Semantics for Typed Object Theory

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Interpretations

The following assumes familiarity with the technical definition of *type* and the use of $\bar{\varepsilon}$-terms, as found in the document "The Systems of *Principia Logico-Metaphysica*" (http://mally.stanford.edu/systems.pdf).

$$\mathcal{I} = \langle \mathbf{D}, \mathbf{W}, \boldsymbol{T}, \boldsymbol{F}, \mathbf{ext}_{\boldsymbol{w}}, \mathbf{enc}_{\boldsymbol{w}}, \mathbf{ex}_{\boldsymbol{w}}, \mathbf{V}, \mathbf{C} \rangle,$$

where:

- $\mathbf{D}$ is the general union of non-empty domains $\mathbf{D}_t$, for every type t; i.e., $\mathbf{D} = \bigcup_t \mathbf{D}_t$. We often use $\boldsymbol{o}^t$ as a variable ranging over the elements of $\mathbf{D}_t$; use $\boldsymbol{r}$ as a variable ranging over the elements of $\mathbf{D}_{\langle t_1,\ldots,t_n \rangle}$, where $t_1,\ldots,t_n$ are any types and $n \geq 1$; and use $\boldsymbol{p}$ as a variable ranging over the elements of $\mathbf{D}_{\langle\rangle}$,
- $\mathbf{W}$ is a non-empty set of possible worlds with a *distinguished element* $\boldsymbol{w}_0$; we use $\boldsymbol{w}$ as a variable ranging over the elements of $\mathbf{W}$,
- $\boldsymbol{T}$ is the truth-value The True,
- $\boldsymbol{F}$ is the truth-value The False,
- $\mathbf{ext}_{\boldsymbol{w}}$ is a binary *exemplification extension* function indexed to its second argument; $\mathbf{ext}_{\boldsymbol{w}}$ maps each relation $\boldsymbol{r}$ in $\mathbf{D}_{\langle t_1,\ldots,t_n \rangle}$ ($n \geq 1$) and world $\boldsymbol{w}$ to a set of n-tuples whose elements have types $t_1,\ldots,t_n$, respectively, so that $\mathbf{ext}_{\boldsymbol{w}}(\boldsymbol{r})$ serves as the exemplification extension of $\boldsymbol{r}$ at $\boldsymbol{w}$,[1]
- $\mathbf{enc}_{\boldsymbol{w}}$ is a binary *encoding extension* function indexed to its second argument; $\mathbf{enc}_{\boldsymbol{w}}$ maps each relation $\boldsymbol{r}$ in $\mathbf{D}_{\langle t_1,\ldots,t_n \rangle}$ ($n \geq 1$) and world $\boldsymbol{w}$ to a set of n-tuples whose elements have types $t_1,\ldots,t_n$, respectively, so that $\mathbf{enc}_{\boldsymbol{w}}(\boldsymbol{r})$ serves as the encoding extension of $\boldsymbol{r}$ at $\boldsymbol{w}$,
- $\mathbf{ex}_{\boldsymbol{w}}$ is a binary *extension* function indexed to its second argument; $\mathbf{ex}_{\boldsymbol{w}}$ maps each proposition $\boldsymbol{p}$ in $\mathbf{D}_{\langle\rangle}$ and world $\boldsymbol{w}$ to one of the truth-values ($\boldsymbol{T}$ or $\boldsymbol{F}$) so that $\mathbf{ex}_{\boldsymbol{w}}(\boldsymbol{p})$ serves as the extension of $\boldsymbol{p}$ at $\boldsymbol{w}$,
- $\mathbf{V}$ is an interpretation function that assigns each the primitive constant of type t to an element of the domain $\mathbf{D}_t$, and
- $\mathbf{C}$ is a choice function that takes, as argument, any semantic formula $\boldsymbol{A}$ having a single free variable that ranges over some domain $\mathbf{D}_t$, for $t \neq i$, and returns an arbitrary but determinate value in $\mathbf{D}_t$ that satisfies $\boldsymbol{A}$ if there is one, and is undefined otherwise. If the semantic $\bar{\varepsilon}$-term has the form $\bar{\varepsilon}\boldsymbol{r}^n\boldsymbol{A}$, where $\boldsymbol{r}^n$ is a semantic variable that ranges over the n-ary relations ($n \geq 0$) in the domain $\mathbf{D}_{\langle t_1,\ldots,t_n \rangle}$, then the object $\mathbf{C}(\boldsymbol{A})$ is an entity of type $\langle t_1,\ldots,t_n \rangle$ that serves as the value of the term. For example, if $\boldsymbol{A}$ has $\boldsymbol{r}$ free and $\boldsymbol{r}$ ranges over relations in $\mathbf{D}_{\langle i,i \rangle}$ (i.e., ranges over binary relations among individuals), then the semantic term $\bar{\varepsilon}\boldsymbol{r}^n\boldsymbol{A}$ denotes $\mathbf{C}(\boldsymbol{A})$, where the latter is an arbitrary but determinate relation in $\mathbf{D}_{\langle i,i \rangle}$ that satisfies $\boldsymbol{A}$, if there is one. Similarly, if $\boldsymbol{A}$ has $\boldsymbol{p}$ free, where $\boldsymbol{p}$ ranges over $\mathbf{D}_{\langle\rangle}$, then $\bar{\varepsilon}\boldsymbol{p}\boldsymbol{A}$ denotes $\mathbf{C}(\boldsymbol{A})$, where the latter is an arbitrary but determinate proposition in $\mathbf{D}_{\langle\rangle}$ that satisfies $\boldsymbol{A}$, if there is one.

