiv q$. By definitions (365) and (216), it suffices to show $s_0[\lambda y\, q] \equiv q$. But from ($\vartheta$), we can infer:

$$s_0[\lambda y\, q] \equiv \exists p(p \,\&\, [\lambda y\, q] = [\lambda y\, p])$$

By the definition of proposition identity, this implies:

$$(\zeta)\quad s_0[\lambda y\, q] \equiv \exists p(p \,\&\, q = p)$$

But we independently know:

$$(\xi)\quad \exists p(p \,\&\, q = p) \equiv q$$

by (86.8).[286] Hence, by a biconditional syllogism, (ζ) and (ξ) imply $s_0[\lambda y q] \equiv q$.

To show (B), we have to show, by (386), that $\forall p(s_0 \models p \to p)$. But this is already done, since in the left-to-right direction of (A) above, we showed, for an arbitrary proposition q, that $s_0 \models q \to q$. Hence, by GEN, $\forall p(s_0 \models p \to p)$. ⋈

To show (C), it suffices by GEN to show $(\mathit{PossibleWorld}(s) \,\&\, \mathit{Actual}(s)) \to s = s_0$. Assume $\mathit{PossibleWorld}(s) \,\&\, \mathit{Actual}(s)$ and suppose, for *reductio*, $s \neq s_0$. Since s and s_0 are distinct situations, it follows by (368) that there is a proposition q true in one but not in the other. Without loss of generality, assume that $s \models q$ and $\neg s_0 \models q$. Since we know by (A) above that s_0 is a possible world, it follows by theorem (408) that it is maximal. So $s_0 \models \neg q$. But s is actual by hypothesis and s_0 is actual by (B) above. Hence, by definition of *Actual* (386), q and $\neg q$ are both true. Contradiction. ⋈

(423) By applying the Rule of Actualization to theorem (421), we may conclude:

$$\mathcal{A}\exists! w \mathit{Actual}(w)$$

So by (179.2), it follows that:

$$\exists y(y = \iota w \mathit{Actual}(w)) \qquad ⋈$$

(425.1)⋆ We begin by first proving some useful facts about $\boldsymbol{w}_\alpha$ and $\top$. By definition (424) and our theory of definitions (207.2), it follows that:

$$\boldsymbol{w}_\alpha = \iota w \mathit{Actual}(w)$$

[286] In case this isn't immediately obvious, note that (86.8) has the following instance when we set φ to q, β to p, and α to q (so that φ^β_α becomes p): $q \equiv \exists p(p = q \,\&\, p)$. By the symmetry of conjunction and identity, this is equivalent to $q \equiv \exists p(p \,\&\, q = p)$, which by the commutativity of the biconditional is equivalent to $\exists p(p \,\&\, q = p) \equiv q$.

If we use one of the methods described in Remark (401) for eliminating the restricted variable w in the definite description, this identity becomes:

$$\boldsymbol{w}_\alpha = \imath x(\mathit{PossibleWorld}(x) \,\&\, \mathit{Actual}(x))$$

Hence, by (101.2)★, it follows that:

(ϑ) $\mathit{PossibleWorld}(\boldsymbol{w}_\alpha) \,\&\, \mathit{Actual}(\boldsymbol{w}_\alpha)$

Moreover, from the facts that truth-values are situations (364) and $\top$ is a truth-value (224.1)★, it follows that:

(ξ) $\mathit{Situation}(\top)$

With facts (ϑ) and (ξ) in hand, we can now show $\boldsymbol{w}_\alpha = \top$. From ($\vartheta$) it follows by the definition of *PossibleWorld* that $\mathit{Situation}(\boldsymbol{w}_\alpha)$. From this and ($\xi$), we know $\boldsymbol{w}_\alpha$ and $\top$ are situations, and so it suffices, in virtue of a fact about the identity of situations (368), to show $\forall p(\top \models p \equiv \boldsymbol{w}_\alpha \models p)$. And by GEN, it suffices to show $\top \models p \equiv \boldsymbol{w}_\alpha \models p$.

($\rightarrow$) Assume $\top \models p$. By the definition of $\models$ (365), it follows that $\top \Sigma p$. So by a theorem we just cited, (227.3)★, it follows that p. Now assume, for reductio, that $\neg \boldsymbol{w}_\alpha \models p$. Then, since $\boldsymbol{w}_\alpha$ is a possible world, it is maximal (408). Hence by the definition of *Maximal*, $\boldsymbol{w}_\alpha \models \neg p$. But since $\boldsymbol{w}_\alpha$ is actual, it follows that $\neg p$. Contradiction.

($\leftarrow$) Assume $\boldsymbol{w}_\alpha \models p$. Now by the second conjunct of (ϑ), we know that $\mathit{Actual}(\boldsymbol{w}_\alpha)$. Hence, by the definition of *Actual* (386), it follows that p. But theorem (227.3)★ tells us that $p \equiv \top \Sigma p$. Hence $\top \Sigma p$. But since $\top$ is a situation, it follows by (365) that $\top \models p$. ⋈

(425.2)★ (Exercise)

(426)★ By (227.3)★, we know $p \equiv \top \Sigma p$. But by the previous theorem (425.1)★, we know $\top = \boldsymbol{w}_\alpha$. Hence, $p \equiv \boldsymbol{w}_\alpha \Sigma p$. But since $\boldsymbol{w}_\alpha$ is a situation by definition, it follows that $p \equiv \boldsymbol{w}_\alpha \models p$, by definition (365). ⋈

(427.1) By definition (424) and the method of eliminating restricted w variables noted in Remark (401), we know:

$$\boldsymbol{w}_\alpha = \imath x(\mathit{PossibleWorld}(x) \,\&\, \mathit{Actual}(x))$$

Moreover, as an instance of (107.3), we know:

$$\boldsymbol{w}_\alpha = \imath x(\mathit{PossibleWorld}(x) \& \mathit{Actual}(x)) \rightarrow \mathcal{A}(\mathit{PossibleWorld}(\boldsymbol{w}_\alpha) \& \mathit{Actual}(\boldsymbol{w}_\alpha))$$

Hence $\mathcal{A}(\mathit{PossibleWorld}(\boldsymbol{w}_\alpha) \,\&\, \mathit{Actual}(\boldsymbol{w}_\alpha))$. By (95.2), it follows that:

$$\mathcal{A}\mathit{PossibleWorld}(\boldsymbol{w}_\alpha) \,\&\, \mathcal{A}\mathit{Actual}(\boldsymbol{w}_\alpha))$$

So by (405.4), the first conjunct implies $PossibleWorld(\boldsymbol{w}_\alpha)$. $\bowtie$

(427.2) By (427.1) we know $PossibleWorld(\boldsymbol{w}_\alpha)$. Hence by (408), $Maximal(\boldsymbol{w}_\alpha)$. $\bowtie$

(428) [Given the proof strategy to be employed, it is more convenient to prove the right-to-left direction first.[287]] ($\leftarrow$) Assume $\boldsymbol{w}_\alpha \models p$. Since $\boldsymbol{w}_\alpha$ is a possible world (427.1), it follows by the right-to-left direction of theorem (405.4) that:

(ϑ) $\mathcal{A}\boldsymbol{w}_\alpha \models p$

Now, in the reasoning for (427.1), we independently established $\mathcal{A}Actual(\boldsymbol{w}_\alpha)$. It follows from the latter, by definition (386), that $\mathcal{A}\forall q(\boldsymbol{w}_\alpha \models q \rightarrow q)$. Hence by axiom (31.3), $\forall q\mathcal{A}(\boldsymbol{w}_\alpha \models q \rightarrow q)$. Instantiating to p yields $\mathcal{A}(\boldsymbol{w}_\alpha \models p \rightarrow p)$. So by axiom (31.2), it follows that:

(ξ) $(\mathcal{A}\boldsymbol{w}_\alpha \models p) \rightarrow \mathcal{A}p$

Hence from (ϑ) and (ξ), it follows that $\mathcal{A}p$. ($\rightarrow$) Assume $\mathcal{A}p$. For reductio, suppose $\neg\boldsymbol{w}_\alpha \models p$. By (427.2), we know $\boldsymbol{w}_\alpha$ is maximal. Hence, by definition, $\boldsymbol{w}_\alpha \models \neg p$. But by the right-to-left direction of the present theorem, we know $\boldsymbol{w}_\alpha \models \neg p \rightarrow \mathcal{A}\neg p$. Hence, $\mathcal{A}\neg p$. But it follows from this by axiom (31.1) that $\neg\mathcal{A}p$, which contradicts our assumption. $\bowtie$

(429)$\star$ Assume $w \neq \boldsymbol{w}_\alpha$ but suppose, for reductio, that $Actual(w)$. Since w and $\boldsymbol{w}_\alpha$ are distinct and both situations, it follows by (368) that there is a proposition q true in one but not in the other. Without loss of generality, suppose $w \models q$ and $\neg\boldsymbol{w}_\alpha \models q$. From the actuality of w and $w \models q$, it follows that q. But by (426)$\star$ and $\neg\boldsymbol{w}_\alpha \models q$, it follows that $\neg q$. Contradiction. $\bowtie$

(430)$\star$ ($\rightarrow$) Assume $Actual(s)$. To show $s \trianglelefteq \boldsymbol{w}_\alpha$, it suffices, by definition (369) and GEN, to show: $s \models p \rightarrow \boldsymbol{w}_\alpha \models p$. So assume $s \models p$. Since s is actual, p is true. But by (426)$\star$, all and only true propositions are true at $\boldsymbol{w}_\alpha$. Hence, $\boldsymbol{w}_\alpha \models p$. ($\leftarrow$) Assume $s \trianglelefteq \boldsymbol{w}_\alpha$. To show $Actual(s)$, it suffices by definition (386) and GEN to show: $s \models p \rightarrow p$. So assume $s \models p$. Then it follows from $s \trianglelefteq \boldsymbol{w}_\alpha$ and the definition of $\trianglelefteq$ (369) that $\boldsymbol{w}_\alpha \models p$. But we know by (426)$\star$ that all and only true propositions are true in $\boldsymbol{w}_\alpha$. Hence, p is true. $\bowtie$

(431.1)$\star$ ($\rightarrow$) Follow the beginning of the proof of (425.1)$\star$ to establish $Actual(\boldsymbol{w}_\alpha)$. Hence by theorem (391.1), it follows that $\boldsymbol{w}_\alpha \models p \rightarrow [\lambda y\ p]\boldsymbol{w}_\alpha$. ($\leftarrow$) Assume $[\lambda y\ p]\boldsymbol{w}_\alpha$. Then, by β-Conversion, p is true. So, by (426)$\star$, $\boldsymbol{w}_\alpha \models p$. $\bowtie$

(431.2)$\star$ ($\rightarrow$) Suppose p. Then by β-Conversion, $[\lambda y\ p]\boldsymbol{w}_\alpha$. But note that $[\lambda y\ p]\boldsymbol{w}_\alpha$ is a 0-place relation term and so may be instantiated into the universal generalization of (426)$\star$ to obtain:

[287] I'm indebted to Uri Nodelman for suggesting the proof strategy of the left-to-right direction.

$$[\lambda y\ p]\boldsymbol{w}_\alpha \equiv \boldsymbol{w}_\alpha \models [\lambda y\ p]\boldsymbol{w}_\alpha$$

Hence, $\boldsymbol{w}_\alpha \models [\lambda y\ p]\boldsymbol{w}_\alpha$. ($\leftarrow$) By reverse reasoning. ⋈

(432.1) To show $\Diamond p \rightarrow \Diamond\exists w(w \models p)$, it suffices to show $\Box(p \rightarrow \exists w(w \models p))$, since an instance of theorem (117.5) asserts:

$$\Box(p \rightarrow \exists w(w \models p)) \rightarrow (\Diamond p \rightarrow \Diamond\exists w(w \models p))$$

But to show $\Box(p \rightarrow \exists w(w \models p))$, it suffices, by the Rule of Necessitation, to give a modally strict proof of $p \rightarrow \exists w(w \models p)$. So assume p, to show $\exists w(w \models p)$. Now we know, independently, that it follows *a fortiori* from (421) that:

$$\exists w Actual(w)$$

Suppose w_0 is an arbitrary such object, so that we know:

(ϑ) $Actual(w_0)$

It follows from (ϑ) by the definition of *Actual* that:

(ξ) $\forall q(w_0 \models q \rightarrow q)$

Now, for reductio, assume $\neg w_0 \models p$. But since w_0 is a possible world, it is maximal (408). So by the definition of *Maximal*, $w_0 \models \neg p$. Hence, by (ξ), it follows that $\neg p$. This contradicts our assumption that p. So $w_0 \models p$. Hence $\exists w(w \models p)$, which is what we were trying to show. ⋈

(432.2) Our assumption is $\Diamond\exists w(w \models p)$ and we want to show $\exists w(w \models p)$. If we eliminate the restricted variable w in our assumption, it becomes:

$$\Diamond\exists x(PossibleWorld(x)\ \&\ x \models p).$$

By the BF$\Diamond$ (122.3), it follows from our assumption that:

$$\exists x\Diamond(PossibleWorld(x)\ \&\ x \models p)$$

Let b be an arbitrary such object, so that we have: $\Diamond(PossibleWorld(b)\ \&\ b \models p)$. Since the conjuncts of a possibly true conjunction are possible (117.8), it follows that:

(ξ) $\Diamond PossibleWorld(b)\ \&\ \Diamond b \models p$

Now by lemma (405.2), the first conjunct of (ξ) implies *PossibleWorld*(b). Since *PossibleWorld*(b), the facts (406) about the rigidity of truth at a possible world apply. So by (406.2), the second conjunct of (ξ) implies $b \models p$. Since we've established that $PossibleWorld(b)\ \&\ b \models p$, it follows that $\exists x(PossibleWorld(x)\ \&\ x \models p)$, i.e., $\exists w(w \models p)$. ⋈

(432.3) ($\rightarrow$) Assume p. By GEN, it suffices to show $\forall q(s \models q \equiv q) \rightarrow s \models p$. So assume $\forall q(s \models q \equiv q)$. As an instance of this latter assumption, we know $s \models p \equiv p$. Hence $s \models p$. ⋈

(432.4) Since (432.3) is modally strict, it follows by RN that:

$$\Box(p \to \forall s(\forall q(s \models q \equiv q) \to s \models p))$$

Hence by the K axiom (32.1), it follows that $\Box p \to \Box\forall s(\forall q(s \models q \equiv q) \to s \models p)$. ⋈

(432.5) To show $\Box\forall s\varphi \to \forall s\Box\varphi$, we have to show:

$$\Box\forall x(Situation(x) \to \varphi) \to \forall x(Situation(x) \to \Box\varphi)$$

So assume $\Box\forall x(Situation(x) \to \varphi)$. By the Converse Barcan Formula (122.2), it follows that:

(ϑ) $\forall x\Box(Situation(x) \to \varphi)$

Now to show $\forall x(Situation(x) \to \Box\varphi)$, it suffices by GEN to show $Situation(x) \to \Box\varphi$. So assume $Situation(x)$, to show $\Box\varphi$. By the rigidity of the notion of situation (366.1), it follows that:

(ξ) $\Box Situation(x)$

Moreover, it follows by applying $\forall$E to (ϑ) that:

(ζ) $\Box(Situation(x) \to \varphi)$

Hence, from (ξ) and (ζ) it follows that $\Box\varphi$, by the relevant instance of the K axiom (32.1). ⋈

(432.6) Assume $\forall w(w \models p)$, i.e., $\forall x(PossibleWorld(x) \to x \models p)$. For reductio, assume, $\neg\Box\forall w(w \models p)$, i.e., $\Diamond\neg\forall w(w \models p)$. Eliminating the restricted variable, we know:

$$\Diamond\neg\forall x(PossibleWorld(x) \to x \models p)$$

By modally strict theorems of quantification theory and the Rule of Substitution, it follows that:

$$\Diamond\exists x(PossibleWorld(x)\ \&\ \neg x \models p)$$

By the BF$\Diamond$ schema (122.3), we therefore know:

$$\exists x\Diamond(PossibleWorld(x)\ \&\ \neg x \models p)$$

So let b be an arbitrary such object, so that we know:

$$\Diamond(PossibleWorld(b)\ \&\ \neg b \models p)$$

Since each conjunct of a possibly true conjunction is possibly true (117.8), it follows that:

(ξ) $\Diamond PossibleWorld(b)\ \&\ \Diamond\neg b \models p$

But by a previous lemma (405.2), we know that the first conjunct of (ξ) implies $PossibleWorld(b)$. Since our initial assumption was that $\forall x(PossibleWorld(x) \rightarrow x \models p)$, it follows that $b \models p$. But the second conjunct of (ξ) is equivalent to $\neg\Box b \models p$. Since $PossibleWorld(b)$, the facts about the rigidity of truth at a world apply. Hence by (406.3), it follows that $\neg\Diamond b \models p$, i.e., $\Box\neg b \models p$. By the T schema, $\neg b \models p$. Contradiction. ⋈

(432.7) By the K axiom, it suffices to show $\Box(\forall w(w \models p) \rightarrow p)$. And by RN, it suffices to show $\forall w(w \models p) \rightarrow p$ by a modally strict proof. So assume $\forall w(w \models p)$, to show p. If we eliminate the restricted variable from our assumption, we know:

(ϑ) $\forall x(PossibleWorld(x) \rightarrow x \models p)$

Now by (421), we know *a fortiori* that $\exists x(PossibleWorld(x) \& Actual(x))$. Suppose a is an arbitrary such object, so that we know:

(ξ) $PossibleWorld(a) \& Actual(a)$

From (ϑ) and the first conjunct of (ξ), it follows that $a \models p$. From this and the second conjunct of (ξ), it follows by the definition of *Actual* that p. ⋈

(433.1) ($\rightarrow$) By hypothetical syllogism from (432.1) and (432.2). ($\leftarrow$) Assume $\exists w(w \models p)$. Let w_1 be an arbitrary such possible world, so that we know $w_1 \models p$. Furthermore, by the definition of possible world, $\Diamond\forall q(w_1 \models q \equiv q)$. By the Buridan$\Diamond$ formula (123.2), it follows that $\forall q\Diamond(w_1 \models q \equiv q)$. Hence, $\Diamond(w_1 \models p \equiv p)$ and, *a fortiori*, $\Diamond(w_1 \models p \rightarrow p)$. But we also know $w_1 \models p$ and so, by the rigidity of *truth at* (406.1), $\Box w_1 \models p$. But from $\Diamond(w_1 \models p \rightarrow p)$ and $\Box w_1 \models p$, it follows that $\Diamond p$, by the left-to-right direction of (117.9), i.e., the left-to-right direction of $\Diamond(\varphi \rightarrow \psi) \equiv (\Box\varphi \rightarrow \Diamond\psi)$. ⋈

(433.2) The following proof uses (432.4) – (432.7). However, we also provide an alternative, simpler proof below that appeals to (433.1) instead. ($\rightarrow$) Assume $\Box p$. It follows from this and (432.4) that:

$$\Box\forall s(\forall q(s \models q \equiv q) \rightarrow s \models p)$$

From this last fact and the version of the Converse Barcan Formula restricted to situations (432.5), it follows that:

(ξ) $\forall s\Box(\forall q(s \models q \equiv q) \rightarrow s \models p)$

Now we want to show: $\forall w(w \models p)$, i.e., $\forall s(PossibleWorld(s) \rightarrow s \models p)$ (treating w as doubly-restricted). By GEN, it suffices to show: $PossibleWorld(s) \rightarrow s \models p$. So assume $PossibleWorld(s)$. By the rigidity of *truth at* (367.2) for situations, it suffices to show $\Diamond s \models p$. By definition of *PossibleWorld* (400), we know:

$$\Diamond\forall q(s \models q \equiv q)$$

But since (ξ) holds for every situation, it follows from (ξ) in particular that:

$$\Box(\forall q(s \models q \equiv q) \to s \models p)$$

But then, from our last two results it follows that $\Diamond s \models p$, by theorem (117.5), i.e., by the fact that $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$.

($\leftarrow$) By hypothetical syllogism from (432.6) and (432.7).

(433.2) (Alternative) Here is a simpler proof that goes by way of the previous theorem (433.1) and the theorem that every possible world is coherent (414):

1.	$\Diamond\neg q \equiv \exists w(w \models \neg q)$	Instance of (433.1)
2.	$\Diamond\neg q \equiv \exists w\neg(w \models q)$	From 1, by (414), (411), and Substitution
3.	$\neg\Diamond\neg q \equiv \neg\exists w\neg(w \models q)$	From 2, by (63.5.d)
4.	$\Box q \equiv \forall w(w \models q)$	From 3, by (117.3) and (86.3) ⋈

(433.3) (Exercise)

(433.4) (Exercise)

(435.1) Assume $\exists p\mathit{ContingentlyTrue}(p)$. Let p_1 be an arbitrary such proposition, so that, by definition, we know both p_1 and $\Diamond\neg p_1$. Now if we instantiate Fundamental Theorem of Possible World Theory (433.1) to $\neg p_1$, we know $\Diamond\neg p_1 \equiv \exists w(w \models \neg p_1)$. So from this and the second conjunct of our assumption, it follows that $\exists w(w \models \neg p_1)$. Let w_1 be such a possible world, so that we know $w_1 \models \neg p_1$. Now suppose, for reductio, that $\mathit{Actual}(w_1)$. Then by definition of *Actual*, every proposition true at w_1 is true. Hence $\neg p_1$. Contradiction. ⋈

(435.2) (Exercise)

(435.3) (The proof is given in the text.)

(435.4) Since (421) established $\exists!w\mathit{Actual}(w)$, it follows *a fortiori* that $\exists w\mathit{Actual}(w)$. Suppose w_1 is such a possible world, so that we know $\mathit{Actual}(w_1)$. Independently, by (435.3), we know $\exists w\neg\mathit{Actual}(w)$. Suppose w_2 is such a possible world, so that we know $\neg\mathit{Actual}(w_2)$. Since w_1 is actual and w_2 is not, it follows that $w_1 \neq w_2$. Hence $\exists w\exists w'(w \neq w')$. ⋈

(438) By β-Conversion (36.2), we know that $[\lambda y\, p]x \equiv p$ is a modally strict theorem. Hence, by RN, $\Box([\lambda y\, p]x \equiv p)$, and by (111.4), $\Box([\lambda y\, p]x \to p)$ and $\Box(p \to [\lambda y\, p]x)$. Hence by definition of $\Rightarrow$ (415), we know both:

(a) $[\lambda y\, p]x \Rightarrow p$

(b) $p \Rightarrow [\lambda y\, p]x$

We now establish our theorem by arguing for both directions. ($\to$) Assume $w \models p$. Then by (b) and the modal closure of possible worlds (419), it follows that $w \models [\lambda y\, p]x$. ($\leftarrow$) Assume $w \models [\lambda y\, p]x$. Then by (a) and the modal closure of possible worlds (419), it follows that $w \models p$. ⋈

(439.1) Assume, for reductio, $\exists w \exists p(w \models (p \,\&\, \neg p))$. Let w_1 and p_1 be an arbitrary such world and proposition, respectively, so that we know $w_1 \models (p_1 \,\&\, \neg p)$. Hence, $\exists w(w \models (p_1 \,\&\, \neg p))$. Then by fundamental theorem (433.1), it follows that $\diamond(p_1 \,\&\, \neg p_1)$, contradicting the fact, provable from (109), (63.1.a), and (117.4), that $\neg\diamond(p_1 \,\&\, \neg p_1)$. Hence $\neg\exists w \exists p(w \models p \,\&\, w \models \neg p)$. ⋈

(439.2): By (439.1), we know that $\neg\exists w \exists p(w \models (p \,\&\, \neg p))$. Hence, $\forall w \neg\exists p(w \models (p \,\&\, \neg p))$, and so by $\forall$E, $\neg\exists p(w \models (p \,\&\, \neg p))$. Hence, by (58.3), $\exists p(w \models (p \,\&\, \neg p)) \rightarrow \mathit{Trivial}(w)$. ⋈

(440.1) ($\rightarrow$) In what follows, we use $w \not\models p$ to abbreviate $\neg w \models p$. For conditional proof, assume:

(a) $w \models (p \vee q)$

Suppose, for reductio, that $\neg(w \models p \vee w \models q)$, i.e.,

(b) $(w \not\models p) \,\&\, (w \not\models q)$

Then, by the maximality of possible worlds, it follows from the first conjunct of (b) that:

(c) $w \models \neg p$

Independently, starting with disjunctive syllogism (64.4.b), we may derive $((p \vee q) \,\&\, \neg p) \rightarrow q$ by a modally strict proof. Hence, by RN:

(d) $\Box(((p \vee q) \,\&\, \neg p) \rightarrow q)$

By definition of $\Rightarrow$ (415), (d) implies:

(e) $((p \vee q) \,\&\, \neg p) \Rightarrow q$

But by the fact that worlds are closed under necessary implication (419), it follows from (a), (c), and (e), that $w \models q$. But this contradicts the second conjunct of (b). ($\leftarrow$) (Exercise) ⋈

(440.2) (Exercise)

(443.1) (Exercise)

(443.2) (Exercise)

(445.1) Let φ be $\exists q(w \models (q \equiv p) \,\&\, F = [\lambda y\, q])$. We want to show:

$$\Box\forall F(\varphi \rightarrow \Box\varphi)$$

By RN and GEN, it suffices to show: $\varphi \rightarrow \Box\varphi$. So assume φ, i.e.,

$$\exists q(w \models (q \equiv p) \ \& \ F = [\lambda y\, q])$$

Let q_1 be an arbitrary such proposition, so that we know:

(ϑ) $w \models (q_1 \equiv p) \ \& \ F = [\lambda y\, q_1]$

Now recall that we are trying to show:

$$\Box \exists q(w \models (q \equiv p) \ \& \ F = [\lambda y\, q])$$

By the Buridan schema (123), it suffices to show:

$$\exists q \Box(w \models (q \equiv p) \ \& \ F = [\lambda y\, q])$$

By $\exists$I, it suffices to show $\Box(w \models (q_1 \equiv p) \ \& \ F = [\lambda y\, q_1])$. Finally, by (111.3), it suffices to show: $\Box w \models (q_1 \equiv p) \ \& \ \Box F = [\lambda y\, q_1]$. But this is easy: from the first conjunct of (ϑ) it follows by the rigidity of truth at a world (406.1) that $\Box w \models (q_1 \equiv p)$, and by the second conjunct of (ϑ), it follows by the necessity of identity (75) that $\Box F = [\lambda y\, q_1]$. ⋈

(445.2) By definition of p°_w (442), theorem (189.2), and the fact that the formula $\exists q(w \models (q \equiv p) \ \& \ F = [\lambda y\, q])$ is a rigid condition on properties (445.1). ⋈

(445.3) (Exercise)

(445.4) Before we begin, we note that since $\Box(p \equiv p)$ is a theorem, it follows by a Fundamental Theorem of Possible World Theory that $\forall w(w \models (p \equiv p))$. Hence $w \models (p \equiv p)$, and so $w \models (p \equiv p) \ \& \ [\lambda y\, p] = [\lambda y\, p]$. So by $\exists$I:

(ϑ) $\exists q(w \models (q \equiv p) \ \& \ [\lambda y\, p] = [\lambda y\, q])$

Now by (445.2), we know:

$$A!p^\circ_w \ \& \ \forall F(p^\circ_w F \equiv \exists q(w \models (q \equiv p) \ \& \ F = [\lambda y\, q]))$$

But from the right conjunct of this last conclusion and (ϑ), it follows that:

$$p^\circ_w[\lambda y\, p]$$

And since p°_w is abstract, it follows by (216) that $p^\circ_w \Sigma p$. ⋈

(448.1) Let φ be $\exists q(w \models q \ \& \ F = [\lambda y\, q])$. We want to show:

$$\Box \forall F(\varphi \rightarrow \Box \varphi)$$

By RN and GEN, it suffices to show: $\varphi \rightarrow \Box \varphi$. So assume φ, i.e.,

$$\exists q(w \models q \ \& \ F = [\lambda y\, q])$$

Let q_1 be an arbitrary such proposition, so that we know:

(ϑ) $w \models q_1 \ \& \ F = [\lambda y\, q_1]$

Now we have to show:

$$\Box\exists q(w \models q \;\&\; F = [\lambda y\, q])$$

By the Buridan schema (123), it suffices to show:

$$\exists q\Box(w \models q \;\&\; F = [\lambda y\, q])$$

By $\exists$I, it suffices to show $\Box(w \models q_1 \;\&\; F = [\lambda y\, q_1])$. Finally, by (111.3), it suffices to show: $\Box w \models q_1 \;\&\; \Box F = [\lambda y\, q_1]$. But this is easy: from the first conjunct of (ϑ) it follows by the rigidity of truth at a world (406) that $\Box w \models q_1$, and by the second conjunct of (ϑ), it follows by the necessity of identity (75) that $\Box F = [\lambda y\, q_1]$. Hence $\Box w \models q_1 \;\&\; \Box F = [\lambda y\, q_1]$. ⋈

(448.2) (Exercise)

(448.3) (Exercise)

(448.4) (Exercise)

(449.1) Before we prove our theorem, note that we established in (448.1) and the subsequent discussion that $\top_w$ is strictly canonical and so subject to theorem (189.2). It therefore follows that:

$$\forall F(\top_w F \equiv \exists q(w \models q \;\&\; F = [\lambda y\, q]))$$

If we instantiate this to $[\lambda y\, p]$, we obtain:

$$\top_w[\lambda y\, p] \equiv \exists q(w \models q \;\&\; [\lambda y\, p] = [\lambda y\, q]))$$

Since $\top_w$ is abstract, it follows by (216) and the Rule of Substitution that:

$$(\vartheta) \quad \top_w \Sigma p \equiv \exists q(w \models q \;\&\; [\lambda y\, p] = [\lambda y\, q])$$

We now prove both directions of our theorem.

($\rightarrow$) Assume $\top_w \Sigma p$. Then by (ϑ),

$$\exists q(w \models q \;\&\; [\lambda y\, p] = [\lambda y\, q])$$

Suppose q_1 is an arbitrary such proposition, so that we know:

$$w \models q_1 \;\&\; [\lambda y\, p] = [\lambda y\, q_1]$$

The right conjunct, by the definition of proposition identity, ensures $p = q_1$. Hence, the left conjunct imlies $w \models p$.

($\leftarrow$) Assume $w \models p$. Then, $w \models p \;\&\; [\lambda y\, p] = [\lambda y\, p]$. So $\exists q(w \models q \;\&\; [\lambda y\, p] = [\lambda y\, q])$. So by ($\vartheta$), $\top_w \Sigma p$. ⋈

(449.2) (Exercise)

(449.3) By (449.1), we already know:

(ϑ) $\top_w \Sigma p \equiv w \models p$

Independently, we know that $\top_w$ is a situation—by (448.3), one can easily derive that $\top_w$ is abstract and that every property it encodes is propositional. So by definition (365) of $\models$, we know that:

(ζ) $\top_w \models p \ \equiv \ \top_w \Sigma p$

Hence from (ζ) and (ϑ), it follows that $\top_w \models p \equiv w \models p$. So by GEN:

$$\forall p(\top_w \models p \ \equiv \ w \models p)$$

But since w is also a situation (exercise), it follows by definition of situation identity (368) that $\top_w = w$. ⋈

(450.1) By (445.2), we know:

(ϑ) $A!p^\circ_w \ \& \ \forall F(p^\circ_w F \equiv \exists q(w \models (q \equiv p) \ \& \ F = [\lambda y\, q]))$

Moreover, theorem (448.3) is:

(ξ) $A!\top_w \ \& \ \forall F(\top_w F \equiv \exists q(w \models q \ \& \ F = [\lambda y\, q]))$

We now prove both directions of our biconditional theorem.

($\rightarrow$) Assume $w \models p$. As noted in a previous theorem, it is easy to show, from (ϑ) and (ξ) respectively, that p°_w and $\top_w$ are situations, since it remains only for one to show that every property they encode is propositional (exercise). So to show they are identical, it suffices by (368) to show $\forall r(p^\circ_w \models r \ \equiv \ \top_w \models r)$. By GEN, it suffices to show $p^\circ_w \models r \ \equiv \ \top_w \models r$.

