

Definitions

Operators and Terms

$\&$, \vee , \equiv , \exists , and \diamond are all defined in the usual way

$O! \equiv_{df} [\lambda x \diamond E!x]$ ('being ordinary')

$A! \equiv_{df} [\lambda x \neg \diamond E!x]$ ('being abstract')

Existence (\downarrow) (defined by cases)

$x \downarrow \equiv_{df} \exists F Fx$

$F^n \downarrow \equiv_{df} \exists x_1 \dots \exists x_n (x_1 \dots x_n F^n)$ ($n \geq 1$)

$p \downarrow \equiv_{df} [\lambda x p] \downarrow$

Identity ($=$) (defined by cases)

$x = y \equiv_{df} (O!x \& O!y \& \square \forall F (Fx \equiv Fy)) \vee (A!x \& A!y \& \square \forall F (xF \equiv yF))$

$F^1 = G^1 \equiv_{df} \square \forall x (xF^1 \equiv xG^1)$

$F^n = G^n \equiv_{df}$ (where $n > 1$)

$$\begin{aligned} \forall x_1 \dots \forall x_{n-1} ([\lambda y F^n y x_1 \dots x_{n-1}] = [\lambda y G^n y x_1 \dots x_{n-1}] \& \\ [\lambda y F^n x_1 y x_2 \dots x_{n-1}] = [\lambda y G^n x_1 y x_2 \dots x_{n-1}] \& \dots \& \\ [\lambda y F^n x_1 \dots x_{n-1} y] = [\lambda y G^n x_1 \dots x_{n-1} y]) \end{aligned}$$

$p = q \equiv_{df} [\lambda y p] = [\lambda y q]$

Axioms

A *closure* of a formula φ is the result of prefacing any string of quantifiers $\forall \alpha$, necessity operators \square , or actuality operators A to φ . We take, as axioms, the closures (modal, universal, actualizations) of all (the instances of) the following axioms (axiom schemata), with the exception of the axiom schema $\mathcal{A}\varphi \rightarrow \varphi$, which we take only the universal closures of the instances:

Axioms for Negations and Conditionals:

- $\varphi \rightarrow (\psi \rightarrow \varphi)$
- $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
- $(\neg \varphi \rightarrow \neg \psi) \rightarrow ((\neg \varphi \rightarrow \psi) \rightarrow \varphi)$

Axioms for Free Logic of Complex Terms:

- $\forall \alpha \varphi \rightarrow (\tau \downarrow \rightarrow \varphi_\alpha^\tau)$, provided τ is substitutable for α in φ
- $\tau \downarrow$, provided τ is primitive constant, a variable, or a λ -expression in which encoding conditions aren't directly built into the exemplification conditions of τ .¹
- $\forall \alpha (\varphi \rightarrow \psi) \rightarrow (\forall \alpha \varphi \rightarrow \forall \alpha \psi)$
- $\varphi \rightarrow \forall \alpha \varphi$, provided α doesn't occur free in φ
- $\Pi^n \kappa_1 \dots \kappa_n \rightarrow (\Pi^n \downarrow \& \kappa_1 \downarrow \& \dots \& \kappa_n \downarrow)$ ($n \geq 0$)
- $\kappa_1 \dots \kappa_n \Pi^n \rightarrow (\Pi^n \downarrow \& \kappa_1 \downarrow \& \dots \& \kappa_n \downarrow)$ ($n \geq 1$)

¹Formally, this can be defined as: $\tau \downarrow$ if τ is $[\lambda x_1 \dots x_n \varphi]$ and none of the x_i bound by the λ occur as a *primary* term in an encoding formula *subterm* of the matrix φ . For the definitions of *primary* term and *subterm*, see item (7) of *Principia Logico-Metaphysica*, at <http://mally.stanford.edu/principia.pdf>.