Assignments to Variables

Given such a structure $\mathcal{I}$, let $\boldsymbol{w}$ range over the primitive possible worlds in $\mathbf{W}$, and let f be a *assignment function* relative to $\mathcal{I}$ that assigns to each variable α^t an element of the domain $\mathbf{D}_t$. (For ease of readability, we always omit the index on f that relativizes it to $\mathcal{I}$.)

[1] By convention, $\mathbf{ext}_{\boldsymbol{w}}$ maps each relation unary relation $\boldsymbol{r}$ in $\mathbf{D}_{\langle t \rangle}$ ($n \geq 1$) and world $\boldsymbol{w}$ to a subset of $\mathbf{D}_t$.

$d_{\mathcal{I},f}(\tau)$ and $w \models_{\mathcal{I},f} \varphi$ Defined Simultaneously

Then we shall assign denotations to the terms and truth conditions to the formulas by defining the following notions simultaneously:

$d_{\mathcal{I},f}(\tau)$, i.e., the denotation of τ relative to $\mathcal{I}$ and f

$w \models_{\mathcal{I},f} \varphi$, i.e., *under* $\mathcal{I}$ and f, φ *is true at* w

The definitions are given in full below but note that, in what follows, we are re-purposing the symbol $\models$ for the semantics. When we use $\models$ in a semantic context in what follows, it is to be understood as representing a semantic notion, and not the object-theoretic notion p is true in s ($s \models p$) defined in object-theoretic situation theory.

Intuitively, $d_{\mathcal{I},f}$ is a partial denotation function which, relative to an interpretation $\mathcal{I}$ and variable assignment f, assigns to every term τ of type t an element of the domain $\mathbf{D}_t$ if τ is significant, and nothing otherwise. And, $w \models_{\mathcal{I},f} \varphi$ states the truth conditions of φ at world w, relative to $\mathcal{I}$ and f. Now let:

- $\mathcal{I}$ be any interpretation and f be any assignment function,
- $\mathbf{V}$ be the interpretation function of $\mathcal{I}$,
- $f[\alpha^t/o^t]$ be the variable assignment just like f except that it assigns the entity o^t to the variable α^t,[2] and
- $f[\alpha^{t_i}/o^{t_i}]_{i=1}^n$ be the variable assignment just like f but which assigns the entities $o^{t_1},\ldots,o^{t_n}$, respectively, to the variables $\alpha^{t_1},\ldots,\alpha^{t_n}$, for $1 \leq i \leq n$