> ($\rightarrow$) Assume $p^\circ_w \models r$. Then by definitions (365) and (216), $p^\circ_w[\lambda y\, r]$. Hence by the right conjunct of (ϑ), it follows that $\exists q(w \models (q \equiv p) \& [\lambda y\, r] = [\lambda y\, q])$. Let q_1 be an arbitrary such proposition, so that we know $w \models (q_1 \equiv p)$ & $[\lambda y\, r] = [\lambda y\, q_1]$. The second conjunct implies $r = q_1$, by the definition of proposition identity, and so the first conjunct implies $w \models (r \equiv p)$, by Rule SubId. But from this last fact, our assumption that $w \models p$, and the easily established fact that $((r \equiv p) \ \& \ p) \Rightarrow r$, it follows that $w \models r$, by the modal closure of possible worlds (419). Hence, by (449.1), $\top_w \Sigma r$, which implies $\top_w \models r$, by (365) and the fact that $\top_w$ is a situation.
>
> ($\leftarrow$) Assume $\top_w \models r$. Then by (449.1), $w \models r$. But from this last fact, our assumption that $w \models p$, and the easily established fact that $(r \ \& \ p) \Rightarrow (r \equiv p)$, it follows that $w \models (r \equiv p)$, again by the modal closure of possible worlds. Hence $w \models (r \equiv p)$ & $[\lambda y\, r] = [\lambda y\, r]$. So by $\exists$I, $\exists q(w \models (q \equiv p)$ & $[\lambda y\, r] = [\lambda y\, q])$. It follows by the right conjunct of (ϑ) that $p^\circ_w[\lambda y\, r]$. So by definitions (365) and (216) and the fact that p°_w is a situation, it follows that $p^\circ_w \models r$.

(←) Assume $p^{\circ}_{w} = \top_{w}$. Independently, by (445.4), we know $p^{\circ}_{w}\Sigma p$. And we also know that p°_{w} is a situation, by an exercise earlier in the proof. So by (365), it follows that $p^{\circ}_{w} \models p$. Hence, by applying Rule SubId to this and our initial assumption, it follows that $\top_{w} \models p$. But then by (449.1), $w \models p$. ⋈

(450.2) (Exercise)

(451.1) By applying GEN to our recent theorem (450.1), we know:

$$\forall w(w \models p \equiv p^{\circ}_{w} = \top_{w})$$

It follows from this, by a law of quantification theory (83.3), that:

$$(\vartheta)\ \ \forall w(w \models p) \equiv \forall w(p^{\circ}_{w} = \top_{w})$$

But it is a Fundamental Theorem of Possible World Theory (433.2) that:

$$\Box p \equiv \forall w(w \models p)$$

Hence from (433.2) and (ϑ), it follows that $\Box p \equiv \forall w(p^{\circ}_{w} = \top_{w})$. ⋈

(451.2) (Exercise)

(451.3) (Exercise)

(451.4) (Exercise)

(453.1) (Exercise)

(453.2) (Exercise)

(455.1) By RN and GEN, it suffices to show $\varphi \to \Box\varphi$. So assume φ, i.e., $w \models \forall y(Fy \equiv Gy)$. Then by the rigidity of truth at a world (406), it follows that $\Box w \models \forall y(Fy \equiv Gy)$, i.e., $\Box\varphi$.

(455.2) By definition of ϵG_{w}, (452), theorem (189.2) and the fact that $w \models \forall y(Fy \equiv Gy)$ is a rigid condition on properties (455.1). ⋈

(455.3) (Exercise)

(455.4) (Exercise)

(456.1) Assume $\mathit{ExtensionOfAt}(x, G, w)$ and $\mathit{ExtensionOfAt}(y, H, w)$. By the definition of *ExtensionOfAt* (452), we know:

(a) $A!x \,\&\, \forall F(xF \equiv w \models \forall z(Fz \equiv Gz))$

(b) $A!y \,\&\, \forall F(yF \equiv w \models \forall z(Fz \equiv Hz))$

(→) Assume $x = y$. Then by Rule SubId, it follows from (a) that:

(c) $A!y \,\&\, \forall F(yF \equiv w \models \forall z(Fz \equiv Gz))$

Hence, by (83.11) and (83.10), the right conjuncts of (b) and (c) imply:

(d) $\forall F[w \models \forall z(Fz \equiv Hz) \equiv w \models \forall z(Fz \equiv Gz)]$

Now if we instantiate (d) to G, we know:

(e) $w \models \forall z(Gz \equiv Hz) \equiv w \models \forall z(Gz \equiv Gz)$

But the right side of (e) is easily derived: from the tautology $Gz \equiv Gz$, we obtain $\forall z(Gz \equiv Gz)$, by GEN. Since this is a modally strict theorem, we obtain $\Box\forall z(Gz \equiv Gz)$. So by a Fundamental Theorem of Possible World Theory (433.2), it follows that $\forall w'(w' \models \forall z(Gz \equiv Gz))$. Instantiating to w, we obtain the right side of (e). So the left side of (e), $w \models \forall z(Gz \equiv Hz)$, follows by biconditional syllogism.

($\leftarrow$) Assume:

(f) $w \models \forall z(Gz \equiv Hz)$

Since we know $A!x$ and $A!y$ by the left conjuncts of (a) and (b), it suffices by theorem (172.1) to show $\forall F(xF \equiv yF)$, and by GEN, that $xF \equiv yF$:

> ($\rightarrow$) Assume xF. Then by the right conjunct of (a), it follows that
>
> (g) $w \models \forall z(Fz \equiv Gz)$
>
> Now, independently, we leave it as an exercise to prove:
>
> (h) $(\forall z(Fz \equiv Gz) \,\&\, \forall z(Gz \equiv Hz)) \Rightarrow \forall z(Fz \equiv Hz)$
>
> Hence, from (g), (f), and (h), it follows by the modal closure of possible worlds (419) that $w \models \forall z(Fz \equiv Hz)$. Hence, by the right conjunct of (b), yF.
>
> ($\leftarrow$) (Exercise) ⋈

(456.2) Since ϵF_w and ϵG_w are logically proper, we can instantiate them into world-relativized pre-Law V (456.1). Simultaneously substituting ϵF_w for x, F for G, ϵG_w for y, and G for H, we obtain:

$$(\mathit{ExtensionOfAt}(\epsilon F_w, F, w) \,\&\, \mathit{ExtensionOfAt}(\epsilon G_w, G, w)) \rightarrow \\ (\epsilon F_w = \epsilon G_w \equiv w \models \forall z(Fz \equiv Gz))$$

But we also know both conjuncts of the antecedent, by (455.3). Hence, $\epsilon F_w = \epsilon G_w \equiv w \models \forall z(Fz \equiv Gz)$. ⋈

(461) (Exercise)

(462) It follows from (377.4) and (374.2) that:

(ϑ) $\forall p(\boldsymbol{s_V} \models p)$

We have to show: (a) $\boldsymbol{s_V}$ is maximal and (b) $\boldsymbol{s_V}$ fails to be possible. But clearly, if every proposition is true in $\boldsymbol{s_V}$, then for every proposition q, either q is true in $\boldsymbol{s_V}$ or $\neg q$ is true in $\boldsymbol{s_V}$. So (a) holds, by definition of *Maximal* (407). And just as clearly, if every proposition is true in $\boldsymbol{s_V}$, then for every (and hence, for some) proposition q, both q is true in $\boldsymbol{s_V}$ and $\neg q$ is true in $\boldsymbol{s_V}$. So by definition (392), $\neg Consistent(\boldsymbol{s_V})$, and by theorem (398.1), it follows that (b).[288] ⋈

(463)★ Our theorem requires us to show:

(a) $Situation(\bot)$

(b) $Maximal(\bot)$

(c) $\neg Possible(\bot)$

(d) $\neg Trivial(\bot)$

(a) follows from the facts that $\bot$ is a truth-value (224.2)★ and that truth-values are situations (364).

(b) By GEN, it suffices to show $\bot \models p \vee \bot \models \neg p$. We reason by cases from the tautology that $p \vee \neg p$. (i) If p, then $\neg\neg p$ and so by (227.5)★, $\bot \Sigma \neg p$. Since $\bot$ is a situation, it follows that $\bot \models \neg p$. So $\bot \models p \vee \bot \models \neg p$. (ii) If $\neg p$, then by (227.5)★, it follows that $\bot\ \Sigma p$, i.e., $\bot \models p$. So $\bot \models p \vee \bot \models \neg p$.

(c) Now where q_1 is any arbitrarily chosen proposition, we know that $\neg(q_1 \& \neg q_1)$. Hence, by an appropriate instance of (227.5)★, it follows that $\bot\ \Sigma (q_1 \ \&\ \neg q_1)$. Since $\bot$ is a situation, the last fact implies $\bot \models (q_1 \ \&\ \neg q_1)$. Since it is a theorem that $\neg\Diamond(q_1 \ \&\ \neg q_1)$, it follows by an appropriate instance of (396.2) that $\neg Possible(\bot)$.

(d) To see that $\bot$ isn't trivial, consider the proposition p_0, which was defined in (145) as $\forall x(E!x \rightarrow E!x)$. p_0 is clearly a theorem, and so known to be true. Hence, by (227.4)★, $\neg \bot \Sigma p_0$, and so from the fact that $\bot$ is a situation, $\neg\bot \models p_0$. Consequently, $\exists q \neg(\bot \models q)$, i.e., $\neg\forall q(\bot \models q)$. So, by definition of *Trivial* (374.2), $\neg Trivial(\bot)$. ⋈

(464) The formula $\exists q(\neg q \ \&\ F = [\lambda y\ q])$ is easily established as a condition on propositional properties (exercise). So by the Comprehension for Situations (381.1), there is a situation that encodes all and only those propositional properties $[\lambda y\ q]$ constructed from some false proposition q:

[288] To see precisely how these arguments for (a) and (b) can be made precise, we can reason as follows. (a) To show maximality, we have to show $\forall p(\boldsymbol{s_V} \models p \vee \boldsymbol{s_V} \models \neg p)$, and so, by GEN, that $\boldsymbol{s_V} \models p \vee \boldsymbol{s_V} \models \neg p$. But by $\forall$E, the first disjunct follows from (ϑ) and by $\vee$I, $\boldsymbol{s_V} \models p \vee \boldsymbol{s_V} \models \neg p$. (b) Let p_1 be any proposition. Now axiom (29.2) asserts, as an instance, $\exists q(q = (p_1 \ \&\ \neg p_1))$, and so we can instantiate the 0-place relation term $p_1 \ \&\ \neg p_1$ in (ϑ) and obtain $\boldsymbol{s_V} \models (p_1 \ \&\ \neg p_1)$. But, it is a theorem that $\neg\Diamond(p_1 \ \&\ \neg p_1)$, by (63.1.a), (109), and (117.4). Hence by an appropriate instance of (396.2), it follows that $\neg Possible(\boldsymbol{s_V})$.

$$\exists s \forall F(sF \equiv \exists q(\neg q \,\&\, F = [\lambda y\, q]))$$

Let s_1 be an arbitrary such object, so that we know:

(ϑ) $\forall F(s_1 F \equiv \exists q(\neg q \,\&\, F = [\lambda y\, q]))$

Since s_1 is, by definition, a situation, it remains to show that:

(a) $Maximal(s_1)$

(b) $\neg Possible(s_1)$

(c) $\neg Trivial(s_1)$

Before we begin, we establish the following fact about s_1:

Fact. $\forall p(s_1 \models p \equiv \neg p)$

Proof. By GEN, it suffices to show $s_1 \models p \equiv \neg p$. ($\rightarrow$) Assume $s_1 \models p$. Then by (365), $s_1 \Sigma p$ and by (216), $s_1[\lambda y\, p]$. So by (ϑ), $\exists q(\neg q \,\&\, [\lambda y\, p] = [\lambda y\, q])$. Suppose q_1 is an arbitrary such proposition, so that we know both that $\neg q_1$ and that $[\lambda y\, p] = [\lambda y\, q_1]$. Then by definition of proposition identity, $p = q_1$. Hence $\neg p$, since $\neg q_1$. ($\leftarrow$) Assume $\neg p$. Then $\neg p \,\&\, [\lambda y\, p] = [\lambda y\, p]$. So $\exists q(\neg q \,\&\, [\lambda y\, p] = [\lambda y\, q])$. So by ($\vartheta$), $s_1[\lambda y\, p]$. By (216) and (365), $s_1 \models p$.

Now we can establish (a), (b), and (c).

(a) Let q_1 be an arbitrarily chosen proposition. If we can show $s_1 \models q_1 \vee s_1 \models \neg q_1$, then by $\forall$I, we may conclude $\forall q(s_1 \models q_1 \vee s_1 \models \neg q_1)$, which establishes that s_1 is maximal. We reason by cases from the tautology that $q_1 \vee \neg q_1$. (i) Suppose q_1. Then $\neg\neg q_1$ and so by the above *Fact*, $s_1 \models \neg q_1$. So $s_1 \models q_1 \vee s_1 \models \neg q_1$. (ii) Suppose $\neg q_1$. Then by the above *Fact*, it follows that $s_1 \models q_1$. So $s_1 \models q_1 \vee s_1 \models \neg q_1$.

(b) Now let q_2 be any arbitrarily chosen proposition. Then we know that $\neg(q_2 \,\&\neg q_2)$. Hence, by the above *Fact*, implies $s_1 \models (q_2 \,\&\, \neg q_2)$. Since it is a theorem that $\neg\Diamond(q_2 \,\&\, \neg q_2)$, it follows by an appropriate instance of (396.2) that $\neg Possible(s_1)$.

(c) To see that s_1 isn't trivial, consider the proposition p_0, which was defined in (145) as $\forall x(E!x \rightarrow E!x)$. p_0 is clearly a theorem, and so known to be true. Hence, by the above *Fact*, $\neg s_1 \models p_0$ Consequently, $\exists q \neg(s_1 \models q)$, i.e., $\neg\forall q(s_1 \models q)$. So, by definition of *Trivial* (374.2), $\neg Trivial(s_1)$. ⋈

(465): We begin by following the proof of the previous theorem: the Comprehension for Situations (381.1) asserts that there is a situation, say s_1, that encodes all and only those propositional properties $[\lambda y\, q]$ constructed from some false proposition q. We showed that s_1 is governed by:

Fact. $\forall p(s_1 \models p \equiv \neg p)$

For the present theorem, however, we have to show:

(a) $Maximal(s_1)$

(b) $\neg Possible(s_1)$

(c) $\neg ModallyClosed(s_1)$

The proofs of (a) and (b) are identical to the proofs of (a) and (b) in the previous theorem (464). For (c), it suffices to show, by definition (417), that there exist propositions p and q such that $s_1 \models p$, $p \Rightarrow q$, and $\neg s_1 \models q$. Let our witness for p be $p_0 \,\&\, \neg p_0$, where p_0 is $\forall x(E!x \rightarrow E!x)$, and let our witness for q be p_0. So we have to show:

(i) $s_1 \models (p_0 \,\&\, \neg p_0)$

(ii) $(p_0 \,\&\, \neg p_0) \Rightarrow p_0$

(iii) $\neg s_1 \models p_0$

(i) Since $p_0 \,\&\, \neg p_0$ is a contradiction, we know $\neg(p_0 \,\&\, \neg p_0)$ is a theorem (63.1.a). Hence, by the *Fact* proved in the previous theorem, it follows that $s_1 \models (p_0 \& \neg p_0)$.

(ii) Since $\neg(p_0 \,\&\, \neg p_0)$ is a theorem, it follows by RN that $\Box\neg(p_0 \,\&\, \neg p_0)$. Hence by (111.2), it follows that $\Box((p_0 \,\&\, \neg p_0) \rightarrow p_0)$. So by definition of $\Rightarrow$ (339.1), it follows that $(p_0 \,\&\, \neg p_0) \Rightarrow p_0$.

(iii) It follows from our previously established *Fact* that $s_1 \models p_0 \equiv \neg p_0$. But p_0 is a easy theorem and so $\neg\neg p_0$. Hence, $\neg s_1 \models p_0$. ⋈

(467.1): By RN and GEN, it suffices to show $\varphi \rightarrow \Box\varphi$. So assume φ, i.e.,

$$sF \vee F = [\lambda y\, p]$$

Note that both disjuncts imply their own necessity: sF implies $\Box sF$ by (367.1), and $F = [\lambda y\, p]$ implies $\Box F = [\lambda y\, p]$ by (75). Hence, by disjunctive syllogism:

$$\Box sF \vee \Box F = [\lambda y\, p]$$

But by (117.7), it follows that $\Box(sF \vee F = [\lambda y\, p])$. ⋈

(467.2): (Exercise)

(467.3): Assume $s \models q$. Then $s[\lambda y\, q]$, by (365). Hence by $\vee$I, $s[\lambda y\, q] \vee [\lambda y\, q] = [\lambda y\, p]$. So by (467.2), it follows that $s^{+p}[\lambda y\, q]$. Hence, $s^{+p} \models q$. ⋈

(467.4): (Exercise)

(468): We first follow the proof in (421) to establish some useful facts. By Comprehension for Situations (381.1), we know there exists a situation that encodes all and only those propositional properties $[\lambda y\, q]$ for which q is a true proposition; i.e.,

$$\exists s \forall F(sF \equiv \exists q(q \,\&\, F = [\lambda y\, q]))$$

Let s_0 be an arbitrary such situation, so that we know:

(ϑ) $\forall F(s_0 F \equiv \exists q(q \,\&\, F = [\lambda y\, q]))$

In the proof of (421), we showed, among other things:

(ξ) $PossibleWorld(s_0)$

With these constructions in hand, we now use conditional proof to establish the present theorem. Assume that $\neg\Diamond p$. Then consider the p-extension of s_0, i.e., ${s_0}^{+p}$, as this is defined in (466). Since ${s_0}^{+p}$ is a situation, to show $\exists i(\neg TrivialSituation(i) \,\&\, i \models p)$, it suffices by $\exists$I to show:

(a) $Maximal({s_0}^{+p})$

(b) ${s_0}^{+p} \models p$

(c) $\neg Possible({s_0}^{+p})$

(d) $\neg Trivial({s_0}^{+p})$

(a) Since s_0 is, by (ξ), a possible world, we know that it is maximal (408). But by (467.3), we know that every proposition true at s_0 is true at ${s_0}^{+p}$. Hence ${s_0}^{+p}$ is maximal, by disjunctive syllogism from the maximality of s_0.

(b) By (467.4), we know ${s_0}^{+p} \models p$.

(c) It follows from our initial assumption that $\Box\neg p$. But as an instance of fundamental theorem (433.2), we know $\Box\neg p \equiv \forall w(w \models \neg p)$. Hence, $\forall w(w \models \neg p)$. But by ($\xi$), s_0 is a possible world. Hence $s_0 \models \neg p$. So by (467.3), ${s_0}^{+p} \models \neg p$. But by (b), ${s_0}^{+p} \models p$. Hence, by definition, $\neg Consistent({s_0}^{+p})$, and so by (398.1), $\neg Possible({s_0}^{+p})$.

(d) To show that $\neg Trivial({s_0}^{+p})$, we have to show $\exists q \neg({s_0}^{+p} \models q)$, i.e., find a proposition that isn't true at ${s_0}^{+p}$. The witness can't be a true proposition, since ${s_0}^{+p}$ will encode it given that s_0 does (467.3). Nor can the witness be an arbitrary falsehood, since the arbitrary choice might be p itself, which is true in ${s_0}^{+p}$ despite being necessarily false. The witness can't be some necessary falsehood distinct from p because we haven't proved that there is such.[289] But if our witness is a contingent falsehood, then by (150.6), it is distinct from p and we can show that it fails to be true in ${s_0}^{+p}$, as follows.

By (150.2), $ContingentlyFalse(q_0) \vee ContingentlyFalse(\overline{q}_0)$, where q_0 is defined as $\exists x(E!x \,\&\, \Diamond\neg E!x)$. We may reason by cases to the conclusion $\exists q \neg({s_0}^{+p} \models q)$ as follows:

[289] Though our system allows us to assert, without contradiction, that there are distinct necessary falsehoods, our axioms thus far don't guarantee that there are such. We can't prove, for example, that when q is a necessary falsehood, $q \neq (q \,\&\, \neg q)$. See the discussion in footnote 203.

- Suppose *ContingentlyFalse*(q_0). Now before we show $\neg s_0{}^{+p} \models q_0$, note that in the discussion following (467.1) we showed generally that s^{+p} is strictly canonical. So $s_0{}^{+p}$ is. Hence, it follows from definition (466) and theorem (189) that:

  (ζ) $\forall F(s_0{}^{+p}F \equiv s_0F \vee F = [\lambda y\, p])$

  Now to show $\neg s_0{}^{+p} \models q_0$, we have to show $\neg s_0{}^{+p}[\lambda y\, q_0]$. So by ($\zeta$), we have to show that neither $s_0[\lambda y\, q_0]$ nor $[\lambda y\, q_0] = [\lambda y\, p]$. Since by assumption q_0 is contingently false, it is false (148.2), and so by (ϑ), it follows that $\neg s_0[\lambda y\, q_0]$, by now familiar reasoning. Moreover, since q_0 is contingently false and p is necessarily false (i.e., impossible), it follows from (150.6) that $q_0 \neq p$. Hence, by definition of proposition identity, $[\lambda y\, q_0] \neq [\lambda y\, p]$. Since we've shown $\neg s_0{}^{+p} \models q_0$, it follows that $\exists q \neg(s_0{}^{+p} \models q)$.

- Suppose *ContingentlyFalse*$(\overline{q}_0)$. Then the reasoning is analogous to the previous case.

Hence $\neg$*Trivial*$(s_0{}^{+p})$. ⋈

(469.1) In the proof of (464), we established, by comprehension for situations, that there is a situation in which all and only falsehoods are true. We supposed that s_1 is an arbitrary such situation and then showed that s_1 is an impossible world. Now recall that we've defined p_0 as the necessary truth $\forall x(E!x \rightarrow E!x)$. We know that $p_0 \,\&\, \neg p_0$ is a falsehood and so true in s_1. But since p_0 is a truth, it fails to be encoded by s_1. Hence there is an impossible world i and propositions p (= p_0) and q (= p_0) such that $p \,\&\, \neg p$ is true at i but q fails to be true at i. ⋈

(469.2) Again let p_0 be the proposition $\forall x(E!x \rightarrow E!x)$ and consider $\overline{p_0}$, which by (137.7) is identical to $\neg\forall x(E!x \rightarrow E!x)$. Clearly, $\neg\Diamond\overline{p_0}$. Now recall that we began the proof of the Fundamental Theorem of Impossible Worlds (468) by establishing that there is a (non-trivial) situation in which every true proposition is true and we let s_0 be an arbitrary such situation. Consider the $\overline{p_0}$-extension of s_0, namely, $s_0{}^{+\overline{p_0}}$. By reasoning analogous to the proof of (468), it follows that:

- *ImpossibleWorld*$(s_0{}^{+\overline{p_0}})$
- $s_0{}^{+\overline{p_0}} \models \overline{p_0}$
- $s_0{}^{+\overline{p_0}} \models \neg\overline{p_0}$ (since $\neg\overline{p_0}$ is true)

Moreover, by reasoning analogous to part (d) of the proof of (468), there is a contingently false proposition that fails to be true at $s_0{}^{+\overline{p_0}}$. Hence, there is an impossible world i and there are propositions p and q such that both p and $\neg p$ are true at i and q fails to be true at i. ⋈

(470) In the proofs of (468) and (469.2), we established that there is a situation, s_0, such that:

(ϑ) $\forall F(s_0 F \equiv \exists p(p \,\&\, F = [\lambda y\ p]))$

Where p_0 is $\forall x(E!x \to E!x)$ and $\overline{p_0}$ is the negation of p_0 as defined in (136.2), we established in that same proof that ${s_0}^{+\overline{p_0}}$ obeys the principle:

(ξ) $\forall F({s_0}^{+\overline{p_0}} F \equiv s_0 F \vee F = [\lambda y\ \overline{p_0}])$

We further showed in (469.2) that ${s_0}^{+\overline{p_0}}$ is an impossible world. Finally, by theorem (150.4), there is a contingently false proposition, say r_0. With these facts in hand, it suffices by $\exists$I to prove the following to establish our theorem:

(a) ${s_0}^{+\overline{p_0}} \models (p_0 \vee r_0)$

(b) ${s_0}^{+\overline{p_0}} \models \neg p_0$

(c) $\neg {s_0}^{+\overline{p_0}} \models r_0$

To prove these claims, we shall, on occasion, rehearse some of the steps in the proof of (468).

(a) Since s_0 encodes all and only true propositions, we know *PossibleWorld*(s_0); a proof was given in (468), which in turn cited (421). Since p_0 is a necessary truth, it follows by a Fundamental Theorem of Possible World Theory (433.2) that $\forall w(w \models p_0)$. Hence $s_0 \models p_0$. Since possible worlds are modally closed (419), it follows that $s_0 \models (p_0 \vee r_0)$. But by (467.3), we know that every proposition true in s_0 is true in ${s_0}^{+\overline{p_0}}$. Hence ${s_0}^{+\overline{p_0}} \models (p_0 \vee r_0)$.

(b) As as instance of (469.3), we know ${s_0}^{+\overline{p_0}} \models \overline{p_0}$. But by theorem (137.7), $\overline{p_0} = \neg p_0$. Hence ${s_0}^{+\overline{p_0}} \models \neg p_0$.

(c) By reasoning analogous to part (d) of the proof of (468) – the cases are analogous because in part (d) of the proof of (468), the proposition in question, p, was necessarily false by hypothesis. In the present case $\overline{p_0}$ is provably necessarily false. ⋈

(473.1) Recall that axiom (32.4) asserts $\Diamond q_0 \,\&\, \Diamond\neg q_0$, where q_0 is the proposition $\exists x(E!x \,\&\, \Diamond\neg E!x)$. Note that $\neg\Diamond\Box q_0$. For suppose otherwise, i.e., that $\Diamond\Box q_0$. Then by 5$\Diamond$ (119.1), it follows $\Box q_0$, i.e., $\neg\Diamond\neg q_0$, which contradicts the second conjunct of axiom (32.4). So $\Box q_0$ is a necessary falsehood.

Now it follows *a fortiori* from (32.4) that $\Diamond q_0$. So by the Fundamental Theorem of Possible World Theory (433.1), it follows that $\exists w(w \models q_0)$. Suppose w_1 is an arbitrary such possible world, so that we know $w_1 \models q_0$.

Composing definitions (472.1) and (472.2), consider:

$({w_1}^{-\Rightarrow\Diamond\neg q_0})^{+\overline{\Rightarrow\Diamond\neg q_0}}$

Call this situation s_1. s_1 is the result of replacing the necessary consequences of $\Diamond\neg q_0$ in w_1 with their negations. We now work our way towards a general principle that states what s_1 encodes. Note that since every term of the form $s^{+\overline{\Rightarrow p}}$ denotes a strictly canonical situation, we know by definition (472.2) and theorem (189.2) that:

> s_1 encodes a property F just in case either $w_1{}^{-\Rightarrow\Diamond\neg q_0}$ encodes F or F is constructed from the negation of a proposition p necessarily implied by $\Diamond\neg q_0$, i.e.,

(ϑ) $s_1F \;\equiv\; w_1{}^{-\Rightarrow\Diamond\neg q_0}F \vee \exists p((\Diamond\neg q_0 \Rightarrow p)\,\&\, F = [\lambda y\, \neg p])$

Independently, since terms of the form $s^{-\Rightarrow p}$ denote strictly canonical situations, we know by definition (472.1) and theorem (189.2) that:

(ζ) $w_1{}^{-\Rightarrow\Diamond\neg q_0}F \;\equiv\; w_1F \,\&\, \neg\exists p((\Diamond\neg q_0 \Rightarrow p)\,\&\, F = [\lambda y\, p]))$

So by the Rule of Substitution, we can substitution the right side of (ζ) for the left where the latter occurs in (ϑ) to obtain the following general principle stating what s_1 encodes:

(ξ) $$s_1F \;\equiv\; (w_1F \,\&\, \neg\exists p((\Diamond\neg q_0 \Rightarrow p)\,\&\, F = [\lambda y\, p])) \vee \exists p((\Diamond\neg q_0 \Rightarrow p)\,\&\, F = [\lambda y\, \neg p])$$

Now to establish our theorem, it suffices by the definition of impossible world, &I and ∃I, to show:

(a) $\Box q_0$, which is known to be a necessary falsehood, is true at s_1,

(b) s_1 is maximal,

(c) s_1 is not possible,

(d) s_1 is consistent, and

(e) s_1 is consistent*.

We show (a) – (e) by reasoning from (ξ):

(a) Since $\Diamond\neg q_0$ is a necessary consequence of itself, when $F = [\lambda y\, \neg\Diamond\neg q_0]$, the second disjunct of the disjunction on the right side of (ξ) is true. Hence it follows from (ξ) that s_1 encodes $[\lambda y\, \neg\Diamond\neg q_0]$, i.e., $s_1 \models \neg\Diamond\neg q_0$, i.e., $s_1 \models \Box q_0$.

(b) To see that s_1 is maximal, note that since w_1 is a possible world, it is maximal, i.e., for any proposition p, either $w_1 \models p$ or $w_1 \models \neg p$. But s_1 is obtained from w_1 by replacing deleted propositions (i.e., the necessary consequences of $\Diamond\neg q_0$) with their negations. So the maximality condition is preserved.

(c) To show that s_1 is not possible, note that by (a), we know $s_1 \models \Box q_0$. But previously, we established $\neg\Diamond\Box q_0$. Hence by (396.2), we know $\neg \mathit{Possible}(s_1)$.

(d) To show that s_1 is consistent, suppose, for reductio, that $\neg Consistent(s_1)$. Then by definition (392), there is a proposition, say r_1, such that $s_1 \models r_1$ and $s_1 \models \neg r_1$. Now independently, we know that w_1 is maximal given that it is a possible world; so either $w_1 \models r_1$ or $w_1 \models \neg r_1$. Without loss of generality, suppose $w_1 \models r_1$. Then, since w_1 is also consistent, $\neg w_1 \models \neg r_1$. Since $s_1 \models \neg r_1$, we know $s_1[\lambda y\, \neg r_1]$. Hence from ($\xi$), it follows that:

$$(w_1[\lambda y\, \neg r_1] \;\&\; \neg\exists p((\Diamond\neg q_0 \Rightarrow p) \;\&\; [\lambda y\, \neg r_1] = [\lambda y\, p])) \vee \\ \exists p((\Diamond\neg q_0 \Rightarrow p) \;\&\; [\lambda y\, \neg r_1] = [\lambda y\, \neg p])$$

But we know it can't be the left disjunct, because its left conjunct, namely, $w_1[\lambda y\, \neg r_1]$, is known to be false, given that we know $\neg w_1 \models \neg r_1$. So it follows that:

$$\exists p((\Diamond\neg q_0 \Rightarrow p) \;\&\; [\lambda y\, \neg r_1] = [\lambda y\, \neg p])$$

Suppose r_2 is an arbitrary such proposition, so that we know $(\Diamond\neg q_0 \Rightarrow r_2)\,\&\, [\lambda y\, \neg r_1] = [\lambda y\, \neg r_2]$. From the second conjunct, it follows that $\neg r_1 = \neg r_2$, by the definition of proposition identity. By theorem (137.9), which asserts that $p = q \rightarrow \neg p = \neg q$, it follows that $\neg\neg r_1 = \neg\neg r_2$. Now independently, it is easy to establish both $\neg\neg r_1 \Rightarrow r_1$ and $r_2 \Rightarrow \neg\neg r_2$. So from the established fact that $\Diamond\neg q_0 \Rightarrow r_2$ and $r_2 \Rightarrow \neg\neg r_2$, it follows that $\Diamond\neg q_0 \Rightarrow \neg\neg r_2$. From this and the identity of $\neg\neg r_2$ and $\neg\neg r_1$, it follows that $\Diamond\neg q_0 \Rightarrow \neg\neg r_1$. And from this and the fact that $\neg\neg r_1 \Rightarrow r_1$, it follows that $\Diamond\neg q_0 \Rightarrow r_1$, i.e., r_1 is a necessary consequence of $\Diamond\neg q_0$. But we also know that $w_1 \models \Diamond\neg q_0$, because we introduced w_1 as an arbitrary possible world such that $w_1 \models \neg q_0$ and possible worlds are closed under necessary consequences. But then since we've established that r_1 is a necessary consequence of $\Diamond\neg q_0$, it similarly follows that that $w_1 \models \neg r_1$, which is a contradiction.