Axioms for the Substitution of Identicals:

- $\alpha = \beta \rightarrow (\varphi \rightarrow \varphi')$, whenever β is substitutable for α in φ , and φ' is the result of replacing zero or more free occurrences of α in φ with occurrences of β

Axioms for Actuality:

- $\mathcal{A}\varphi \rightarrow \varphi$ (only universal closures)
- $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$
- $\mathcal{A}(\varphi \rightarrow \psi) \equiv (\mathcal{A}\varphi \rightarrow \mathcal{A}\psi)$
- $\mathcal{A}\forall\alpha\varphi \equiv \forall\alpha\mathcal{A}\varphi$
- $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$

Axioms for Necessity:

- $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- $\Box\varphi \rightarrow \varphi$
- $\Diamond\varphi \rightarrow \Box\Diamond\varphi$
- $\Diamond\exists x(E!x \ \& \ \neg\mathcal{A}E!x)$

Axioms for Necessity and Actuality:

- $\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi$
- $\Box\varphi \equiv \mathcal{A}\Box\varphi$

Axioms for Definite Descriptions:

- $x = ix\varphi \equiv \forall z(\mathcal{A}\varphi_x^z \equiv z = x)$, provided z is substitutable for x in φ and doesn't occur free in φ

Axioms for Relations (λ -Calculus for Relations):

- $[\lambda v_1 \dots v_n \varphi] \downarrow \rightarrow [\lambda v_1 \dots v_n \varphi] = [\lambda v_1 \dots v_n \varphi]'$ ($n \geq 0$)
($[\lambda v_1 \dots v_n \varphi]'$ an alphabetic variant)
- $[\lambda x_1 \dots x_n \varphi] \downarrow \rightarrow ([\lambda x_1 \dots x_n \varphi]x_1 \dots x_n \equiv \varphi)$ ($n \geq 1$)
- $[\lambda x_1 \dots x_n F^n x_1 \dots x_n] = F^n$ ($n \geq 0$)
- $([\lambda x_1 \dots x_n \varphi] \downarrow \ \& \ \Box \forall x_1 \dots \forall x_n (\varphi \equiv \psi)) \rightarrow [\lambda x_1 \dots x_n \psi] \downarrow$ ($n \geq 1$)

Axioms for Encoding:

- $x_1 \dots x_n F^n \equiv$
 $x_1[\lambda y F^n y x_2 \dots x_n] \ \& \ x_2[\lambda y F^n x_1 y x_3 \dots x_n] \ \& \ \dots \ \& \ x_n[\lambda y F^n x_1 \dots x_{n-1} y]$
- $xF \rightarrow \Box xF$
- $O!x \rightarrow \neg \exists F xF$
- $\exists x(A!x \ \& \ \forall F(xF \equiv \varphi))$, provided x doesn't occur free in φ

Deductive Systems

Rule of Inference: Modus Ponens

Derivations and Theoremhood:

- There are two derivability systems: \vdash and \vdash_{\Box} .
- $\Gamma \vdash \varphi$ (derivations) and $\vdash \varphi$ (theorems) defined in the usual way: these are derivations (theorems) from inferred from *any* axioms.
- $\Gamma \vdash_{\Box} \varphi$ (*modally strict* derivations) and $\vdash_{\Box} \varphi$ (*modally strict* theorems): these are derivations (theorems) that don't depend on the axiom $\mathcal{A}\varphi \rightarrow \varphi$.
 - $\mathcal{A}\varphi \rightarrow \varphi$ is a 'modally fragile' axiom and can't be necessitated.
 - We mark non-modally strict derivations and theorems with a \star .
 - The system is therefore set-up for additional axioms whose necessitations aren't asserted.
- Derived Metarules GEN:
 - If $\Gamma \vdash \varphi$ and α doesn't occur free in any formula in Γ , then $\Gamma \vdash \forall \alpha \varphi$.
 - If $\Gamma \vdash_{\Box} \varphi$ and α doesn't occur free in any formula in Γ , then $\Gamma \vdash_{\Box} \forall \alpha \varphi$.
- Derived Metarules RN, where $\Box\Gamma$ is $\{\Box\psi \mid \psi \in \Gamma\}$:
 - If $\Gamma \vdash_{\Box} \varphi$, then $\Box\Gamma \vdash_{\Box} \Box\varphi$
 - If $\Gamma \vdash \varphi$, then $\Box\Gamma \vdash \Box\varphi$
- Derived Metarules RA, where $\mathcal{A}\Gamma$ is $\{\mathcal{A}\psi \mid \psi \in \Gamma\}$:
 - If $\Gamma \vdash \varphi$, then $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$.
 - If $\Gamma \vdash_{\Box} \varphi$, then $\mathcal{A}\Gamma \vdash_{\Box} \mathcal{A}\varphi$.