And let us adopt the convention of omitting the type index on a symbol after its first use in a semantic formula whenever it can be done without ambiguity. Then the simultaneous definition of denotation and world-relative truth, relative to $\mathcal{I}$ and f, proceeds as follows:

Base Clauses

D1. If τ is a constant of type t, then $d_{\mathcal{I},f}(\tau) = \mathbf{V}(\tau)$

D2. If τ is a variable of type t, then $d_{\mathcal{I},f}(\tau) = f(\tau)$

T1. If φ is a formula in $Base^{\langle\rangle}$, i.e., if φ is a constant, variable, or description of type $\langle\rangle$, then $w \models_{\mathcal{I},f} \varphi$ if and only if $\exists p^{\langle\rangle}(p = d_{\mathcal{I},f}(\varphi) \,\&\, \mathbf{ex}_w(p) = T)$

T2. If φ is a formula of the form $\Pi^{\langle t_1,\ldots,t_n\rangle}\tau^{t_1}\ldots\tau^{t_n}$ $(n \geq 1)$, then $w \models_{\mathcal{I},f} \varphi$ if and only if $\exists r^{\langle t_1,\ldots,t_n\rangle}\exists o^{t_1}\ldots\exists o^{t_n}(r = d_{\mathcal{I},f}(\Pi) \,\&\, o^{t_1} = d_{\mathcal{I},f}(\tau^{t_1}) \,\&\, \ldots \,\&\, o^{t_n} = d_{\mathcal{I},f}(\tau^{t_n}) \,\&\, \langle o^{t_1},\ldots,o^{t_n}\rangle \in \mathbf{ext}_w(r))$

T3. If φ is a formula of the form $\tau^{t_1}\ldots\tau^{t_n}\Pi^{\langle t_1,\ldots,t_n\rangle}$ $(n \geq 1)$, then $w \models_{\mathcal{I},f} \varphi$ if and only if $\exists o^{t_1}\ldots\exists o^{t_n}\exists r^{\langle t_1,\ldots,t_n\rangle}(o^{t_1} = d_{\mathcal{I},f}(\tau^{t_1}) \,\&\, \ldots \,\&\, o^{t_n} = d_{\mathcal{I},f}(\tau^{t_n}) \,\&\, r = d_{\mathcal{I},f}(\Pi) \,\&\, \langle o^{t_1},\ldots,o^{t_n}\rangle \in \mathbf{enc}_w(r))$

Recursive Clauses

T4. If φ is a formula of the form $[\lambda\,\psi]$, then $w \models_{\mathcal{I},f} \varphi$ if and only if $w \models_{\mathcal{I},f} \psi$

T5. If φ is a formula of the form $\neg\psi$, then $w \models_{\mathcal{I},f} \varphi$ if and only if it is not the case that $w \models_{\mathcal{I},f} \psi$, i.e., iff $w \not\models_{\mathcal{I},f} \psi$

[2] This can be defined formally in one of two ways, suppressing the type index. If an assignment function f is represented as a set of ordered pairs, then where α is a variable and o is an entity from the domain over which α ranges:

$$f[\alpha/o] = (f \sim \langle\alpha, f(\alpha)\rangle) \cup \{\langle\alpha, o\rangle\}$$

I.e., $f[\alpha/o]$ is the result of removing the pair $\langle\alpha, f(\alpha)\rangle$ from f and replacing it with the pair $\langle\alpha, o\rangle$.