(e) To show that s_1 is consistent*, suppose, for reductio, that $\neg Consistent^*(s_1)$. So for some proposition, say r_1, $s_1 \models (r_1 \,\&\, \neg r_1)$. Hence $s_1[\lambda y\, (r_1 \,\&\, \neg r_1)]$. So by ($\xi$):

$$(\omega)\quad (w_1[\lambda y\, (r_1 \,\&\, \neg r_1)] \;\&\; \neg\exists p((\Diamond\neg q_0 \Rightarrow p) \;\&\; [\lambda y\, (r_1 \,\&\, \neg r_1)] = [\lambda y\, p])) \vee \\ \exists p((\Diamond\neg q_0 \Rightarrow p) \;\&\; [\lambda y\, (r_1 \,\&\, \neg r_1)] = [\lambda y\, \neg p])$$

Now we know from the Exercise in (404) that possible worlds are consistent*. So w_1 is consistent*, and hence $r_1 \,\&\, \neg r_1$ isn't true at w_1, i.e., $\neg w_1 \models (r_1 \,\&\, \neg r_1)$. So $\neg w_1[\lambda y\, (r_1 \,\&\, \neg r_1)]$. Hence by ($\omega$), it follows that:

$$\exists p((\Diamond\neg q_0 \Rightarrow p) \;\&\; [\lambda y\, (r_1 \,\&\, \neg r_1)] = [\lambda y\, \neg p])$$

Suppose r_2 is an arbitrary such proposition, so that we know $(\Diamond\neg q_0 \Rightarrow r_2)\,\&\, [\lambda y\, (r_1 \,\&\, \neg r_1)] = [\lambda y\, \neg r_2]$. Exercise: Complete the proof by using reasoning analogous to that in (d). ⋈

(473.2) (Exercise)

(473.3) (Exercise)

(478.1) (Exercise)

(478.2) We know that $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$ is logically proper. Hence we may instantiate it into the theorem $\forall z(z = z)$ to obtain:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi)) = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

So by definition (477) and our theory of definition (207.3), we may infer:

$$\imath x(C!x \,\&\, \forall F(xF \equiv \varphi)) = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

⋈

(479.1) – **(479.3)** (Exercises)

(481.1) – **(481.3)** (Exercises)

(483.1) – **(483.3)** (Exercises)

(485.1) By RN and GEN, it suffices to show $(dF \vee eF) \to \Box(dF \vee eF)$. So assume $dF \vee eF$. Since axiom (37) applies to every individual whatsoever, it applies to concepts and so each disjunct of our assumption implies its own necessitation. Hence, by disjunctive syllogism, $\Box dF \vee \Box eF$. By (117.7), it follows that $\Box(dF \vee eF)$. ⋈

(485.2) Let φ be the formula $dF \vee eF$. Then by (485.1), it is a rigid condition on properties. So by (189.2), we know:

$$y = \imath x(A!x \,\&\, \forall F(xF \equiv dF \vee eF)) \to (A!y \,\&\, \forall F(yF \equiv dF \vee eF))$$

From the identity (478.2), it follows by Rule SubId that:

$$y = \imath x(C!x \,\&\, \forall F(xF \equiv dF \vee eF)) \to (A!y \,\&\, \forall F(yF \equiv dF \vee eF))$$

Moreover, by definition of $C!$ (477), it follows from this last result by Rule SubId that:

$$y = \imath x(C!x \,\&\, \forall F(xF \equiv dF \vee eF)) \to (C!y \,\&\, \forall F(yF \equiv dF \vee eF))$$

Furthermore, by our conventions for restricted variables, this previous result can be expressed as:

$$y = \imath c\forall F(cF \equiv dF \vee eF) \to (C!y \,\&\, \forall F(yF \equiv dF \vee eF))$$

But by the definition of $d \oplus e$ (484), we know:

$$d \oplus e = \imath c\forall F(cF \equiv dF \vee eF)$$

Hence $C!d\oplus e \,\&\, \forall F(d\oplus eF \equiv dF \vee eF)$. ⋈

(485.3) (Exercise)

(486.1) Since c is a concept (by hypothesis) and $c\oplus c$ is a concept (by the first conjunct of (485.2)), they are both abstract (477). Hence, by (172.1) and GEN, it suffices to show: $c\oplus cF \equiv cF$. But this follows quickly:

$$\begin{array}{rcll} c\oplus cF & \equiv & cF \vee cF & \text{by (485.2)} \\ & \equiv & cF & \text{by idempotence of } \vee \text{ (63.3.d)} ⋈ \end{array}$$

(486.2) (Exercise)

(486.3) In the following proof, we sometimes assert encoding formulas $\kappa\Pi$ in which κ is either $(c_1\oplus c_2)\oplus c_3$ or $c_1\oplus(c_2\oplus c_3)$ and Π is F.

Let c_1, c_2, and c_3 be any arbitrarily chosen concepts. By now familiar reasoning, it suffices to show $(c_1\oplus c_2)\oplus c_3$ and $c_1\oplus(c_2\oplus c_3)$ encode the same properties.

$$\begin{array}{rcll} (c_1\oplus c_2)\oplus c_3F & \equiv & c_1\oplus c_2F \vee c_3F & \text{by (485.2)} \\ & \equiv & (c_1F \vee c_2F) \vee c_3F & \text{by (485.2), RN, and (112.2)} \\ & \equiv & c_1F \vee (c_2F \vee c_3F) & \text{by associativity of } \vee \text{ (63.3.f)} \\ & \equiv & c_1F \vee c_2\oplus c_3F & \text{by (485.2), RN, and (112.2)} \\ & \equiv & c_1\oplus(c_2\oplus c_3)F & \text{by (485.2), RN, and (112.2)} \quad ⋈ \end{array}$$

(488.1) Assume $c = d$. Since $c\oplus e$ is logically proper, we know by (70.1) that $c\oplus e = c\oplus e$. Hence by Rule SubId, $c\oplus e = d\oplus e$. ⋈

(488.2) Assume $c = d \,\&\, e = f$. Since $c\oplus e$ is a well-defined term, we know by (70.1) that $c\oplus e = c\oplus e$. Hence by Rule SubId and the first conjunct of our assumption, $c\oplus e = d\oplus e$. And by the second conjunct of our assumption and Rule Subid, $c\oplus e = d\oplus f$. ⋈

(490.1) (Exercise)

(490.2) Suppose $c \preceq d$ and $c \neq d$. To show that $d \npreceq c$, we must establish that there is a property that d encodes but which c doesn't encode. Now since c and d are both concepts, they are both abstract objects. Since they are distinct, it follows by (172.1) that either there is a property c encodes that d doesn't, or there is a property d encodes that c doesn't. But, since $c \preceq d$ it follows by definition of $\preceq$ (489.1) that it must be the latter. ⋈

(490.3) (Exercise)

(491.1) (Exercise)

(491.2) ($\rightarrow$) Exercise. ($\leftarrow$) Assume $\forall e(e \preceq c \equiv e \preceq d)$, to show that $c = d$. Since concepts are abstract objects, we have to show that $cF \equiv dF$:

($\rightarrow$) Assume cF, and for reductio, assume $\neg dF$. So $c \not\preceq d$. But it follows from our initial hypothesis that $c \preceq c \equiv c \preceq d$. Since we know $c \preceq c$ by the reflexivity of concept inclusion (490.1), it follows that $c \preceq d$. Contradiction.

($\leftarrow$) By analogous reasoning. ⋈

(491.3) ($\rightarrow$) Exercise. ($\leftarrow$) Assume $\forall e(c \preceq e \equiv d \preceq e)$, to show that $c = d$. By now familiar reasoning, we have to show that $cF \equiv dF$:

($\rightarrow$) Assume cF, and for reductio, assume $\neg dF$. So $c \not\preceq d$. But it follows from our initial hypothesis that $c \preceq d \equiv d \preceq d$. Since we know $d \preceq d$ by the reflexivity of concept inclusion (490.1), it follows that $c \preceq d$. Contradiction.

($\leftarrow$) By analogous reasoning. ⋈

(492.1) – **(492.3)** (Exercises)

(492.4) Note that the following is an immediate consequence of the modally-strict theorem (485.2) about sums:

(ϑ) $c \oplus dF \equiv (cF \vee dF)$

($\rightarrow$) Assume $c \oplus d \preceq e$, to show both (a) $c \preceq e$ and (b) $d \preceq e$. Our assumption, by definition of $\preceq$, becomes:

$\forall F(c \oplus dF \rightarrow eF)$

From this last fact and the modal strictness of (ϑ), it follows by the Rule of Substitution that:

(ξ) $\forall F((cF \vee dF) \rightarrow eF)$

Now to show (a), it suffices by GEN to show: $cG \rightarrow eG$. Assume cG. Then by $\vee$I, $cG \vee dG$. Hence by (ξ), eG. To show (b), use analogous reasoning.

($\leftarrow$) Assume $c \preceq e \,\&\, d \preceq e$. To show $c \oplus d \preceq e$, it suffices by GEN to show: $c \oplus dF \rightarrow eF$. Assume $c \oplus dF$. Then by (ϑ), it follows that $cF \vee dF$. Reasoning by cases: if cF, then by the first conjunct of our assumption, it follows that eF, and if dF, then by the second conjunct of our assumption, it follows that eF. Hence, eF. ⋈

(492.5) (Exercise)

(493) ($\rightarrow$) Assume $c \preceq d$. We prove this direction by cases, with the two cases being: (a) $c = d$ and (b) $c \neq d$. (a) Suppose $c = d$. By the idempotency of $\oplus$, $c \oplus c = c$, in which case, $c \oplus c = d$. Therefore, $\exists e(c \oplus e = d)$. (b) Suppose $c \neq d$.

Then since $c \preceq d$, we know there must be a property encoded by d which is not encoded by c. By Comprehension for Concepts, we know there exists a concept that encodes those properties F that d encodes and c doesn't:

$$\exists c' \forall F(c'F \equiv dF \,\&\, \neg cF)$$

Let c_1 be an arbitrary such concept, so that we know:

(ϑ) $\forall F(c_1 F \equiv dF \,\&\, \neg cF)$

To complete the proof of (b), it suffices by $\exists$I to show $c \oplus c_1 = d$, i.e., that $c \oplus c_1$ and d encode the same properties:

($\rightarrow$) Assume $c \oplus c_1 G$ (to show: dG). By (485.2), it follows that $cG \vee c_1 G$. If cG, then by the fact that $c \preceq d$, it follows that dG. On the other hand, if $c_1 G$, then by (ϑ), it follows that $dG \,\&\, \neg cG$, and hence dG. So we've established dG by cases.

($\leftarrow$) Assume dG (to show $c \oplus c_1 G$). This time our proof by cases begins from cG or $\neg cG$. If cG, then $cG \vee c_1 G$, so by (485.2), $c \oplus c_1 G$. Alternatively, if $\neg cG$, then we have $dG \,\&\, \neg cG$. So by (ϑ), $c_1 G$, and hence $cG \vee c_1 G$. So by (485.2), it follows that $c \oplus c_1 G$.

($\leftarrow$) Assume $\exists e(c \oplus e = d)$. Let c_2 be an arbitrary such concept, so that we know $c \oplus c_2 = d$. To show $c \preceq d$, assume cG (to show dG). Then, $cG \vee c_2 G$, which by (485.2) entails that $c \oplus c_2 G$. But by hypothesis, $c \oplus c_1 = d$. So dG. $\bowtie$

(494) ($\rightarrow$) Assume $c \preceq d$. So $\forall F(cF \rightarrow dF)$. To show that $c \oplus d = d$, it suffices, by now familiar reasoning, to show that $c \oplus d$ and d encode the same properties:

($\rightarrow$) So assume $c \oplus dG$. Then, by (485.2), $cG \vee dG$. To show dG, it suffices by disjunctive syllogism to show only that $cG \rightarrow dG$. But if cG, then by the fact that $c \preceq d$, it follows that dG.

($\leftarrow$) Assume dG. Then $cG \vee dG$. So by (485.2), $c \oplus dG$.

($\leftarrow$) Assume that $c \oplus d = d$. It follows that $\exists e(c \oplus e = d)$. So by (493), $c \preceq d$. $\bowtie$

(495.1) Assume $(c \npreceq d) \,\&\, (d \npreceq c))$. Now let e_1 be the concept $c \oplus d$. By &I and $\exists$I, it suffices to show $e_1 \neq c$, $e_1 \neq d$, and $c \oplus e_1 = c \oplus d$. We leave it as an exercise to show that $e_1 \neq c$ and $e_1 \neq d$. To show $c \oplus e_1 = c \oplus d$, we may reason as follows:

$c \oplus e_1$	$=$	$c \oplus e_1$	Rule ReflId
	$=$	$c \oplus (c \oplus d)$	by Rule SubId, $e_1 = c \oplus d$
	$=$	$(c \oplus c) \oplus d$	by associativity (486.3)
	$=$	$c \oplus d$	by idempotence (486.1) $\bowtie$

(495.2) ($\rightarrow$) Assume $c \preceq d$ and $d \npreceq c$. Now define a concept as follows:

$e_1 = \imath c'(c'F \equiv dF \,\&\, \neg cF)$

We leave it as an exercise to show that this is strictly canonical. Hence, by (189.2), it follows that:

(ϑ) $e_1F \equiv dF \,\&\, \neg cF$

Now by &I and $\exists$I, it remains only to show (a) $e_1 \not\preceq c$, and (b) $c \oplus e_1 = d$:

(a) Since $d \not\preceq c$ by assumption, $\exists F(dF \,\&\, \neg cF)$. Let P be an arbitrary such property, so that we know $dP \,\&\, \neg cP$. Then by (ϑ), it follows that e_1P. Hence, we've established $e_1P \,\&\, \neg cP$. So $e_1 \not\preceq c$.

(b) Since concepts are abstract objects, it suffices by (172.1) and GEN to show $c \oplus e_1F \equiv dF$. ($\rightarrow$) Assume $c \oplus e_1F$. Then $cF \vee e_1F$. So we may reason by cases. If cF, then since $c \preceq d$, dF. If e_1F, then by (ϑ), it follows that $dF \,\&\, \neg cF$. *A fortiori*, dF. ($\leftarrow$) Assume dF. Now we reason by cases from the tautology $cF \vee \neg cF$. If cF, then $cF \vee e_1F$, and hence $c \oplus e_1F$. If $\neg cF$, then $dF \,\&\, \neg cF$, and hence e_1F. Therefore, $cF \vee e_1F$, and so $c \oplus e_1F$.

($\leftarrow$) Assume $\exists e(e \not\preceq c \,\&\, c \oplus e = d)$. Let e_2 be such a concept, so that we know:

(ξ) $e_2 \not\preceq c \,\&\, c \oplus e_2 = d$

The right conjunct of (ξ) implies $\exists e(c \oplus e = d)$. So by (493), $c \preceq d$. Hence it remains to show $d \not\preceq c$, i.e., that $\exists F(dF \,\&\, \neg cF)$. Note that the first conjunct of (ξ) implies $\exists F(e_2F \,\&\, \neg cF)$. Suppose Q is such a property, so that we know both e_2Q and $\neg cQ$. From the former, it follows that $c \oplus e_2Q$, by the second conjunct of (485.2). Hence, by the second conjunct of (ξ) and Rule SubId, it follows that dQ. So we have established $dQ \,\&\, \neg cQ$. Hence $\exists F(dF \,\&\, \neg cF)$, which is what we had to show. ⋈

(497.1) – **(497.3)** (Exercises)

(499.1) – **(499.3)** (Exercises)

(500.1) – **(500.3)** (Exercises)

(501.1) It suffices to show $c \oplus (c \otimes d)$ encodes a property F if and only if c encodes F. ($\rightarrow$) Assume $c \oplus (c \otimes d)$ encodes F. Then by (485.2), we know either c encodes F or $c \otimes d$ encodes F. Reasoning by cases, if c encodes F, we're done. If $c \otimes d$ encodes F then by (499.2), both c encodes F and d encodes F. So, again, we're done. ($\rightarrow$) Assume c encodes F. Then either c encodes F or $c \otimes d$ encodes F. Hence, by (485.2), $c \oplus (c \otimes d)$ encodes F. ⋈

(501.2) (Exercise)

(502.1) It suffices to show that $c \oplus \boldsymbol{a}_\varnothing$ encodes F if and only if c encodes F. ($\rightarrow$) Assume $c \oplus \boldsymbol{a}_\varnothing$ encodes F. Then, by theorem (485.2), either c encodes F

or $\boldsymbol{a}_\varnothing$ encodes F. But by theorem (194.3) and definition (191.1), we know $\boldsymbol{a}_\varnothing$ doesn't encode any properties. Hence c encodes F, by disjunctive syllogism. ($\rightarrow$) Assume c encodes F. Then either c encodes F or $\boldsymbol{a}_\varnothing$ encodes F. Hence, by theorem (485.2), $c \oplus \boldsymbol{a}_\varnothing$ encodes F. ⋈

(502.2) It suffices to show that $c \otimes \boldsymbol{a_V}$ encodes F if and only if c encodes F. ($\rightarrow$) Assume $c \otimes \boldsymbol{a_V}$ encodes F. Then, by theorem (499.2), both c encodes F and $\boldsymbol{a_V}$ encodes F. But then c encodes F, and we're done. ($\rightarrow$) Assume c encodes F. But by theorem (194.4) and definition (191.2), we know $\boldsymbol{a_V}$ encodes every property. So $\boldsymbol{a_V}$ encodes F. Hence both c encodes F and $\boldsymbol{a_V}$ encodes F. So by (499.2), $c \otimes \boldsymbol{a_V}$ encodes F. ⋈

(502.3) (Exercise)

(502.4) (Exercise)

(504.1) It suffices to show that $c \oplus (d \otimes e)$ encodes F if and only if $(c \oplus d) \otimes (c \oplus e)$ encodes F:

$$\begin{array}{rcll}
c\oplus(d\otimes e)F & \equiv & cF \vee d\otimes eF & \text{by (485.2)} \\
 & \equiv & cF \vee (dF \,\&\, eF) & \text{by (499.2), RN, and (112.2)} \\
 & \equiv & (cF \vee dF) \,\&\, (cF \vee eF) & \text{by (63.7.b)} \\
 & \equiv & (c\oplus dF) \,\&\, (c\oplus eF) & \text{by (485.2), RN, and (112.2)} \\
 & \equiv & (c\oplus d)\otimes(c\oplus e)F & \text{by (499.2), RN, and (112.2)} \qquad \bowtie
\end{array}$$

(504.2) (Exercise)

(506.1) – **(506.3)** (Exercises)

(508.1) By RN and GEN, it suffices to show $\neg dF \rightarrow \Box\neg dF$. But as an instance of theorem (126.7), we know $\neg dF \equiv \Box\neg dF$. *A fortiori*, $\neg dF \rightarrow \Box\neg dF$. ⋈

(508.2) (Exercise)

(508.3) (Exercise)

(509.1) Theorem (194.4), is that *Universal*$(\boldsymbol{a_V})$. By definition (191.2), it follows that $\forall F(\boldsymbol{a_V}F)$. So to show $c \oplus -c$ and $\boldsymbol{a_V}$ are identical, i.e., encode the same properties, it suffices to show that $\forall F(c \oplus -cF)$. Now it is a tautology that $cF \vee \neg cF$. Since $\neg cF \equiv -cF$ is a consequence of (508.3) by the commutativity of $\equiv$, it follows that $cF \vee -cF$. But from this it follows by (485.2) that $c\oplus -cF$. Since we've proved this result without any assumptions, it follows by GEN that $\forall F(c\oplus -cF)$. ⋈

(509.2) (Exercise)

(510.1) It suffices to show $-(-c)$ and c encode the same properties. Before we prove this, note that as an instance of (508.3), we know:

$$-cF \equiv \neg cF$$

Hence it follows that:

(ϑ) $\neg{-}cF \equiv \neg\neg cF$

So we may prove our theorem as follows:

$$\begin{aligned} -(-c)F &\equiv \neg{-}cF && \text{by (508.3), } -c \text{ substituted for } d \\ &\equiv \neg\neg cF && \text{by } (\vartheta) \\ &\equiv cF && \text{by } \neg\neg\text{E (64.8)} \end{aligned}$$ ⋈

(510.2) It suffices to show $-c \oplus -d$ encodes F if and only if $-(c \otimes d)$ encodes F:

$$\begin{aligned} -c \oplus -dF &\equiv -cF \vee -dF && \text{by (485.2)} \\ &\equiv \neg cF \vee -dF && \text{by (508.3), RN, (112.2)} \\ &\equiv \neg cF \vee \neg dF && \text{by (508.3), RN, (112.2)} \\ &\equiv \neg(cF \;\&\; dF) && \text{by (63.6.c)} \\ &\equiv \neg c{\otimes}dF && \text{by (499.2), RN, (112.2)} \\ &\equiv -(c \otimes d)F && \text{by (508.3)} \end{aligned}$$ ⋈

(510.3) (Exercise)

(515.1) Assume for reductio that $c \prec c$. Then by definition of $\prec$ (514), it follows that $c \preceq c$ and $c \neq c$. But by Rule ReflId, $c = c$. Contradiction. ⋈

(515.2) Assume $c \prec d$ and $d \prec e$. By (514), it follows from these two assumptions, respectively, that:

(ϑ) $c \preceq d \;\&\; c \neq d$

(ξ) $d \preceq e \;\&\; d \neq e$

Now to show $c \prec e$, we have to show $c \preceq e \& c \neq e$, by (514). The first conjuncts of (ϑ) and (ξ) jointly imply $c \preceq e$, by the transitivity of $\preceq$ (490.3). So it remains to show $c \neq e$. Assume, for reductio, that $c = e$. Then substituting e for c into the first conjunct of (ϑ), it follows that $e \preceq d$. But from this and the first conjunct of (ξ), it follows that $e = d$, by the right-to-left direction of (491.1). But by the symmetry of identity, this contradicts the second conjunct of (ξ). ⋈

(515.3) Assume $c \prec d$. For reductio, assume $d \prec c$. Then by (515.2), $c \prec c$, contradicting (515.1). ⋈

(517.1) By definition (516.1), we have to show $\forall d(\boldsymbol{a}_\varnothing \preceq d)$. It suffices, by GEN, to show $\boldsymbol{a}_\varnothing \preceq d$, and so by definition (489.1), we must show $\forall F(\boldsymbol{a}_\varnothing F \rightarrow dF)$. Again, by GEN, we show $\boldsymbol{a}_\varnothing F \rightarrow dF$. But independently, by theorem (194.3) and definition (191.1), we know that $\forall F(\neg \boldsymbol{a}_\varnothing F)$. Instantiating to F, it follows that $\neg \boldsymbol{a}_\varnothing F$. Hence $\boldsymbol{a}_\varnothing F \rightarrow dF$, by failure of the antecedent. ⋈

(517.2) Since we know $Bottom(\boldsymbol{a}_\varnothing)$, it remains only to show $\forall c(Bottom(c) \rightarrow c = \boldsymbol{a}_\varnothing)$, since our theorem will then follow by &I, ∃I, and the definition of the unique existence quantifier. By GEN, it suffices to show $Bottom(c) \rightarrow c = \boldsymbol{a}_\varnothing$, so assume $Bottom(c)$. Since c and $\boldsymbol{a}_\varnothing$ are both abstract, we must show $\forall F(cF \equiv \boldsymbol{a}_\varnothing F)$. Now, in the proof of the previous theorem, we saw $\forall F(\neg \boldsymbol{a}_\varnothing F)$, i.e., $\neg\exists F(\boldsymbol{a}_\varnothing F)$. So by (86.12), we need only show $\neg\exists FcF$. For reductio, assume $\exists FcF$. Let P be an arbitrary such property, so that we know cP. But c is, by hypothesis, a bottom concept, and so $\forall d(c \preceq d)$. Since $\boldsymbol{a}_\varnothing$ is a concept, it follows that $c \preceq \boldsymbol{a}_\varnothing$. Hence, $\forall F(cF \rightarrow \boldsymbol{a}_\varnothing F)$. So $\boldsymbol{a}_\varnothing P$, given that cP, and thus $\exists F\boldsymbol{a}_\varnothing F$. Contradiction. ⋈

(517.3) (Exercise)

(517.4) Since the previous theorem is that $\boldsymbol{a}_\varnothing \prec \boldsymbol{a}_G$, the present theorem will follow by &I, ∃I, and the definition of the unique existence quantifier if we can establish that $\forall c(c \prec \boldsymbol{a}_G \rightarrow c = \boldsymbol{a}_\varnothing)$. So, by GEN, assume $c \prec \boldsymbol{a}_G$ and, for reductio, suppose $c \neq \boldsymbol{a}_\varnothing$. Since $\boldsymbol{a}_\varnothing$ encodes no properties, the non-identity implies $\exists F(cF \;\&\; \neg\boldsymbol{a}_\varnothing F)$. Suppose P is an arbitrary such property, so that we know cP and $\neg\boldsymbol{a}_\varnothing P$. Since $c \prec \boldsymbol{a}_G$ by assumption, it follows by definition of $\prec$ (514) both that:

(ϑ) $c \preceq \boldsymbol{a}_G$

(ξ) $c \neq \boldsymbol{a}_G$

From (ϑ) and cP, it follows that $\boldsymbol{a}_G P$, and so $P = G$, by the right conjunct of (323.1). Hence,

(ζ) cG

But (ϑ) and (ξ) jointly imply that there is a property that $\boldsymbol{a}_G$ encodes that c fails to encode. Suppose Q is such that (i) $\boldsymbol{a}_G Q$ and (ii) $\neg cQ$. Then from (i) and the right conjunct of (323.1), we know $Q = G$, and so from (ii), it follows that $\neg cG$, which contradicts (ζ). ⋈

(517.5) Assume $Bottom(c)$. So by definition (516.1),

(ϑ) $\forall d(c \preceq d)$

To show $Atom(c)$, we have to show $\neg\exists d(d \prec c)$, by (516.2). For reductio, assume $\exists d(d \prec c)$. Suppose d_1 is an arbitrary such concept, so that we know $d_1 \prec c$. Then by definition of $\prec$ (514), it follows that (a) $d_1 \preceq c$ and (b) $d_1 \neq c$. But if we instantiate d_1 into (ϑ), it follows that $c \preceq d_1$. But from (a) and this last fact, it follows by (491.1) that $d_1 = c$, contradicting (b). ⋈

(517.6) (Exercise)

(517.7) Since we've established that $Atom(\boldsymbol{a}_\varnothing)$, it suffices, by &I, ∃I, and the definition of the unique existence quantifier, to show $\forall c(Atom(c) \rightarrow c = \boldsymbol{a}_\varnothing)$. By GEN, it suffices to show $Atom(c) \rightarrow c = \boldsymbol{a}_\varnothing$. So assume $Atom(c)$ and, for reductio, assume $c \neq \boldsymbol{a}_\varnothing$. From the atomicity of c, it follows that:

(ϑ) $\neg\exists d(d \prec c)$

Now $\boldsymbol{a}_\varnothing$ is a bottom concept (517.1), and so $\forall d(\boldsymbol{a}_\varnothing \preceq d)$, by definition (516.1). In particular, $\boldsymbol{a}_\varnothing \preceq c$. But from our reductio assumption, we also know $\boldsymbol{a}_\varnothing \neq c$, by symmetry of identity. Hence, $\boldsymbol{a}_\varnothing \prec c$, and so $\exists d(d \prec c)$, which contradicts (ϑ). ⋈

(519.1) Assume $Overlap(c,d)$. Then by (518.1), $\exists F(cF \,\&\, dF)$. Assume P is such a property, so that we know $cP \,\&\, dP$. Now to show $Overlap^*(c,d)$, we have to show $\exists e(e \preceq c \,\&\, e \preceq d)$. So consider the Thin Form of P, namely, $\boldsymbol{a}_P$. If we can show both $\boldsymbol{a}_P \preceq c$ and $\boldsymbol{a}_P \preceq d$, then by &I and ∃I, we're done. To show that $\boldsymbol{a}_P \preceq c$, we have to show $\forall F(\boldsymbol{a}_P F \rightarrow cF)$, and by GEN, $\boldsymbol{a}_P F \rightarrow cF$. So assume $\boldsymbol{a}_P F$. Then by the second conjunct of (323.1), it follows that $F = P$. Since cP, it follows that cF. By analogous reasoning, $\boldsymbol{a}_P \preceq d$. Conjoining our results and quantifying, it follows that ⋈

(519.2) Choose the witnesses for c and d, respectively, to be the null concept, $\boldsymbol{a}_\varnothing$, and the Thin Form of P, $\boldsymbol{a}_P$, where P is any arbitrarily chosen property. So by ∃I, we have to show both (a) $Overlap^*(\boldsymbol{a}_\varnothing, \boldsymbol{a}_P)$ and (b) $\neg Overlap(\boldsymbol{a}_\varnothing, \boldsymbol{a}_P)$. (a) Since we know by (517.1) that $Bottom(\boldsymbol{a}_\varnothing)$, it follows by definition that $\forall d(\boldsymbol{a}_\varnothing \preceq d)$ and so, in particular, that:

(ϑ) $\boldsymbol{a}_\varnothing \preceq \boldsymbol{a}_\varnothing$

We also know, indepdently by (194.3) and (191.1), that $\neg\exists F \boldsymbol{a}_\varnothing F$. So by failure of the antecedent, it follows that $\boldsymbol{a}_\varnothing F \rightarrow \boldsymbol{a}_P F$. So by GEN, $\forall F(\boldsymbol{a}_\varnothing F \rightarrow \boldsymbol{a}_P F)$. Hence, by definition:

(ξ) $\boldsymbol{a}_\varnothing \preceq \boldsymbol{a}_P$

So if we conjoin our (ϑ) and (ξ) and quantify, we have established $\exists e(e \preceq \boldsymbol{a}_\varnothing \,\&\, e \preceq \boldsymbol{a}_P)$, i.e., $Overlap^*(\boldsymbol{a}_\varnothing, \boldsymbol{a}_P)$. (b) We've already established $\neg\exists F \boldsymbol{a}_\varnothing F$. Hence, $\neg\exists F(\boldsymbol{a}_\varnothing F \,\&\, \boldsymbol{a}_P F)$. So $\neg Overlap(\boldsymbol{a}_\varnothing, \boldsymbol{a}_P)$. ⋈

(519.3) – **(519.7)** (Exercises)

(519.8) Since $Bottom(\boldsymbol{a}_\varnothing)$ by (517.1), it follows by definition (516.1) that $\boldsymbol{a}_\varnothing$ is a part of every concept. Hence, we know both $\boldsymbol{a}_\varnothing \preceq c$ and $\boldsymbol{a}_\varnothing \preceq d$, for any concepts c and d. So, $\exists e(e \preceq c \& e \preceq d)$. Thus, $Overlap^*(c,d)$, by definition (518.2). ⋈

(519.9) We may prove our theorem in one of two ways, namely, by showing that either $d \ominus c$ or $\boldsymbol{a}_\varnothing$ may serve as the witness. Recall that $d \ominus c$ was defined in

Exercise (511.5) as $\imath e\mathit{DifferenceOf}(e,d,c)$, where $\mathit{DifferenceOf}(e,d,c)$ was defined in (511.1) as $\forall F(eF \equiv dF \mathbin{\&} \neg cF)$. By &I and ∃I, it suffices to show:

(a) $d \ominus c \preceq d$

(b) $\neg \mathit{Overlap}(d \ominus c, c)$

For (a), we need to show $d{\ominus}cF \to dF$. So assume $d{\ominus}cF$. Then by Exercise (511.6), it follows that $dF \mathbin{\&} \neg cF$. So dF. For (b), proceed by reductio. Assume $\mathit{Overlap}(d{\ominus}c,c)$. Then $\exists F(d{\ominus}cF \mathbin{\&} cF)$. Suppose P is an arbitrary such property, so that we know $d{\ominus}cP$ and cP. From the former, it follows by Exercise (511.6) that $dP \mathbin{\&} \neg cP$. So $\neg cP$. Contradiction.