See <http://mally.stanford.edu/principia.pdf>.

Some Distinctive Theorems Governing Existence and Identity

The principles (theorems) of classical propositional logic and the principles of free predicate logic are all preserved. But the following \vdash_{\Box} theorems governing existence and identity are distinctive – the numbers refer to the numbered items the May 5, 2022 version of *Principia Logico-Metaphysica*, at URL in red noted above.

- | | | |
|---------|--|---|
| (105.2) | $\varphi \downarrow$ | (for any formula φ) |
| (107) | $\tau \downarrow \rightarrow \Box \tau \downarrow$ | (logical existence implies necessary logical existence) |
| (108.1) | $\tau = \sigma \rightarrow \tau \downarrow$ | |
| (108.2) | $\tau = \sigma \rightarrow \sigma \downarrow$ | |
| (112.2) | $[\lambda \varphi] \equiv \varphi$ | ("that- φ is true iff φ ") |
| (118.1) | $\alpha = \alpha$ | |
| (118.2) | $\alpha = \beta \rightarrow \beta = \alpha$ | |
| (118.3) | $(\alpha = \beta \ \& \ \beta = \gamma) \rightarrow \alpha = \gamma$ | |
| (122.1) | $\tau \downarrow \equiv \exists \beta (\beta = \tau)$ | (provided that β doesn't occur free in τ) |
| (126.1) | $\alpha = \beta \rightarrow \Box \alpha = \beta$ | (necessity of identity) |

Typed Object Theory

(Latest unpublished version)

Language

Types:

- i is a type.
- If t_1, \dots, t_n are any types ($n \geq 0$), $\langle t_1, \dots, t_n \rangle$ is a type.

BNF:

δ^t primitive constants of type t
 α^t variables of type t
 τ^t terms of type t
 φ formulas

$$\begin{aligned}
 \delta^t &::= a_1^t, a_2^t, \dots \quad (E!^{\langle t \rangle} \text{ a distinguished constant, for every } t) \\
 \alpha^t &::= x_1^t, x_2^t, \dots \\
 \text{Base}^t &::= \delta^t \mid \alpha^t \mid i\alpha^t\varphi \\
 \tau^i &::= \text{Base}^i \\
 (n \geq 1) \tau^{\langle t_1, \dots, t_n \rangle} &::= \text{Base}^{\langle t_1, \dots, t_n \rangle} \mid [\lambda\alpha^{t_1} \dots \alpha^{t_n} \varphi] \quad (\alpha^{t_1} \dots \alpha^{t_n} \text{ pairwise distinct}) \\
 \varphi &::= \text{Base}^{\langle \rangle} \mid \tau^{\langle t_1, \dots, t_n \rangle} \tau^{t_1} \dots \tau^{t_n} \quad (n \geq 1) \mid \tau^{t_1} \dots \tau^{t_n} \tau^{\langle t_1, \dots, t_n \rangle} \quad (n \geq 1) \mid \\
 &\quad [\lambda\varphi] \mid (\neg\varphi) \mid (\varphi \rightarrow \varphi) \mid \forall\alpha^t\varphi \mid (\Box\varphi) \mid (\mathcal{A}\varphi) \\
 \tau^{\langle \rangle} &::= \varphi
 \end{aligned}$$