Alternatively, we can define $f[\alpha/o]$ functionally, where β is a variable ranging over the same domain as α, as:

$$f[\alpha/o](\beta) = \begin{cases} f(\beta), & \text{if} \beta \neq \alpha \\ o, & \text{if} \beta = \alpha \end{cases}$$

T6. If φ is a formula of the form $\psi \to \chi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if either it is not the case that $\boldsymbol{w} \models_{\mathcal{I},f} \psi$ or it is the case that $\boldsymbol{w} \models_{\mathcal{I},f} \chi$, i.e., iff either $\boldsymbol{w} \not\models_{\mathcal{I},f} \psi$ or $\boldsymbol{w} \models_{\mathcal{I},f} \chi$

T7. If φ is a formula of the form $\forall \alpha^t \psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\forall \boldsymbol{o}^t(\boldsymbol{w} \models_{\mathcal{I},f[\alpha/\boldsymbol{o}]} \psi)$

T8. If φ is a formula of the form $\Box\psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\forall \boldsymbol{w}'(\boldsymbol{w}' \models_{\mathcal{I},f} \psi)$

T9. If φ is a formula of the form $\mathcal{A}\psi$, then $\boldsymbol{w} \models_{\mathcal{I},f} \varphi$ if and only if $\boldsymbol{w}_0 \models_{\mathcal{I},f} \psi$.

D3. If τ is a description of the form $\imath\alpha^t\varphi$, then

$$\boldsymbol{d}_{\mathcal{I},f}(\tau) = \begin{cases} \boldsymbol{o}^t, \text{ if } \boldsymbol{w}_0 \models_{\mathcal{I},f[\alpha/\boldsymbol{o}]} \varphi \;\&\; \forall \boldsymbol{o}'(\boldsymbol{w}_0 \models_{\mathcal{I},f[\alpha/\boldsymbol{o}']} \varphi \to \boldsymbol{o}' = \boldsymbol{o}) \\ \text{undefined, otherwise} \end{cases}$$

where $\boldsymbol{o}'$ also ranges over the entities in $\mathbf{D}_t$

D4. If τ is an n-ary λ-expression ($n \geq 1$) of the form $[\lambda \alpha^{t_1} \dots \alpha^{t_n} \varphi]$, then

$$\boldsymbol{d}_{\mathcal{I},f}(\tau) = \begin{cases} \bar{\varepsilon}\boldsymbol{r}^{\langle t_1,\dots,t_n\rangle} \forall \boldsymbol{w} \forall \boldsymbol{o}^{t_1} \dots \forall \boldsymbol{o}^{t_n}(\langle \boldsymbol{o}^{t_1},\dots,\boldsymbol{o}^{t_n}\rangle \in \mathbf{ext}_{\boldsymbol{w}}(\boldsymbol{r}) \equiv \boldsymbol{w} \models_{\mathcal{I},f[\alpha^{t_i}/\boldsymbol{o}^{t_i}]_{i=1}^n} \varphi), \\ \quad \text{if there is one} \\ \text{undefined, otherwise} \end{cases}$$

where $\bar{\varepsilon}\boldsymbol{r}\boldsymbol{A} = \mathbf{C}(\boldsymbol{A})$ and $\mathbf{C}$ is the choice function of the interpretation.

D5. If τ is an 0-ary λ-expression of the form $[\lambda\, \varphi]$, then

$$\boldsymbol{d}_{\mathcal{I},f}(\tau) = \bar{\varepsilon}\boldsymbol{p}^{\langle\rangle} \forall \boldsymbol{w}(\mathbf{ex}_{\boldsymbol{w}}(\boldsymbol{p}) = \boldsymbol{T} \equiv \boldsymbol{w} \models_{\mathcal{I},f} \varphi)$$

where $\bar{\varepsilon}\boldsymbol{p}\boldsymbol{A} = \mathbf{C}(\boldsymbol{A})$ and $\mathbf{C}$ is the choice function of the interpretation.