Alternatively, we can show $\boldsymbol{a}_\varnothing$ is available as the witness by showing:

(a) $\boldsymbol{a}_\varnothing \preceq d$

(b) $\neg \mathit{Overlap}(\boldsymbol{a}_\varnothing, c)$

(a) follows immediately from the fact that $\boldsymbol{a}_\varnothing$ is a bottom concept, and so is a part of every concept, by (517.1) and (516.1). For (b), we know that $\neg\exists F \boldsymbol{a}_\varnothing F$. Hence, it follows *a fortiori* that $\neg\exists F(\boldsymbol{a}_\varnothing F \mathbin{\&} cF)$. So $\neg \mathit{Overlap}(\boldsymbol{a}_\varnothing, c)$. ⋈

[Note that if d is a part of c, the $d \ominus c$ is $\boldsymbol{a}_\varnothing$, and so the witnesses in the two proofs are the same.]

(519.10) (Complete the proof sketched in the text.)

(520.1) (Exercise)

(520.2) (Exercise)

(520.3) Assume $c \prec d$. By &I and EI, it suffices to show both (a) $\boldsymbol{a}_\varnothing \prec d$ and (b) $\neg \mathit{Overlap}(\boldsymbol{a}_\varnothing, c)$. (a) By definition, we have to show both $\boldsymbol{a}_\varnothing \preceq d$ and $\boldsymbol{a}_\varnothing \neq d$. The former is a simple consequence of the fact that $\boldsymbol{a}_\varnothing$ is a bottom concept (517.1) and thus by definition (516.1) a part of everthing. For the latter, it suffices to show that d encodes a property, since we know, by theorem (194.3) and definition (191.1), that $\boldsymbol{a}_\varnothing$ doesn't encode any. Note that from our initial hypothesis, it follows that $c \preceq d$ and $c \neq d$. By the former, $\forall F(cF \to dF)$. But by the latter, one of c and d encodes a property the other fails to encode. Hence, we know $\exists F(dF \mathbin{\&} \neg cF)$. *A fortiori*, $\exists F dF$.

(b) By now familiar reasoning, since $\boldsymbol{a}_\varnothing$ encodes no properties, there is no property that both $\boldsymbol{a}_\varnothing$ encodes and c encodes. Hence $\neg \mathit{Overlap}(\boldsymbol{a}_\varnothing, c)$. ⋈

(522.1) – **(522.2)** (Exercises)

(526.1) Clearly, since $\boldsymbol{a}_\varnothing$ fails to be a non-null concept, it fails to be a non-null bottom. ⋈

(526.2) Suppose, for reductio, that $\exists \underline{c}\mathit{Bottom}^+(\underline{c})$. Let $\underline{c}_1$ be such a non-null concept, so that we know $\mathit{Bottom}^+(\underline{c}_1)$, i.e., by (525), that $\forall \underline{d}(\underline{c}_1 \preceq \underline{d})$, i.e., that:

(ϑ) $\forall \underline{d} \forall F(\underline{c}_1 F \rightarrow \underline{d} F)$

Since $\underline{c}_1$ is non-null, $\exists F \underline{c}_1 F$. Suppose $\underline{c}_1 P$. Then let us define the following canonical concept:

(ζ) $\boldsymbol{c}_0 =_{df} \imath c \forall F(cF \equiv F = \overline{P})$

$\boldsymbol{c}_0$ is well-defined by (480.3) and is *strictly canonical* (by the necessity of identity). Since $\overline{P} = \overline{P}$, it follows by modally strict reasoning from (ζ) that $\boldsymbol{c}_0 \overline{P}$. So $\boldsymbol{c}_0$ is non-null. Hence (ϑ) implies $\forall F(\underline{c}_1 F \rightarrow \boldsymbol{c}_0 F)$. So since $\underline{c}_1 P$, it follows that $\boldsymbol{c}_0 P$. But $P \neq \overline{P}$ (137.5), and so it follows by modally strict reasoning from (ζ) that $\neg \boldsymbol{c}_0 P$. Contradiction. ⋈

(528.1) We begin by first proving the following fact about $\boldsymbol{a}_G$, namely, that if a non-null concept is a part of $\boldsymbol{a}_G$, it fails to be a proper part of $\boldsymbol{a}_G$:

(ϑ) $\forall \underline{d}(\underline{d} \preceq \boldsymbol{a}_G \rightarrow \neg \underline{d} \prec \boldsymbol{a}_G)$

> *Proof*. By GEN, we need only show $\underline{d} \preceq \boldsymbol{a}_G \rightarrow \neg \underline{d} \prec \boldsymbol{a}_G)$. So suppose $\underline{d} \preceq \boldsymbol{a}_G$. Then $\forall F(\underline{d} F \rightarrow \boldsymbol{a}_G F)$. Since $\underline{d}$ is non-null, it encodes some property. Suppose P is such a property, so that we know dP. Hence, $\boldsymbol{a}_G P$. So by (323.1), it follows that $P = G$. So $\underline{d} G$. Since (323.1) also implies that $\boldsymbol{a}_G$ encodes only G and no other property, it follows that $\underline{d}$ encodes every property that $\boldsymbol{a}_G$ encodes. Hence $\neg \underline{d} \prec \boldsymbol{a}_G$, for otherwise there would be a property that $\boldsymbol{a}_G$ encodes that $\underline{d}$ doesn't.

Now by definition of *Atom*$^+$, we have to show $\neg \exists \underline{d}(\underline{d} \prec \boldsymbol{a}_G)$. Suppose, for reductio, that $\exists \underline{d}(\underline{d} \prec \boldsymbol{a}_G)$. Let $\underline{d}_1$ be such a non-null concept, so that we know $\underline{d}_1 \prec \boldsymbol{a}_G$. Then the definition of $\prec$ implies $\underline{d}_1 \preceq \boldsymbol{a}_G$. But then by ($\vartheta$), $\neg \underline{d}_1 \prec \boldsymbol{a}_G$. Contradiction. ⋈

(528.2) For reductio, assume $\exists \underline{c} \exists ! \underline{d}(\underline{d} \prec \underline{c})$. Suppose $\underline{c}_1$ is such an non-null concept, so that we know $\exists ! \underline{d}(\underline{d} \prec \underline{c}_1)$. By definition of the unique existence quantifier, it follows that $\exists \underline{d}(\underline{d} \prec \underline{c}_1 \ \& \ \forall \underline{e}(\underline{e} \prec \underline{c}_1 \rightarrow \underline{e} = \underline{d}))$. Suppose $\underline{d}_1$ is such a non-null concept, so that we know:

(ϑ) $\underline{d}_1 \prec \underline{c}_1 \ \& \ \forall \underline{e}(\underline{e} \prec \underline{c}_1 \rightarrow \underline{e} = \underline{d}_1)$

By definition of $\prec$, the first conjunct of (ϑ) implies:

(ξ) $\underline{d}_1 \preceq \underline{c}_1 \ \& \ \underline{d}_1 \neq \underline{c}_1$

By now familiar reasoning, (ξ) implies $\exists F(\underline{c}_1 F \ \& \ \neg \underline{d}_1 F)$. Suppose P is such a property, so that we know both $\underline{c}_1 P$ and $\neg \underline{d}_1 P$. Now consider the Thin Form of P, $\boldsymbol{a}_P$. Clearly, we can conclude:

(ζ) $\boldsymbol{a}_P \neq \underline{d}_1$

given $\boldsymbol{a}_P P$ (330.1) and the recently established fact that $\neg\underline{d}_1 P$. Now if we can establish that $\boldsymbol{a}_P \prec \underline{c}_1$, then we will have reached our contradiction, since this would imply, by the second conjunct of (ϑ), that $\boldsymbol{a}_P = \underline{d}_1$, contradicting ($\zeta$). So it remains only to show $\boldsymbol{a}_P \prec \underline{c}_1$, i.e., (a) $\boldsymbol{a}_P \preceq \underline{c}_1$ and (b) $\boldsymbol{a}_P \neq \underline{c}_1$:

(a) To show $\boldsymbol{a}_P \preceq \underline{c}_1$, we have to show $\forall F(\boldsymbol{a}_P F \rightarrow \underline{c}_1 F)$. So, by GEN, assume $\boldsymbol{a}_P F$. Then by (323.1), $F = P$. But we've established $\underline{c}_1 P$. So $\underline{c}_1 F$.

(b) To show $\boldsymbol{a}_P \neq \underline{c}_1$, assume, for reductio, $\boldsymbol{a}_P = \underline{c}_1$. From this and the first conjunct of (ϑ), it follows that $\underline{d}_1 \prec \boldsymbol{a}_P$. So $\exists \underline{d}(\underline{d} \prec \boldsymbol{a}_P)$, which contradicts the atomicity$^+$ of $\boldsymbol{a}_P$ (528.1), given definition (527). ⋈

(528.3) Assume $Atom^+(\underline{c})$. Then, by definition:

(ϑ) $\neg\exists\underline{d}(\underline{d} \prec \underline{c})$

Assume $\underline{c}F \,\&\, \underline{c}G$ and, for reductio, $F \neq G$. Consider, then, the Thin Form of F, $\boldsymbol{a}_F$. If we can establish both that (a) $\boldsymbol{a}_F \preceq \underline{c}$ and (b) $\boldsymbol{a}_F \neq \underline{c}$, then it would follow that $\boldsymbol{a}_F \prec \underline{c}$, by definition (514), and therefore that $\exists\underline{d}(\underline{d} \prec \underline{c})$ (since $\boldsymbol{a}_F$ is a non-null concept), contradicting (ϑ):

(a) We have to show $\forall H(\boldsymbol{a}_F H \rightarrow \underline{c}H)$. So by GEN, assume $\boldsymbol{a}_F H$. Then since $\boldsymbol{a}_F H \equiv H = F$ (323.1), it follows that $H = F$. Since we know $\underline{c}F$, it follows that $\underline{c}H$.

(b) (323.1) also implies $\boldsymbol{a}_F G \equiv G = F$. So it follows from our assumption that $F \neq G$ that $\neg\boldsymbol{a}_F G$. Since we know $\underline{c}G$, it follows that $\boldsymbol{a}_F \neq \underline{c}$. ⋈

(528.4) (Exercise)

(529) ($\rightarrow$) Assume $Overlap(\underline{c},\underline{d})$, i.e., that $\exists F(\underline{c}F \,\&\, \underline{d}F)$. Suppose P is such a property, so that we know both $\underline{c}P$ and $\underline{d}P$. Then choose our witness for e to be the Thin Form of P, i.e., $\boldsymbol{a}_P$. Since $\boldsymbol{a}_P$ encodes the single property P, it clearly follows that $\boldsymbol{a}_P \preceq \underline{c}$ and $\boldsymbol{a}_P \preceq \underline{d}$. And since $\boldsymbol{a}_P$ encodes at least one property, it is a non-null concept. Consequently, $\exists\underline{e}(\underline{e} \preceq \underline{c} \,\&\, \underline{e} \preceq \underline{d})$. ($\leftarrow$) Assume $\exists\underline{e}(\underline{e} \preceq \underline{c} \,\&\, \underline{e} \preceq \underline{d})$. Let $\underline{e}_1$ be such a concept, so that we know both $\underline{e}_1 \preceq \underline{c}$ and $\underline{e}_1 \preceq \underline{d}$. Since $\underline{e}$ is non-null, $\exists F\underline{e}_1 F$. Suppose P is such a property, so that $\underline{e}_1 P$. Then since both $\underline{e}_1 \preceq \underline{c}$ and $\underline{e}_1 \preceq \underline{d}$, it follows from each, respectively, that $\underline{c}P$ and $\underline{d}P$. So $\exists F(\underline{c}F \,\&\, \underline{d}F)$. ⋈

(531.1) Let P and Q be any two distinct properties and consider the Thin Forms of P and Q, $\boldsymbol{a}_P$ and $\boldsymbol{a}_Q$. Since both encode a property, both are non-null concepts. But there is no property that they both encode. Hence $\neg Overlap(\boldsymbol{a}_P,\boldsymbol{a}_Q)$. So $\exists\underline{c}\exists\underline{d}\neg Overlap(\underline{c},\underline{d})$, i.e., $\neg\forall\underline{c}\forall\underline{d}Overlap(\underline{c},\underline{d})$. ⋈

(531.2) (Exercise)

(531.3) (Exercise)

(531.4) Consider any three distinct properties P, Q, and R (we know there are such by a previous theorem). Now consider the Thin Form of P ($\boldsymbol{a}_P$), the (strictly canonical) concept c_1 that encodes just the two properties P and Q and the (strictly canonical) concept c_2 that encodes just the two properties Q and R, i.e.,

$$c_1 = \imath c\forall F(cF \equiv F = P \vee F = Q)$$

$$c_2 = \imath c\forall F(cF \equiv F = Q \vee F = R)$$

Since $\boldsymbol{a}_P$, c_1, and c_2 are all non-null concepts, it suffices by &I and ∃I to establish:

(a) $Overlap(\boldsymbol{a}_P, c_1)$

(b) $Overlap(c_1, c_2)$

(c) $\neg Overlap(\boldsymbol{a}_P, c_2)$

(a) Since $\boldsymbol{a}_P$ encodes just the property P and c_1 encodes just P and Q, there is a property they both encode, and so $Overlap(\boldsymbol{a}_P, c_1)$. (b) Since c_1 and c_2 both encode Q, $Overlap(c_1, c_2)$. (c) Since $\boldsymbol{a}_P$ encodes just P and c_2 encodes just Q and R, then there is no property they encode in common. Hence $\neg Overlap(\boldsymbol{a}_P, c_2)$. ⋈

(531.5) By applications of quantifier negation theorems (86.3) and (86.4), the present theorem is equivalent to:

$$\exists \underline{c}\exists \underline{d}\forall \underline{e}\neg(\underline{e} \preceq \underline{d} \;\&\; \neg Overlap(\underline{e}, \underline{c}))$$

So by (63.5.a), we have to show:

(ϑ) $\exists \underline{c}\exists \underline{d}\forall \underline{e}(\underline{e} \preceq \underline{d} \rightarrow Overlap(\underline{e}, \underline{c}))$

Let P and Q be any two distinct properties and consider the following strictly canonical concept:

$$c_1 =_{df} \imath c\forall F(cF \equiv F = P \vee F = Q)$$

Since both c_1 and the Thin Form of P, $\boldsymbol{a}_P$, are non-null, we may pick the former as the witness to the existential quantifier $\exists \underline{c}$ in (ϑ) and the latter as the witness to the existential quantifier $\exists \underline{d}$ in (ϑ). So we have to show:

$$\forall \underline{e}(\underline{e} \preceq \boldsymbol{a}_P \rightarrow Overlap(\underline{e}, c_1))$$

By GEN, it suffices to show $\underline{e} \preceq \boldsymbol{a}_P \to Overlap(\underline{e}, c_1)$. So assume $\underline{e} \preceq \boldsymbol{a}_P$. Now to show $Overlap(\underline{e}, c_1)$, we have to show $\exists F(\underline{e}F \,\&\, c_1F)$. By $\exists$I, it suffices to show $\underline{e}P \& c_1P$. But c_1P follows immediately from the definition of c_1. So it remains to show eP. Since $\underline{e}$ is non-null, $\exists F\underline{e}F$. Let R be such a property, so that we know eR. But since $\underline{e}$ is by hypothesis a part of $\boldsymbol{a}_P$, it follows that $\boldsymbol{a}_PR$, by definition of $\preceq$. But then $R = P$ (323.1). So eP. ⋈

(531.6) (Complete the proof sketch in the text.)

(532.1) Assume $\underline{c} \prec \underline{d}$. Then $\underline{c} \preceq \underline{d} \,\&\, \underline{c} \neq \underline{d}$. So $\exists F(\underline{d}F \,\&\, \neg\underline{c}F)$. Let P be such a property, so that we know $\underline{d}P \,\&\, \neg\underline{c}P$. Then consider the Thin Form of P, $\boldsymbol{a}_P$. Since $\boldsymbol{a}_P$ encodes just P and no other properties, every property it encodes is a property $\underline{d}$ encodes. Hence, $\boldsymbol{a}_P \preceq \underline{d}$. Moreover, since $\boldsymbol{a}_P$ encodes just P and $\underline{c}$ doesn't encode P, there is no property that they encode in common. Hence $\neg Overlap(\boldsymbol{a}_P, \underline{c})$. Since $\boldsymbol{a}_P$ is a non-null concept, we may conjoin $\boldsymbol{a}_P \preceq \underline{d}$ and $\neg Overlap(\boldsymbol{a}_P, \underline{c})$ and generalize, to conclude that there is a non-null concept that is both a part of $\underline{d}$ and that doesn't overlap with $\underline{c}$. ⋈

(532.2) Assume $\underline{d} \npreceq \underline{c}$. Then $\exists F(\underline{d}F \,\&\, \neg\underline{c}F)$. The proof now reduces the proof of the previous theorem. ⋈

(532.3) Assume $\underline{c} \prec \underline{d}$. So by definition of $\prec$, we know both:

(ϑ) $\underline{c} \preceq \underline{d}$

(ζ) $\underline{c} \neq \underline{d}$

Now consider $\underline{d} \ominus \underline{c}$. By &I and $\exists$I, it suffices to show:

(a) $\underline{d} \ominus \underline{c}$ is a non-null concept.

(b) $\underline{d} \ominus \underline{c} \prec \underline{d}$

(c) $\neg Overlap(\underline{d} \ominus \underline{c}, \underline{c})$

(a) From (ϑ) and (ζ), it follows that $\exists F(\underline{d}F \& \neg\underline{c}F)$. Suppose P is such a property, so that we know $\underline{d}P \& \neg\underline{c}P$. Then by a fact about $\ominus$ (511.6), it follows that $\underline{d} \ominus \underline{c}P$. Hence, $\exists F(\underline{d} \ominus \underline{c}F)$ and so $\underline{d} \ominus \underline{c}$ is non-null.

(b) To show $\underline{d} \ominus \underline{c} \prec \underline{d}$, we have to show (i) $\underline{d} \ominus \underline{c} \preceq \underline{d}$, and (ii) $\underline{d} \ominus \underline{c} \neq \underline{d}$:

(i) Assume $\underline{d} \ominus \underline{c}F$. Then by (511.6), $\underline{d}F \,\&\, \neg\underline{c}F$. So $\underline{d}F$.

(ii) Since $\underline{c}$ is non-null, it encodes some property. So suppose $\underline{c}Q$. From this it follows that $\underline{d}Q$, by (ϑ), and that $\neg\underline{d} \ominus \underline{c}Q$, by (511.6). Hence $\underline{d} \ominus \underline{c} \neq \underline{d}$.

(c) (511.6) implies $\forall F(\underline{d} \ominus \underline{c}F \equiv \underline{d}F \,\&\, \neg\underline{c}F)$. *A fortiori*, $\forall F(\underline{d} \ominus \underline{c}F \to \underline{d}F \,\&\, \neg\underline{c}F)$. *A fortiori*, $\forall F(\underline{d} \ominus \underline{c}F \to \neg\underline{c}F)$. Hence, $\neg\exists F(\underline{d} \ominus \underline{c}F \,\&\, \underline{c}F)$, i.e., $\neg Overlap(\underline{d} \ominus \underline{c}, \underline{c})$. ⋈

(535.1) – **(535.3)** (Exercises)

(537.1) – (537.2) (Exercises)

(538.1) Let us define:

c_1 to be $\imath c \forall F(cF \equiv G \Rightarrow F \vee H \Rightarrow F)$

Thus, c_1 is a canonical concept. Now let φ be the condition $G \Rightarrow F \vee H \Rightarrow F$. Then we leave it as a simple exercise to show that φ is a rigid condition on properties and, hence, that c_1 is strictly canonical. From this, it follows by theorem (189.2) that:

(ϑ) $\forall K(c_1 K \equiv G \Rightarrow K \vee H \Rightarrow K)$

Now we want to show that $\boldsymbol{c}_G \oplus \boldsymbol{c}_H$ and c_1 encode the same properties. Let P be an arbitrarily chosen property. Then:

$$\begin{array}{lcll} \boldsymbol{c}_G \oplus \boldsymbol{c}_H P & \equiv & \boldsymbol{c}_G P \vee \boldsymbol{c}_H P & \text{by (485.2)} \\ & \equiv & G \Rightarrow P \vee \boldsymbol{c}_H P & \text{by (537.1), RN, and (112.2)} \\ & \equiv & G \Rightarrow P \vee H \Rightarrow P & \text{by (537.1), RN, and (112.2)} \\ & \equiv & c_1 P & \text{by } (\vartheta) \qquad \bowtie \end{array}$$

(538.2) (Exercise)

(539) (Exercise)

(541) (Exercise)

(543.1) – (543.3) (Exercises)

(546.1)★ By (544) and (543.3), $\boldsymbol{c}_u$ is logically proper. So we may instantiate $\boldsymbol{c}_u$ for z in (101.2)★ to produce the following instance:

$\boldsymbol{c}_u = \imath c ConceptOf(c, u) \rightarrow ConceptOf(\boldsymbol{c}_u, u)$

The antecedent holds by definition (544). So $ConceptOf(\boldsymbol{c}_u, u)$, and by definition (542), $\forall F(\boldsymbol{c}_u F \equiv Fu)$. Hence, $\boldsymbol{c}_u G \equiv Gu$. ⋈

(546.2)★ ($\rightarrow$) Assume $\boldsymbol{c}_u G$. By definition of $\succeq$, we need to show $\boldsymbol{c}_G \preceq \boldsymbol{c}_u$, and by definition of $\preceq$, show $\forall F(\boldsymbol{c}_G F \rightarrow \boldsymbol{c}_u F)$. So assume $\boldsymbol{c}_G F$. Then by (537.1), we know $G \Rightarrow F$, i.e., that $\Box\forall x(Gx \rightarrow Fx)$. By the T schema, $\forall x(Gx \rightarrow Fx)$. Now from our initial assumption that $\boldsymbol{c}_u G$, it follows by (546.1)★ that Gu. So from Gu and $\forall x(Gx \rightarrow Fx)$ it follows that Fu. So, again by (546.1)★, it follows that $\boldsymbol{c}_u F$. ($\leftarrow$) Assume $\boldsymbol{c}_u \succeq \boldsymbol{c}_G$, i.e., $\boldsymbol{c}_G \preceq \boldsymbol{c}_u$, i.e., $\forall F(\boldsymbol{c}_G F \rightarrow \boldsymbol{c}_u F)$. Now independently, by (537.2), we know $\boldsymbol{c}_G G$. Hence, $\boldsymbol{c}_u G$. ⋈

(548.1)★ (Exercise)

(548.2)★ By (548.1)★, $Complete(\boldsymbol{c}_u)$. So, by definition, $\forall F(\boldsymbol{c}_u F \vee \boldsymbol{c}_u \overline{F})$, and by $\forall$E, $\boldsymbol{c}_u F \vee \boldsymbol{c}_u \overline{F}$. But each disjunct implies its own necessitation, by axiom (37).

So, by disjunctive syllogism, $\square \boldsymbol{c}_u F \vee \square \boldsymbol{c}_u \overline{F}$. Hence $\square(\boldsymbol{c}_u F \vee \boldsymbol{c}_u \overline{F})$, by (117.7). And by GEN, $\forall F \square(\boldsymbol{c}_u F \vee \boldsymbol{c}_u \overline{F})$. So by BF, $\square \forall F(\boldsymbol{c}_u F \vee \boldsymbol{c}_u \overline{F})$. Hence, by definition, $\square Complete(\boldsymbol{c}_u)$. ⋈

(551.1)★ (Exercise)

(551.2)★ (Exercise)

(551.3)★ $(\rightarrow)$ Assume $\boldsymbol{c}_{\forall G} F$. By the definitions of $\succeq$ and $\preceq$, we have to show $\forall H(\boldsymbol{c}_F H \rightarrow \boldsymbol{c}_{\forall G} F)$. So, by GEN, assume $\boldsymbol{c}_F H$, to show $\boldsymbol{c}_{\forall G} H$. By definition of $\boldsymbol{c}_{\forall G}$ and the Abstraction Principle (184)★ in its form governing concepts (as discussed in footnote 216), we have to show $\forall x(Gx \rightarrow Hx)$. But from $\boldsymbol{c}_F H$, we know by (537.1) that $F \Rightarrow H$, from which it follows by the T schema that $\forall x(Fx \rightarrow Hx)$. But our initial assumption implies, by now familiar reasoning, $\forall x(Gx \rightarrow Fx)$. By predicate logic, it then follows that $\forall x(Gx \rightarrow Hx)$. $(\leftarrow)$ Assume $\boldsymbol{c}_{\forall G} \succeq \boldsymbol{c}_F$. By the definitions of $\succeq$ and $\preceq$, this means $\forall H(\boldsymbol{c}_F H \rightarrow \boldsymbol{c}_{\forall G} H)$. As an instance of this latter, we know $\boldsymbol{c}_F F \rightarrow \boldsymbol{c}_{\forall G} F$, from which it follows, by (537.2), that $\boldsymbol{c}_{\forall G} F$. ⋈

(551.4)★ $(\rightarrow)$ Assume $\boldsymbol{c}_{\exists G} F$. By the definitions of $\succeq$ and $\preceq$, we have to show $\forall H(\boldsymbol{c}_F H \rightarrow \boldsymbol{c}_{\exists G} H)$. So assume $\boldsymbol{c}_F H$. By the Abstraction Principle (184)★ applied to concepts, have to show $\exists x(Gx \,\&\, Hx)$. But from $\boldsymbol{c}_F H$, we know by (537.1) that $F \Rightarrow H$, from which it follows by the T schema that $\forall x(Fx \rightarrow Hx)$. But our initial assumption implies, by now familiar reasoning, $\exists x(Gx \& Fx)$. By predicate logic, it then follows that $\exists x(Gx \,\&\, Hx)$. $(\leftarrow)$ Assume $\boldsymbol{c}_{\exists G} \succeq \boldsymbol{c}_F$. By the definitions of $\succeq$ and $\preceq$, our assumption implies $\forall H(\boldsymbol{c}_F H \rightarrow \boldsymbol{c}_{\exists G} H)$. As an instance of this latter, we know $\boldsymbol{c}_F F \rightarrow \boldsymbol{c}_{\exists G} F$, from which it follows, by (537.2), that $\boldsymbol{c}_{\exists G} F$. ⋈

(553.1)★ By the commutativity of the biconditional, it follows from (546.1)★ that $Gu \equiv \boldsymbol{c}_u G$. But from this and (546.2)★, and it follows that $Gu \equiv \boldsymbol{c}_u \succeq \boldsymbol{c}_G$ by a biconditional syllogism. ⋈

(553.2)★ – **(553.3)**★ (Exercises)

(559.1) Assume $RealizesAt(u,c,w)$ and $RealizesAt(u,d,w)$, Then, by the definition of realization (557), we know both $\forall F(w \models Fu \equiv cF)$ and $\forall F(w \models Fu \equiv dF)$. So, by the laws of quantified biconditionals, it follows that $\forall F(cF \equiv dF)$. Since c and d are concepts, they are abstract (477). So, by theorem (172.1), $c = d$. ⋈

(559.2) Assume $RealizesAt(u,c,w)$ and $RealizesAt(v,c,w)$, Then, by the definition of realization (557), we know both $\forall F(w \models Fu \equiv cF)$ and $\forall F(w \models Fv \equiv cF)$. So by the laws of quantified biconditionals, we know: $\forall F(w \models Fu \equiv w \models Fv)$. Instantiate this to the property $[\lambda y\, u =_E y]$, to yield the following fact:

(ϑ) $w \models [\lambda y\, u =_E y]u \equiv w \models [\lambda y\, u =_E y]v$

Now, independently, by the facts that E-identity is an equivalence relation on ordinary objects (168.1) and that u is ordinary, we know $u =_E u$. From this it follows that $\Box u =_E u$, by the necessity of E-identity (157.1). We also know, as a instance of β-Conversion (128), that $\Box([\lambda y\, u =_E y]u \equiv u =_E u)$. So it follows by (111.6) that $\Box[\lambda y\, u =_E y]u$, and by a fundamental theorem of possible world theory (433.2), it follows that $\forall w'(w' \models [\lambda y\, u =_E y]u)$. In particular, $w \models [\lambda y\, u =_E y]u$. So it follows from ($\vartheta$) that:

(ξ) $w \models [\lambda y\, u =_E y]v$

Note separately that by an instance of β-Conversion (128) and RN we know:

(ζ) $\Box([\lambda y\, u =_E y]v \to u =_E v)$

Thus, since possible worlds are modally closed (419), it follows from (ϑ) and (ξ) that $w \models u =_E v$. Hence, by the first form of the Fundamental Theorem of Possible World Theory (433.1), it follows that $\Diamond u =_E v$. So, by theorem (157.2), it follows that $u =_E v$. Hence, by the definition of identity (15), $u = v$. ⋈

(559.3) Assume $RealizesAt(u,c,w)$ and $RealizesAt(u,c,w')$. So we know, by the definition of realization (557), that $\forall F(w \models Fu \equiv cF)$ and $\forall F(w' \models Fu \equiv cF)$. So, by the laws of quantified biconditionals, we know:

(ϑ) $\forall F(w \models Fu \equiv w' \models Fu)$

Suppose, for reductio, that $w \neq w'$. Then, since w and w' are possible worlds, and hence situations, we know by (368) that there must be a proposition, say q, true at one and not at the other. Without loss of generality, assume $w \models q$ and $w' \not\models q$. From the former, it follows by a useful fact about possible worlds (438), that $w \models [\lambda y\, q]u$. So, in light of ($\vartheta$), it follows that $w' \models [\lambda y\, q]u$. So again by (438), $w' \models q$. Contradiction. ⋈

(561) Assume $AppearsAt(c,w)$. By the definition of appearance (560), it follows that some ordinary individual, say a, is such that $RealizesAt(a,c,w)$. It therefore remains to show uniqueness, i.e., that $\forall v(RealizesAt(v,c,w) \to v = a)$. So, by GEN, assume $RealizesAt(v,c,w)$. Then by (559.2), $v = a$. ⋈

(562) Assume $AppearsAt(c,w)$. By the definition of appearance (560), it follows that some ordinary individual, say a, is such that $RealizesAt(a,c,w)$. So, by definition (557):

(ϑ) $\forall F(w \models Fa \equiv cF)$

Since we know $O!a$, we therefore know $\Box O!a$, by (153.1). Hence by a fundamental theorem of world theory (433.2), $\forall w'(w' \models O!a)$. In particular, $w \models O!a$. But by (ϑ), we also know $w \models O!a \equiv cO!$. Hence, $cO!$. ⋈

(564) Suppose $AppearsAt(c,w)$. So some ordinary object, say a, realizes c at w, i.e.,

(ϑ) $\forall F(w \models Fa \equiv cF)$

By (563) and GEN, we have to show that $c\Sigma p \equiv w \models p$:

$$\begin{array}{lcll} c\Sigma p & \equiv & c[\lambda y\, p] & \text{by (216) and (477)} \\ & \equiv & w \models [\lambda y\, p]a & \text{by } (\vartheta) \\ & \equiv & w \models p & \text{by (438)} \end{array}$$ ⋈

(565) Assume $AppearsAt(c,w)$ and $AppearsAt(c,w')$. So, by (564), it follows, respectively, that $Mirrors(c,w)$ and $Mirrors(c,w')$. We may infer from these, respectively, by the definition of mirroring (563), that $\forall p(c\Sigma p \equiv w \models p)$ and $\forall p(c\Sigma p \equiv w' \models p)$. So by the laws of quantified biconditionals, we know $\forall p(w \models p \equiv w' \models p)$. But since w and w' are both possible worlds, and hence situations, it follows by a fact about the identity of situations (368), that $w = w'$.