Definitions

- (.1) $\varphi \& \psi \equiv_{df} \neg(\varphi \rightarrow \neg\psi)$
- (.2) $\varphi \vee \psi \equiv_{df} \neg\varphi \rightarrow \psi$
- (.3) $\varphi \equiv \psi \equiv_{df} (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$
- (.4) $\exists x\varphi \equiv_{df} \neg\forall x\neg\varphi$ x any type
- (.5) $\diamond\varphi \equiv_{df} \neg\Box\neg\varphi$
- (.6.a) $x\downarrow \equiv_{df} \exists F Fx$ x has type i
- (.6.b) $p\downarrow \equiv_{df} \exists F Fp$ p has type $\langle \rangle$
- (.6.c) $F\downarrow \equiv_{df} \exists x_1 \dots \exists x_n (x_1 \dots x_n F)$ F has type t_1, \dots, t_n ($n \geq 1$)
- (.7) $O! \equiv_{df} [\lambda x \diamond E!x]$ x has any type
- (.8) $A! \equiv_{df} [\lambda x \neg\diamond E!x]$ x has any type
- (.9) $x = y \equiv_{df} (O!x \& O!y \& \Box\forall F(Fx \equiv Fy)) \vee (A!x \& A!y \& \Box\forall F(xF \equiv yF))$ x, y have type i
- (.10) $F = G \equiv_{df} (O!F \& O!G \& \Box\forall x(xF \equiv xG)) \vee (A!F \& A!G \& \Box\forall \mathcal{H}(F\mathcal{H} \equiv G\mathcal{H}))$ F, G have type $\langle t \rangle$
- (.11) $F = G \equiv_{df}$ F, G have type $\langle t_1, \dots, t_n \rangle$
- $$\begin{aligned}
 &O!F \& O!G \& \forall x_2 \dots \forall x_n ([\lambda x_1 Fx_1 \dots x_n] = [\lambda x_1 Gx_1 \dots x_n]) \& \\
 &\forall x_1 \forall x_3 \dots \forall x_n ([\lambda x_2 Fx_1 \dots x_n] = [\lambda x_2 Gx_1 \dots x_n]) \& \dots \& \\
 &\forall x_1 \dots \forall x_{n-1} ([\lambda x_n Fx_1 \dots x_n] = [\lambda x_n Gx_1 \dots x_n]) \vee \\
 &A!F \& A!G \& \Box\forall \mathcal{H}(F\mathcal{H} \equiv G\mathcal{H})
 \end{aligned}$$
- (.12) $p = q \equiv_{df} (O!p \& O!q \& [\lambda x p] = [\lambda x q]) \vee (A!p \& A!q \& \Box\forall \mathcal{H}(p\mathcal{H} \equiv q\mathcal{H}))$ p, q have type $\langle \rangle$

Axioms

Negations and Conditionals.

- (.1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
- (.2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
- (.3) $(\neg\varphi \rightarrow \neg\psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \varphi)$

Quantification and Logical Existence.

- (.4) $\forall x\varphi \rightarrow (\tau\downarrow \rightarrow \varphi_x^\tau)$, provided τ is substitutable for x in φ x, τ have type t
- (.5) $\tau\downarrow$, whenever τ is either a primitive constant, a variable, or a core λ -expression
- (.6) $\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)$ x any type
- (.7) $\varphi \rightarrow \forall x\varphi$, provided x doesn't occur free in φ x any type
- (.8) (a) $\Pi\tau_1 \dots \tau_n \rightarrow (\Pi\downarrow \& \tau_1\downarrow \& \dots \& \tau_n\downarrow)$ $(n \geq 0)$
(b) $\tau_1 \dots \tau_n \Pi \rightarrow (\Pi\downarrow \& \tau_1\downarrow \& \dots \& \tau_n\downarrow)$ $(n \geq 1)$

Substitution of Identicals.