D6. If τ is a term of type $\langle\rangle$, i.e., if τ is a formula φ, then:

- if φ is a formula in $Base^{\langle\rangle}$ $\boldsymbol{d}_{\mathcal{I},f}(\tau)$ is given by D1 – D3
- if φ is a formula of the form $[\lambda\, \varphi]$, then $\boldsymbol{d}_{\mathcal{I},f}(\tau)$ is given by D5
- if φ is a formula of any other form, then $\boldsymbol{d}_{\mathcal{I},f}(\tau) = \boldsymbol{d}_{\mathcal{I},f}([\lambda\, \varphi])$

Definitions of Truth, Logical Truth (Validity), and Logical Consequence

Now where $\mathcal{I}$ and f are given and $\boldsymbol{w}_0$ is the distinguished actual world of the domain of possible worlds $\mathbf{W}$ in $\mathcal{I}$, we say that φ is *true under* $\mathcal{I}$ and f ('true$_{\mathcal{I},f}$') if and only if under $\mathcal{I}$ and f, φ is *true at* $\boldsymbol{w}_0$. That is, using the formal notation $\models_{\mathcal{I},f} \varphi$ for the definiendum, we have:

$\models_{\mathcal{I},f} \varphi$ if and only if $\boldsymbol{w}_0 \models_{\mathcal{I},f} \varphi$

And we now say that φ is *true under* $\mathcal{I}$ just in case for every f, φ is true under $\mathcal{I}$ and f:

$\models_{\mathcal{I}} \varphi \;=_{df}\; \forall f(\models_{\mathcal{I},f} \varphi)$

Thus, if φ is not true under $\mathcal{I}$, then some assignment f is such that $\boldsymbol{w}_0 \not\models_{\mathcal{I},f} \varphi$ and we write $\not\models_{\mathcal{I}} \varphi$. We say that a formula φ is *false under* $\mathcal{I}$ if and only if no assignment function f is such that $\models_{\mathcal{I},f} \varphi$, i.e., iff no assignment function f is such that $\boldsymbol{w}_0 \models_{\mathcal{I},f} \varphi$. So open formulas may be neither true under $\mathcal{I}$ nor false under $\mathcal{I}$, whereas a sentence (i.e., a closed formula) will be either true under $\mathcal{I}$ or false under $\mathcal{I}$.

In the usual manner, we say that φ is *valid* or *logically true* if and only if φ is true under every interpretation $\mathcal{I}$, i.e.,

$\models \varphi \;=_{df}\; \forall \mathcal{I}(\models_{\mathcal{I}} \varphi)$

Clearly, given our previous definitions, it follows that:

> $\models \varphi$ if and only if for every $\mathcal{I}$ and f, $\models_{\mathcal{I},f} \varphi$, i.e.,
>
> $\models \varphi$ if and only if for every $\mathcal{I}$ and f, $\boldsymbol{w}_0 \models_{\mathcal{I},f} \varphi$

In what follows, when we say that a schema is valid, we mean that all of its instances are valid. Clearly, if a formula φ is not valid, then for some interpretation $\mathcal{I}$ and assignment f, $\boldsymbol{w}_0 \not\models_{\mathcal{I},f} \varphi$.

Finally, we conclude the definitions for a general interpretation with several more traditional definitions:

- φ is *satisfiable* if and only if there is some interpretation $\mathcal{I}$ and assignment f such that φ is true$_{\mathcal{I},f}$, i.e., iff $\exists \mathcal{I} \exists f (\models_{\mathcal{I},f} \varphi)$.
- φ *logically implies* ψ (or ψ is a *logical consequence* of φ) just in case, for every interpretation $\mathcal{I}$ and assignment f, if φ is true$_{\mathcal{I},f}$, then ψ is true$_{\mathcal{I},f}$:

  $$\varphi \models \psi \ =_{df} \ \forall \mathcal{I} \forall f (\models_{\mathcal{I},f} \varphi \rightarrow \models_{\mathcal{I},f} \psi)$$

- φ and ψ are *logically equivalent* just in case both $\varphi \models \psi$ and $\psi \models \varphi$
- φ is a *logical consequence* of a set of formulas Γ just in case, for every interpretation $\mathcal{I}$ and assignment f, if every member of Γ is true$_{\mathcal{I},f}$, then φ is true$_{\mathcal{I},f}$:

  $$\Gamma \models \varphi \ =_{df} \ \forall \mathcal{I} \forall f [\forall \psi (\psi \in \Gamma \rightarrow \models_{\mathcal{I},f} \psi) \rightarrow \models_{\mathcal{I},f} \varphi]$$