(566) We prove only the left-to-right direction, since the right-to-left direction is an instance of the T schema. Only one step of the proof below requires special commentary. By the reasoning in Remark (402), $w \models Fa$ is equivalent to $w[\lambda y\, Fa]$, by modally strict reasoning. Hence, the following instance of theorem (126.6):

$$(w[\lambda y\, Fa] \equiv cF) \equiv \Box(w[\lambda y\, Fa] \equiv cF)$$

implies:

(ϑ) $(w \models Fa \equiv cF) \equiv \Box(w \models Fa \equiv cF)$

by the Rule of Substitution. (ϑ) is modally strict and so we may substitute the right condition for the left whenever the left occurs as a subformula. Hence we may reason as follows:

$$\begin{array}{lcll} AppearsAt(c,w) & \rightarrow & \exists u RealizesAt(u,c,w) & \text{By df (560)} \\ & \rightarrow & RealizesAt(a,c,w) & \text{where `}a\text{' is arbitrary} \\ & \rightarrow & \forall F(w \models Fa \equiv cF) & \text{by df (557)} \\ & \rightarrow & \forall F\Box(w \models Fa \equiv cF) & \text{by } (\vartheta)\text{, Rule of Substitution} \\ & \rightarrow & \Box\forall F(w \models Fa \equiv cF) & \text{by BF (122.1)} \\ & \rightarrow & \Box RealizesAt(a,c,w) & \text{by df (557)} \\ & \rightarrow & \exists u\Box RealizesAt(u,c,w) & \text{by } \exists\text{I and } \exists\text{E} \\ & \rightarrow & \Box\exists u RealizesAt(u,c,w) & \text{by Buridan (123.1)} \\ & \rightarrow & \Box AppearsAt(c,w) & \text{by df (560)} \end{array}$$

(567) Assume $RealizesAt(u,c,w)$ and $RealizesAt(v,c,w')$. By two applications of $\exists$I, it follows that $AppearsAt(c,w)$ and $AppearsAt(c,w')$. Hence by (565), $w = w'$. From this and the second of our assumptions, we know $RealizesAt(v,c,w)$. From this, and the first of our hypotheses, it follows, by (559.2), that $u = v$. ⋈

(568.1)⋆ By the commutativity of the biconditional, (426)⋆ implies $\boldsymbol{w}_\alpha \models Fu \equiv Fu$. Also, by the commutativity of the biconditional, (546.1)⋆ implies $Fu \equiv \boldsymbol{c}_u F$. Hence, by a biconditional syllogism, $\boldsymbol{w}_\alpha \models Fu \equiv \boldsymbol{c}_u F$. By GEN, it follows that $\forall F(\boldsymbol{w}_\alpha Fu \equiv \boldsymbol{c}_u F)$. So by the definition of realization (557), $Realizes(u, \boldsymbol{c}_u, \boldsymbol{w}_\alpha)$. ⋈

(568.2)⋆ (Exercise)

(568.3)⋆ (Exercise)

(570) Assume $\exists u ConceptOf(c,u)$. Suppose a is such an ordinary object, so that we know $ConceptOf(c,a)$. Then by definition (542), it follows that:

(ϑ) $\forall F(cF \equiv Fa)$

By (569), we have to show $\exists w AppearsAt(c,w)$. By (560), we have to show $\exists w \exists u RealizesAt(u,c,w)$. And by (557), we have to show: $\exists w \exists u \forall F(w \models Fu \equiv cF)$. So if we can find witnesses to both existential quantifiers in this claim, we're done.

By Comprehension for Situations (381):

$$\exists s \forall F(sF \equiv \exists p(F = [\lambda y\, p] \,\&\, Fa))$$

Let s_0 be such a situation, so that we know:

(ξ) $\forall F(s_0 F \equiv \exists p(F = [\lambda y\, p] \,\&\, Fa))$

If we can show (a) s_0 is a possible world, and (b) $\forall F(s_0 \models Fa \equiv cF)$, then s_0 and a are the two witnesses we need.

(a) Since s_0 is a situation, all we have to do to show s_0 is a possible world is to show $\Diamond \forall q(s_0 \models q \equiv q)$. By the T$\Diamond$ schema, it suffices to show $\forall q(s_0 \models q \equiv q)$. By GEN, it suffices to show $s_0 \models q \equiv q$. ($\rightarrow$) Assume $s_0 \models q$. Then by now familiar reasoning, we know $s_0[\lambda y\, q]$. Hence by (ξ), $\exists p([\lambda y\, q] = [\lambda y\, p] \& [\lambda y\, q]a)$. Suppose p_1 is such a proposition, so that we know $[\lambda y\, q] = [\lambda y\, p_1] \,\&\, [\lambda y\, q]a$. Then by &E, $[\lambda y\, q]a$, and by β-Conversion, q. ($\leftarrow$) Assume q. Then by β-Conversion, $[\lambda y\, q]a$. And by (74.2) and &I, $[\lambda y\, q] = [\lambda y\, q] \& [\lambda y\, q]a$. Hence $\exists p([\lambda y\, q] = [\lambda y\, p] \& [\lambda y\, q]a)$. So by ($\xi$), $s_0[\lambda y\, q]$, and by now familiar reasoning, $s_0 \models q$.

(b) By GEN, we have to show $s_0 \models Fa \equiv cF$. To do this, we begin by noting that as part of (a), we established $\forall q(s_0 \models q \equiv q)$. If we instantiate this to the term Fa, it follows that $s_0 \models Fa \equiv Fa$. So by GEN, $\forall F(s_0 \models Fa \equiv Fa)$. But ($\vartheta$) is equivalent to $\forall F(Fa \equiv cF)$. So by the laws of quantified biconditionals, $\forall F(s_0 \models Fa \equiv cF)$. ⋈

(571)⋆ By (101.2)⋆ and the definition of $\boldsymbol{c}_u$ (544) as $\iota c ConceptOf(c,u)$, it follows that $ConceptOf(\boldsymbol{c}_u, u)$. Hence by $\exists$I and definition (570), $IndividualConcept(\boldsymbol{c}_u)$. ⋈

(572) We know by (174.2) and the T schema that ordinary objects exist. So let a be ordinary. Then by (543), $\exists c ConceptOf(c,a)$. So $\exists u \exists c ConceptOf(c,a)$. By (86.14), $\exists c \exists u ConceptOf(c,a)$. So suppose c_1 is such a concept, so that we have $\exists u ConceptOf(c_1,u)$. Then, by (570), $IndividualConcept(c_1)$. But then $\exists c IndividualConcept(c)$. ⋈

(573.1) Assume $IndividualConcept(c)$. So, by definition (569), there is some possible world, say w_1, such that $AppearsAt(c,w_1)$. It therefore remains to show that $\forall w(AppearsAt(c,w) \to w = w_1)$, and by GEN, that $AppearsAt(c,w) \to w = w_1$. So assume that $AppearsAt(c,w)$. Then by a fact about appearance (565), it follows that $w = w_1$. ⋈

(573.2) It follow *a fortiori* from theorem (566) that:

$$AppearsAt(c,w) \to \Box AppearsAt(c,w)$$

Since this holds for arbitrary concepts, it holds for individual concepts:

$$AppearsAt(\hat{c},w) \to \Box AppearsAt(\hat{c},w)$$

So by GEN:

$$\forall w(AppearsAt(\hat{c},w) \to \Box AppearsAt(\hat{c},w))$$

Hence, from this and the the previous theorem (573.1), i.e.,

$$\exists! w AppearsAt(\hat{c},w)$$

it follows that:

$$\exists! w \Box AppearsAt(\hat{c},w)$$

by theorem (88). ⋈

(573.3) (Exercise)

(575) As an instance of (108.1), we know:

$$\exists! w \Box AppearsAt(\hat{c},w) \to \forall y(y = \imath w(AppearsAt(\hat{c},w)) \to AppearsAt(\hat{c},y))$$

So by (573.2), it follows that:

$$\forall y(y = \imath w(AppearsAt(\hat{c},w)) \to AppearsAt(\hat{c},y))$$

Instantiating to $\boldsymbol{w}_{\hat{c}}$, we have:

$$\boldsymbol{w}_{\hat{c}} = \imath w(AppearsAt(\hat{c},w) \to AppearsAt(\hat{c},\boldsymbol{w}_{\hat{c}})$$

But then by definition (574), it follows that $AppearsAt(\hat{c}, \boldsymbol{w}_{\hat{c}})$. ⋈

(576.1) By (575) and (564). ⋈

(576.2) By GEN and the definitions of $\geq$ (489.2) and $\preceq$ (489.1) we have to show: $\boldsymbol{w}_{\hat{c}}F \rightarrow \hat{c}F$. So assume $\boldsymbol{w}_{\hat{c}}F$. Since $\boldsymbol{w}_{\hat{c}}$ is a situation, every property that it encodes is a propositional property, and so for some proposition, say q_1, $F = [\lambda y\, q_1]$. Thus, from our assumption, it follows that $\boldsymbol{w}_{\hat{c}}[\lambda y\, q_1]$. By definition, this implies that $\boldsymbol{w}_{\hat{c}} \models q_1$. Now, independently, we know by (576.1) that $Mirrors(\hat{c}, \boldsymbol{w}_{\hat{c}})$. So by definition of mirroring, $\forall p(\hat{c}\Sigma p \equiv \boldsymbol{w}_{\hat{c}} \models p)$. Since we've established $\boldsymbol{w}_{\hat{c}} \models q_1$, it follows that $\hat{c}\Sigma q_1$. By definition, this implies $\hat{c}[\lambda y\, q_1]$, i.e., $\hat{c}F$. ⋈

(577) ($\rightarrow$) Assume $\hat{c}G$. Before we show $\hat{c} \succeq \boldsymbol{c}_G$, note that since $\hat{c}$ is an individual concept, it follows by definition (569) that for some possible world, say w_1, $AppearsAt(\hat{c}, w_1)$. Further, by the definition of appearance (560), for some ordinary object, say a, we know $RealizesAt(a, \hat{c}, w_1)$. So by definition (557), this implies:

(ϑ) $\forall H(w_1 \models Ha \equiv \hat{c}H)$

Now to show $\hat{c} \succeq \boldsymbol{c}_G$, we have to show $\forall F(\boldsymbol{c}_G F \rightarrow \hat{c}F)$, by the definition of containment (489.2) and inclusion (489.1). So, by GEN, assume $\boldsymbol{c}_G F$, to show $\hat{c}F$. It then follows that $\Box\forall x(Gx \rightarrow Fx)$, by (537.1) and (339). So by a Fundamental Theorem of Possible World Theory (433.2), we have $\forall w(w \models \forall x(Gx \rightarrow Fx))$ and, in particular, $w_1 \models \forall x(Gx \rightarrow Fx)$. But since $\hat{c}G$ is our global assumption, it follows from (ϑ) that $w_1 \models Ga$. Since the conjunction of $\forall x(Gx \rightarrow Fx)$ and Ga necessarily implies Fa, and possible worlds are closed under necessary implication (419), it follows that $w_1 \models Fa$. Hence, by (ϑ), it follows that $\hat{c}F$.

($\leftarrow$) Assume $\hat{c} \succeq \boldsymbol{c}_G$. Then, by definition of containment (489.2) and inclusion (489.1), we know $\forall F(\boldsymbol{c}_G F \rightarrow \hat{c}F)$. But we also know $\boldsymbol{c}_G G$ (537.2). So $\hat{c}G$. ⋈

(578.1) Before we begin the proof proper, we establish two preliminary facts. By definition of individual concept (569), we know that $\exists w(AppearsAt(\hat{c}, w)$. Suppose w_1 is such a world, so that we know $AppearsAt(\hat{c}, w_1)$. Then, by definition of appears at (560), $\exists u(RealizesAt(u, \hat{c}, w_1)$, and by definition of realizes at (557), $\exists u \forall F(w_1 \models Fu \equiv \hat{c}F)$. Suppose a is such an ordinary object. So as our first preliminary fact, we know:

(ϑ) $\forall F(w_1 \models Fa \equiv \hat{c}F)$

Our second preliminary fact is obtained by observing that it is a modally strict fact that $Ga \equiv \neg\overline{G}a$, by commuting an appropriate instance of (137.2). Hence, by RN, $\Box(Ga \equiv \neg\overline{G}a)$. So by definition (415.2):

(ξ) $Ga \Leftrightarrow \neg\overline{G}a$

With these two facts, we may reason as follows:

$$\begin{array}{lcll} \hat{c}G & \equiv & w_1 \models Ga & \text{by } (\vartheta) \text{ and commutativity of} \equiv \\ & \equiv & w_1 \models \neg\overline{G}a & \text{by } (\xi) \text{ and the modal closure of } w_1 \text{ (417)} \\ & \equiv & \neg w_1 \models \overline{G}a & \text{by the coherency of } w_1 \text{ (414)} \\ & \equiv & \neg\hat{c}\overline{G} & \text{by } (\vartheta) \qquad\qquad \bowtie \end{array}$$

(578.2)

(578.3) By (577), we know both:

(ϑ) $\hat{c}G \equiv \hat{c} \succeq \boldsymbol{c}_G$

(ξ) $\hat{c}\overline{G} \equiv \hat{c} \succeq \boldsymbol{c}_{\overline{G}}$

So we may reason as follows:

$$\begin{array}{lcll} \hat{c} \succeq \boldsymbol{c}_G & \equiv & \hat{c}G & \text{by } (\vartheta) \text{ and commutativity of} \equiv \\ & \equiv & \neg\hat{c}\overline{G} & \text{by (578.1)} \\ & \equiv & \hat{c} \not\succeq \boldsymbol{c}_{\overline{G}} & \text{by } (\xi) \qquad\qquad \bowtie \end{array}$$

(578.4) (Exercise)

(579) By the definition of completeness (547) and GEN, we have to show $\hat{c}F \vee \hat{c}\overline{F}$. Since $\hat{c}$ is an individual concept, we know by the definitions of *Individual-Concept* (569) and *AppearsAt* (560) that some ordinary object, say a, realizes $\hat{c}$ at some possible world, say w_1. So by definition of *RealizesAt* (557), we know:

(ϑ) $\forall G(w_1 \models Ga \equiv \hat{c}G)$

Now, independently, note that from easily-proved theorem $\square(Fa \vee \neg Fa)$ and the instance $\overline{F}a \equiv \neg Fa$ of the modally strict theorem (137.1), it follows by the Rule of Substitution that $\square(Fa \vee \overline{F}a)$. Then by a Fundamental Theorem of Possible World Theory (433.2), it follows that $w_1 \models (Fa \vee \overline{F}a)$. But then by (440.1), it follows that:

(ξ) $w_1 \models Fa \vee w_1 \models \overline{F}a$

So, we may reason by disjunctive syllogism from the disjuncts of (ξ): if $w_1 \models Fa$, then by (ϑ), then $\hat{c}F$, and if $w_1 \models \overline{F}a$, then by ($\vartheta$), $\hat{c}\overline{F}$. Hence, $\hat{c}F \vee \hat{c}\overline{F}$. $\bowtie$

(581) ($\rightarrow$) Assume $\hat{c}$ and $\hat{e}$ are compossible individual concepts. Then by definition (580), there is a possible world, say w_1, such that both *AppearsAt*$(\hat{c}, w_1)$ and *AppearsAt*$(\hat{e}, w_1)$. But we also know, by (575), that *AppearsAt*$(\hat{c}, \boldsymbol{w}_{\hat{c}})$ and *AppearsAt*$(\hat{e}, \boldsymbol{w}_{\hat{e}})$. Since individual concepts appear at a unique possible world (573.1), it follows, respectively, that $w_1 = \boldsymbol{w}_{\hat{c}}$ and $w_1 = \boldsymbol{w}_{\hat{e}}$. Hence $\boldsymbol{w}_{\hat{c}} = \boldsymbol{w}_{\hat{e}}$. ($\leftarrow$) Assume $\boldsymbol{w}_{\hat{c}} = \boldsymbol{w}_{\hat{e}}$. Independently, by (575), we know both *AppearsAt*$(\hat{c}, \boldsymbol{w}_{\hat{c}})$ and

$AppearsAt(\hat{e}, \boldsymbol{w}_{\hat{e}})$. But from our assumption and the second of these, it follows that $AppearsAt(\hat{e}, \boldsymbol{w}_{\hat{c}})$. But then, there is a world, namely $\boldsymbol{w}_{\hat{c}}$, where both $\hat{c}$ and $\hat{e}$ appear. So $Compossible(\hat{c}, \hat{e})$. ⋈

(582.1) By the definition of an individual concept (569), $\exists w(Appear(\hat{c}, w)$. So by predicate logic and the commutativity of conjunction, $\exists w(Appear(\hat{c}, w) \,\&\, Appear(\hat{c}, w))$. Hence, by the definition of compossibility (580), $Compossible(\hat{c}, \hat{c})$. ⋈

(582.2) Suppose $Compossible(\hat{c}, \hat{e})$. Then, by definition (580):

$$\exists w(Appear(\hat{c}, w) \,\&\, Appear(\hat{e}, w))$$

So, by predicate logic and the laws of conjunction:

$$\exists w(Appear(\hat{e}, w) \,\&\, Appear(\hat{c}, w))$$

Hence, $Compossible(\hat{e}, \hat{c})$, again by the definition of compossibility (580). ⋈

(582.3) Suppose $Compossible(\hat{c}, \hat{d})$ and $Compossible(\hat{d}, \hat{e})$. Then, by a previous theorem (581), $w_{\hat{c}} = w_{\hat{d}}$ and $w_{\hat{d}} = w_{\hat{e}}$. So, by transitivity of identity, $w_{\hat{c}} = w_{\hat{e}}$. Hence, again by (581), $Compossible(\hat{c}, \hat{e})$. ⋈

(584.1) – **(584.3)** (Exercises)

(586.1) – **(586.2)** (Exercises)

(587.1) By the second conjunct of (586.2), we know $\forall F(\boldsymbol{c}_u^w F \equiv w \models Fu)$. So, by commutativity of the biconditional and the definition of realization (557), it follows that $RealizesAt(u, \boldsymbol{c}_u^w, w)$. ⋈

(587.2) By applying GEN to (587.1), we know that $\forall u RealizesAt(u, \boldsymbol{c}_u^w, w)$. But we also know that there are ordinary objects, by (174.2) and the T schema. Hence $\exists u RealizesAt(u, \boldsymbol{c}_u^w, w)$. So, by the definition of appearance (560), it follows that $AppearsAt(\boldsymbol{c}_u^w, w)$. ⋈

(587.3) By applying GEN to (587.2), we know $\forall w AppearsAt(\boldsymbol{c}_u^w, w)$. But we know that there are possible worlds. Hence $\exists w AppearsAt(\boldsymbol{c}_u^w, w)$. So by definition (569), $IndividualConcept(\boldsymbol{c}_u^w)$. ⋈

(587.4) This follows immediately from (587.2) and (564). ⋈

(587.5) (Exercise)

(588.1) We may reason biconditionally as follows:

$$\begin{array}{lll} IndividualConcept(c) & \equiv \exists w AppearsAt(c, w) & \text{by definition (569)} \\ & \equiv \exists w \exists u RealizesAt(u, c, w) & \text{by definition (560)} \\ & \equiv \exists u \exists w RealizesAt(u, c, w) & \text{by theorem (86.14)} \\ & \equiv \exists u \exists w \forall F(w \models Fu \equiv cF) & \text{by definition (557)} \\ & \equiv \exists u \exists w \forall F(cF \equiv w \models Fu) & \text{by (63.3.g), RN, (112.2)} \\ & \equiv \exists u \exists w ConceptOfAt(c, u, w) & \text{by definition (583)} \quad ⋈ \end{array}$$

(588.2) (→) Assume *IndividualConcept*(c), to show $\exists u \exists w(c = \boldsymbol{c}_u^w)$. Then by the definition of individual concepts (569), we know:

$$\exists w AppearsAt(c, w)$$

Suppose w_1 is an arbitrary such world, so that we know $AppearsAt(c, w_1)$. By the definition of appearance (560), we may infer that some ordinary individual, say b, is such that $\exists u RealizesAt(b, c, w_1)$. By the definition of realization (557), it follows that: $\forall F(w_1 \models Fb \equiv cF)$. Now we also know, by the second conjunct of lemma (586.2), that $\forall F(\boldsymbol{c}_b^{w_1}F \equiv w_1 \models Fb)$. So by the logic of quantified biconditionals, we know: $\forall F(cF \equiv \boldsymbol{c}_b^{w_1}F)$. Since both c and $\boldsymbol{c}_b^{w_1}$ are concepts and, hence, abstract, it follows that $c = \boldsymbol{c}_b^{w_1}$. *A fortiori*, $\exists u \exists w(c = \boldsymbol{c}_u^w)$.

(←) Assume $\exists u \exists w(c = \boldsymbol{c}_u^w)$, to show *IndividualConcept*(c). So $c = \boldsymbol{c}_b^{w_1}$, for some arbitrary ordinary object b and possible world w_1. Now we independently know that *IndividualConcept*($\boldsymbol{c}_b^{w_1}$), by (587.3). So *IndividualConcept*(c). ⋈

(589) By definitions (544), (542), and (477), we know:

$$\boldsymbol{c}_u = \imath x(A!x \,\&\, \forall F(xF \equiv Fu))$$

Moreover, as an instance of Actualized Abstraction (186), we know:

$$\imath x(A!x \,\&\, \forall F(xF \equiv Fu))G \equiv \mathcal{A}Gu$$

From these two results, it follows that:

$$\boldsymbol{c}_u G \equiv \mathcal{A}Gu$$

But as an instance of (428), we know:

$$\mathcal{A}Gu \equiv \boldsymbol{w}_\alpha \models Gu$$

Hence:

$$\boldsymbol{c}_u G \equiv \boldsymbol{w}_\alpha \models Gu$$

But since $\boldsymbol{w}_\alpha$ is a possible world (427.1), it follows from the second conjunct of (586.2) that $\boldsymbol{c}_u^{\boldsymbol{w}_\alpha} G \equiv \boldsymbol{w}_\alpha \models Gu$, which by commutativity yields:

$$\boldsymbol{w}_\alpha \models Gu \equiv \boldsymbol{c}_u^{\boldsymbol{w}_\alpha} G$$

Hence $\boldsymbol{c}_u G \equiv \boldsymbol{c}_u^{\boldsymbol{w}_\alpha} G$, i.e., $\boldsymbol{c}_u^{\boldsymbol{w}_\alpha} G \equiv \boldsymbol{c}_u G$. So by GEN, $\forall G(\boldsymbol{c}_u^{\boldsymbol{w}_\alpha} G \equiv \boldsymbol{c}_u G)$. Since both $\boldsymbol{c}_u^{\boldsymbol{w}_\alpha}$ and $\boldsymbol{c}_u$ are abstract, it follows that $\boldsymbol{c}_u^{\boldsymbol{w}_\alpha} = \boldsymbol{c}_u$. ⋈

(590.1) By (587.3) $\boldsymbol{c}_u^w$ is an individual concept. So by (577), it follows that $\boldsymbol{c}_u^w G \equiv \boldsymbol{c}_u^w \succeq \boldsymbol{c}_G$. ⋈

(590.2) From the second conjunct of (586.2), it follows that $\boldsymbol{c}_u^w G \equiv w \models Gu$, and so by commutativity of the biconditional, $(w \models Gu) \equiv \boldsymbol{c}_u^w G$. But we just established (590.1) that $\boldsymbol{c}_u^w G \equiv \boldsymbol{c}_u^w \succeq \boldsymbol{c}_G$. Hence, $(w \models Gu) \equiv \boldsymbol{c}_u^w \succeq \boldsymbol{c}_G$. ⋈

(590.3) Assume $\boldsymbol{c}_u^w = \boldsymbol{c}_v^w$. Independently, we know both $RealizesAt(u, \boldsymbol{c}_u^w, w)$ and $RealizesAt(v, \boldsymbol{c}_v^w, w)$, by (587.1). So we may substitute $\boldsymbol{c}_u^w$ for $\boldsymbol{c}_v^w$ in the latter, to obtain $RealizesAt(v, \boldsymbol{c}_u^w, w)$. But from $RealizesAt(u, \boldsymbol{c}_u^w, w)$ and $RealizesAt(v, \boldsymbol{c}_u^w, w)$, it follows from a fact about realization (559.2), that $u = v$. ⋈

(590.4) Assume $\boldsymbol{c}_u^w = \boldsymbol{c}_u^{w'}$. Independently, we know both $RealizesAt(u, \boldsymbol{c}_u^w, w)$ and $RealizesAt(u, \boldsymbol{c}_u^{w'}, w')$, by (587.1). So we may substitute $\boldsymbol{c}_u^w$ for $\boldsymbol{c}_u^{w'}$ in the latter, to infer $RealizesAt(u, \boldsymbol{c}_u^w, w')$. But from $RealizesAt(u, \boldsymbol{c}_u^w, w)$ and $RealizesAt(u, \boldsymbol{c}_u^w, w')$, it follows from a fact about realization (559.3), that $w = w'$. ⋈

(590.5) (Exercise)

(591) From a fact about world-relative concepts of individuals (587.1), we know both $RealizesAt(u, \boldsymbol{c}_u^w, w)$ and $RealizesAt(v, \boldsymbol{c}_v^w, w)$. Since ordinary objects exist, by (174.2) and the T schema, we may infer, respectively:

$$\exists u RealizesAt(u, \boldsymbol{c}_u^w, w)$$
$$\exists v RealizesAt(v, \boldsymbol{c}_v^w, w)$$

So, by definition (560), it follows, respectively, that:

$$AppearsAt(\boldsymbol{c}_u^w, w)$$
$$AppearsAt(\boldsymbol{c}_v^w, w)$$

By conjoining these results and quantifying, we have:

$$\exists w'(AppearsAt(\boldsymbol{c}_u^w, w') \,\&\, AppearsAt(\boldsymbol{c}_v^w, w'))$$

By the definition of compossibility (580), then, we have $Compossible(\boldsymbol{c}_u^w, \boldsymbol{c}_v^w)$. ⋈

(595.1) Since $\hat{c}$ is an individual concept, we know, by the definitions of individual concept (569) and appearance (560), that there is an ordinary individual, say b, and a possible world, say w_1, such that $RealizesAt(b, \hat{c}, w_1)$. So, conjoining this fact with itself, we have $RealizesAt(b, \hat{c}, w_1) \,\&\, RealizesAt(b, \hat{c}, w_1)$. By three applications of existential generalization, we have:

$$\exists u \exists w \exists w'(RealizesAt(u, \hat{c}, w) \,\&\, RealizesAt(u, \hat{c}, w')).$$

So, by the definition of counterparts (594), $CounterpartOf(\hat{c}, \hat{c})$. ⋈

(595.2) Assume $CounterpartOf(\hat{e}, \hat{c})$. Then, by applying definitions, we know there is an ordinary object, say b, and there are possible worlds, say w_1 and w_2, such that:

$$RealizesAt(b, \hat{c}, w_1) \,\&\, RealizesAt(b, \hat{e}, w_2)$$

So, reversing the order of the conjuncts, we know:

$$RealizesAt(b, \hat{e}, w_2) \,\&\, RealizesAt(b, \hat{c}, w_1)$$

It follows therefore that:

$$\exists u \exists w \exists w'(RealizesAt(u,\hat{e},w) \,\&\, RealizesAt(u,\hat{c},w'))$$

So by the definition of counterparts (594), $CounterpartOf(\hat{c},\hat{e})$. ⋈

(595.3) Assume $CounterpartOf(\hat{e},\hat{d})$ and $CounterpartOf(\hat{d},\hat{c})$. Then by the first conjunct we know that there is an ordinary object, say a, and there are possible worlds, say w_1 and w_2, such that:

(ϑ) $RealizesAt(a,\hat{d},w_1) \,\&\, RealizesAt(a,\hat{e},w_2)$

And by the second conjunct of our assumption we know that there is an ordinary object, say b, and there are possible worlds, say w_3 and w_4, such that:

(ξ) $RealizesAt(b,\hat{c},w_3) \,\&\, RealizesAt(b,\hat{d},w_4)$

Then, from the first conjunct of (ϑ) and the second conjunct (ξ), it follows by a fact about realization (567) that $w_1 = w_4$ and $a = b$. So after substituting a for b in the first conjunct of (ξ), we may conjoin the result with the second conjunct of (ϑ) to obtain:

$$RealizesAt(a,\hat{c},w_3) \,\&\, RealizesAt(a,\hat{e},w_2)$$

It therefore follows that:

$$\exists u \exists w \exists w'(RealizesAt(u,\hat{c},w) \,\&\, RealizesAt(u,\hat{e},w')),$$

from which it follows that $CounterpartOf(\hat{e},\hat{c})$, by (594). ⋈

(596) ($\rightarrow$) Assume $CounterpartOf(\hat{e},\hat{c})$. Then by the definition of counterpart (594), there is an ordinary object, say a, and there are possible worlds, say, w_1 and w_2, such that:

(ϑ) $RealizesAt(a,\hat{c},w_1) \,\&\, RealizesAt(a,\hat{e},w_2)$

By &I and $\exists$I, it remains only to establish:

$$\forall u((RealizesAt(u,\hat{c},w_1) \,\&\, RealizesAt(u,\hat{e},w_2)) \rightarrow u = a)$$

So by GEN, assume:

(ξ) $RealizesAt(u,\hat{c},w_1) \,\&\, RealizesAt(u,\hat{e},w_2)$

But if we conjoin the first conjunct of (ξ) with the first conjunct of (ϑ), we have $RealizesAt(u,\hat{c},w_1) \,\&\, RealizesAt(a,\hat{c},w_1)$. So $u = a$, by a fact about realization (559.2).

($\leftarrow$) Exercise. ⋈

(597.1) By (587.1), we know both:

$RealizesAt(u, \boldsymbol{c}_u^w, w)$
$RealizesAt(u, \boldsymbol{c}_u^{w'}, w')$

By conjoining these and applying existential generalization several times, we obtain:

$$\exists u \exists w \exists w'(RealizesAt(u, \boldsymbol{c}_u^w, w) \ \&\ RealizesAt(u, \boldsymbol{c}_u^{w'}, w'))$$

So, by the definition of counterparts (594), it follows that $CounterpartOf(\boldsymbol{c}_u^{w'}, \boldsymbol{c}_u^w)$. ⋈

(597.2) (Exercise)

(598.1)★ Assume:

(ϑ) $Fu \ \&\ \Diamond\neg Fu$

The first conjunct and theorem (553.1)★, which asserts $Gu \equiv \boldsymbol{c}_u \succeq \boldsymbol{c}_G$, together imply $\boldsymbol{c}_u \succeq \boldsymbol{c}_F$. So it remains show:

(ξ) $\exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u) \ \&\ \hat{c} \not\succeq \boldsymbol{c}_F \ \&\ \exists w(w \neq \boldsymbol{w}_\alpha \ \&\ AppearsAt(\hat{c}, w)))$

Note that from the second conjunct (ϑ) and a Fundamental Theorem of Possible World Theory (433.1), it follows that $\exists w(w \models \neg Fu)$. So, let w_1 be an arbitrary such possible world, so that we know $w_1 \models \neg Fu$. Now consider the concept of u at w_1, i.e., $\boldsymbol{c}_u^{w_1}$. We know by (587.3) that $IndividualConcept(\boldsymbol{c}_u^{w_1})$. So to show ($\xi$), it suffices by &I and ∃I to show:

(a) $CounterpartOf(\boldsymbol{c}_u^{w_1}, \boldsymbol{c}_u)$

(b) $\boldsymbol{c}_u^{w_1} \not\succeq \boldsymbol{c}_F$

(c) $\exists w(w \neq \boldsymbol{w}_\alpha \ \&\ AppearsAt(\boldsymbol{c}_u^{w_1}, w))$

(a) If we instantiate theorem (597.1) to worlds w_1 and $\boldsymbol{w}_\alpha$, it follows that:

$$CounterpartOf(\boldsymbol{c}_u^{w_1}, \boldsymbol{c}_u^{\boldsymbol{w}_\alpha})$$

But by (589), we know $\boldsymbol{c}_u^{\boldsymbol{w}_\alpha} = \boldsymbol{c}_u$. So it follows that $CounterpartOf(\boldsymbol{c}_u^{w_1}, \boldsymbol{c}_u)$.