- (.9) $x=y \rightarrow (\varphi \rightarrow \varphi')$ x, y have type t

★Actuality (only universal closures).

- (.10) $\mathcal{A}\varphi \rightarrow \varphi$

Actuality (all closures).

- (.11) $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$
- (.12) $\mathcal{A}(\varphi \rightarrow \psi) \equiv (\mathcal{A}\varphi \rightarrow \mathcal{A}\psi)$
- (.13) $\mathcal{A}\forall x\varphi \equiv \forall x\mathcal{A}\varphi$ x any type
- (.14) $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$

Necessity.

- (.15) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
- (.16) $\Box\varphi \rightarrow \varphi$
- (.17) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$
- (.18) $\Diamond\exists x(E!x \& \neg\mathcal{A}E!x)$ x has type i and $E!$ has type $\langle i \rangle$

Necessity and Actuality.

- (.19) $\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi$
- (.20) $\Box\varphi \equiv \mathcal{A}\Box\varphi$

Descriptions.

- (.21) $x = ix\varphi \equiv \forall y(\mathcal{A}\varphi_x^y \equiv y = x)$ x, y have type t

Relations.

$$(.22) [\lambda x_1 \dots x_n \varphi] \downarrow \rightarrow O![\lambda x_1 \dots x_n \varphi] \quad x_1, \dots, x_n \text{ have types } t_1, \dots, t_n, O! \text{ has type } \langle \langle t_1, \dots, t_n \rangle \rangle$$

$$(.23) O!\varphi, \text{ provided } \varphi \text{ is not in } Base^{\langle \rangle}, \text{ i.e., provided } \varphi \text{ is not a constant of type } \langle \rangle, \text{ a variable of type } \langle \rangle, \text{ or a description of type } \langle \rangle$$

$$(.24) A!F \rightarrow \neg \exists x_1 \dots \exists x_n F x_1 \dots x_n \quad x_1, \dots, x_n \text{ have types } t_1, \dots, t_n$$

$$(.25) [\lambda x_1 \dots x_n \varphi] \downarrow \rightarrow [\lambda x_1 \dots x_n \varphi] = [\lambda x_1 \dots x_n \varphi]' \quad (\alpha\text{-Conversion})$$

$$(.26) [\lambda x_1 \dots x_n \varphi] \downarrow \rightarrow ([\lambda x_1 \dots x_n \varphi] x_1 \dots x_n \equiv \varphi) \quad (\beta\text{-Conversion})$$

$$(.27) O!F \rightarrow ([\lambda x_1 \dots x_n F x_1 \dots x_n] = F) \quad (\eta\text{-Conversion})$$

$$(.28) ([\lambda x_1 \dots x_n \varphi] \downarrow \& \Box \forall x_1 \dots \forall x_n (\varphi \equiv \psi)) \rightarrow [\lambda x_1 \dots x_n \psi] \downarrow \quad n \geq 1, x_1, \dots, x_n \text{ any types,}$$

Encoding.

$$(.29) x_1 \dots x_n F \equiv x_1 [\lambda y_1 F y_1 x_2 \dots x_n] \& x_2 [\lambda y_2 F x_1 y_2 x_3 \dots x_n] \& \dots \& x_n [\lambda y_n F x_1 \dots x_{n-1} y_n] \quad (x_i, y_i \text{ have the same type})$$

$$(.30) xF \rightarrow \Box xF \quad x \text{ any type}$$

$$(.31) O!x \rightarrow \neg \exists F xF \quad x \text{ any type}$$

$$(.32) \exists x (A!x \& \forall F (xF \equiv \varphi)), \text{ where } \varphi \text{ has no free } x \text{ s} \quad x \text{ any type}$$