(b) Since we know $w_1 \models \neg Fu$, it follows by the fact that possible worlds are coherent (414) and the definition of coherency (411), that $\neg w_1 \models Fu$. So by the second conjunct of (586.2), it follows that $\neg \boldsymbol{c}_u^{w_1} F$. But since $\boldsymbol{c}_u^{w_1}$ is known to be an individual concept, it then follows from (590.1) that $\boldsymbol{c}_u^{w_1} \not\succeq \boldsymbol{c}_F$.

(c) By &I and ∃I, it suffices to show both $w_1 \neq \boldsymbol{w}_\alpha$ and $AppearsAt(\boldsymbol{c}_u^{w_1}, w_1)$. Note that by a theorem of possible world theory (426)★, the first conjunct of (ϑ) is equivalent to $\boldsymbol{w}_\alpha \models Fu$. But given this last fact and the previously established fact that $\neg w_1 \models Fu$, there is a proposition, namely Fu, that is true at $\boldsymbol{w}_\alpha$ but not true at w_1. Since worlds are situations, it follows by (368) that $w_1 \neq \boldsymbol{w}_\alpha$. So it remains to show $AppearsAt(\boldsymbol{c}_u^{w_1}, w_1)$. But this is immediate from a fact about appearance and world-relative concepts of individuals (587.2). ⋈

(598.2)★ (Exercise)

(599.1)★ (→) This direction follows *a fortiori* from the left-to-right direction of (598.1)★. (←) Assume the antecedent:

$$\boldsymbol{c}_u \succeq \boldsymbol{c}_F \ \&\ \exists \hat{c}(CounterpartOf(\hat{c}, \boldsymbol{c}_u) \ \&\ \hat{c} \not\succeq \boldsymbol{c}_F)$$

From the first conjunct and (553.1)★, it follows that Fu. So it remains to show $\Diamond\neg Fu$. By a fundamental theorem of possible worlds (433.1), it suffices to show $\exists w(w \models \neg Fu)$. So our proof strategy is to find a witness to this latter claim.

Suppose $\hat{c}_1$ is a witness to the second conjunct of our assumption, so that we know:

(ϑ) $CounterpartOf(\hat{c}_1, \boldsymbol{c}_u) \ \&\ \hat{c}_1 \not\succeq \boldsymbol{c}_F$

Now the first conjunct of (ϑ) implies by definition (594) that:

$$\exists v \exists w \exists w'(RealizesAt(v, \boldsymbol{c}_u, w) \ \&\ RealizesAt(v, \hat{c}_1, w'))$$

Assume ordinary object b and possible worlds w_1 and w_2 are witnesses to this existential claim. Then we know:

(ξ) $RealizesAt(b, \boldsymbol{c}_u, w_1) \ \&\ RealizesAt(b, \hat{c}_1, w_2)$

Now independently by (568.1)★, we know $RealizesAt(u, \boldsymbol{c}_u, \boldsymbol{w}_\alpha)$. This and the first conjunct of (ξ) imply, by (567), that $w_1 = \boldsymbol{w}_\alpha$ and $b = u$. But from the latter and the second conjunct of (ξ), it follows that $RealizesAt(u, \hat{c}_1, w_2)$. This and theorem (587.1), which yields as an instance that $RealizesAt(u, \boldsymbol{c}_u^{w_2}, w_2)$, jointly imply $\hat{c}_1 = \boldsymbol{c}_u^{w_2}$, by (559.1). This and the second conjunct (ϑ) imply $\boldsymbol{c}_u^{w_2} \not\succeq \boldsymbol{c}_F$. This in turn implies, by theorem (578.4), that $\boldsymbol{c}_u^{w_2} \succeq \boldsymbol{c}_{\overline{F}}$. By (590.2), this last fact implies:

(ζ) $w_2 \models \overline{F}u$

Now, independently, $\overline{F}u \to \neg Fu$ follows *a fortiori* from an appropriate instance of the modally strict theorem (137.1), and so by RN, we know $\Box(\overline{F}u \to \neg Fu)$, i.e., $\overline{F}u \Rightarrow \neg Fu$. This fact and ($\zeta$) imply, by the modal closure of possible worlds (419) and definition of modal closure (417), that $w_2 \models \neg Fu$. So $\exists w(w \models \neg Fu)$ and, hence, by a fundamental theorem of possible worlds, $\Diamond\neg Fu$. ⋈

(599.2)★ (Exercise)

(600.1) Assume $w \models (Fu \ \&\ \Diamond\neg Fu)$. Then since the laws of conjunction hold with respect to truth at a possible world (420), it follows that:

(ϑ) $w \models Fu \ \&\ w \models \Diamond\neg Fu$

Now the first conjunct of (ϑ) and the second conjunct of theorem (586.2), which asserts $\forall F(\boldsymbol{c}_u^w F \equiv w \models Fu)$, together imply $\boldsymbol{c}_u^w F$. This implies, by (590.1), that $\boldsymbol{c}_u^w \succeq \boldsymbol{c}_F$. We've therefore established the first conjunct of our desired conclusion. So it remains show:

(ξ) $\exists \hat{c}(\mathit{CounterpartOf}(\hat{c}, \boldsymbol{c}_u^w) \,\&\, \hat{c} \not\succeq \boldsymbol{c}_F \,\&\, \exists w'(w' \neq w \,\&\, \mathit{AppearsAt}(\hat{c}, w')))$

Note that from the second conjunct of (ϑ), it follows that $\exists w'(w' \models \Diamond\neg Fu)$. So by a Fundamental Theorem of Possible World Theory (433.1), it follows that $\Diamond\Diamond\neg Fu$. This implies $\Diamond\neg Fu$, by the 4$\Diamond$ schema (119.8). So by the same Fundamental Theorem, $\exists w'(w' \models \neg Fu)$. Now let w_1 be an arbitrary such possible world, so that we know $w_1 \models \neg Fu$, and consider the concept of u at w_1, i.e., $\boldsymbol{c}_u^{w_1}$. We know by (587.3) that $\mathit{IndividualConcept}(\boldsymbol{c}_u^{w_1})$. So to show ($\xi$), it suffices by &I and $\exists$I to show:

(a) $\mathit{CounterpartOf}(\boldsymbol{c}_u^{w_1}, \boldsymbol{c}_u^w)$

(b) $\boldsymbol{c}_u^{w_1} \not\succeq \boldsymbol{c}_F$

(c) $\exists w'(w' \neq w \,\&\, \mathit{AppearsAt}(\boldsymbol{c}_u^{w_1}, w'))$

(a) This is an instance of theorem (597.1).

(b) Since we know $w_1 \models \neg Fu$, it follows by the fact that possible worlds are coherent (414) and the definition of coherency (411), that $\neg w_1 \models Fu$. So by the second conjunct of (586.2), it follows that $\neg \boldsymbol{c}_u^{w_1} F$. But since $\boldsymbol{c}_u^{w_1}$ is known to be an individual concept, it then follows from (590.1) that $\boldsymbol{c}_u^{w_1} \not\succeq \boldsymbol{c}_F$.

(c) By &I and $\exists$I, it suffices to show both $w_1 \neq w$ and $\mathit{AppearsAt}(\boldsymbol{c}_u^{w_1}, w_1)$. By the first conjunct of (ϑ), we know $w \models Fu$. But we've previously established $\neg w_1 \models Fu$. So there is a proposition, namely Fu, that is true at w but not true at w_1. Since worlds are situations, it follows by (368) that $w_1 \neq w$. So it remains to show $\mathit{AppearsAt}(\boldsymbol{c}_u^{w_1}, w_1)$. But this is an instance of a fact about appearance and world-relative concepts of individuals (587.2). ⋈

(600.2) (Exercise)

(600.3) ($\rightarrow$) This direction follows *a fortiori* from the left-to-right direction of (600.1). ($\leftarrow$) Assume the antecedent:

$$\boldsymbol{c}_u^w \succeq \boldsymbol{c}_F \,\&\, \exists \hat{c}(\mathit{CounterpartOf}(\hat{c}, \boldsymbol{c}_u^w) \,\&\, \hat{c} \not\succeq \boldsymbol{c}_F)$$

To show $w \models (Fu \,\&\, \Diamond\neg Fu)$, it suffices to show both $w \models Fu$ and $w \models \Diamond\neg Fu$, in virtue of the right-to-left condition of (420.1). But from the first conjunct of our assumption and (590.2), it follows that $w \models Fu$. So it remains to show $w \models \Diamond\neg Fu$. Suppose $\hat{c}_1$ is a witness to the second conjunct of our assumption, so that we know:

(ϑ) $\mathit{CounterpartOf}(\hat{c}_1, \boldsymbol{c}_u^w)$ & $\hat{c}_1 \not\geq \boldsymbol{c}_F$

Now the first conjunct of (ϑ) implies by definition (594) that:

$$\exists v \exists w' \exists w''(\mathit{RealizesAt}(v, \boldsymbol{c}_u^w, w') \,\&\, \mathit{RealizesAt}(v, \hat{c}_1, w''))$$

Assume ordinary object b and possible worlds w_1 and w_2 are witnesses to this existential claim. Then we know:

(ξ) $\mathit{RealizesAt}(b, \boldsymbol{c}_u^w, w_1) \,\&\, \mathit{RealizesAt}(b, \hat{c}_1, w_2)$

Now independently by (587.1), we know $\mathit{RealizesAt}(u, \boldsymbol{c}_u^w, w)$. This and the first conjunct of (ξ) imply, by (567), that $w_1 = w$ and $b = u$. But from the latter and the second conjunct of (ξ), it follows that $\mathit{RealizesAt}(u, \hat{c}_1, w_2)$. This and theorem (587.1), which yields as an instance that $\mathit{RealizesAt}(u, \boldsymbol{c}_u^{w_2}, w_2)$, jointly imply $\hat{c}_1 = \boldsymbol{c}_u^{w_2}$, by (559.1). This and the second conjunct (ϑ) imply $\boldsymbol{c}_u^{w_2} \not\geq \boldsymbol{c}_F$. This in turn implies, by theorem (578.4), that $\boldsymbol{c}_u^{w_2} \geq \boldsymbol{c}_{\overline{F}}$. By (590.2), this last fact implies:

(ζ) $w_2 \models \overline{F}u$

Independently, by now familiar reasoning, we know $\overline{F}u \Rightarrow \neg Fu$. This and ($\zeta$) imply $w_2 \models \neg Fu$, by the modal closure of possible worlds. So $\exists w(w \models \neg Fu)$ and, hence, by a fundamental theorem of possible worlds (433.1), $\Diamond \neg Fu$. Then by the 5 schema (32.3), $\Box\Diamond\neg Fu$, and so by another fundamental theorem of possible worlds (433.2), $\forall w'(w' \models \Diamond\neg Fu)$. But then $w \models \Diamond\neg Fu$. ⋈

(600.4) (Exercise)

(606) [Since the number of different variables needed for the proof is relatively large, we shall use the constants $a, b, c, \ldots$ as arbitrary names and apply ∀I (80).] (→) Assume $R : F \xleftrightarrow{\text{1-1}} G$. Then by (604) we know:

(A) $\forall x(Fx \to \exists! y(Gy \,\&\, Rxy))$

(B) $\forall x(Gx \to \exists! y(Fy \,\&\, Ryx))$

We need to show:

(a) $R : F \to G$

(b) $R : F \xrightarrow{\text{1-1}} G$

(c) $R : F \underset{\text{onto}}{\longrightarrow} G$

(a) From fact (A) and definition (605.1).

(b) By (a) and definition (605.2), it suffices to show $\forall x \forall y \forall z((Fx \,\&\, Fy \,\&\, Gz) \to (Rxz \,\&\, Ryz \to x = y)$. So by ∀I, assume Fa, Fb, and Gc, and further that Rac and Rbc (to show $a = b$). From Gc and (B), we know that some object, say d, is such that:

(ϑ) $Fd \mathbin{\&} Rdc \mathbin{\&} \forall z(Fz \mathbin{\&} Rzc \to z = d)$

Since we know Fa and Rac by hypothesis, it follows from the third conjunct of (ϑ) that $a = d$. Similarly, since we know Fb and Rbc by hypothesis, it follows from the third conjunct of (ϑ) that $b = d$. Hence, $a = b$.

(c) By (a) and definition (605.3), it suffices to show $\forall x(Gx \to \exists y(Fy \mathbin{\&} Ryx))$. But this follows *a fortiori* from (B).

($\leftarrow$) Assume $R : F \xrightarrow[\text{onto}]{\text{1-1}} G$. Then by (605.2) and (605.3), we know:

(C) $R : F \to G$

(D) $\forall x \forall y \forall z((Fx \mathbin{\&} Fy \mathbin{\&} Gz) \to (Rxz \mathbin{\&} Ryz \to x = y))$

(E) $\forall x(Gx \to \exists y(Fy \mathbin{\&} Ryx))$

We have to show:

(d) $\forall x(Fx \to \exists! y(Gy \mathbin{\&} Rxy))$

(e) $\forall x(Gx \to \exists! y(Fy \mathbin{\&} Ryx))$

(d) This follows from (C) and definition (605.1).

(e) Assume Ga, by $\forall$I, to show $\exists! y(Fy \mathbin{\&} Rya)$. By (E), it follows that $\exists y(Fy \mathbin{\&} Rya)$. So suppose e is such an object, so that we know Fe and Rea. Then by &I, $\exists$I and the definition of the uniqueness quantifier, it remains only to show $\forall z(Fz \mathbin{\&} Rza \to z = e)$. So assume $Fz \mathbin{\&} Rza$, to show $z = e$. But since we now know Fz, Fe, Ga, Rza, and Rea, we may infer $z = e$ by (D). ⋈

(610.1) To show that equinumerosity$_E$ is reflexive, we have to show, by definitions (609.3) and (609.2), that:

$$\exists R[\forall u(Fu \to \exists! v(Fv \mathbin{\&} Ruv)) \mathbin{\&} \forall u(Fu \to \exists! v(Fv \mathbin{\&} Rvu))]$$

If we propose $=_E$ as our witness, then we have to show:

(a) $\forall u(Fu \to \exists! v(Fv \mathbin{\&} u =_E v))$

(b) $\forall u(Fu \to \exists! v(Fv \mathbin{\&} v =_E u))]$

Remember that the unique existence quantifiers in the consequent of both (a) and (b) are specially defined for ordinary objects in terms of $=_E$, by (609.1).

Since (b) follows from (a) by a modally strict proof (exercise) that appeals to the symmetry of $=_E$ (168.2), it suffices to show (a). Then by GEN, it suffices to show $Fu \to \exists! v(Fv \mathbin{\&} u =_E v)$. So assume Fu. We have to show, by the definition of the unique existence quantifier for ordinary objects (609.1):

$$\exists v((Fv \mathbin{\&} u =_E v) \mathbin{\&} \forall v'(Fv' \mathbin{\&} u =_E v' \to v' =_E v))$$

But if we propose that u is a witness to this claim, then we have to show:

$$(Fu \mathbin{\&} u =_E u) \mathbin{\&} \forall v'(Fv' \mathbin{\&} u =_E v' \rightarrow v' =_E u)$$

We have Fu and $u =_E u$ is a theorem. So it remains only to show:

$$\forall v'(Fv' \mathbin{\&} u =_E v' \rightarrow v' =_E u)$$

But this is trivial, by the symmetry of $=_E$. ⋈

(610.2) To show that that equinumerosity$_E$ is symmetric, assume that $F \approx_E G$ and suppose R is a witness to this fact, so that we know:

$$(\vartheta)\ \ \forall u(Fu \rightarrow \exists!v(Gv \mathbin{\&} Ruv)) \mathbin{\&} \forall u(Gu \rightarrow \exists!v(Fv \mathbin{\&} Rvu))$$

Now we want to show that there is a one-to-one relation R' that is a function from the ordinary Gs onto the ordinary Fs. Consider the converse of R, namely $[\lambda xy\, Ryx]$, which we may call R^{-1}. If we can show:

(a) $\forall u(Gu \rightarrow \exists!v(Fv \mathbin{\&} R^{-1}uv))$

(b) $\forall u(Fu \rightarrow \exists!v(Gv \mathbin{\&} R^{-1}vu))$

we're done.

To show (a), assume Gu, by GEN. But from this, the second conjunct of (ϑ) implies there is a unique ordinary object v that exemplifies F and such that Rvu. Now by β-Conversion, the definition of R^{-1} implies that for ordinary objects u and v:

$$R^{-1}uv \equiv Rvu$$

So there is a unique ordinary object v that exemplifies F and such that $R^{-1}uv$.

The proof of (b) is analogous. ⋈

(610.3) To show that equinumerosity$_E$ is transitive, assume both that $F \approx_E G$ and $G \approx_E H$. Suppose R_1 and R_2 are relations that bear witness to these facts, respectively, so that we know:

$$(\vartheta)\ \ \forall u(Fu \rightarrow \exists!v(Gv \mathbin{\&} R_1uv)) \mathbin{\&} \forall u(Gu \rightarrow \exists!v(Fv \mathbin{\&} R_1vu))$$

$$(\xi)\ \ \forall u(Gu \rightarrow \exists!v(Hv \mathbin{\&} R_2uv)) \mathbin{\&} \forall u(Hu \rightarrow \exists!v(Gv \mathbin{\&} R_2vu))$$

Now let R be the relation: $[\lambda u_1u_2\, \exists v(Gv \mathbin{\&} R_1u_1v \mathbin{\&} R_2vu_2)]$.[290] To show that R bears witness to the equinumerosity$_E$ of F and H, we must show:

[290] By our conventions for bound occurrences of restricted variables, this abbreviates:

$$[\lambda xy\, O!x \mathbin{\&} O!y \mathbin{\&} \exists z(O!z \mathbin{\&} Gz \mathbin{\&} R_1xz \mathbin{\&} R_2zy)]$$

By (254.4), we've eliminated bound occurrences of u_1 and u_2 with bound occurrences of x and y, and by (254.2), we've eliminated bound occurrences of v with bound occurrences of z.

(a) $\forall u(Fu \rightarrow \exists!v(Hv \,\&\, Ruv))$

(b) $\forall u(Hu \rightarrow \exists!v(Fv \,\&\, Rvu))$

To show (a), assume Fu, by GEN. By the first conjunct of (ϑ) and (609.1), there is an ordinary object, say b, such that both:

(c) $Gb \,\&\, R_1ub$

(d) $\forall v((Gv \,\&\, R_1uv) \rightarrow v =_E b)$

Then, instantianting b in (ξ), we know, given the first conjunct of (c), that there is an ordinary object, say c, such that both:

(e) $Hc \,\&\, R_2bc$

(f) $\forall v((Hv \,\&\, R_2bv) \rightarrow v =_E c)$

Now to prove that that there is a unique ordinary object exemplifying H to which u bears R, it suffices, by $\exists$I, the definition of the uniqueness quantifier for ordinary objects (609.1), and the fact that c is ordinary, to show that:

(g) $Hc \,\& Ruc$

(h) $\forall v((Hv \,\&\, Ruv) \rightarrow v =_E c)$

To show (g), note that Hc is the first conjunct of (e). And by the definition of R and β-Conversion, we can establish Ruc if we can show $\exists v(Gv \,\&\, R_1uv \,\&\, R_2vc)$. But (c) and (d) above establish that b is such a v. To show (h), assume $Hv \& Ruv$, to show $v =_E c$. Since Ruv, we know by the definition of R and β-Conversion that there is an ordinary object, say e, such that $Ge \,\&\, R_1ue \,\&\, R_2ev$. But the first two conjuncts imply, by (d), that $e =_E b$. From the third conjunct, R_2ev, it then follows that R_2bv. So we know $Hv \,\&\, R_2bv$. Then by (f), it follows that $v =_E c$.

To show (b), assume Hu. Then by the 2nd conjunct of (ξ), we know, for some ordinary object, say c:

(i) $Gc \,\&\, R_2cu \,\&\, \forall v((Gv \,\&\, R_2vu) \rightarrow v =_E c)$

Now if we instantiate c into the second conjunct of (ϑ), then from the first conjunct of (i), we know that for some ordinary object, say d:

(j) $Fd \,\&\, R_1dc \,\&\, \forall v((Fv \,\&\, R_1vc) \rightarrow v =_E d)$

To complete the proof of (b), it suffices, by $\exists$I and the definition of the uniqueness quantifier, to show:

$$Fd \,\&\, Rdu \,\&\, \forall v((Fv \,\&\, Rvu) \rightarrow v =_E d)$$

We have Fd as the first conjunct of (j). Moreover, we may use β-Conversion and the definition of R to conclude Rdu, for there is an ordinary object, i.e., c, such that Gc, R_1dc, and R_2cu. Since it now remains only to show $\forall v((Fv \,\&\, Rvu) \rightarrow v =_E d)$, assume $Fv \,\&\, Rvu$. From Rvu and the definition of R, it follows that some ordinary object, say e, is such that $Ge \,\&\, R_1ve \,\&\, R_2eu$. From Ge, R_2eu, and the third conjunct of (i), it follows that $e =_E c$. But then from the third conjunct of (j) it follows that $\forall v((Fv \,\&\, R_1ve) \rightarrow v =_E d)$. Instantiating to v yields $(Fv \,\&\, R_1ve) \rightarrow v =_E d$. But we already know both Fv and R_1ve. Hence $v =_E d$. ⋈

(610.4) $(\rightarrow)$ Assume $F \approx_E G$. Then by GEN, it suffices to show $H \approx_E F \equiv H \approx_E G$:

$(\rightarrow)$ Assume $H \approx_E F$, then by transitivity (610.3), $H \approx_E G$.

$(\leftarrow)$ Assume $H \approx_E G$. Then, by symmetry (610.2), our initial assumption implies $G \approx_E F$. So by transitivity, $H \approx_E F$.

$(\leftarrow)$ Assume $\forall H(H \approx_E F \equiv H \approx_E G)$. By instantiating to F we obtain: $F \approx_E F \equiv F \approx_E G$. So by reflexivity (610.1), $F \approx_E G$. ⋈

(613.1) – **(613.2)** (Exercises)

(614.1) Assume $\neg\exists uFu$ and $\neg\exists vHv$. Then pick any relation R you please. By failures of the antecedent, we know both:

$$\forall u(Fu \rightarrow \exists!v(Hv \,\&\, Ruv))$$

$$\forall u(Hu \rightarrow \exists!v(Fv \,\&\, Rvu))]$$

Hence, by definition of $\approx_E$, $F \approx_E H$. ⋈

(614.2) Assume $\exists uFu$ and $\neg\exists vHv$. Given the former, suppose b is such an ordinary object, so that we know Fb. Now assume, for reductio, that $F \approx_E H$. Then by definition of $\approx_E$, it follows that:

(ϑ) $\exists R[\forall u(Fu \rightarrow \exists!v(Hv \,\&\, Ruv)) \,\&\, \forall u(Hu \rightarrow \exists!v(Fv \,\&\, Rvu))]$

Suppose R_1 is such an R. Then from the first conjunct of (ϑ) and Fb, it follows that $\exists!v(Hv \,\&\, Ruv)$. But, clearly, this implies $\exists vHv$. Contradiction. ⋈

(616) [Note: In what follows, we use r, s, and t as additional restricted variables for ordinary objects, so that r, s, t, u, v all range over ordinary objects.]

Assume that $F \approx_E G$, Fu, and Gv. The first assumption implies, by definitions (609.3) and (609.2), that there is a relation, say R, that is a one-to-one correspondence$_E$ between the ordinary objects of F and the ordinary objects of G, i.e.,

Fact 1: $\forall r(Fr \rightarrow \exists!s(Gs \,\&\, Rrs)) \,\&\, \forall r(Gr \rightarrow \exists!s(Fs \,\&\, Rsr))$

So by Exercise (611.4), it follows that R is a one-to-one function$_E$ from the ordinary objects of F onto the ordinary objects of G. By definition (611.2), this entails:

Fact 2: $\forall r \forall s \forall t((Fr \,\&\, Fs \,\&\, Gt) \rightarrow (Rrt \,\&\, Rst \rightarrow r =_E s))$

Now we want to show that $F^{-u} \approx_E G^{-v}$. By definitions (609.3) and (609.2), we have to show:

Claim: There is a relation R' such that:

(A) $\forall r(F^{-u}r \rightarrow \exists!s(G^{-v}s \,\&\, R'rs))$

(B) $\forall r(G^{-v}r \rightarrow \exists!s(F^{-u}s \,\&\, R'sr))$

We proceed by showing that the **Claim** holds in each of two mutually exclusive and jointly exhaustive cases: Ruv and $\neg Ruv$.

Case 1: Ruv. Then we show R itself is the witness to the **Claim**, i.e., that both (A) and (B) hold with respect to R. In what follows, we use a, b, c, d as constants for ordinary objects.

(A) By $\forall$I, suppose $F^{-u}a$, to show $\exists!s(G^{-v}s \,\&\, Ras))$. Then Fa and $a \neq_E u$, by the definition of F^{-u} (615). But since Fa and R is a witness to the equinumerosity$_E$ of F and G, we know that there is a unique ordinary object that exemplifies G and to which a bears R. Let b be such an object, so that we know:

(ζ) $Gb \,\&\, Rab \,\&\, \forall t(Gt \,\&\, Rat \rightarrow t =_E b)$

We can now show, using b as a witness, that $\exists!s(G^{-v}s \,\&\, Ras)$. Since we now know Fa, Fu, Gb, Rab, Ruv, and $a \neq_E u$, it follows by **Fact 2** that $b \neq v$, on pain of contradiction. Hence $b \neq_E v$, by (169). Since we have that Gb and $b \neq_E v$, it follows that $G^{-v}b$. So we've established $G^{-v}b$ and Rab. It remains to show b is unique. By GEN, it suffices to show $(G^{-v}t \,\&\, Rat) \rightarrow t =_E b$. So suppose $G^{-v}t$ and Rat. Then by definition of G^{-v}, it follows that Gt. But then $t =_E b$, by the last conjunct of (ζ).

(B) By $\forall$I, assume $G^{-v}d$, to show $\exists!s(F^{-u}s \,\&\, Rsd)$. Then, by definition of G^{-v} (615), we know Gd and $d \neq_E v$. From Gd and the second conjunct of **Fact 1** (R witnesses the equinumerosity$_E$ of F and G), we know $\exists!s(Fs \,\&\, Rsd)$. So suppose c is such an object, so that we know:

(ω) $Fc \,\&\, Rcd \,\&\, \forall t(Ft \,\&\, Rtd \rightarrow t =_E c)$

We can now show, using c as a witness, that $\exists!s(F^{-u}s \,\&\, Rsr)$. Since we now know Fc, Gd, Gv, Rcd, Ruv, and $d \neq_E v$, it follows by the first conjunct of **Fact 1** (i.e., R is a function$_E$ from the ordinary Fs to the ordinary Gs), that $c \neq u$, on pain of contradiction. Hence $c \neq_E u$ by (169) and, given Fc, we have $F^{-u}c$. So we have established $F^{-u}c$ and Rcd. It then remains to prove c is unique. By GEN, it suffices to show $(F^{-u}t \,\&\, Rtd) \to t =_E c$. So suppose $F^{-u}t$ and Rtd. Then Ft, by definition of F^{-u}. So by the last conjunct of (ω), $t =_E c$.

Case 2: $\neg Ruv$. Since we've assumed Fu and Gv, we therefore know by **Fact 1** both:

> $\exists! s(Gs \,\&\, Rus)$, i.e., there is a unique ordinary object that exemplifies G to which u bears R, and
>
> $\exists! s(Fs \,\&\, Rsv)$, i.e., there is a unique ordinary object that exemplifies F and that bears R to v.

Let b be a witness to the first and a be a witness to the second, so that we know both:

(ξ) $Gb \,\&\, Rub \,\&\, \forall t(Gt \,\&\, Rut \to t =_E b)$

(ϑ) $Fa \,\&\, Rav \,\&\, \forall t(Ft \,\&\, Rtv \to t =_E a)$

Now let R_1 be the relation:

$$[\lambda rs\ (r \neq_E u \,\&\, s \neq_E v \,\&\, Rrs) \vee (r =_E a \,\&\, s =_E b)]$$

Since this λ-expression is well-formed, we know R_1 exists. So we now show that R_1 is a witness to our **Claim** by showing both (A) and (B) hold with respect to R_1.

(A) By $\forall$I, assume $F^{-u}c$. Then by definition of F^{-u} (615), we know that Fc and $c \neq_E u$. Now we have to show:

> **Subclaim 1**: There is an ordinary object s such that:
>
> (i) $G^{-v}s$
>
> (ii) $R_1 cs$
>
> (iii) $\forall t(G^{-v}t \,\&\, R_1 ct \to t =_E s)$

We now show that **Subclaim 1** holds by finding a witness in each of two subcases, Rcv and $\neg Rcv$:

> Subcase 1: Rcv. Since we now know Fc and Rcv, it follows that $c =_E a$, by the last conjunct of (ϑ). Then we show b is a witness to **Subclaim 1**:
>
> (i)′ Since we know Gb from (ξ), all we have to do to establish $G^{-v}b$ is to show $b \neq_E v$. But we also have Rub from (ξ) and we know $\neg Ruv$ (Case 2). Hence, $b \neq v$, on pain of contradiction, and so $b \neq_E v$, by (169).

(ii)′ To show R_1cb, we need to establish, by β-Conversion:

$$(c \neq_E u \,\&\, b \neq_E v \,\&\, Rcb) \vee (c =_E a \,\&\, b =_E b)$$

But the conjuncts of the right disjunct are true: $c =_E a$ was established in the initial reasoning in Subcase 1, and $b =_E b$ and by the laws of identity$_E$ and the fact that b is ordinary.

(iii)′ By GEN, it suffices to show $(G^{-v}t \,\&\, R_1ct) \rightarrow t =_E b$. So suppose $G^{-v}t$ and R_1ct. The first implies both Gt and $t \neq_E v$, and the second implies, by β-Conversion:

$$(c \neq_E u \,\&\, t \neq_E v \,\&\, Rct) \vee (c =_E a \,\&\, t =_E b)$$

Suppose, for reductio, that the left disjunct is true. Then its second conjunct and third conjuncts, i.e., $t \neq_E v$ and Rct, and the known facts that Fc, Gt, Gv, and Rcv jointly contradict the first conjunct of **Fact 1** (that R is a function$_E$ from the ordinary objects of F to the ordinary objects of G). So the right disjunct is true and entails $t =_E b$.

Subcase 2: $\neg Rcv$. Recall that the assumption of (A) is $F^{-u}c$, which implies Fc and $c \neq_E u$. So we know by the definition of R and the fact that Fc that there is a unique ordinary object which exemplifies G and to which c bears R. Suppose d is such an object, so that we know:

(a) $Gd \,\&\, Rcd \,\&\, \forall t(Gt \,\&\, Rct \rightarrow t =_E d)$

We now show that d is a witness to **Subclaim 1**:

(i)′ Since Gd, then to establish that $G^{-v}d$ we show $d \neq_E v$. But Rcd by (a), and $\neg Rcv$ in Subcase 2. So $d \neq v$, on pain of contradiction and, hence, $d \neq_E v$ by (169).

(ii)′ To show R_1cd, we need to establish, by β-Conversion:

$$(c \neq_E u \,\&\, d \neq_E v \,\&\, Rcd) \vee (c =_E a \,\&\, d =_E b)$$

But the conjuncts of the left disjunct are true: $c \neq_E u$ follows from our assumption in (A) that $F^{-u}c$; we proved $d \neq_E v$ in (i)′; and Rcd holds by (a).

(iii)′ By GEN, it suffices to show $(G^{-v}t \,\&\, R_1ct) \rightarrow t =_E d$. So assume $G^{-v}t$ and R_1ct. The first implies both Gt and $t \neq_E v$, and the second implies, by β-Conversion:

$$(c \neq_E u \,\&\, t \neq_E v \,\&\, Rct) \vee (c =_E a \,\&\, t =_E b)$$

Suppose, for reductio, that the right disjunct is true. Then its first conjunct, $c =_E a$, implies Rcv by the 2nd conjunct of (ϑ), thus contradicting $\neg Rcv$ (Subcase 2). So the left disjunct is true. Since we now have Gt and Rct, we know that $t =_E d$, by (a).

(B) By $\forall$I, assume $G^{-v}d$. Then by definition of G^{-v}, we know Gd and $d \neq_E v$. We want to show:

Subclaim 2: There is an ordinary object s such that:

(i) $F^{-u}s$

(ii) $R_1 sd$

(iii) $\forall t(F^{-u}t \,\&\, R_1 td \rightarrow t =_E s)$

We now show that **Subclaim** 2 holds by finding a witness in each of two subcases, Rud and $\neg Rud$:

Subcase 1: Rud. Then since we now know both Gd and Rud, it follows that $d =_E b$, by the last conjunct of (ξ). We now show that a is a witness to **Subclaim** 2:

(i)′ Since we know Fa by (ϑ), all we have to do to establish $F^{-u}a$ is to show $a \neq_E u$. But we also know Rav by (ϑ), and we know $\neg Ruv$ (Case 2). So, on pain of contradiction, $a \neq u$. Hence, $a \neq_E u$ by now familiar reasoning.

(ii)′ To show $R_1 ad$, we need to establish, by β-Conversion:

$$(a \neq_E u \,\&\, d \neq_E v \,\&\, Rad) \vee (a =_E a \,\&\, d =_E b)$$

But the conjuncts of the right disjunct are true: the first follows by the laws of identity$_E$ and the fact that a is ordinary, and the second follows from $d =_E b$, which we established in the initial reasoning in Subcase 1.

(iii)′ By GEN, it suffices to show $(F^{-u}t \,\&\, R_1 td) \rightarrow t =_E a$. So suppose $F^{-u}t$ and $R_1 td$. Then by the first we know Ft and $t \neq_E u$, and by the second, it follows from β-Conversion that:

$$(t \neq_E u \,\&\, d \neq_E v \,\&\, Rtd) \vee (t =_E a \,\&\, d =_E b)$$

Suppose, for reductio, that the the left disjunct holds. Then the first and third conjuncts, $t \neq_E u$ and Rtd, and the facts that Ft, Fu, Gd, and Rud jointly contradict **Fact** 2. So the right disjunct is true, from which it follows that $t =_E a$.

Subcase 2: $\neg Rud$. We know from our initial reasoning in (B) that Gd. So by the second conjunct of **Fact 1**, $\exists!s(Fs \,\&\, Rsd)$. Let c be such an ordinary object, so that we know:

(b) $Fc \,\&\, Rcd \,\&\, \forall t(Ft \,\&\, Rtd \rightarrow t =_E c)$

We now show that c is a witness to **Subclaim** 2:

(i)′ Since we know Fc, all we have to do to establish that $F^{-u}c$ is to show $c \neq_E u$. But Rcd by (b), and $\neg Rud$ (Subcase 2). So it follows that $c \neq u$, on pain of contradiction. Hence, $c \neq_E u$, by now familiar reasoning.

(ii)′ To show R_1cd, we need to establish, by β-Conversion:

$$(c \neq_E u \,\&\, d \neq_E v \,\&\, Rcd) \vee (c =_E a \,\&\, d =_E b)$$

But the conjuncts of the left disjunct are true: $c \neq_E u$ holds by the reasoning in (i)′; $d \neq_E v$ was established in the initial reasoning of Subcase 2; and Rcd holds by (b).

(iii)′ Suppose $F^{-u}t$ and R_1td, to show $t =_E c$. The first implies Ft and $t \neq_E u$, while the second implies, by β-Conversion, that:

$$(t \neq_E u \,\&\, d \neq_E v \,\&\, Rtd) \vee (t =_E a \,\&\, d =_E b)$$

Suppose, for reductio, that the right disjunct true. Then its second conjunct implies $d = b$, by (169). So by the second conjunct of (ξ), it follows that Rud, contradicting $\neg Rud$ (Subcase 2). So the left disjunct is true. Since we now have Ft and Rtd, it follows from (b) that $t =_E c$. ⋈

(617) [In what follows, we continue to use r, s, and t as additional restricted variables for ordinary objects, so that r, s, t, u, v all range over ordinary objects.] Assume that $F^{-u} \approx_E G^{-v}$, Fu, and Gv. Suppose R is a witness to the equinumerosity$_E$ of F^{-u} and G^{-v}. Then by now familiar reasoning, we know:

Fact 1: $\forall r(F^{-u}r \rightarrow \exists!s(G^{-v}s \,\&\, Rrs)) \,\&\, \forall r(G^{-v}r \rightarrow \exists!s(F^{-u}s \,\&\, Rsr))$

Fact 2: $\forall r \forall s \forall t((F^{-u}r \,\&\, F^{-u}s \,\&\, G^{-v}t) \rightarrow (Rrt \,\&\, Rst \rightarrow r =_E s))$

We want to show $F \approx_E G$, i.e.,

Claim: There is a relation R' such that:

(A) $\forall r(Fr \rightarrow \exists!s(Gs \,\&\, R'rs))$

(B) $\forall r(Gr \rightarrow \exists!s(Fs \,\&\, R'sr))$

Consider the following relation R_2:

$$[\lambda xy\,(F^{-u}x \,\&\, G^{-v}y \,\&\, Rxy) \vee (x =_E u \,\&\, y =_E v)]$$

Since the λ-expression is well-formed, we know R_2 exists. We now establish that R_2 is a witness to the **Claim**.

(A) Assume Fr. We want to show:

Subclaim 1: $\exists!s(Gs \,\&\, R_2rs)$

We show **Subclaim 1** holds in the two cases $r =_E u$ and $r \neq_E u$.

Case 1. $r =_E u$. Then we establish that v is a witness to **Subclaim 1**:

(i) Gv is true by hypothesis.

(ii) Since $r =_E u$ is true in the present case and $v =_E v$ is true by the laws of identity$_E$, we have $r =_E u \,\&\, v =_E v$. If we disjoin this conjunction with the falsehood $F^{-u}r \,\&\, G^{-v}v \,\&\, Rrv$, we have, by $\vee$I:

$$(F^{-u}r \,\&\, G^{-v}v \,\&\, Rrv) \vee (r =_E u \,\&\, v =_E v)$$

So by β-Conversion and the definition of R_2, it follows that R_2rv.

(iii) Suppose Gt and R_2rt (to show $t =_E v$). Then by definition of R_2 we know:

$$(F^{-u}r \,\&\, G^{-v}t \,\&\, Rrt) \vee (r =_E u \,\&\, t =_E v)$$

Suppose, for reductio, that the first disjunct is true. Then $F^{-u}r$ implies Fr and $r \neq_E u$, which contradicts the present case ($r =_E u$). So the right disjunct holds, and we know $t =_E v$.

Case 2. $r \neq_E u$. Now we know by hypothesis that Fr. So by the first conjunct of **Fact 1**, it follows that $\exists!s(G^{-v}s \,\&\, Rrs)$. Suppose b is such an ordinary object, so that we know:

(ϑ) $G^{-v}b \,\&\, Rrb \,\&\, \forall t(G^{-v}t \,\&\, Rrt \rightarrow t =_E b)$

We then show that b is a witness to **Subclaim 1**:

(i) Since $G^{-v}b$, it follows that Gb.

(ii) Since we know Fr (by assumption) and $r \neq_E u$ (present case), it follows that $F^{-u}r$. From this and the first two conjuncts of (ϑ), we therefore know $F^{-u}r \,\&\, G^{-v}b \,\&\, Rrb$. We can therefore disjoin this conjunction with the falsehood $r =_E u \,\&\, b =_E v$, to obtain by $\vee$I:

$$(F^{-u}r \,\&\, G^{-v}b \,\&\, Rrb) \vee (r =_E u \,\&\, b =_E v)$$

So by β-Conversion and the definition of R_2, it follows that R_2rb.

(iii) Assume Gt and R_2rt (to show $t =_E b$). Hence, by definition of R_2, we know:

$$(F^{-u}r \,\&\, G^{-v}t \,\&\, Rrt) \vee (r =_E u \,\&\, t =_E v)$$

But the right disjunct implies $r =_E u$, which contradicts the present case. Hence the left disjunct is true. But its second and third conjuncts imply, by the third conjunct of (ϑ), that $t =_E b$.

(B) Assume Gr. We want to show:

Subclaim 2: $\exists! s(Fs \,\&\, R_2sr)$

We show that **Subclaim 2** holds in the two cases $r =_E v$ and $r \neq_E v$:

Case 1. $r =_E v$. Then we can establish that u is a witness to **Subclaim 2**:

(i) Fu is true by hypothesis.

(ii) Since $u =_E u$ by the laws of identity$_E$, we know $u =_E u \,\&\, r =_E v$. We may disjoin this with the falsehood $F^{-u}u \,\&\, G^{-v}r \,\&\, Rur$ to establish:

$$(F^{-u}u \,\&\, G^{-v}r \,\&\, Rur) \vee (u =_E u \,\&\, r =_E v)$$

Hence by β-Conversion and the definition of R_2, it follows that R_2ur.

(iii) Assume Ft and R_2tr (to show: $t =_E u$). Then by the definition of R_2, we know:

$$(F^{-u}t \,\&\, G^{-v}r \,\&\, Rtr) \vee (t =_E u \,\&\, r =_E v)$$

Assume, for reductio, that the left disjunct is true. Then $G^{-v}r$ implies Gr and $r \neq_E v$, which contradicts the the present case ($r =_E v$). So the right disjunct holds. Hence $t =_E u$.

Case 2. $r \neq_E v$. Now since Gr, it follows by the second conjunct of **Fact 1** that $\exists! s(F^{-u}s \,\&\, Rsr)$. Suppose a is such an ordinary object, so that we know:

(ξ) $F^{-u}a \,\&\, Rar \,\&\, \forall t(F^{-u}t \,\&\, Rtr \rightarrow t =_E a)$

We now establish that a is a witness to **Subclaim 2**:

(i) Fa is true, given that $F^{-u}a$.

(ii) Now Gr holds by hypothesis, and in the present case, $r \neq_E v$. Hence $G^{-v}r$. So by (ξ), we know $F^{-u}a \,\&\, G^{-v}r \,\&\, Rar$. We may disjoin this with the falsehood $a =_E u \,\&\, r =_E v$ to infer:

$$(F^{-u}a \,\&\, G^{-v}r \,\&\, Rar) \vee (a =_E u \,\&\, r =_E v)$$

So by β-Conversion and the definition of R_2, it follows that R_2ar.

(iii) Suppose that Ft and R_2tr (to show that $t =_E a$). Then by the definition of R_2tr and β-Conversion, we know:

$$(F^{-u}t \,\&\, G^{-v}r \,\&\, Rtr) \vee (t =_E u \,\&\, r =_E v)$$

But the right disjunct is inconsistent with the present case. So the left disjunct holds. Its first and third conjuncts imply $t =_E a$, by the third conjunct of (ξ). ⋈

(619.1) Assume $G \approx_E H$. We want to show $Numbers(x, G) \equiv Numbers(x, H)$.
($\rightarrow$) Assume $Numbers(x, G)$. Then, by definition (618), we know both $A!x$ and:

(ϑ) $\forall F(xF \equiv F \approx_E G)$

So to show $Numbers(x, H)$, it suffices, by GEN, to show $xF \equiv F \approx_E H$. Now (ϑ) implies $xF \equiv F \approx_E G$. And our first assumption, $G \approx_E H$, implies, by (610.4) that $F \approx_E G \equiv F \approx_E H$. Hence, by a biconditional syllogism, $xF \equiv F \approx_E H$.
($\leftarrow$) By analogous reasoning. ⋈

(619.2) Assume $Numbers(x, G)$ and $Numbers(x, H)$. Then by definition (618), these imply, respectively:

$\forall F(xF \equiv F \approx_E G)$

$\forall F(xF \equiv F \approx_E H)$

Hence, by (83.11) and (83.10), it follows that $\forall F(F \approx_E G \equiv F \approx_E H)$. So by (610.4), $G \approx_E H$. ⋈

(619.3) Assume $G \equiv_E H$. Then, by (613.1), $G \approx_E H$. Hence, our conclusion follows by a previous theorem (619.1). ⋈

(620) Assume both $Numbers(x, G)$ and $Numbers(y, H)$. By definition (618), we know, respectively:

(a) $A!x \,\&\, \forall F(xF \equiv F \approx_E G))$

(b) $A!y \,\&\, \forall F(yF \equiv F \approx_E H))$

($\rightarrow$) Assume $x = y$. Then by Rule SubId, it follows from (a) that:

(c) $A!y \,\&\, \forall F(yF \equiv F \approx_E G))$

Hence, by (83.11) and (83.10), the right conjuncts of (b) and (c) imply:

(d) $\forall F(F \approx_E H \equiv F \approx_E G)$

From (d) it follows by (610.4) that $H \approx_E G$. So by the symmetry of $\approx_E$, it follows that $G \approx_E H$.
($\leftarrow$) Assume:

(e) $G \approx_E H$

Since we know $A!x$ and $A!y$ by the left conjuncts of (a) and (b), it suffices by theorem (172.1) to show $\forall F(xF \equiv yF)$, and by GEN, that $xF \equiv yF$:

($\rightarrow$) Assume xF. Then by the right conjunct of (a), it follows that $F \approx_E G$. But from this and (e), it follows that $F \approx_E H$, by the transitivity of $\approx_E$ (610.3). Hence, by the right conjunct of (b), it follows that yF.

($\leftarrow$) Assume yF. Then by the right conjunct of (b), it follows that $F \approx_E H$. Now since $\approx_E$ is symmetric (610.2), (e) implies that $H \approx_E G$. But then by transitivity of $\approx_E$, it follows from $F \approx_E H$ and $H \approx_E G$ that $F \approx_E G$. So by the right conjunct of (a), xF. ⋈

(622) Assume $\exists u \exists v(u \neq v)$. Let c and d be such ordinary objects, so that we know $c \neq d$. Then consider the properties $[\lambda x\, x =_E c]$ and $[\lambda x\, x =_E d]$. By Comprehension for Abstract Objects (39), we know that both:

$$\exists x(A!x \,\&\, \forall F(xF \equiv F \approx_E [\lambda x\, x =_E c])$$

$$\exists x(A!x \,\&\, \forall F(xF \equiv F \approx_E [\lambda x\, x =_E d])$$

Let a and b be such objects, so that we know, respectively:

$$A!a \,\&\, \forall F(aF \equiv F \approx_E [\lambda x\, x =_E c])$$

$$A!b \,\&\, \forall F(bF \equiv F \approx_E [\lambda x\, x =_E d])$$

Hence by (618), we know, respectively:

(ϑ) $Numbers(a, [\lambda x\, x =_E c])$

(ξ) $Numbers(b, [\lambda x\, x =_E d])$

Our first goal is to show $a = b$. But by the modally strict Fact underlying Hume's Principle (620), it now follows from (ϑ) and (ξ) that:

$$a = b \equiv [\lambda x\, x =_E c] \approx_E [\lambda x\, x =_E d]$$

But, clearly, $[\lambda x\, x =_E c] \approx_E [\lambda x\, x =_E d]$: if we pick the witness R to be $=_E$ it is straightforward to show that R is a one-to-one correspondence$_E$ between the ordinary objects exemplifying $[\lambda x x =_E c]$ and the ordinary objects exemplifying $[\lambda x\, x =_E d]$ (exercise). Hence $a = b$, and so it follows from (ξ) that:

(ζ) $Numbers(a, [\lambda x\, x =_E d])$

So, given (ϑ) and (ζ), it remains, by $\exists$I, to show $\neg[\lambda x\, x =_E c] \equiv_E [\lambda x\, x =_E d]$. So we have to show that some ordinary object exemplifies one but not the other. We leave it as an exercise to show either that c exemplifies $[\lambda x\, x =_E c]$ but not $[\lambda x\, x =_E d]$, or that d exemplifies $[\lambda x\, x =_E d]$ but not $[\lambda x\, x =_E c]$. ⋈

(624) Assume *NaturalCardinal*(x). Then, by definition (623), there is a property, say P, such that *Numbers*(x,P). Hence, by definition (618):

(ϑ) $A!x \,\&\, \forall G(xG \equiv G \approx_E P)$

By GEN, it suffices to show $xF \equiv Numbers(x,F)$. ($\rightarrow$) Assume xF. By definition (618), we have to show $A!x \,\&\, \forall H(xH \equiv H \approx_E F)$. Since we know $A!x$, it remains, by GEN, to show $xH \equiv H \approx_E F$:

($\rightarrow$) Assume xH. Then by the second conjunct of (ϑ), $H \approx_E P$. But since xF by assumption, it also follows from the second conjunct of (ϑ) that $F \approx_E P$. Hence $H \approx_E F$, by symmetry and transitivity of equinumerosity$_E$.

($\leftarrow$) Assume $H \approx_E F$. But xF by assumption, which by the second conjunct of (ϑ) implies $F \approx_E P$. Hence, $H \approx_E P$. So again by the second conjunct of (ϑ), xH.

($\leftarrow$) Assume *Numbers*(x,F). Then by definition (618), it follows that $\forall G(xG \equiv G \approx_E F)$. But since $F \approx_E F$, it follows that xF. ⋈

(625.1) – **(625.3)** (Exercises)

(627) We adapt reasoning developed in (241). $\overline{L}$ is the negation of L, where L was defined as $[\lambda x\, E!x \rightarrow E!x]$ (140). Now abbreviate $[\lambda x\, E!x \,\&\, \Diamond\neg E!x]$ as P. Then by axiom (32.4), β-Conversion, and the Rule of Substitution, we know:

(ϑ) $\Diamond\exists xPx \,\&\, \Diamond\neg\exists xPx$

Since $\overline{L}$ is an impossible property (i.e., necessarily, nothing exemplifies $\overline{L}$), we can establish both of the following:

(ζ) $\Diamond\overline{L} \approx_E P$
Proof. We first establish $\neg\exists xPx \rightarrow \overline{L} \approx_E P$. So assume $\neg\exists xPx$. *A fortiori*, $\neg\exists uPu$. Now we know, by definition of $\overline{L}$, that $\neg\exists x\overline{L}x$. Again, *a fortiori*, $\neg\exists u\overline{L}u$. So we may invoke (614.1) to conclude $P \approx_E \overline{L}$, which by the symmetry of $\approx_E$ (610.3), yields $\overline{L} \approx_E P$. So by Conditional Proof, $\neg\exists xPx \rightarrow \overline{L} \approx_E P$, and by RM$\Diamond$, $\Diamond\neg\exists xPx \rightarrow \Diamond\overline{L} \approx_E P$. But we know $\Diamond\neg\exists xPx$, by (ϑ). Hence $\Diamond\overline{L} \approx_E P$.

(ξ) $\Diamond\overline{L} \not\approx_E P$, i.e., $\Diamond\neg\overline{L} \approx_E P$
Proof. We begin by first proving $\exists uPu \rightarrow \neg\overline{L} \approx_E P$. So assume $\exists uPu$. But we also know $\neg\exists x\overline{L}x$, which implies, *a fortiori*, that $\neg\exists u\overline{L}u$. So we may

invoke (614.2) to conclude $\neg P \approx_E \overline{L}$, which by symmetry yields $\neg\overline{L} \approx_E P$. Hence, $\exists u Pu \to \neg\overline{L} \approx_E P$. By RM$\Diamond$, this implies $\Diamond\exists u Pu \to \Diamond\neg\overline{L} \approx_E P$. But we know $\Diamond\exists x Px$, by (ϑ). But if $\Diamond\exists x Px$, then $\Diamond\exists u Pu$ (exercise). Hence $\Diamond\neg\overline{L} \approx_E P$.

If we conjoin the (ζ) and (ξ) and apply (119.12), it follows that:

$$\Diamond(\overline{L} \approx_E P \,\&\, \Diamond\neg\overline{L} \approx_E P) \qquad \bowtie$$

(629) By the reasoning in (627), when P is $[\lambda x E!x \& \Diamond\neg E!x]$ and $\overline{L}$ is the negation of $[\lambda x\, E!x \to E!x]$, then both:

(ϑ) $\Diamond\overline{L} \approx_E P$

(ζ) $\Diamond\neg\overline{L} \approx_E P$

Now, by (625.1), we know $\exists x Numbers(x, P)$. So suppose a is such an object, so that we know $Numbers(a, P)$. If we can show $\neg\Box Numbers(a, P)$, then by &I and two applications of $\exists$I, we're done. For reductio, assume $\Box Numbers(a, P)$. Then by definition (618), $\Box(A!a \,\&\, \forall F(aF \equiv F \approx_E P))$. By (111.3), $\Box A!a$ and $\Box\forall F(aF \equiv F \approx_E P)$. By CBF (122.2), the latter implies $\forall F\Box(aF \equiv F \approx_E P)$. Instantiating to $\overline{L}$, we then know $\Box(a\overline{L} \equiv \overline{L} \approx_E P)$. By (111.4), this implies both:

(A) $\Box(a\overline{L} \to \overline{L} \approx_E P)$,

(B) $\Box(\overline{L} \approx_E P \to a\overline{L})$

By the modally strict laws of contraposition and the Rule of Subsititution, (A) implies (C):

(C) $\Box(\neg\overline{L} \approx_E P \to \neg a\overline{L})$

But, then (C) and (ζ) imply $\Diamond\neg a\overline{L}$, by (117.5). So $\neg a\overline{L}$, by (126.8). Analogously, (B) and (ϑ) imply $\Diamond a\overline{L}$, by (117.5). So $a\overline{L}$, by (126.3). Contradiction. $\bowtie$

(630.1)$\star$ – **(630.4)**$\star$ (Exercises) [Hint: Follow the examples set in (217)$\star$ and (242)$\star$.]

(632)$\star$ It follows from (620) by several applications of GEN that:

$$\forall G\forall H\forall x\forall y[(Numbers(x, G) \,\&\, Numbers(y, H)) \to (x = y \equiv G \approx_E H)]$$

If instantiate $\forall G$ to F, $\forall H$ to G, $\forall x$ to $\#F$, and $\forall y$ to $\#G$, we obtain:

$$(Numbers(\#F, F) \,\&\, Numbers(\#G, G)) \to (\#F = \#G \equiv F \approx_E G)$$

By (630.2)★, we know both $Numbers(\#F, F)$ and $Numbers(\#G, G)$. So by &I and MP, $\#F = \#G \equiv F \approx_E G$. ⋈

(633)★ Assume $F \equiv_E G$. Then $F \approx_E G$, by (613.1). So $\#F = \#G$, by (632)★. ⋈

(634)★ Assume $NaturalCardinal(x)$. Then, by (624), $\forall F(xF \equiv Numbers(x, F))$. But, by applying GEN to (630.1)★ and taking an alphabetic variant of the result, we have $\forall F(Numbers(x, F) \equiv x = \#F)$. Hence, by the laws of biconditionals (83.10), $\forall F(xF \equiv x = \#F)$. ⋈

(635) (Exercise)

(637)★ By (630.2)★, we know $Numbers(\#[\lambda u\, u \neq_E u], [\lambda u\, u \neq_E u])$. Hence:

$$\exists G(Numbers(\#[\lambda u\, u \neq_E u], G))$$

So by definition of Zero (636), $\exists G Numbers(0, G)$. Hence by definition (623), $NaturalCardinal(0)$. ⋈

(638)★ (→) Assume $0F$. Then, by definition of 0, $\#[\lambda u\, u \neq_E u]$ encodes F. So by (630.3)★, $F \approx_E [\lambda u\, u \neq_E u]$. Now assume, for reductio, that $\exists u F u$. Then since we know $\neg\exists y([\lambda u\, u \neq_E u]y)$ by (635), it follows *a fortiori* that no ordinary object exemplifies this property, i.e., that $\neg\exists v([\lambda u\, u \neq_E u]v)$. Hence it follows from (614.2) (by substituting $[\lambda u\, u \neq_E u]$ for H), that $\neg(F \approx_E [\lambda u\, u \neq_E u])$. Contradiction.

(←) Suppose $\neg\exists u F u$. Then, as we've just seen, since nothing whatsoever exemplifies $[\lambda u\, u \neq_E u]$ (635), no ordinary objects exemplify it. So $\neg\exists v([\lambda u\, u \neq_E u]v)$. Hence by (614.1), $F \approx_E [\lambda u\, u \neq_E u]$. It then follows by (630.3)★, that $\#[\lambda u\, u \neq_E u]$ encodes F. So, by definition, $0F$. ⋈

(639.1)★ (→) Assume $\neg\exists u F u$. By definition (618), we have to show:

(a) $A!0$

(b) $\forall G(0G \equiv G \approx_E F)$

(a) By (630.2)★, we know $Numbers(\#[\lambda u\, u \neq_E u], [\lambda u\, u \neq_E u])$. So, by definition of Zero, $Numbers(0, [\lambda u\, u \neq_E u])$. So by definition (618), it follows that $A!0$.

(b) By GEN, it suffices to show $0G \equiv G \approx_E F$:

(→) Assume $0G$. Then by (638)★, $\neg\exists u G u$. From this and our global assumption that $\neg\exists u F u$, it follows by (614.1) that $G \approx_E F$.

(←) Assume $G \approx_E F$. Assume, for reductio, that $\neg 0G$. Then by (638)★, biconditional syllogism, and double-negation rules, $\exists u G u$. But if we conjoin this with our global assumption $\neg\exists u F u$, then it follows by theorem (614.2) that $\neg G \approx_E F$. Contradiction.

($\leftarrow$) By (637)★, Zero is a natural cardinal. So it follows from (634)★ that $0F \equiv Numbers(0,F)$, and by commutativity of the biconditional, $Numbers(0,F) \equiv 0F$. This and the previous theorem (638)★ imply $Numbers(0,F) \equiv \neg\exists uFu$, by a biconditional syllogism. *A fortiori*, $Numbers(0,F) \to \neg\exists uFu$. ⋈

(639.2)★ By (639.1)★, we know $\neg\exists uFu \equiv Numbers(0,F)$. And as an instance of (630.1)★, we know that $Numbers(0,F) \equiv 0 = \#F$. So by biconditional syllogism, $\neg\exists uFu \equiv 0 = \#F$. But then by the modally strict symmetry of identity and the Rule of Substitution, $\neg\exists uFu \equiv \#F = 0$. ⋈

(642) Assume that both x and y precede z. By the definition of predecessor (640), it follows that there are properties and ordinary objects, say P, Q, a, b, such that:

(ϑ) $Pa \,\&\, Numbers(z,P) \,\&\, Numbers(x,P^{-a})$

(ξ) $Qb \,\&\, Numbers(z,Q) \,\&\, Numbers(y,Q^{-b})$

The second conjuncts of (ϑ) and (ξ) jointly yield $P \approx_E Q$, by (619.2). Since we also know Pa and Qb, it follows by lemma (616) that $P^{-a} \approx_E Q^{-b}$. But, separately, the third conjuncts of (ϑ) and (ξ) jointly imply $x = y \equiv P^{-a} \approx_E Q^{-b}$, by Fact (620) underlying Hume's Principle. Hence $x = y$. ⋈

(643) Assume that x precedes both y and z. By the definition of predecessor (640), it follows that there are properties and ordinary objects, say P, Q, a, b, such that:

(ϑ) $Pa \,\&\, Numbers(y,P) \,\&\, Numbers(x,P^{-a})$

(ξ) $Qb \,\&\, Numbers(z,Q) \,\&\, Numbers(x,Q^{-b})$

Now the third conjuncts of (ϑ) and (ξ) jointly imply $P^{-a} \approx_E Q^{-b}$, by (619.2). Since we also know Pa and Qb, it follows by lemma (617) that $P \approx_E Q$. But independently, the second conjuncts of (ϑ) and (ξ) jointly imply $y = z \equiv P \approx_E Q$, by Fact (620) underlying Hume's Principle. Hence $y = z$. ⋈

(647.1) Assume Rxy. By (645), we have to show:

$$\forall F[\forall z(Rxz \to Fz) \,\&\, Hereditary(F,R) \to Fy]$$

So by GEN, assume $\forall z(Rxz \to Fz)$ and $Hereditary(F,R)$. Instantiate the first of these to y, to obtain $Rxy \to Fy$. But then by our first assumption, Fy. ⋈

(647.2) Assume $R^*(x,y)$, $\forall z(Rxz \to Fz)$, and $Hereditary(F,R)$. Then by the first assumption and the definition of R^* (645), we know:

$$\forall G[(\forall z(Rxz \to Gz) \,\&\, Hereditary(G,R)) \to Gy]$$

But we may instantiate this to F to obtain:

$$(\forall z(Rxz \to Fz) \,\&\, Hereditary(F,R)) \to Fy$$

But both conjuncts of the antecedent hold by assumption. ⋈

(647.3) Assume Fx, $R^*(x,y)$, and that $Hereditary(F,R)$. Then by (647.2), it suffices to show only $\forall z(Rxz \to Fz)$. So assume Rxz. Since F is R-hereditary, we know by definition (644) that $\forall x \forall y(Rxy \to (Fx \to Fy))$. In particular, $Rxz \to (Fx \to Fz)$. But both Rxz and Fx both hold by assumption. ⋈

(647.4) Assume Rxy and $R^*(y,z)$. To prove $R^*(x,z)$, further assume $\forall z(Rxz \to Fz)$ and $Hereditary(F,R)$. Our first and third assumptions imply Fy. But from this, $R^*(y,z)$, and $Hereditary(F,R)$, it follows that Fz, by (647.3). ⋈

(647.5) Assume $R^*(x,y)$. Now, by (647.2) and GEN, we know:

$$\forall F[(R^*(x,y) \,\&\, \forall z(Rxz \to Fz) \,\&\, Hereditary(F,R)) \to Fy]$$

Instantiate to $[\lambda y' \, \exists x' Rx'y']$ and we obtain:

$$(R^*(x,y) \,\&\, \forall z(Rxz \to [\lambda y' \, \exists x' Rx'y']z) \,\&\, Hereditary([\lambda y' \, \exists x' Rx'y'], R)) \to [\lambda y' \, \exists x' Rx'y']y$$

By β-Conversion and the Rule of Substitution, this reduces to:

$$(\vartheta) \quad (R^*(x,y) \,\&\, \forall z(Rxz \to \exists x' Rx'z) \,\&\, Hereditary([\lambda y' \, \exists x' Rx'y'], R)) \to \exists x' Rx'y$$

The consequent of (ϑ), $\exists x' Rx'y$, is an alphabetic variant of what we have to show. So since $R^*(x,y)$ by assumption, it remains to show the second and third conjuncts of (ϑ). But these are quickly obtained. For the second conjunct, assume Rxz, by GEN. Then $\exists x' Rx'z$, by $\exists$I. For the third conjunct, we have to show, by (644):

$$\forall z \forall z'(Rzz' \to ([\lambda y' \, \exists x' Rx'y']z \to [\lambda y' \, \exists x' Rx'y']z'))$$

Again, by β-Conversion and the Rule of Substitution, this means we have to show:

$$\forall z \forall z'(Rzz' \to (\exists x' Rx'z \to \exists x' Rx'z'))$$

By GEN, it suffices to show $Rzz' \to (\exists x' Rx'z \to \exists x' Rx'z')$. So assume Rzz' and $\exists x' Rx'z$. But the first of these assumptions yields $\exists x' Rx'z'$. ⋈

(649.1) Assume Rxy. Then by (647.1), $R^*(x,y)$. But then by $\vee$I, $R^*(x,y) \vee x = y$. Hence, $R^+(x,y)$, by (648). ⋈

(649.2) Assume Fx, $R^+(x,y)$, and $Hereditary(F,R)$. The second assumption implies, by definition (648), that either $R^*(x,y)$ or $x = y$. If the former, then Fy, by (647.3) and the first and third assumptions. If the latter, then Fy, by the first assumption and substitution of identicals. ⋈

(649.3) Assume $R^+(x,y) \& Ryz$. Then, by definition (648), $(R^*(x,y) \vee x=y) \& Ryz$. So by a variant of (63.7.a), we know either $R^*(x,y) \& Ryz$ or $x=y \& Ryz$. We show $R^*(x,z)$ holds in both cases:

Case 1: $R^*(x,y)$ and Ryz. To show $R^*(x,z)$, assume by GEN that $\forall z'(Rxz' \rightarrow Fz')$ and $Hereditary(F,R)$, to show Fz. From these assumptions and the fact that $R^*(x,y)$ in the present case, it follows that Fy, by the definition of R^* (645). But since Ryz also holds in Case 1, the result Fy and the assumption $Hereditary(F,R)$ jointly imply that Fz, by definition (644).

Case 2: $x=y$ and Ryz. Then Rxz, and so by (647.1), it follows that $R^*(x,z)$.

⋈

(649.4) Assume $R^*(x,y)$ and Ryz. Then from the first assumption, it follows by $\vee$I that $R^*(x,y) \vee x=y$. So by definition (648), $R^+(x,y)$. From this and our second assumption, it follows by (649.3) that $R^*(x,z)$. So, again by $\vee$I, $R^*(x,z) \vee x=z$, and hence $R^+(x,z)$, by definition (648). ⋈

(649.5) Assume Rxy and $R^+(y,z)$. By the latter and definition (648), either $R^*(y,z) \vee y=z$. We reason by cases from the two disjuncts. From the first disjunct and our first assumption, it follows that $R^*(x,z)$, by (647.4). From the second disjunct and the first assumption, it follows that Rxz, in which case, $R^*(x,z)$, by (647.1). ⋈

(649.6) To avoid clash of variables, we prove $R^*(a,b) \rightarrow \exists x(R^+(a,x) \& Rxb)$, where a and b are any arbitrarily chosen objects. So assume $R^*(a,b)$. Note that the following is an instance of (647.2):

$$(R^*(a,b) \& \forall z(Raz \rightarrow Fz) \& Hereditary(F,R)) \rightarrow Fb$$

Instantiate F in the above to the property $[\lambda y\, \exists x(R^+(a,x) \& Rxy)]$ to obtain:

$$\begin{aligned}&(R^*(a,b) \& \forall z(Raz \rightarrow [\lambda y\, \exists x(R^+(a,x) \& Rxy)]z) \,\& \\ &\quad Hereditary([\lambda y\, \exists x(R^+(a,x) \& Rxy)],R)) \rightarrow [\lambda y\, \exists x(R^+(a,x) \& Rxy)]b\end{aligned}$$

By applying β-Conversion and the Rule of Substitution, this reduces to:

$$\begin{aligned}&(R^*(a,b) \& \forall z(Raz \rightarrow \exists x(R^+(a,x) \& Rxz)) \,\& \\ &\quad Hereditary([\lambda y\, \exists x(R^+(a,x) \& Rxy)],R)) \rightarrow \exists x(R^+(a,x) \& Rxb)\end{aligned}$$

By applying the definition of hereditary (644) and applying β-Conversion and the Rule of Substitution to the result, this becomes:

$$\begin{aligned}&(R^*(a,b) \& \forall z(Raz \rightarrow \exists x(R^+(a,x) \& Rxz)) \,\& \\ &\quad \forall y \forall z(Ryz \rightarrow (\exists x(R^+(a,x) \& Rxy) \rightarrow \exists x(R^+(a,x) \& Rxz)))) \rightarrow \\ &\qquad \exists x(R^+(a,x) \& Rxb)\end{aligned}$$

Since the consequent is the desired conclusion, it remains only to show the three conjuncts of the antecedent. The first is true by assumption. For the second, by GEN, assume Raz. But, it is a simple consequence of definition (648) that $R^+(a,a)$. So, from $R^+(a,a) \,\&\, Raz$, it follows that $\exists x(R^+(a,x) \,\&\, Rxz)$. For the third conjunct, assume Ryz and $\exists x(R^+(a,x) \,\&\, Rxy)$, by GEN. By the second assumption, suppose c is such an object, so that we know $R^+(a,c) \,\&\, Rcy$. Then by (649.3), it follows that $R^*(a,y)$. But, then $R^+(a,y)$, by now familiar reasoning. Hence we know $R^+(a,y) \,\&\, Ryz$. So by $\exists$I, $\exists x(R^+(a,x) \,\&\, Rxz)$. ⋈

(652.1)★ Suppose, for reductio, that something, say a, precedes 0. Then, by the definition of predecessor (640), it follows that there is a property, say P, and an ordinary object, say b, such that:

$$Pb \,\&\, Numbers(0,P) \,\&\, Numbers(a,P^{-b})$$

From Pb it follows that $\exists uPu$. But from $Numbers(0,P)$, it follows by (639.1)★ that $\neg\exists uPu$. Contradiction. ⋈

(652.2)★ Assume, for reductio, $\exists x Precedes^*(x,0)$. Suppose a is such an object, so that we know $Precedes^*(a,0)$. By (647.5), it follows that $\exists x Precedes(x,0)$. But this contradicts (652.1)★. ⋈

(652.3)★ (Exercise)

(654) (Exercise)

(655)★ By (652.1)★, nothing precedes Zero. *A fortiori*, no natural number precedes Zero. ⋈

(656) By (642), *Predecessor* is one-to-one *tout court*. *A fortiori*, it is one-to-one with respect to the natural numbers. ⋈

(659) Assume $Precedes(n,x)$. Since n is a natural number, $Precedes^+(0,n)$. So by (649.3), it follows that $Precedes^*(0,x)$. By the definition of weak ancestral (648), it follows that $Precedes^+(0,x)$; i.e., $\mathbb{N}x$. ⋈

(660)★ Let n be a natural number. Then, by definition (653), $Precedes^+(0,n)$. By definition of R^+ (648), it follows that either $Precedes^*(0,n)$ or $0=n$. We now reason by cases from this disjunction.

If $Precedes^*(0,n)$, then there is an object, say a, such that $Precedes(a,n)$, by (647.5). So by the definition of *Precedes* it follows that there is a property, say P, and an ordinary object, say b, such that:

$$Pb \,\&\, Numbers(n,P) \,\&\, Numbers(a,P^{-b})$$

The second conjunct implies $\exists G(Numbers(n,G))$ and so it follows by definition (623) that n is a natural cardinal.

Alternatively, if $0 = n$, then since 0 is a natural cardinal, by (637)★, it follows that n is a natural cardinal. ⋈

(662) Before we begin the proof proper, we establish some preliminary facts. First, note that from the premise that $\mathcal{A}Gu$, we may derive, by modally strict means, that $\forall u(\mathcal{A}Gu \rightarrow u \neq_E v) \rightarrow u \neq_E v$. That is, it is a fact that:

$$\mathcal{A}Gu \vdash_{\Box} \forall u(\mathcal{A}Gu \rightarrow u \neq_E v) \rightarrow u \neq_E v$$

Proof. Assume $\mathcal{A}Gu$. For conditional proof, assume $\forall u(\mathcal{A}Gu \rightarrow u \neq_E v)$. Then $\mathcal{A}Gu \rightarrow u \neq_E v$, by ∀E (77). But then $u \neq_E v$, by MP.

Hence it follows by RN that:

(ϑ) $\Box\mathcal{A}Gu \vdash \Box(\forall u(\mathcal{A}Gu \rightarrow u \neq_E v) \rightarrow u \neq_E v)$

Now for the proof proper, assume:

(a) $\Diamond\forall u(\mathcal{A}Gu \rightarrow u \neq_E v)$

We want to show $\forall u(\mathcal{A}Gu \rightarrow u \neq_E v)$. So by GEN, assume:

(b) $\mathcal{A}Gu$

From (b) and theorem (95.7), it follows that $\Box\mathcal{A}Gu$. Hence, by (ϑ):

$$\Box(\forall u(\mathcal{A}Gu \rightarrow u \neq_E v) \rightarrow u \neq_E v)$$

But from this and (a) it follows by K◇ (117.5) that $\Diamond u \neq_E v$. But this implies $u \neq_E v$, by theorem (160.2).[291] ⋈

(663)★ To show $\exists! m Precedes(n,m)$, it suffices, in virtue of (643), to show that $\exists m Precedes(n,m)$.[292] Furthermore, in virtue of (659), it suffices to show that $\exists y Precedes(n,y)$. Now by assumption, $\mathbb{N}n$. Not only does this imply $\mathcal{A}\mathbb{N}n$, by (30)★, but it also implies *NaturalCardinal*(n), by (660)★. The latter, by definition (623), implies $\exists G(Numbers(n,G))$. Let Q be such a property, so that we know *Numbers*(n,Q). Then again by (30)★, $\mathcal{A}Numbers(n,Q)$. Since we have now established both $\mathcal{A}\mathbb{N}n$ and $\mathcal{A}Numbers(n,Q)$, it follows by (95.2) that:

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[291] The following, alternative proof, based on possible world theory, was proposed by Jonas Raab. Assume $\Diamond\forall u(\mathcal{A}Gu \rightarrow u \neq_E v)$. Then by a fundamental theorem of possible world theory (433.1), it follows that: $\exists w(w \models \forall u(\mathcal{A}Gu \rightarrow u \neq_E v))$. Suppose w_1 is such a world, so that we know:

(ϑ) $w_1 \models \forall u(\mathcal{A}Gu \rightarrow u \neq_E v)$

Now we want to show $\forall u(\mathcal{A}Gu \rightarrow u \neq_E v)$. So, by GEN, assume $\mathcal{A}Gu$. Then by theorem (95.7), $\Box\mathcal{A}Gu$. So by a fundamental theorem of world theory, $\forall w(w \models \mathcal{A}Gu)$, and so:

(ξ) $w_1 \models \mathcal{A}Gu$

But by the fact that possible worlds are modally closed, (ϑ) and (ξ) imply $w_1 \models u \neq_E v$. Hence $\exists w(w \models u \neq_E v)$, and by a fundamental theorem, $\Diamond u \neq_E v$. Hence $u \neq_E v$, by (160.2).

[292] To see why, suppose we do indeed establish $\exists m Precedes(n,m)$. Then let m_1 be such a natural number, so that we know *Precedes*(n,m_1). By the definition of the uniqueness quantifier, we have to show $\forall o(Precedes(n,o) \rightarrow o = m_1)$. By GEN, it suffices to show $Precedes(n,o) \rightarrow o = m_1$. So assume *Precedes*($n,o$). Then given both *Precedes*(n,o) and *Precedes*(n,m_1) it follows by (643) that $o = m_1$.

$$\mathcal{A}(\mathbb{N}n \,\&\, Numbers(n, Q))$$

So, by ∃I:

$$\exists x \mathcal{A}(\mathbb{N}x \,\&\, Numbers(x, Q))$$

Thus, by (95.11):

$$\mathcal{A}\exists x(\mathbb{N}x \,\&\, Numbers(x, Q))$$

Our modal axiom (661) therefore implies:

$$\Diamond\exists y(E!y \,\&\, \forall u(\mathcal{A}Qu \to u \neq_E y))$$

By BF$\Diamond$ (122.3), this implies:

$$\exists y\Diamond(E!y \,\&\, \forall u(\mathcal{A}Qu \to u \neq_E y))$$

Let a be such an object, so that we know:

$$\Diamond(E!a \,\&\, \forall u(\mathcal{A}Qu \to u \neq_E a))$$

By (117.8), it follows that:

$$\Diamond E!a \,\&\, \Diamond\forall u(\mathcal{A}Qu \to u \neq_E a)$$

From the first conjunct of this last result, it follows that $O!a$. Since a is ordinary, the second conjunct becomes an instance of the antecedent of Lemma (662) and so we may conclude:

$$(\vartheta)\quad \forall u(\mathcal{A}Qu \to u \neq_E a)$$

Now consider the property $[\lambda z\, Qz \vee z =_E a]$, henceforth written Q^{+a} and which we know exists (the λ-expression is well-formed). By (625.1), we know that something numbers Q^{+a}. So suppose b is such an object, so that we know $Numbers(b, Q^{+a})$. If we can show $Precedes(n, b)$, we are done. So we have to show:

$$\exists F\exists u(Fu \,\&\, Numbers(b, F) \,\&\, Numbers(n, F^{-u}))$$

If we propose that Q^{+a} and a are witnesses to the above, it remains to show:

(i) $Q^{+a}a$

(ii) $Numbers(b, Q^{+a})$

(iii) $Numbers(n, Q^{+a-a})$

(i) We've established that a is ordinary. Hence $a =_E a$, by (168.1). So by $\vee$I, it follows that $Qa \vee a =_E a$. By β-Conversion, $[\lambda z\, Qz \vee z =_E a]a$. Hence, by the definition of Q^{+a}, it follows that $Q^{+a}a$.

(ii) $Numbers(b, Q^{+a})$ is already known.

(iii) Since we know $Numbers(n, Q)$, it suffices to show $Q \approx_E Q^{+a-a}$, for by (619.1), these facts imply $Numbers(n, Q^{+a-a})$. Now to show $Q \approx_E Q^{+a-a}$, it suffices by (613.1) to show $Q \equiv_E Q^{+a-a}$. By the definition of F^{-u} (615), this means we have to show:

$$Q \equiv_E [\lambda z\, Q^{+a}z \,\&\, z \neq_E a]$$

By definition of $\equiv_E$ (612) and GEN, we have to show:

$$Qv \equiv [\lambda z\, Q^{+a}z \,\&\, z \neq_E a]v$$

We argue both directions. ($\rightarrow$) Assume Qv. By &I and β-Conversion, it suffices to show both (1) $Q^{+a}v$ and (2) $v \neq_E a$. (1) By definition of Q^{+a}, we have to show $[\lambda z\, Qz \vee z =_E a]v$, and so by β-Conversion, show $Qv \vee v =_E a$. But we know Qv by hypothesis. Hence $Qv \vee v =_E a$. (2) Since we're still under the assumption that Qv, it follows by axiom (30)$\star$ that $\mathcal{A}Qv$. So by (ϑ), it follows that $v \neq_E a$. ($\leftarrow$) Assume $[\lambda z\, Q^{+a}z \,\&\, z \neq_E a]v$. Then by β-Conversion and &E, $Q^{+a}v$ and $v \neq_E a$. By definition of Q^{+a}, the first of these implies $[\lambda z\, Qz \vee z =_E a]v$. So by λ-Conversion, $Qv \vee v =_E a$. But the second, $v \neq_E a$, allows us to conclude, by disjunctive syllogism, that Qv. $\bowtie$

(666) Consider any relation R and its weak ancestral R^+:

(ϑ) $Fz \,\&\, \forall x \forall y((R^+(z,x) \,\&\, R^+(z,y)) \rightarrow (Rxy \rightarrow (Fx \rightarrow Fy)))$

By GEN, we have to show that $R^+(z,x) \rightarrow Fx$. So assume $R^+(z,x)$. To show Fx, we appeal to lemma (649.2). Instantiate the variables F, x, and y in this lemma to $[\lambda y\, Fy \,\&\, R^+(z,y)]$, z, and x, respectively, so that we know, after β-Conversion:

(ξ) $[Fz \,\&\, R^+(z,z) \,\&\, R^+(z,x) \,\&\, Hereditary([\lambda y\, Fy \,\&\, R^+(z,y)], R)] \rightarrow (Fx \,\&\, R^+(z,x))$

So if we can establish the antecedent of (ξ), we may conclude Fx. We know the first conjunct of the antecedent of (ξ) is true, by the first conjunct of (ϑ). We know that the second conjunct of the antecedent of (ξ) is true, by the reflexivity of R^+, which immediately follows from its definition. We know that the third conjunct of the antecedent is true, by further assumption. So if we can establish:

$$Hereditary([\lambda y\, Fy \,\&\, R^+(z,y)], R),$$

we are done. By the definition of *hereditary* and λ-Conversion, we have to show:

$$\forall x' \forall y'(Rx'y' \rightarrow ((Fx' \,\&\, R^+(z,x')) \rightarrow (Fy' \,\&\, R^+(z,y'))))$$

To prove this claim, we assume $Rx'y'$, Fx', and $R^{+}(z,x')$, to show $Fy' \& R^{+}(z,y')$). The second conjunct follows easily: from the facts that $R^{+}(z,x')$ and $Rx'y'$, it follows from (649.3) that $R^{*}(z,y')$, which implies $R^{+}(z,y')$, by the definition of R^{+}. So it remains to show Fy'. Since we now have $R^{+}(z,x')$, $R^{+}(z,y')$, $Rx'y'$, and Fx', it follows from the second conjunct of (ϑ) that Fy'. ⋈

(668) By GEN, it suffices to show $F0 \;\&\; \forall n \forall m(\mathit{Precedes}(n,m) \rightarrow (Fn \rightarrow Fm)) \rightarrow \forall n Fn$. Now if we substitute, 0 for z and *Precedes* for R in (666), we have the following instance:

$$[F0 \,\&\, \forall x \forall y((\mathit{Precedes}^{+}(0,x) \,\&\, \mathit{Precedes}^{+}(0,y)) \rightarrow (\mathit{Precedes}(x,y) \rightarrow (Fx \rightarrow Fy)))] \rightarrow \forall x(\mathit{Precedes}^{+}(0,x) \rightarrow Fx)$$

By definition (653) of $\mathbb{N}$, this reduces to:

$$[F0 \;\&\; \forall x \forall y((\mathbb{N}x \,\&\, \mathbb{N}y) \rightarrow (\mathit{Precedes}(x,y) \rightarrow (Fx \rightarrow Fy)))] \rightarrow \forall x(\mathbb{N}x \rightarrow Fx)$$

By employing our restricted variables n and m, this can be written:

$$[F0 \;\&\; \forall n \forall m(\mathit{Precedes}(n,m) \rightarrow (Fn \rightarrow Fm))] \rightarrow \forall n Fn \qquad ⋈$$

(671.1)★ By (630.2)★, $\mathit{Numbers}(\#O!, O!)$. So by ∃I, $\exists G(\mathit{Numbers}(\#O!, G)$. Hence, by definition (623), $\mathit{NaturalCardinal}(\#O!)$. ⋈

(671.2)★ By (669.2), we have to show $\neg \mathit{Finite}(\#O!)$. So by definition (669.1), we have to show $\neg \mathbb{N}\#O!$. For reductio, assume $\mathbb{N}\#O!$. So it follows that $\mathcal{A}\mathbb{N}\#O!$, by (30)★. Since we also know, by (630.2)★, that $\mathit{Numbers}(\#O!, O!)$, we can use (30)★ again to infer $\mathcal{A}\mathit{Numbers}(\#O!, O!)$. Thus, by the reasoning similar to that used in the proof of (663)★, we may derive the following from our results $\mathcal{A}\mathbb{N}\#O!$ and $\mathcal{A}\mathit{Numbers}(\#O!, O!)$:

$$\mathcal{A}\exists x(\mathbb{N}x \,\&\, \mathit{Numbers}(x, O!))$$

Hence, by our modal axiom (661):

$$\Diamond \exists y(E!y \,\&\, \forall u(\mathcal{A}O!u \rightarrow u \neq_E y))$$

By BF◇ (122.3):

$$\exists y \Diamond (E!y \,\&\, \forall u(\mathcal{A}O!u \rightarrow u \neq_E y))$$

Suppose b is such an object, so that we know:

$$\Diamond (E!b \,\&\, \forall u(\mathcal{A}O!u \rightarrow u \neq_E b))$$

Then by (117.8):

$$(\vartheta) \quad \Diamond E!b \,\&\, \Diamond \forall u(\mathcal{A}O!u \rightarrow u \neq_E b)$$

The first conjunct of (ϑ) implies $O!b$, and so $\mathcal{A}O!b$, by (30)★. The second conjunct of (ϑ) implies, by (662), $\forall u(\mathcal{A}O!u \rightarrow u \neq_E b)$. Instantiating to b and applying the fact that $\mathcal{A}O!b$, it follows that $b \neq_E b$. But this contradicts the fact that $b =_E b$ (168.1). ⋈

(671.3)★ (Exercise)

(673.1)★ – **(673.5)★** (Exercises)

(675.1)★ By theorem (242.2)★, we know $ExtensionOf(\epsilon O!, O!)$. So by definition (246):

(ϑ) $ClassOf(\epsilon O!, O!)$

Independently, theorem (671.1)★ is $NaturalCardinal(\#O!)$, theorem (671.2)★ is $Infinite(\#O!)$, and theorem (630.1)★ is $Numbers(\#O!, O!)$. Assembling these last results and generalizing, we have:

$$\exists x(NaturalCardinal(x) \,\&\, Infinite(x) \,\&\, Numbers(x, O!))$$

i.e.,

$$\exists \kappa(Infinite(\kappa) \,\&\, Numbers(\kappa, O!))$$

Conjoining (ϑ) with this last result and we have:

$$ClassOf(\epsilon O!, O!) \,\&\, \exists \kappa(Infinite(\kappa) \,\&\, Numbers(\kappa, O!))$$

Hence:

$$\exists G(ClassOf(\epsilon O!, G) \,\&\, \exists \kappa(Infinite(\kappa) \,\&\, Numbers(\kappa, G)))$$

So by by definition (674), $InfiniteClass(\epsilon O!)$. ⋈

(675.2)★ (Exercise)

(679.1) We reason biconditionally as follows:

$$\begin{array}{lcll}
\exists!_1 uFu & \equiv & \exists u(Fu \,\&\, \exists!_0 vF^{-u}v) & \text{by (678.2)} \\
& \equiv & \exists u(Fu \,\&\, \neg\exists vF^{-u}v) & \text{df } \exists!_0 vF^{-u}v \text{ (678.1)} \\
& \equiv & \exists u(Fu \,\&\, \neg\exists v[\lambda u'\, Fu' \,\&\, u' \neq_E u]v) & \text{df } F^{-u} \text{ (615)} \\
& \equiv & \exists u(Fu \,\&\, \neg\exists v(Fv \,\&\, v \neq_E u)) & \text{by } \beta\text{-Conversion} \\
& \equiv & \exists u(Fu \,\&\, \forall v(Fv \rightarrow v =_E u)) & \text{by predicate logic} \\
& \equiv & \exists! uFu & \text{by (87.1)} \quad ⋈
\end{array}$$

(679.2) (Exercise)

(680.1)★ We prove this by induction. *Base Case*: $n = 0$. We want to show:

$$0 = \iota x(A!x \,\&\, \forall F(xF \equiv \exists!_0 uFu))$$

By (678.1), we have to show:

$$0 = \iota x(A!x \,\&\, \forall F(xF \equiv \neg\exists u Fu))$$

It suffices to show that 0 and $\iota x(A!x \& \forall F(xF \equiv \neg\exists u Fu))$ encode the same properties, i.e., by GEN, that $0G \equiv \iota x(A!x \,\&\, \forall F(xF \equiv \neg\exists u Fu))G$. We do this as follows:

$$\begin{array}{rcll} 0G & \equiv & \neg\exists u Gu & \text{by (638)}\bigstar \\ & \equiv & \iota x(A!x \,\&\, \forall F(xF \equiv \neg\exists u Fu))G & \text{by (184)}\bigstar \end{array}$$

Inductive Case: Our Inductive Hypothesis is:

$$n = \iota x(A!x \,\&\, \forall F(xF \equiv \exists!_n u Fu))$$

We want to show:

$$n' = \iota x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} u Fu))$$

To do this, we have to show that the objects flanking the identity sign encode the same properties, i.e., by GEN, that:

$$n'G \equiv \iota x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} u Fu))G$$

So we show both directions.

($\rightarrow$) Assume that $n'G$. By (184)$\bigstar$, we have to show:

$$\exists!_{n'} u Gu$$

i.e., by definition (678.2):

$$\exists u(Gu \,\&\, \exists!_n v G^{-u} v)$$

By $\exists$I, it suffices to find a witness to this existential claim. Now by (101.2)$\bigstar$, it is an immediate consequence of the definition of n' (676) that $Precedes(n, n')$. So there is some property, say Q and some ordinary object, say a, such that:

(ϑ) $Qa \,\&\, Numbers(n', Q) \,\&\, Numbers(n, Q^{-a})$

Since $Numbers(n', Q)$, it follows by definition (618) that $\forall F(n'F \equiv F \approx_E Q)$. Since $n'G$ by assumption, we have $G \approx_E Q$ and, by symmetry, $Q \approx_E G$. So, by (609.3), there is a relation R which is a one-to-one correspondence between Q and G, and by (609.2) and Qa, there is a (unique) ordinary object, say b, such that both Gb and Rab. Now if we can show:

$$Gb \,\&\, \exists!_n v G^{-b} v$$

then we are done. But since we already know Gb, it remains only to show $\exists!_n v G^{-b} v$. Now from the known facts that $Q \approx_E G$, Qa, and Gb it follows by (616) that $Q^{-a} \approx_E G^{-b}$. So by (619.1), it follows from the last conjunct of (ϑ), namely $Numbers(n, Q^{-a})$, that $Numbers(n, G^{-b})$. By definition (618), this implies $\forall F(nF \equiv F \approx_E G^{-b})$. Since equinumerosity$_E$ is reflexive, it follows that:

(ξ) nG^{-b}

But, recall that our inductive hypothesis is:

$$n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n uFu))$$

This implies, by (101.2)★, that:

$$\forall F(nF \equiv \exists!_n vFv)$$

But this and (ξ) imply $\exists!_n vG^{-b}v$.

($\leftarrow$) Assume $\imath x(A!x \,\&\, \forall F(xF \equiv \exists!_{n'} uFu))G$. By (184)★, it follows that:

$$\exists!_{n'} uGu$$

By definition (678), it follows that:

$$\exists u(Gu \,\&\, \exists!_n vG^{-u}v)$$

Suppose a is such an ordinary object, so that we know:

(ϑ) $Ga \,\&\, \exists!_n vG^{-a}v$

Now the second conjunct of (ϑ) implies, by (184)★:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n uFu))G^{-a}$$

But our IH is:

$$n = \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n uFu))$$

So our last two displayed facts imply nG^{-a}. Since natural numbers are natural cardinals (660)★, it follows by (624) that:

(ζ) $Numbers(n, G^{-a})$

Independently, the definition of n' (676) implies $Precedes(n, n')$, by (101.2)★. So by the definition of *Precedes*, $\exists F \exists u(Fu \,\&\, Numbers(n', F) \,\&\, Numbers(n, F^{-u}))$. Let P and b be such a property and ordinary object, so that we know:

(ξ) $Pb \,\&\, Numbers(n', P) \,\&\, Numbers(n, P^{-b})$

Then, from (ζ) and the third conjunct of (ξ), it follows that $G^{-a} \approx_E P^{-b}$, by (619.2). But this last fact, together with Ga (ϑ) and Pb (ξ), imply $G \approx_E P$, by (617). So $P \approx_E G$ by symmetry. From this and the second conjunct of (ξ), it follows by (619.1) that $Numbers(n', G)$. But since n is a natural cardinal, this last fact implies $n'G$, by (624). ⋈

(680.2)★ By (184)★ and the commutativity of the biconditional, we know:

$$\exists!_n uGu \equiv \imath x(A!x \,\&\, \forall F(xF \equiv \exists!_n uFu))G$$

Hence by (680.1)★, it follows that $\exists!_n uGu \equiv nG$. ⋈

(680.3)★ Since natural numbers are natural cardinals (660)★, we know by (634)★ that $nG \equiv n = \#G$. Hence from this and (680.2)★ it follows that $\exists!_n uGu \equiv n = \#G$. ⋈

(683.1) Assume R is one-to-one, Rxy, and $R^*(z,y)$. By the latter and (649.6), it follows that there is some object, say a, such that $R^+(z,a)$ and Ray. Since we now know that R is one-to-one, Rxy and Ray, it follows by definition (682) that $x = a$. So $R^+(z,x)$. ⋈

(683.2) Assume R is one-to-one. By GEN, it suffices to show $Rxy \rightarrow (\neg R^*(x,x) \rightarrow \neg R^*(y,y))$. So assume Rxy. We'll prove the contrapositive, so further assume $R^*(y,y)$. Now, independently, we know that the following is an instance (683.1) by setting z in that theorem to y:

$$\mathit{1\text{-}1}(R) \,\&\, Rxy \,\&\, R^*(y,y) \rightarrow R^+(y,x)$$

So, $R^+(y,x)$. Similarly, we know that the following is an instance of (649.5) if we set z in that theorem to x:

$$Rxy \,\&\, R^+(y,x) \rightarrow R^*(x,x)$$

Hence $R^*(x,x)$. ⋈

(683.3) Assume that R is one-to-one and $\neg R^*(x,x)$. Further assume $R^+(x,y)$. Now independently, if we instantiate F in (649.2) to $[\lambda z\, \neg R^*(z,z)]$ and apply β-Conversion, we have:

$$(\neg R^*(x,x) \,\&\, R^+(x,y) \,\&\, \mathit{Hereditary}([\lambda z\, \neg R^*(z,z)], R)) \rightarrow \neg R^*(y,y)$$

Since the consequent is what we want to show, we establish the antecedent. The first two conjuncts of the antecedent are true by assumption. So by definition (644) and β-Conversion, it remains to show:

$$\forall x' \forall y' [Rx'y' \rightarrow (\neg R^*(x',x') \rightarrow \neg R^*(y',y'))]$$

But since R is one-to-one, this follows from an instance of (683.2), by two applications of GEN. ⋈

(685.1)★ By (683.3), we know the following about the *Precedes* relation and its ancestrals:

$$(\mathit{1\text{-}1}(\mathit{Precedes}) \,\&\, \neg \mathit{Precedes}^*(0,0)) \rightarrow (\mathit{Precedes}^+(0,n) \rightarrow \neg \mathit{Precedes}^*(n,n))$$

Now by (642), *Precedes* is one-to-one. And by (652.3)★, $\neg \mathit{Precedes}^*(0,0)$. Hence:

$$\mathit{Precedes}^+(0,n) \rightarrow \neg \mathit{Precedes}^*(n,n)$$

But n is natural number. So by (653), $Precedes^{+}(0,n)$. Hence $\neg Precedes^{*}(n,n)$ and so by definition (684.1), $\neg(n<n)$. ⋈

(685.2)★ By (685.1)★, definition (684.1), and (647.1). ⋈

(685.3)★ By (685.2)★, $\neg Precedes(n,n')$. But as an instance of (101.2)★, we know:

$$n = \imath m(Precedes(n,m)) \rightarrow Precedes(n,n)$$

Hence $n \neq \imath m(Precedes(n,m))$. So by (676), $n \neq n'$.

(687) Assume $Precedes(x,i)$. Since i is a restricted variable ranging over positive integers, we know by hypothesis and definition (686) that $\mathbb{N}i$ and $i \neq 0$. It follows from the former, by definition (653), that $Precedes^{+}(0,i)$, i.e., by definition (648), that either $Precedes^{*}(0,i)$ or $0=i$. But we know $i \neq 0$, and so we may conclude $Precedes^{*}(0,i)$. Now we also know *Precedes* is one-to-one (642), so we can assemble the results we've established to conclude:

$$\mathit{1\text{-}1}(Precedes) \,\&\, Precedes(x,i) \,\&\, Precedes^{*}(0,i)$$

Hence it follows from (683.1) that $Precedes^{+}(0,x)$, i.e., $\mathbb{N}x$. ⋈

(688)★ Since i is a positive integer, definition (686) implies $\mathbb{N}i$ and $i \neq 0$. From the former, it follows by definition (653) that $Precedes^{+}(0,i)$. By the definition of weak ancestral (648), it follows that either $Precedes^{*}(0,i)$ or $0=i$. But we know $i \neq 0$, so $Precedes^{*}(0,i)$. So by (647.5), $\exists z Precedes(z,i)$. Let a be such an object, so that we know $Precedes(a,i)$. Then by (687), it follows that $\mathbb{N}a$. Since we now know a is a natural number that precedes i, it remains to show that every natural number that precedes i is identical to a. So, by GEN, assume $\mathbb{N}x$ & $Precedes(x,i)$. Then by the one-to-one character of *Precedes* (642), it follows from the facts that $Precedes(x,i)$ and $Precedes(a,i)$ that $x=a$. ⋈

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